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ASYMPTOTIC APPROXIMATIONS FOR BROWNIAN MOTION BOUNDARY HITTING TIMES

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The problem of approximating boundary hitting times for diffusion processes, and in particular Brownian motion, is considered. Using a combination of probabilistic and function-analytic techniques, approximations for conditioned diffusion distributions are obtained. These lead to approximations for the distribution of the hitting time itself. The approximations are split into three cases depending on whether the boundary is upper case, approximation square root or lower case, and one-sided boundaries are also considered separately.

1. Introduction. The calculation of the distribution of time-dependent boundary hitting times for Brownian motion has been found to be intractable, the simplest of problems leading to complex partial differential equations, and solutions at best being given in terms of implicit eigenfunctions. However, very often we are merely interested in studying the asymptotic behaviour of such hitting times.

In this article we attempt to describe the asymptotic properties of certain classes of these boundary hitting times. The approach is to consider these different types of boundary: approximate square-root boundaries [i.e., boundaries of the form $f(t) = t^{1/2}a(t)$, where $a(t)$ has a positive limit as $t \to \infty$], lower-case boundaries [i.e., boundaries of the form $f(t) = t^{1/2}a(t)$, where $a(t)\downarrow 0$ as $t \to \infty$] and upper-case boundaries [$f(t) = t^{1/2}a(t)$, where $a(t)\uparrow\infty$ as $t \to \infty$]. We also treat one-sided boundaries and two-sided boundaries separately.

The existence of power moments for Brownian motion hitting times of exact square-root boundaries was first considered by Breiman (1967) and Shepp (1971). They established separately the following result. Let

$$\tau = \inf\{t \geq 1; |B_t| \geq ct^{1/2}\}, \tag{1.1}$$

where B_t is Brownian motion, then

$$\mathbb{E}[\tau^p] < \infty \Leftrightarrow c < c(p), \tag{1.2}$$

where $c(p)$ is the smallest positive root of the pth confluent hypergeometric

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function:

$$F_p(x) = \sum_{m=0}^{\infty} \frac{(-2x^2)^m p(p-1)\cdots(p-m+1)}{(2m)!}. \tag{1.3}$$

More recently, Taksar (1982) considered the approximate square-root boundary case, and showed that if

$$\tau = \inf\{t \geq 1; |B_t| \geq a(t)t^{1/2}\},$$

where $a(t) \uparrow c(p)$ as $t \to \infty$, then

$$\mathbb{E}[\tau^p] < \infty \quad \text{if} \int_1^{\infty} t^{p-1-m(a(t))}\, dt < \infty,$$

where $m(\cdot)$ is the inverse function of $c(\cdot)$. Taksar's methods involve approximating the boundary at each time point by a boundary for which the problem becomes more tractable. This characterises modern methodology in this area. In particular, Strassen's tangent approximation techniques involve the approximation of the density of the hitting time by the density of a straight-line boundary hitting time where the straight line is tangent to f at some point. Tangent approximations can be used to approximate asymptotic properties of both upper- and lower-case boundary hitting times; see Bass and Cranston (1983) and Lerche (1986) for an excellent account of these methods.

However, we advocate a different approach. Instead of approximating the boundary, we approximate the distribution of Brownian motion conditioned not to hit the boundary. It turns out that this method can lead to powerful techniques of approximation for boundary hitting times which are applicable for a large class of diffusions, as well as Brownian motion.

Most of the results to transform distributional estimates for the conditioned process to approximations for the distribution of the hitting time are proved in Section 2. Stochastic inequalities for the conditioned Brownian motion are derived by coupling-type arguments, and the existence of limit distributions for time-homogeneous systems is established. This leads to a proof of the asymptotic exponential decay rate of hitting times of constant boundaries for time-homogeneous stochastically monotone Markov processes. It is interesting to compare these proofs with their associated derivations by the more function-analytic methods of Jacka and Roberts (1987) and Pinsky (1985), and results from Jacka and Roberts (1987) are also used extensively in this article.

The nature of the problem makes it necessary to consider different classes of boundaries separately despite the similarities in the techniques used. Therefore Sections 3, 4 and 5 consider the three cases: approximate square-root boundaries, lower-case boundaries and upper-case boundaries respectively, and in Section 6, we set up an analogous theory to the function-analytic setup of Jacka and Roberts (1987) in the context of a semi-infinite domain. Then one-sided boundary hitting time approximations are given for certain classes of these boundaries.

2. Notation and preliminaries.

2.1. *Stochastic approximation results.* Let B_t be a standard Brownian motion, and let X_t be the associated Ornstein–Uhlenbeck process defined by

$$X_t = \frac{B(e^t)}{e^{t/2}}. \tag{2.1.1}$$

Then X_t satisfies

$$dX_t = dB'_t - \tfrac{1}{2}X_t\,dt, \tag{2.1.2}$$

where B' is a Brownian motion. Define

$$_c\tau = \inf\{t \ge 1;\ |B_t| \ge ct^{1/2}\} \tag{2.1.3}$$

and

$$\tau_c = \inf\{t \ge 0;\ |X_t| \ge c\}; \tag{2.1.4}$$

then $\tau_c = \log_c \tau$.

More generally, suppose $f(\cdot)$ is a positive function; then

$$_f\tau = \inf\{t \ge 1;\ |B_t| \ge f(t)\}, \tag{2.1.5}$$

$$\tau_f = \inf\{t \ge 0;\ |X_t| \ge f(e^t)e^{-t/2}\}, \tag{2.1.6}$$

and we will write $f(t) = a(t)t^{1/2}$ for the approximate square-root case. So, in particular, $\tau_{a(t_1)}$ will denote the hitting time of the exact square-root boundary, $f(t) = a(t_1)t^{1/2}$.

Suppose μ is a probability distribution with support contained in $(-c, c)$. Then we define

$$\begin{aligned} \mu_t(x) &= \mathbb{P}[X_t \le x \mid \tau_f > t,\ X_0 \text{ has law } \mu] \\ &= \int \mathbb{P}[x_t \le x \mid \tau_f > t,\ X_0 = y]\ d\mu(y). \end{aligned} \tag{2.1.7}$$

and

$$\mu_t(x) = \int_{-c}^{x} d\mu_t(y). \tag{2.1.8}$$

Also denote by $\bar{\mu}$ the modulus law of μ,

$$\bar{\mu}(x) = \int_{-x}^{x} d\mu(y). \tag{2.1.9}$$

We will make extensive use of stochastic order relations and write

$$\mu_1 \overset{st.}{\le} \mu_2 \quad \text{if and only if } \mu_1(x) \ge \mu_2(x), \qquad \forall\, x \in \mathbb{R}.$$

We then say μ_1 is stochastically less than μ_2.

The correspondence between Brownian motion and the Ornstein–Uhlenbeck process is fundamental to the main results of this article. However, the preliminary results in this section are true in greater generality than just for the Ornstein–Uhlenbeck process. So for the rest of this section, we will assume X is a time-homogeneous symmetric diffusion process,

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,$$

where $\sigma(x) = \sigma(-x)$ and $b(x) = -b(-x)$. We also impose regularity conditions on σ and b to ensure that the solution to the S.D.E. is unique in law. We assume therefore that σ and b are bounded and measurable; see, for example, Stroock and Varadhan (1969). Also σ and b are such that the scale function of X is bounded on bounded intervals, and σ is bounded above and below by positive constants (at least on bounded intervals). The symmetry of the problem allows us to look at $|X|$ and X interchangeably and most of the following results are stochastic inequalities for $|X|$.

