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PROBABILITY LAWS WITH 1-STABLE MARGINALS ARE 1-STABLE[1]

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We show that if $\mathbf{X} = (X_1, \dots, X_d)$ is a vector in $\mathbb{R}^d$ and all linear combinations $\sum_{i=1}^d C_i X_i$ are 1-stable random variables, then $\mathbf{X}$ is itself 1-stable. More generally, a probability measure μ on a vector space whose univariate marginals are 1-stable is itself 1-stable. This settles an outstanding problem of Dudley and Kanter.

1. Introduction. A probability measure Q on $\mathbb{R}^d$ is stable if for any $A > 0$, $B > 0$, there is a $C > 0$ and a $\mathbf{D} \in \mathbb{R}^d$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} =_d C\mathbf{X} + \mathbf{D}, \tag{1.1}$$

where $\mathbf{X}, \mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots$ is a sequence of i.i.d. random vectors with distribution Q. Equivalently, Q is stable if for any $n \geq 1$, there is an $a_n > 0$ and a $\mathbf{b}_n \in \mathbb{R}^d$ such that

$$\mathbf{X} =_d a_n^{-1}(\mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)}) + \mathbf{b}_n. \tag{1.2}$$

In fact, $a_n = n^{1/\alpha}$ for some $0 < \alpha \leq 2$. The probability law Q satisfying (1.2) with $a_n = n^{1/\alpha}$ is called α-stable. Obviously, the α-stability of Q implies that for any vector $\mathbf{C} \in \mathbb{R}^d$, the scalar product $(\mathbf{C}, \mathbf{X})$ of $\mathbf{C}$ and $\mathbf{X}$ is α-stable as well. For some time it was believed that this is an "if and only if" statement, that is, α-stability of the real-valued random variable $Y(\mathbf{C}) = (\mathbf{C}, \mathbf{X})$ for every $\mathbf{C} \in \mathbb{R}^d$ implies that Q is an α-stable probability law on $\mathbb{R}^d$. In fact, Dudley and Kanter (1974) stated this as a theorem, but then de Acosta and Kuelbs noticed that their argument works, in general, only when $\alpha > 1$. A student of Dudley's, Marcus (1983) has constructed a counterexample (with $d = 2$) showing that in the case $0 < \alpha < 1$, the α-stability of $Y(\mathbf{C})$ for every $\mathbf{C} \in \mathbb{R}^d$ does not imply the α-stability of Q. Marcus' proof was simplified by Samotij and Żak (1989).

It is the purpose of this note to settle the long outstanding remaining case, $\alpha = 1$.

In Section 2 we prove that if $Y(\mathbf{C})$ is a 1-stable real-valued random variable for every $\mathbf{C} \in \mathbb{R}^d$, then Q is 1-stable. In Section 3 we generalize this result to the general setting of measurable vector spaces of Dudley and Kanter.

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We close this section by recalling that a real-valued 1-stable random variable Y has characteristic function of the form

$$Ee^{i\theta Y} = \exp\{-\sigma|\theta|(1 + i(2/\pi)\beta\,\mathrm{sgn}(\theta)\ln|\theta|) + i\mu\theta\} \tag{1.3}$$

for some $\sigma \geq 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$. We use the notation $Y \sim S_1(\sigma, \beta, \mu)$.

2. The case of $\mathbb{R}^d$.

THEOREM 2.1. *Let Q be a probability measure on $\mathbb{R}^d$ and let $\mathbf{X} = (X_1, \dots, X_d)$ be a random vector with the distribution Q. If $Y(\mathbf{C}) = (\mathbf{C}, \mathbf{X})$ is 1-stable for every $\mathbf{C} \in \mathbb{R}^d$, then Q is 1-stable.*

PROOF. By assumption, for every vector $\mathbf{C} \in \mathbb{R}^d$,

$$Y(\mathbf{C}) = (\mathbf{C}, \mathbf{X}) \sim S_1(\sigma(\mathbf{C}), \beta(\mathbf{C}), \mu(\mathbf{C})) \tag{2.1}$$

for some $\sigma(\mathbf{C}) \geq 0$, $\beta(\mathbf{C}) \in [-1, 1]$ and $\mu(\mathbf{C}) \in \mathbb{R}$. If σ_i, β_i and μ_i denote, respectively, $\sigma(\mathbf{C})$, $\beta(\mathbf{C})$ and $\mu(\mathbf{C})$, corresponding to the special vector $\mathbf{C} = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the ith position, then

$$X_i \sim S_1(\sigma_i, \beta_i, \mu_i), \qquad i = 1, \dots, d.$$

Let $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots$ be i.i.d. copies of $\mathbf{X}$, and define for $n \geq 1$,

$$\mathbf{S}^{(n)} = \left(\frac{1}{n}\sum_{j=1}^{n} \mathbf{X}^{(j)}\right) - \frac{2}{\pi}(\ln n)(\boldsymbol{\sigma} \times \boldsymbol{\beta}), \tag{2.2}$$