LEMMA 2.1.1. *Let μ_1, μ_2 be distributions on $(-f(1), f(1))$ for some positive function $f\colon [1,\infty) \to \mathbb{R}^+ \cup \{\infty\}$. Suppose $\bar{\mu}_1 \overset{st.}{\leq} \bar{\mu}_2$; then $(\bar{\mu}_1)_t \overset{st.}{\leq} (\bar{\mu}_2)_t$.*

PROOF. Let

$$p_{y,t}(z) = \mathbb{P}\big[|X_t| < y | \tau_f > t,\, X_0 = z\big]. \tag{2.1.10}$$

Then since $\bar{\mu}_i$ has support $[0, f(1))$,

$$\begin{aligned}(\bar{\mu}_i)_t(y) &= \lim_{c \uparrow f(1)} \int_0^c p_{y,t}(z)\, d\bar{\mu}_i(z) \\ &= \lim_{c \uparrow f(1)} \left[p_{y,t}(c)\bar{\mu}_i(c) - \int_0^c \bar{\mu}_i(x)\, dp_{y,t}(x) \right]\end{aligned} \tag{2.1.11}$$

for $i = 1, 2$. Now $\bar{\mu}_1 \overset{st.}{\leq} \bar{\mu}_2$, so

$$\bar{\mu}_1(x) - \bar{\mu}_2(x) \geq 0, \tag{2.1.12}$$

$$\begin{aligned}(\bar{\mu}_1)_t(y) - (\bar{\mu}_2)_t(y) = \lim_{c \uparrow f(1)} \Big[& p_{y,t}(x)(\bar{\mu}_1(c) - \bar{\mu}_2(c)) \\ & + \int_0^c (\bar{\mu}_2(x) - \bar{\mu}_1(x))\, dp_{y,t}(x) \Big],\end{aligned} \tag{2.1.13}$$

and the first term on the right-hand side is clearly 0.

So it remains to show that $p_{y,t}(z)$ is a decreasing function of z. To prove this, we need to show that the conditioned process Y_s, defined by

$$Y_s = \big[X_s | \tau_f > t\big],$$

satisfies the strong Markov property, and has almost surely continuous sample paths. Then we can use a pathwise argument on the process started at two different points to give the result.

The strong Markov property follows easily from that of the parent process; see, for example, Karlin and Taylor (1981), page 261.

Similarly, the almost sure continuity of the sample paths of Y follows from that of X, since

$$(2.1.14)\qquad \begin{aligned} &\mathbb{P}[Y \text{ is discontinuous on } [0,s]] \\ &\quad = \mathbb{P}\big[X \text{ is discontinuous on } [0,s] | \tau_f > t\big] \\ &\quad \le \frac{\mathbb{P}[X \text{ is discontinuous on } [0,s]]}{\mathbb{P}\big[\tau_f > t\big]}. \end{aligned}$$

Now $\mathbb{P}[\tau_f > t] > 0$ for all t since f is strictly nonzero. So Y_s is an a.s. continuous function of time.

Now consider two processes Y^{z_1}, Y^{z_2} started at z_1, z_2 respectively, with $0 \le z_1 \le z_2 \le f(0)$, and let

$$(2.1.15)\qquad \tau^* = \inf\{s;\, |Y_s^{z_1}| = |Y_s^{z_2}|\}.$$

Then

$$(2.1.16)\qquad \mathbb{P}\big[|Y_t^{z_1}| > |Y_t^{z_2}| \,|\tau^* \ge t\big] = 0$$

(by a.s. continuity of the sample paths), and so

$$(2.1.17)\qquad \mathbb{P}\big[|Y_t^{z_1}| < x|\tau^* \ge t\big] \ge \mathbb{P}\big[|Y_t^{z_2}| < x|\tau^* \ge t\big].$$

Also

$$(2.1.18)\qquad \begin{aligned} p_{y,t}(z_i) &= \mathbb{P}\big[|Y_t^{z_i}| < y|\tau^* < t\big]\mathbb{P}[\tau^* < t] \\ &\quad + \mathbb{P}\big[|Y_t^{z_i}| < t|\tau^* \ge t\big]\mathbb{P}[\tau^* \ge t], \qquad i=1,2. \end{aligned}$$

Now

$$(2.1.19)\qquad \mathbb{P}\big[|Y_t^{z_1}| < y|\tau^* < t, Y_{\tau^*}, \tau^*\big] = \mathbb{P}\big[|Y_t^{z_2}| < y|\tau^* < t, Y_{\tau^*}, \tau^*\big]$$

by the strong Markov property. So conditioning on the values of τ^*, Y_{τ^*},

$$(2.1.20)\qquad \mathbb{P}\big[|Y_t^{z_1}| < y|\tau^* < t\big] = \mathbb{P}\big[|Y_t^{z_2}| < y|\tau^* < t\big].$$

Therefore, $p_{y,t}(z_1) \ge p_{y,t}(z_2)$ and hence

$$(2.1.21)\qquad (\bar{\mu}_1)_t \overset{st.}{\le} (\bar{\mu}_2)_t. \qquad \square$$

LEMMA 2.1.2. *Suppose f, g are positive functions*: $\mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ *such that* $f(t) \le g(t)$, $\forall\, t$. *If*

$$(2.1.22)\qquad \bar{\mu}(t,f) = \textit{distribution of } \big[|X_t|\,|\tau_f > t,\, X_0 \sim \mu\big],$$

then

$$(2.1.23)\qquad \bar{\mu}(t,f) \overset{st.}{\le} \bar{\mu}(t,g).$$

PROOF. The idea of the proof is as follows.

We will consider two processes run "on the events" $[\tau_g > t,\ \tau_f > t]$ and $[\tau_g > t,\ \tau_f \le t]$ simultaneously, and will prove a coupling inequality for sample paths where the two processes coincide at some time after the latter hits f for the last time before t. This will follow from the Markov property for X. The "remaining sample paths" satisfy an a.s. inequality due to the a.s. continuity of the two processes.

Denoting by $\bar{\mu}(t, f)(\cdot)$ the distribution function corresponding to $\bar{\mu}(t, f)$,

$$\bar{\mu}(t,g)(y) = \alpha\bar{\mu}(t,f)(y) + (1-\alpha)\mathbb{P}\big[|X_t| < y | \tau_g > t, \tau_f \le t\big], \tag{2.1.24}$$

where

$$\alpha = \frac{\mathbb{P}[\tau_f > t]}{\mathbb{P}[\tau_g > t]}$$

because $\tau_f \le \tau_g$. Define a function h by

$$\begin{aligned} h(s) &= f(s) \quad \text{on } (u,t] \\ &= g(s) \quad \text{on } [0,u], \end{aligned} \tag{2.1.25}$$

and let $\{\mathscr{F}_s,\ s \ge 0\}$ be a filtration rich enough to carry mutually independent processes, Z^1, Z^2, Z^3 and X^1, where

$$Z^1 =_{\mathscr{D}} [X | \tau_g > t, \tau_f \le t], \qquad Z^2 =_{\mathscr{D}} [X | \tau_f > t], \qquad Z^3 =_{\mathscr{D}} [X | \tau_h > t],$$

where the conditional processes are defined in the following way.

Let Z be a process generating a filtration $\{\mathscr{B}_s,\ s \ge 0\}$. Formally we define a probability space $\{\Omega, \mathscr{B}_s, s \ge 0, \mathbb{P}'\}$ for the conditioned process $\{|Z_s|\,|A],\ 0 \le s \le t\}$, where $A \in \mathscr{B}_\infty$ and $\mathbb{P}(A) > 0$, as follows.

Suppose $B \in \mathscr{B}_\infty$, then

$$\mathbb{P}'[B] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}.$$

Also we assume here X^1 is an independent copy of X, and all processes are assumed to start at the same initial value y. We also denote by $\{\mathscr{G}_s,\ s \ge 0\}$ the filtration generated by Z^3, and let $\mathscr{H}$ be the σ-algebra generated by $\{Z_s^1, Z_s^2, X_s^1;\ 0 \le s \le t\}$. Also the stopping times, $\tau_f(X^1)$, $\tau_g(X^1)$ and $\tau_h(X^1)$, will denote the hitting times, τ_f, τ_g and τ_h, as defined earlier, for the specific process X^1.

We want to prove $\mathrm{Law}(|Z_t^1|) \overset{st.}{\ge} \mathrm{Law}(|Z_t^2|)$. Z^3 and X^1 are only used for comparison. Define for any $\mathscr{G} \times \mathscr{H}$ adapted Y, Y_1 and Y_2:

$$\tau'(Y) = \sup\{s \le t;\ |Y_s| \ge f(s)\} \tag{2.1.26}$$

and

$$\tau''(Y^1, Y^2) = \inf\{r \ge \tau'(Y^1),\ |Y_r^1| = |Y_r^2|\} \wedge t. \tag{2.1.27}$$

Now τ' and τ'' are not stopping times and so we must take care about the preservation of the strong Markov property.

Clearly for $\tau''(Z^1, Z^2) = t$, $|Z_t^1| \ge |Z_t^2|$ a.s. due to the almost sure continuity of Z_1 and Z_2, so we concentrate on the case $\tau''(Z^1, Z^2) < t$.

Note that

$$\text{(2.1.28)}\qquad \begin{aligned} &\mathbb{P}\big[Z_w^3 < y | Z_v^3 = x, A\big], && v \le w,\\ &= \mathbb{P}\big[Z_w^3 < y | Z_v^3 = x\big], && \forall A \in \mathscr{G}_v \times \mathscr{H}. \end{aligned}$$

Consider the events, defined for any $\mathscr{G} \times \mathscr{H}$ adapted Y:

$$\text{(2.1.29)}\qquad \begin{aligned} A_1(Y) = \big[&|Y_s| < f(s),\ s \le v;\ |Y_s| < |Z_s^1|,\ u \le s < v;\\ &|Z_u^1| = f(u);\ |Z_v^1| = |Y_v|;\ |Z_s^1| < f(s),\ u < s < t\big], \qquad u \le v, \end{aligned}$$

$$\text{(2.1.30)}\qquad \begin{aligned} A_2(Y) = \big[&|Y_s| < f(s),\ u < s \le t;\ |Y_u| = f(u);\\ &|Z_v^2| = |Y_v|;\ |Y_s| > |Z_s^2|,\ u \le s < v\big], \qquad u \le v. \end{aligned}$$

Clearly $A_1(Z^3), A_2(Z^3) \in \mathscr{G}_v \times \mathscr{H}$; moreover, we can rewrite A_1 and A_2 as

$$A_1(Y) = \big[\tau'(Z^1) = u,\ \tau''(Z^1, Y) = v\big],$$
$$A_2(Y) = \big[\tau'(Y) = u,\ \tau''(Y, Z^2) = v\big].$$

Also by definition

$$\mathbb{P}\big[Z_w^3 < y | Z_v^3 = x, A_1(Z^3)\big] = \mathbb{P}\big[X_w^1 < y | X_v^1 = x, \tau_h(X^1) > t, A_1(X^1)\big], \qquad v \le w,$$

and clearly

$$\big[\tau_h(X^1) > v\big] \supset A_1(X^1),$$
$$\big[\tau_f(X^1) \notin (0, v)\big] \supset A_1(X^1).$$

So

$$\text{(2.1.31)}\qquad \begin{aligned} &\mathbb{P}\big[X_w^1 < y | X_v^1 = x, \tau_h(X^1) > t,\ A_1(X^1)\big]\\ &\quad= \mathbb{P}\big[X_w^1 < y | X_v^1 = x, \tau_f(X^1) > t, A_1(X^1)\big]\\ &\quad= \mathbb{P}\big[Z_w^2 < y | \tau'(Z^1) = u, \tau''(Z^1, Z^2) = v, Z_v^2 = x\big], \qquad t \ge w \ge v \ge u, \end{aligned}$$

by the definition of Z^2.

Similarly we can show that

$$\text{(2.1.32)}\qquad \begin{aligned} &\mathbb{P}\big[Z_w^3 < y | Z_v^3 = x, A_2(Z^3)\big]\\ &\quad= \mathbb{P}\big[X_w^1 < y | X_v^1 = x, \tau_h > t, A_2(X^1)\big]\\ &\quad= \mathbb{P}\big[Z_w^1 < y | \tau'(Z^1) = u, \tau''(Z^1, Z^2) = v, Z_v^1 = x\big], \qquad u \le v \le w \le t. \end{aligned}$$

So (2.1.28) gives us

$$\text{(2.1.33)}\qquad \begin{aligned} &\mathbb{P}\big[Z_w^2 < y | Z_v^2 = x, \tau'(Z^1) = u, \tau''(Z^1, Z^2) = v < t\big]\\ &\quad= \mathbb{P}\big[Z_w^1 < y | Z_v^1 = x, \tau'(Z^1) = u, \tau''(Z^1, Z^2) = v < t\big], \qquad v \le w \le t, \end{aligned}$$

and by the continuity of X and hence of Z^1, Z^2,

$$\tau''(Z^1, Z^2) = t \Rightarrow |Z_t^1| \geq |Z_t^2| \quad \text{a.s.} \tag{2.1.34}$$

So

$$\begin{aligned}
&\mathbb{P}\big[Z_t^2| < y|\, |Z_v^2| = x, \tau''(Z^1, Z^2) = t\big] \\
&\geq \mathbb{P}\big[|Z_t^1| < y|\, |Z_v^1| = x, \tau''(Z^1, Z^2) = t\big]\mathbb{P}\big[|Z_t^1| < y\big] - \mathbb{P}\big[|Z_t^2| < y\big] \qquad (2.1.35)\\
&= \mathbb{P}\big[\tau''(Z^1, Z^2) = t\big]\Big(\mathbb{P}\big[|Z_t^1| < y|\tau''(Z^1, Z^2) = t\big] \\
&\qquad\qquad - \mathbb{P}\big[|Z_t^2| < y|\tau''(Z^1, Z^2) = t\big]\Big) \\
&\quad + \mathbb{P}\big[\tau''(Z^1, Z^2) < t\big]\Big(\mathbb{P}|Z_t^2| < y|\tau''(Z^1, Z^2) < t\big] \\
&\qquad\qquad - \mathbb{P}\big[|Z_t^1| < y|\tau''(Z^1, Z^2) < t\big]\Big) \\
&= -\mathbb{P}\big[\tau''(Z^1, Z^2) = t\big]\mathbb{P}\big[|Z_t^2| < y|\tau''(Z^1, Z^2) = t\big] \quad \text{for } y \leq f(t)
\end{aligned}$$

$$\leq 0. \tag{2.1.36}$$

We have used (2.1.33) and (2.1.34) to get the latter equation. So, for some $\alpha \in [0, 1]$:

$$\bar{\mu}(t, g)(y) \leq \alpha\bar{\mu}(t, f)(y) + (1 - \alpha)\bar{\mu}(t, f)(y) = \bar{\mu}(t, f)(y), \tag{2.1.37}$$

that is,

$$\bar{\mu}(t, f) \overset{st.}{\leq} \bar{\mu}(t, g). \tag{2.1.38}$$

□

LEMMA 2.1.3. *Suppose Z is the process obtained by placing a reflecting boundary at $\pm a$ for the process X, then if ξ_t is the law of Z_t, and the function f is identically the constant a:*

$$\bar{\mu}_t \overset{st.}{\leq} \bar{\xi}_t. \tag{2.1.39}$$

PROOF. Let $Y_t = [Z_t|\tau_a > t]$, then $Y_t = [X_t|\tau_a > t]$ a.s. since

$$\mathbb{P}[Y_t \text{ hits } \pm a] = 0.$$

So we can apply Lemma 2.1.2 to $[Z_t|\tau_\infty > t]$ and $[Z_t|\tau_a > t]$ to give the result. □

THEOREM 2.1.4. *Suppose μ_0, the initial distribution of X, has support on $(-a, a)$ and $f \equiv 0$.*

(i) *The distribution of $[|X_t|\, |\tau_a > t]$ has a limit $\bar{\delta}_\infty = \bar{\delta}_\infty^a$, independent of μ_0, and this convergence is uniform for all μ_0.*

(ii) *$\bar{\delta}_\infty^a$ satisfies the quasistationary relation*

$$\int p_{y,t}(x)\, d\bar{\delta}_\infty^a(x) = \bar{\delta}_\infty^a(y), \qquad \forall\, t, y.$$

Furthermore, δ_∞ *satisfies*

$$\text{If } \bar{\mu} \overset{st.}{\leq} \bar{\delta}^a_\infty, \quad \text{then } \bar{\mu}_t \overset{st.}{\leq} \bar{\delta}^a_\infty; \tag{2.1.40}$$

$$\text{if } \bar{\mu} \overset{st.}{\leq} \bar{\delta}^a_\infty, \quad \text{then } \bar{\mu}_t \overset{st.}{\geq} \bar{\delta}^a_\infty. \tag{2.1.41}$$