where $\boldsymbol{\sigma} \times \boldsymbol{\beta} = (\sigma_1\beta_1, \sigma_2\beta_2, \dots, \sigma_d\beta_d)$. Since $X_i \sim S_1(\sigma_i, \beta_i, \mu_i)$, we have $\sum_{j=1}^{n} X_i^{(j)} \sim S_1(n\sigma_i, \beta_i, n\mu_i)$, $(1/n)\sum_{j=1}^{n} X_i^{(j)} \sim S_1(\sigma_i, \beta_i, \mu_i + (2/\pi)(\ln n)\sigma_i\beta_i)$ and therefore the ith component of the vector $\mathbf{S}^{(n)}$ satisfies $S_i^{(n)} =_d X_i \sim S_1(\sigma_i, \beta_i, \mu_i)$. This means that the sequence $\mathbf{S}^{(n)}$, $n \geq 1$, has marginal distributions that do not depend on n. Therefore the sequence $\mathbf{S}^{(n)}$, $n \geq 1$, is tight, and as such, it has a subsequence $\mathbf{S}^{(n_k)}$, $k \geq 1$, converging weakly to a probability measure on $\mathbb{R}^d$ [Theorem 29.3 of Billingsley (1986).] In particular, $(\mathbf{C}, \mathbf{S}^{(n_k)})$ converges weakly for every vector $\mathbf{C} = (C_1, \dots, C_n)$ in $\mathbb{R}^d$. But (2.1) and (2.2) imply

$$(\mathbf{C}, \mathbf{S}^{(n)}) \sim S_1\left(\sigma(\mathbf{C}), \beta(\mathbf{C}), \mu(\mathbf{C}) + (2/\pi)(\ln n) \times \left(\sigma(\mathbf{C})\beta(\mathbf{C}) - \sum_{i=1}^{d} C_i\sigma_i\beta_i\right)\right). \tag{2.3}$$

In order for $(\mathbf{C}, \mathbf{S}^{(n_k)})$ to converge weakly, the coefficient of $\ln n$ must be zero, that is,

$$\sigma(\mathbf{C})\beta(\mathbf{C}) = \sum_{i=1}^{d} C_i\sigma_i\beta_i \quad \text{for every } \mathbf{C} \in \mathbb{R}^d. \tag{2.4}$$

Now (2.1), (2.3) and (2.4) imply $(\mathbf{C}, \mathbf{S}^{(n)}) =_d (\mathbf{C}, \mathbf{X})$ for every $\mathbf{C} \in \mathbb{R}^d$, every $n \geq 1$. Therefore, $\mathbf{X} =_d \mathbf{S}^{(n)}$ for every $n \geq 1$. Thus, $\mathbf{X}$ satisfies (1.2) with $a_n = n$ and so Q is 1-stable. □

REMARK. A similar proof works in the case $1 < \alpha \leq 2$. Replacing the denominator n in (2.2) by $n^{1/\alpha}$, the term $(2/\pi)(\ln n)(\boldsymbol{\sigma} \times \boldsymbol{\beta})$ by $n^{1-1/\alpha}\boldsymbol{\mu} - \boldsymbol{\mu}$, we get

$$(\mathbf{C}, \mathbf{S}^{(n)}) \sim S_\alpha\left(\sigma(\mathbf{C}), \beta(\mathbf{C}), n^{1-1/\alpha}\left(\mu(\mathbf{C}) - \sum_{i=1}^{d} C_i\mu_i\right) + \sum_{i=1}^{d} C_i\mu_i\right).$$

The result follows from the fact that $n^{1-1/\alpha} \to \infty$ as $n \to \infty$.

3. Infinite-dimensional spaces. Let S be a real vector space and F a vector space of linear forms on S. Let $\mathscr{S}(F)$ be the σ-algebra generated by F on S. The definition of a stable probability measure on $(S, \mathscr{S}(F))$ is completely analogous to the definition (1.1) in $\mathbb{R}^d$.

Choose a topology τ on F which makes (F, τ) a topological vector space such that a linear form ϕ on F is τ-continuous if and only if for every $f \in F$, $\phi(f) = f(s)$ for some $s \in S$. A pair $(\mathscr{S}, F)$ is called *semifull* if every sequentially τ-continuous linear form on F is of the form $\phi(f) = f(s)$ for some $s \in S$. For an example of a semifull pair, let S be a separable Banach space and $F = S'$ the dual space of S with the weak-star topology. We refer the reader to Dudley and Kanter (1974) and Giné and Hahn (1983) for more details and examples of semifull pairs.

We will use the following consequence of Theorem 2 of Giné and Hahn (1983).

PROPOSITION 3.1 (Giné and Hahn). *If a law on $\mathbb{R}^d$ has all of its two-dimensional projections infinitely divisible and has all its one-dimensional marginals α-stable, then the law must be α-stable.*

The next theorem extends Theorem 2.1 to infinite-dimensional spaces.

THEOREM 3.1. *Let (S, F) be a semifull pair, and let μ be a probability measure on $(S, \mathscr{S}(F))$. If $\mu \circ f^{-1}$ is 1-stable for all $f \in F$, then μ is 1-stable.*

PROOF. By assumption, all one-dimensional marginals $\mu \circ f^{-1}$, $f \in F$, are 1-stable. It also follows from Theorem 2.1 with $d = 2$, that all two-dimensional marginals $\mu \circ (f, g)^{-1}$, $f, g \in F$, are stable and thus, infinitely divisible. Applying Proposition 3.1 we conclude that μ is 1-stable. □

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