PROOF. (i) Let δ^x_t be the distribution of X_t at time t given $|X_0| = x$ and $\tau_a > t$. Then for $0 \leq x_1 \leq x_2 \leq a$,

$$\bar{\delta}^{x_1}_t(y) \geq \bar{\delta}^{x_2}_t(y) \quad \text{by Lemma 2.1.1.} \tag{2.1.42}$$

We need to show that

$$\bar{\delta}^{x_1}_t(y) - \bar{\delta}^{x_2}_t(y) \to 0 \quad \text{as } t \to \infty, \forall\, y. \tag{2.1.43}$$

If we consider two processes X^{x_1}, X^{x_2} started at x_1, x_2 respectively, and define

$$\tau' = \inf\{t \geq 0;\, X^{x_2}_t = 0\}, \tag{2.1.44}$$

we can apply Lemma 2.1.3 to the process $W^{x_2}_t$ defined by

$$W^{x_2}_t = \begin{cases} X^{x_2}_t, & t < \tau', \\ 0, & t \geq \tau'. \end{cases} \tag{2.1.45}$$

Now

$$\begin{aligned} \mathbb{P}[X^{x_2}_s \text{ hits 0 before } t | \tau_a > t] &= \mathbb{P}[|W^{x_2}_t| \leq 0 | \tau_a > t] \\ &= \mathbb{P}[\tau' \leq t | \tau_a > t] \\ &\geq \mathbb{P}[|Y^{x_2}_t| \leq 0] \quad \text{by Lemma 2.1.3,} \end{aligned} \tag{2.1.46}$$

where $Y^{x_2}_t$ is $W^{x_2}_t$ with reflecting boundaries at $\pm a$. However,

$$\begin{aligned} \mathbb{P}[|Y^{x_2}_t| \leq 0] &= \mathbb{P}[Z^{x_2}_s \text{ hits 0 before } t] \\ &\geq \mathbb{P}[Z^a_s \text{ hits 0 before } t], \end{aligned} \tag{2.1.47}$$

where Z^{x_2} is X^{x_2} with reflecting boundaries at $\pm a$ as in the proof of Lemma 2.1.3 and $\mathbb{P}[Z^a_s$ hits 0 before $t] \to 1$ as $t \to \infty$ since Z_s has a bounded scale function and is confined to a compact interval. Then

$$\begin{aligned} \bar{\delta}^{x_1}_t(y) - \bar{\delta}^{x_2}_t(y) = \mathbb{P}[\tau' < t | \tau_a > t]\big(&\mathbb{P}[|X^{x_1}_t| < y | \tau' < t, \tau_a > t] \\ &- \mathbb{P}[|X^{x_2}_t| < y | \tau' < t, \tau_a > t]\big) \\ + \mathbb{P}[\tau' \geq t | \tau_a > t]\big(&\mathbb{P}[|X^{x_1}_t| < y | \tau_a > t, \tau' \geq t] \\ &- \mathbb{P}[X^{x_2}_t < y | \tau' \geq t, \tau_a > t]\big) \end{aligned} \tag{2.1.48}$$

and

$$\begin{aligned} &\mathbb{P}[|X^{x_1}_t| < y | \tau' = s < t, \tau_a > t, |X^{x_1}_s| = x] \\ &\quad = \mathbb{P}[|X^x_{t-s}| < y | \tau' = s < t, \tau_a > t] \end{aligned} \tag{2.1.49}$$

(by time homogeneity and the Markov property)

$$\begin{aligned} &\leq \mathbb{P}[|X^0_{t-s}| < y | \tau' = s < t, \tau_a > t] \quad \text{by Lemma 2.1.1,} \\ &= \mathbb{P}[|X^{x_2}_t| < y | \tau_a > t, \tau' = s < t], \end{aligned} \tag{2.1.50}$$

and so conditioning on the values of τ' and $X_{\tau'}^{x_1}$,

$$\mathbb{P}[|X_t^{x_1}| < y | \tau' < t, \tau_a > t] - \mathbb{P}[|X_t^{x_2}| < y | \tau' < t, \tau_a > t] \le 0. \tag{2.1.51}$$

So

$$\begin{aligned} \bar{\delta}_t^{x_1}(y) - \bar{\delta}_t^{x_2}(y) &\le \mathbb{P}[\tau' \ge t | \tau_a > t] \\ &\to 0 \quad \text{uniformly for } x_1, x_2 \in [-a, a] \text{ as } t \to \infty. \end{aligned} \tag{2.1.52}$$

This follows from (2.1.46) and (2.1.47).

Now all that remains is to prove that the distributional limit exists for some initial distribution $\bar{\mu}$. We look at $\{\bar{\delta}_t^0\}$ and show that it is a stochastically increasing function of t, that is, $\bar{\delta}_{t_1}^0 \overset{st.}{\le} \bar{\delta}_{t_2}^0$ for $t_1 \le t_2$. $\bar{\delta}_t^0(y)$ is a continuous function of t (for $t \ne 0$), because

$$\begin{aligned} &\bar{\delta}_{t+\delta t}^0(y) - \bar{\delta}_t^0(y) \\ &\quad = \frac{\mathbb{P}_0[|X_{t+\delta t}| < y, \tau_a > t + \delta t]}{\mathbb{P}_0[\tau_a > t + \delta t]} - \frac{\mathbb{P}_0[|X_t| < y, \tau_a > t]}{\mathbb{P}_0[\tau_a > t]} \\ &\quad = \frac{(\mathbb{P}_0[|X_{t+\delta t}| < y, \tau_a > t] - \mathbb{P}_0[|X_{t+\delta t}| < y, t < \tau_a \le t + \delta t])\mathbb{P}_0[\tau_a > t]}{\mathbb{P}_0[\tau_a > t](\mathbb{P}_0[\tau_a > t] - \mathbb{P}_0[t < \tau_a \le t + \delta t])} \\ &\qquad - \frac{\mathbb{P}_0[|X_t| < y, \tau_a > t](\mathbb{P}_0[\tau_a > t] - \mathbb{P}_0[t < \tau_a \le t + \delta t])}{\mathbb{P}_0[\tau_a > t](\mathbb{P}_0[\tau_a > t] - \mathbb{P}_0[t < \tau_a \le t + \delta t])}. \end{aligned} \tag{2.1.53}$$

Since $\mathbb{P}_0[t < \tau_a \le t + \delta t] \to 0$ as $\delta t \to 0$, (2.1.53) equals

$$\frac{\mathbb{P}_0[|X_{t+\delta t}| < y, |X_t| \ge y, \tau_a > t]\mathbb{P}_0[\tau_a > t]}{(\mathbb{P}_0[\tau_a > t])^2} + o(1) \tag{2.1.54}$$

as $\delta t \downarrow 0$. Equation (2.1.54) is bounded above by

$$\frac{\mathbb{P}_0[\bigcup_{r \le \delta t}\{|X_{t+r}| < y, |X_t| \ge y, \tau_a > t\}]}{\mathbb{P}_0[\tau_a > t]} + o(1), \tag{2.1.55}$$

where r runs over positive rationals less than or equal to δt. So by the monotone convergence theorem, (2.1.54) converges as $\delta t \downarrow 0$ to

$$\frac{\mathbb{P}_0[|X_t| = y, \tau_a > t]}{\mathbb{P}_0[\tau_a > t]} = 0. \tag{2.1.56}$$

A similar argument can be used to prove left continuity.

Also, $\bar{\delta}_{ns}^0(y) \le \bar{\delta}_{(n-1)s}^0(y)$, $n \in \mathbb{R}^+$ by induction, since if we suppose the inequality holds up to $n = m - 1$, then

$$\begin{aligned} \bar{\delta}_{ms}^0(y) &= \int \bar{\delta}_{(m-1)s}^x(y)\, dv(x) \\ &\le \bar{\delta}_{(m-1)s}^0(y), \end{aligned} \tag{2.1.57}$$

where $v(\cdot)$ is the law of $[X_s | X_0 = 0, \tau_a > ms]$.

Here we have applied Lemma 2.1.1 to $\bar{\delta}^0_{ns}$. Clearly the inductive hypothesis is true for $n = 1$, so $\{\bar{\delta}^0_{ns}(y),\ n \in \mathbb{N}\}$ is a stochastically increasing sequence. However, by letting $s \to 0$, and using the t-continuity of $\bar{\delta}^0_t(y)$, it follows that $\{\bar{\delta}^0_t,\ t \geq 0\}$ is a stochastically ordered set such that

$$\bar{\delta}^0_{t_1} \overset{st.}{\leq} \bar{\delta}^0_{t_2} \quad \text{for } 0 \leq t_1 \leq t_2. \tag{2.1.58}$$

Since $\bar{\delta}^0_t$ is stochastically bounded above by a point mass at a, $\bar{\delta}^0_t$ therefore has a limit $\bar{\delta}_\infty$ which is also the limit for all initial distributions. Moreover, this convergence is uniform by (2.1.52).

(ii) We have

$$\int p_{y,t}(x)\, d\bar{\delta}^0_s(x) = \bar{\delta}^0_{s+t}(y), \qquad \forall\, t, y. \tag{2.1.59}$$

So, taking the limit as $s \to \infty$ (formally the left-hand side is integrated by parts, and then we apply dominated convergence before integating by parts back again), we get

$$\int p_{y,t}(x)\, d\bar{\delta}_\infty(x) = \bar{\delta}_\infty(y), \qquad \forall\, t, y; \tag{2.1.60}$$

thus proving the first part. Furthermore, for $\bar{\mu} \overset{st.}{\leq} \bar{\delta}_\infty$,

$$\begin{aligned} \bar{\mu}_t(y) &= \int p_{y,t}(x)\, d\bar{\mu}(x) \\ &\geq \int p_{y,t}(x)\, d\bar{\delta}_\infty(x) = \bar{\delta}_\infty(y). \end{aligned} \tag{2.1.61}$$

The inequality above follows from the first part of the proof of Lemma 2.1.1.

The second stochastic inequality follows similarly. □

Note that the existence of δ_∞ (the limit distribution of $[X_t | \tau > t]$) is also a trivial consequence of Theorem 2.1.4. This is clear by considering the first hitting time of 0, τ_0, for X. The distribution after τ_0 will be symmetric, and $\mathbb{P}[\tau_0 > t | \tau_a > t] \to 0$ as $t \to \infty$.

Corollary 2.1.5.

$$\bar{\delta}^a_\infty \overset{st.}{\leq} \bar{\delta}^{a+\varepsilon}_\infty \tag{2.1.62}$$

for $\varepsilon > 0$.

Proof. This result follows from Lemma 2.1.2 by taking the limit on both sides of the inequality:

$$\bar{\mu}^a_t \overset{st.}{\leq} \bar{\mu}^{a+\varepsilon}_t \tag{2.1.63}$$

[where $\bar{\mu}^a_t(y) = \mathbb{P}[|X_t| < y | \tau_a > t]$]. □

LEMMA 2.1.6.

$$\mathbb{P}[\tau < t|\mu_1] \leq \mathbb{P}[\tau < t|\mu_2] \quad \text{if } \bar{\mu}_1 \overset{st.}{\leq} \bar{\mu}_2,$$

where τ is a symmetric boundary hitting stopping time (i.e., the hitting time of a boundary symmetric about $x = 0$).

PROOF. Suppose $|x_1| < |x_2|$, then $\mathbb{P}[\tau < t|X_0 \sim x_1] \leq \mathbb{P}[\tau < t|X_0 \sim x_2]$ by a similar argument to that in Lemma 2.1.1.

Define $p_t(x) = \mathbb{P}[\tau < t|X_0 = x]$. Now $p_t(x)$ is an increasing function for $x \geq 0$ and

$$\mathbb{P}[\tau < t|\mu_1] = \int p_t(x)\, d\bar{\mu}_1(x) \leq \int p_t(x)\, d\bar{\mu}_2(x) = \mathbb{P}[\tau < t|\mu_2].$$

The inequality follows from the stochastic ordering of μ_1, μ_2, and the integration by parts argument of Lemma 2.1.1. □

2.2. *Function-analytic preliminaries.* In this section, we will first give a brief review of the results from Jacka and Roberts (1987) that we will use in later sections. Then we go on to prove regularity conditions for eigenvalues in the specific Ornstein–Uhlenbeck case.

Let X_t be a time-homogeneous diffusion process, let τ be the stopping time for X,

$$\tau = \inf\{t \geq 0\colon X_t = -b \text{ or } a\},$$

and define the distributions

$$\delta_\infty = \lim_{t\to\infty} \text{law}[X_t|\tau > t],$$

$$v_\infty = \lim_{t\to\infty} \lim_{s\to\infty} \text{law}[X_t|\tau > s].$$

The existence of these distributions and expressions for them are given in Jacka and Roberts (1987) for the case where X is an Itô diffusion. Specifically, we will write $\delta_\infty(\alpha, a)$ for the limit law δ_∞ when X is an Ornstein–Uhlenbeck process with parameters $(\alpha, 1)$ on the interval $[-a, a]$, where the Ornstein–Uhlenbeck (α, β) process satisfies the S.D.E.,

$$dX_t = \beta\, d\beta_t - \alpha X_t\, dt.$$

Also, we write $m(\alpha, a)$ for the exponential decay rate of τ:

$$\mathbb{P}\big[\tau > t|\, |X_0| \sim \bar{\delta}_\infty(\alpha, a)\big] = e^{-m(\alpha, a)t}.$$

That is, $-m(\alpha, a)$ is the largest eigenvalue of the infinitesimal generator of X. In this case m is related to the confluent hypergeometric function. Theorem 3.2 gives a proof of the exponential behavior of τ.

We must establish the following results about the behaviour of m.

LEMMA 2.2.1.

$$m(\varepsilon, \delta) = 2\varepsilon m\left(\tfrac{1}{2}, \delta(2\varepsilon)^{1/2}\right).$$

PROOF. Suppose X is an O.U. $(\varepsilon, 1)$ process, and

$$\tau = \inf\{t \geq 0; |X_t| \geq \delta\}.$$

Let

$$Y_t = (2\varepsilon)^{1/2} X_{t/2\varepsilon}.$$

Then Y is an O.U. $(\frac{1}{2}, 1)$ process, and if

$$\tau' = \inf\left\{t \geq 0; |Y_t| \geq \delta(2\varepsilon)^{1/2}\right\},$$

then

$$[\tau' = 2\varepsilon t] = [\tau = t].$$

Also it is clear that $\delta_\infty(\frac{1}{2}, \delta(2\varepsilon)^{1/2}) = (2\varepsilon)^{1/2}\delta_\infty(\varepsilon, \delta)$ by this transformation, so

$$\begin{aligned}\mathbb{P}[\tau > t | X_0 \sim \delta_\infty(\varepsilon, \delta)] &= \mathbb{P}\left[\tau' > 2\varepsilon t | Y_0 \sim \delta_\infty\left(\tfrac{1}{2}, \delta(2\varepsilon)^{1/2}\right)\right] \\ &= \exp\left\{-2\varepsilon t m\left(\tfrac{1}{2}, \delta(2\varepsilon)^{1/2}\right)\right\} \\ &= e^{-tm(\varepsilon, \delta)}.\end{aligned}$$

So $m(\varepsilon, \delta) = 2\varepsilon m(\frac{1}{2}, (\delta(2\varepsilon)^{1/2})$. □

Recall from Jacka and Roberts (1987) the following definitions:

1. $V(a) = \{C^2$ functions constrained to be 0 at $\pm a\}$.
2. $\langle \cdot, \cdot \rangle_a$ = natural inner product on $V(a)$ chosen to make an operator G self-adjoint.

In the case $G = 1/2\, d^2/dx^2 - (x/2)\, d/dx$,

$$\langle f, g \rangle_a = \int_{-a}^{a} fg e^{-x^2/2}\, dx.$$

Now fix a and consider points $a + \varepsilon$ for small positive ε. We will denote by ρ_b the eigenfunction corresponding to the largest negative eigenvalue, $-m(\frac{1}{2}, b)$.

$$G\rho_b = -m\left(\tfrac{1}{2}, b\right)\rho_b(x) \text{ for } x \in [-b, b].$$

Also extend ρ_a to its analytic continuation on $[-(a + \varepsilon), a + \varepsilon]$.

LEMMA 2.2.2.

$$\left[m\left(\tfrac{1}{2}, a + \varepsilon\right) - m\left(\tfrac{1}{2}, a\right)\right]\langle \rho_a, \rho_{a+\varepsilon} \rangle_{a+\varepsilon} = -\rho'_{a+\varepsilon}(a + \varepsilon)\rho_a(a + \varepsilon).$$

PROOF. Now $\rho_a \notin V(a+\varepsilon)$, and using the inner product on $[-(a+\varepsilon), a+\varepsilon]$, writing G in terms of its natural scale and speed measures, $G = \frac{1}{2} d/dm\, d/ds$.

$$\begin{aligned}\langle G\rho_a, \rho_{a+\varepsilon}\rangle_{a+\varepsilon} &= \int_{-a-\varepsilon}^{a+\varepsilon} \frac{1}{2}\frac{d^2\rho_a}{ds^2}\rho_{a+\varepsilon}\,ds \\ &= \int_{-a-\varepsilon}^{a+\varepsilon} \frac{1}{2}\frac{d^2\rho_{a+\varepsilon}}{ds^2}\rho_a\,ds - \frac{1}{2}\left[\rho'_{a+\varepsilon}\rho_a\right]_{-a-\varepsilon}^{a+\varepsilon} \\ &= \langle G\rho_{a+\varepsilon}, \rho_a\rangle_{a+\varepsilon} - \rho'_{a+\varepsilon}(a+\varepsilon)\rho_a(a+\varepsilon),\end{aligned}$$

where $\rho'_{a+\varepsilon}(x)$ is the derivative with respect to s. In our case, of course, $ds(x) = e^{x^2/2}\,dx$. The second line follows from two integrations by parts, and the third uses the even nature of $\rho_a, \rho_{a+\varepsilon}$. So

$$\begin{aligned}&\left[m\left(\tfrac{1}{2}a+\varepsilon\right) - m\left(\tfrac{1}{2}, a\right)\right]\langle \rho_a, \rho_{a+\varepsilon}\rangle_{a+\varepsilon} \\ &\quad = -\rho'_{a+\varepsilon}(a+\varepsilon)\rho_a(a+\varepsilon).\end{aligned} \tag{2.2.1}$$

□

Now $\rho_b(x) = k(b)M(-m(\frac{1}{2}, b), \frac{1}{2}, x^2/2)$, where M is a confluent hypergeometric function and $k(b)$ is an L^2 normalising constant chosen so $\langle \rho_b, \rho_b\rangle = 1$. So we need the following lemma about M.

LEMMA 2.2.3.

(i) *$M(\alpha, \delta, x^2/2)$ is a continuous function of α for δ not negative integer valued and this continuity is uniform for $x \in [-A, A]$ for some fixed but arbitrarily large $A > 0$.*

(ii) *$M(-m(\delta, b), \delta, x^2/2)$ is bounded for $(b, x) \in [a, a+R'] \times [-A, A]$ for some constant $R' > 0$.*

PROOF. (i) A power series expansion for the confluent hypergeometric function [see, e.g., Abramowitz and Stegun (1972)] is Kummer's expansion,

$$M(\alpha, \delta, y) = \sum_{i=0}^{\infty} \frac{(\alpha)_i y^i}{(\delta)_i i!},$$

where $(\beta)_i = \beta(\beta+1)(\beta+2)\cdots(\beta+i-1)$ and $(\beta)_0 = 1$.

This series is absolutely convergent for all y and α, provided that δ is not a negative integer. Also the series is eventually monotone for $y > 0$, since all the terms in the expansion for which $i \ge \max\{|\alpha|+1, |\delta|+1\}$ have the same sign.

Now fix $R' > 0$. We choose $I \ge (\delta, b) + 1$ for all $b \in [a, a+R']$. We can certainly do this since $m(\delta, b)$ is nonincreasing as a function of b, so it suffices to take $I \ge m(\delta, a) + 1$.

$$M(\alpha, \delta, y) = \sum_{i=0}^{I-1} \frac{y^i(\alpha)_i}{(\delta)_i i!} + (\alpha)_I \sum_{i=I}^{\infty} \frac{y^i(\alpha+I)_{i-I}}{(\delta)_i i!}.$$

We would like to show that

$$(2.2.2) \qquad |M(\alpha, \delta, y) - M(\alpha - \varepsilon, \delta, y)| \to 0 \quad \text{as } \varepsilon \to 0,$$

uniformly for $y \in [0, R]$, for some positive constant R from which assertion (i) will follow.

The first term in the above expression is a polynomial in y and α, and so satisfies

$$\sum_{i=0}^{I-1} \frac{y^i((\alpha)_i - (\alpha - \varepsilon)_i)}{(\delta)_i i!} = \varepsilon P_1(\alpha\varepsilon, y)$$

for some polynomial P_1.

Similarly, $(\alpha)_I - (\alpha - \varepsilon)_I = \varepsilon\alpha P_2(\alpha\varepsilon)$ for some polynomial P_2. So it remains to consider the terms

$$T(\alpha, y) = \sum_{i=I}^{\infty} \frac{y^i(\alpha + I)_{i-I}}{(\delta)_i i!},$$

$$T(\alpha, y) - T(\alpha - \varepsilon, y) = \sum_{i=I}^{\infty} \frac{y^i[(\alpha + I)_{i-I} - (\alpha + I - \varepsilon)_{i-I}]}{(\delta)_i i!}$$
$$\leq T(\alpha, R) - T(\alpha - \varepsilon, R) \quad \text{for } y \in [0, R],$$

since all the terms are positive. Also,

$$T(\alpha, R) - T(\alpha - \varepsilon, R) \to 0 \quad \text{as } \varepsilon \to \infty,$$

since the individual terms tend to 0, and T is given by an absolutely convergent series. This establishes (2.2.2) and assertion (i) follows by taking $R \geq A^2/2$.

(ii) M is a continuous function of α, y and so is bounded on any compact set. Now $m(\delta, b)$ is a decreasing function of b, so

$$m(\delta, a + R') \leq m(\delta, b) \leq m(\delta, a) \quad \text{for } b \in [a, a + R'].$$

This implies that $M(-m(\delta, b), \delta, x^2/2)$ is bounded for $(b, x) \in [a, a + R'] \times [-A, A]$. □

LEMMA 2.2.4. *$\exists\, k_1, k_2 > 0$ such that for $b \in [a, a + R']$,*

$$k_1 \leq k(b) \leq k_2.$$

PROOF.

$$k(b) = \left[\int_{-b}^{b} M^2\left(-m\left(\tfrac{1}{2}, b\right), \tfrac{1}{2}, x^2/2\right) dm(x)\right]^{-1/2},$$

where $dm(x) = e^{-x^2/2}\,dx$ denotes the speed measure for the O.U. $(\frac{1}{2}, 1)$ process. Define

$$\gamma(b, \lambda) = \int_{-b}^{b} M^2\left(-\lambda, \tfrac{1}{2}, x^2/2\right) dm(x).$$

Now $m(x)$ is absolutely continuous with respect to Lebesgue measure, so Lemma 2.2.3 implies that $\gamma(b, \lambda)$ is continuous for $(b, \lambda) \in [a, a+R] \times [m(\frac{1}{2}, a+R'), m(\frac{1}{2}, a)]$. So γ is bounded in this region and also clearly cannot be 0 anywhere [this would imply $M(-\lambda, \frac{1}{2}, x^2/2) \equiv 0$], and must therefore be bounded away from 0. Hence the existence of k_1, k_2, since $m(\cdot)$ is nonincreasing. □

DEFINITION 2.2.1. We call a function f locally Lipschitz in D if for each point $x \in D$, $\exists$ an open neighbourhood $N(x)$ of x and a constant $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|, \qquad \forall\, y \in N(x).$$

LEMMA 2.2.5. *$m(\frac{1}{2}, a)$ is locally Lipschitz for $a \in (0, \infty)$.*

PROOF.

$$\rho_b'(x) = -2xk(b)m\left(\tfrac{1}{2}, b\right)M\left(-m\left(\tfrac{1}{2}, b\right) + 1, \tfrac{3}{2}, x^2/2\right).$$

So it follows by a simple argument from Lemma 2.2.3(i) that $\rho_b'(x)$ is bounded for $(b, x) \in [a, a+R'] \times [a, a+R']$. Also, $\rho_a(\cdot)$ is clearly locally Lipschitz since it is continuously differentiable. So $\exists\ k_3 > 0$ such that

$$\rho_b'(a + \varepsilon) \leq k_3$$

and

$$|\rho_a(a + \varepsilon)| \leq k_3 \varepsilon$$

for small enough ε. Also, as $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon \downarrow 0} |\langle \rho_a, \rho_{a+\varepsilon} \rangle_{a+\varepsilon}| \geq k_1^2 \int_{-a}^{a} \left(M\left(-m\left(\tfrac{1}{2}, a\right), \tfrac{1}{2}, x^2/2\right)\right)^2 dm(x)$$
$$> 0.$$

So for small enough ε,

$$|\langle \rho_a, \rho_{a+\varepsilon} \rangle_{a+\varepsilon}| \geq k_4 > 0, \quad \text{say}.$$

Finally, from Lemma 2.2.2 we see that

$$\left|m\left(\tfrac{1}{2}, a + \varepsilon\right) - m\left(\tfrac{1}{2}, a\right)\right| \leq \frac{k_3^2}{k_4} \varepsilon \quad \text{as required}.$$

This proves the right-hand local Lipschitz property. The left-hand property follows similarly. □

3. The approximate square-root boundary. In order to prove the main result of this section, it is necessary to look at a geometric partition for the Brownian motion time (corresponding to a uniform partition of time for the Ornstein–Uhlenbeck process). This allows us to make approximations for the approximate square-root case in terms of the exact square-root results obtained by Breiman (1967). So in this section X_t is the Ornstein–Uhlenbeck process corresponding to Brownian motion B_t and we shall use the two processes interchangeably.

Recall that $c(\cdot)$ is given in (1.2), and observe that $m(c) = m(\frac{1}{2}, c)$. First, we need the following lemma.

LEMMA 3.1. *Suppose $m(\cdot)$ is the inverse function of $c(\cdot)$. Then*

$$\mathbb{P}\left[\tau > t | B_1 \sim \bar{\delta}_\infty^c\right] = t^{-m(c)}, \tag{3.1}$$

where $\tau = \inf\{t \geq 1;\ |B_t| \geq c\sqrt{t}\}$ and $\bar{\delta}_\infty^c$ is given in Theorem 2.1.4.

PROOF. Fix $r < 1$ and let $t = r^{-n}$ for some positive integer n. Now we will assume we are conditioning on the event $[\tau > 1,\ B_1 \sim \bar{\delta}_\infty^c]$ in all that follows:

$$\begin{aligned} \mathbb{P}[\tau > t] &= \prod_{i=1}^{n} \mathbb{P}[\tau > r^{-i} | \tau > r^{-(i-1)}] \\ &= \prod_{i=1}^{n} \mathbb{P}\left[\tau > r^{-i} | \tau > r^{-(i-1)},\ X_{-(i-1)\log r} \sim \delta_\infty^c\right] \\ &= [k(r)]^n \\ &= t^{-g(r)}, \quad \text{say}. \end{aligned} \tag{3.2}$$

We will show that g is a constant function and that $\mathbb{P}[\tau > t] = t^{-g(r)}$ for all $t \geq 1$. By a similar argument we can show that

$$\mathbb{P}[\tau > t] = t^{-g(r^{1/m})};$$

hence $g(r) = g(r^{1/m})$, over all $r < 1$, $m \in \mathbb{N}^+$. So we have

$$g(r) = g(r^{m/n}) \quad \text{for positive integers } m, n.$$

Now fix arbitrary $t \geq 0$. We can pick sequences $\{s_i\}, \{t_i\}$, such that for each $i \geq 1$, s_i and t_i can be written in the form $r^{-m/n}$ for some integers m, n and $s_i \uparrow t$ and $t_i \downarrow t$.

$$\mathbb{P}[\tau > t_i] \leq \mathbb{P}[\tau > t] \leq \mathbb{P}[\tau > s_i] \tag{3.3}$$

and

$$t_i^{-g(r)} \leq \mathbb{P}[\tau > t] \leq s_i^{-g(r)}, \qquad \forall\, i \geq 1.$$

So taking limits as $i \uparrow \infty$,

$$\mathbb{P}[\tau > t] = t^{-g(r)}, \qquad \forall\, t \geq 1.$$

But by Breiman (1967), $g(r) = m(c)$. □

Armed with the results of Section 2, we are able to completely describe the asymptotic behaviour of approximate square-root boundaries under certain conditions on the boundary.

DEFINITION. We call f a simple approximate square-root boundary if $f(t) = t^{1/2}a(t)$, where

(A1) $a(t)$ is asymptotically nondecreasing to a limit $a(\infty)$.

(A2) $a(\cdot)$ is differentiable and $a^2(t) + ta'(t)a(t)$ is asymptotically nondecreasing.

We shall also say that a function $p(t)$ is $O(q(t))$ as $t \to \infty$ if $0 < C_1 < |p(t)/q(t)| < C_2 < \infty$ for constants C_1, C_2 and for large enough t.

THEOREM 3.2. *Suppose f is a simple approximate square-root boundary, B_t is a Brownian motion and τ the hitting time,*

$$\tau = \inf\{t \geq 1; |B_t| \geq f(t)\}.$$

Then for any initial distribution

$$\mathbb{P}[\tau > t] = O\left(\exp\left\{-\int^t \frac{m(\frac{1}{2}, a(s))}{s}\, ds\right\}\right)$$

as $t \to \infty$, where $a(t) = f(t)t^{-1/2}$.

PROOF. Assume first that $B_1 = X_0 \sim \delta_\infty^{a(1)}$, then by Corollary 2.5,

$$\bar{\delta}_\infty^{a(1)} \overset{st.}{\leq} \bar{\delta}_\infty^{a(t)}. \tag{3.4}$$

Suppose

$${}_c\tau = \inf\{t \geq 1; |B_t| \geq ct^{1/2}\} \tag{3.5}$$

and

$${}_f\tau = \inf\{t \geq 1; |B_t| \geq f(t)\} \tag{3.6}$$

for $f(t) = a(t)t^{1/2}$ and denote by ${}_c\mu_t$, the law of $[X_{\ln t}|{}_c\tau > t]$ and ${}_f\mu_t$ the law of $[X_{\ln t}|{}_f\tau > t]$, in the usual way. Then it is clear from Lemma 2.1.2 and Theorem 2.1.4(ii) that

$${}_f\bar{\mu}_t \overset{st.}{\leq} {}_{a(t)}\bar{\mu}_t \overset{st.}{\leq} \bar{\delta}_\infty^{a(t)} \tag{3.7}$$

for all $t \geq 1$. It is this powerful distributional inequality that allows us to prove the theorem.

Now let $t = r^{-n}$ for some $r < 1$. Then

$$\mathbb{P}\big[{}_f\tau > t\big] = \prod_{i=1}^{n} \mathbb{P}\big[{}_f\tau > r^{-i}|{}_f\tau > r^{-(i-1)}\big]. \tag{3.8}$$

But

$$\begin{aligned} &\mathbb{P}\big[{}_f\tau > r^{-i}|{}_f\tau > r^{-(i-1)}\big] \\ &\quad= \mathbb{P}\big[{}_f\tau > r^{-i}|{}_f\tau > r^{-(i-1)},\, X_{(i-1)\ln 1/r} \sim {}_f\mu_{r^{-(i-1)}}\big], \\ &\quad\geq \mathbb{P}\big[{}_f\tau > r^{-i}|{}_f\tau > r^{-(i-1)},\, X_{(i-1)\ln 1/r} \sim \delta_\infty^{a(r^{-(i-1)})}\big] \end{aligned} \tag{3.9}$$

from (3.7) and by Lemma 2.1.6. Now define the function $\bar{a}(t)$ as follows:

$$\bar{a}(t) = \begin{cases} a(t), & 1 \le t \le r^{-(i-1)}, \\ a(r^{-(i-1)}), & r^{-(i-1)} \le t \le r^{-i}. \end{cases} \tag{3.10}$$

So $\bar{a}(t) \le a(t)$, $1 \le t \le r^{-n}$, and if $\bar{f}(t) = \bar{a}(t)t^{1/2}$:

$$\begin{aligned} &\mathbb{P}\Big[{}_f\tau > r^{-i} \big| {}_f\tau > r^{-(i-1)}, X_{(i-1)\ln 1/r} \sim \delta_\infty^{a(r^{-(i-1)})}\Big] \\ &\quad \ge \mathbb{P}\Big[{}_{\bar{f}}\tau < r^{-i} \big| {}_{\bar{f}}\tau > r^{-(i-1)}, X_{(i-1)\ln 1/r} \sim \delta_\infty^{a(r^{-(i-1)})}\Big] \\ &\quad = r^{m(a(r^{-(i-1)}))}. \end{aligned} \tag{3.11}$$

So

$$\begin{aligned} \mathbb{P}\big[{}_f\tau > t\big] &\ge \prod_{i=1}^{n} r^{m(a(r^{-(i-1)}))} \\ &= \prod_{i=1}^{n} t^{-m(a(r^{-(i-1)}))/n} = M_r(t), \quad \text{say}. \end{aligned} \tag{3.12}$$

The same argument applies for $r^k t$, $k < n$. So

$$\frac{M_r(t)}{M_r(r^k t)} = \prod_{i=n-k}^{n-1} r^{(m(a(r^{-i}))}$$

and

$$\frac{M_r(t) - M_r(r^k t)}{t(1-r^k)M_r(tr^k)} = \frac{-1 + \prod_{i=n-k}^{n-1} r^{m(a(r^{-i}))}}{t(1-r^k)} = -b(r), \quad \text{say}. \tag{3.13}$$

Now choose $k = \log(1 - \delta/t)/\log r$, for suitable r,

$$-b(r) \triangleq \frac{M_r(t) - M_r(t-\delta)}{\delta M_r(t-\delta)} = \frac{-1 + \prod_{i=n-k}^{n-1} r^{m(a(r^{-i}))}}{\delta}.$$

Now $a(t)\uparrow$ with t, $m(a(t))\downarrow$ with t and

$$\frac{1 - (1-\delta/t)^{m(a(t))}}{\delta} \le b(r) \le \frac{1 - (1-\delta/t)^{m(a(t-\delta))}}{\delta}.$$

Now we define

$$M(t) = \liminf_{r \uparrow 1} M_r(t), \tag{3.14}$$

and so taking limits as $r \uparrow 1$,

$$\begin{aligned} \frac{-1 + (1-\delta/t)^{m(a(t))}}{\delta} &\ge \frac{M(t) - M(t-\delta)}{\delta M(t-\delta)} \\ &\ge \frac{-1 + (1-\delta/t)^{m(a(t-\delta))}}{\delta}, \end{aligned} \tag{3.15}$$

and now letting $\delta \downarrow 0$, we see that $M'(t)$ exists and

$$\frac{M'(t)}{M(t)} \geq \frac{-m(a(t))}{t}, \tag{3.16}$$

$$M(t) \geq \exp\left\{-\int_1^t \frac{m(a(s))}{s}\, ds\right\},$$

that is,

$$\mathbb{P}\left[{}_f\tau > t\right] \geq \exp\left\{-\int_1^t \frac{m(a(s))}{s}\, ds\right\}. \tag{3.17}$$

It remains to show the result holds for all initial distributions μ_0 of X. We will write $\mathbb{P}[\tau > t|\mu_0]$ for $\mathbb{P}[\tau > t|X_0 \sim \mu_0]$ and show that

$$\mathbb{P}[\tau > t|\mu_0] = O\left(\mathbb{P}\left[\tau > t|\delta_\infty^{a(1)}\right]\right) \tag{3.18}$$

for arbitrary μ_0 on $(-a(1), a(1))$. Now $\mathbb{P}[\tau > t|x]$ is a nonincreasing function of x for $x \geq 0$. This can be shown by a simple coupling argument. So

$$\mathbb{P}[\tau > t|\mu_0] \leq \mathbb{P}[\tau > t|0].$$

Also

$$\begin{aligned}\mathbb{P}[\tau > t|x] &\geq \mathbb{P}[\tau > t|x, X \text{ hits 0 before } f(\cdot)]\,\mathbb{P}[X \text{ hits 0 before } f(\cdot),\ X_0 = x] \\ &\geq \mathbb{P}[\tau > t|0]\,\mathbb{P}[X \text{ hits 0 before } f(\cdot),\ X_0 = x].\end{aligned}$$

But since $\mathbb{P}[X$ hits 0 before $f(\cdot)$, $X_0 = x] > 0$ and is independent of t,

$$\mathbb{P}[\tau > t|0] \leq k\mathbb{P}\left[\tau > t|\delta_\infty^{a(1)}\right]$$

for some positive constant k, proving the result.

For the opposite inequality, we define the following deterministic space and time change:

$$X_t = \frac{B_{\alpha(t)}}{g(\alpha(t))},$$

where $g(t) = t^{1/2}a(t)/a(\infty)$ and $\alpha(\cdot)$ is the solution of

$$\alpha'(t) = g^2(\alpha(t)), \tag{3.19}$$

such that $\alpha(0) = 1$. X_t satisfies the S.D.E.

$$dX_t = dB'_t - X_t g(\alpha(t))g'(\alpha(t))\, dt,$$

where B' is another Brownian motion.

Write $r(t) = g(\alpha(t))g'(\alpha(t))$ and let $\beta(\cdot)$ be the inverse function of $\alpha(\cdot)$.

Since f is a simple approximate square-root boundary, by (A2), $r(t)$ is asymptotically nondecreasing. We may assume that r is always nondecreasing without loss of generality since we are only interested in the asymptotic behaviour of it. The idea of the proof is to approximate μ_t by $\delta_\infty(r(t), a(\infty))$.

First, assume $\mu_0 = \delta_\infty(r(0), a(\infty))$; then $r(s) \le r(t)$, $\forall\ s \le t$. We will approximate with the Ornstein–Uhlenbeck $(r(t), 1)$ process, so

$$\bar{\mu}_t \overset{st.}{\ge} \bar{\sigma}_t,$$

where σ_t = distribution of $[X_t' | \gamma > t]$,

$$dX_s' = dB_s' - X_s' r(t)\, ds$$

and $\gamma = \inf\{t \ge 0;\ |X_t'| \ge a(\infty)\}$.

Also

$$\bar{\delta}_\infty(r(t), a(\infty)) \overset{st.}{\le} \bar{\delta}_\infty(r(0), a(\infty)),$$

since $r(t)$ increases by a transformation similar to that given in the proof of Corollary 2.1.5, and so by Theorem 2.1.4

$$\bar{\sigma}_t \overset{st.}{\ge} \bar{\delta}_\infty(r(t), a(\infty))$$

and so

$$\bar{\mu}_t \overset{st.}{\ge} \bar{\delta}_\infty(r(t), a(\infty)). \tag{3.20}$$

Now

$$\begin{aligned}
&\mathbb{P}[\beta(\tau) > t | X_0 \sim \delta_\infty(r(0), a(\infty)), \beta(\tau) > t - \varepsilon] \\
&\quad \le \mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon,\ X_{t-\varepsilon} \sim \delta_\infty(r(t-\varepsilon), a(\infty))] \quad \text{from (3.2)} \\
&\quad \le \mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon,\ X_{t-\varepsilon} \sim \delta_\infty(r(t), a(\infty))] \\
&\quad \le \exp - \{m(r(t), a(\infty))\varepsilon\},
\end{aligned}$$

since $r(s) \le r(t)$ for $s \in [t - \varepsilon, t]$. Thus

$$\frac{\mathbb{P}[\beta(\tau) > t] - \mathbb{P}[\beta(\tau) > t - \varepsilon]}{\mathbb{P}[\beta(\tau) > t - \varepsilon]\varepsilon} \le -\frac{1 - \exp\{-m(r(t), a(\infty))\varepsilon\}}{\varepsilon},$$

just as in the first part of the proof. So in the limit as $\varepsilon \downarrow 0$, $\mathbb{P}[\beta(\tau) > t] \le M(t)$, where $M(t)$ satisfies

$$\frac{M'(t)}{M(t)} = \frac{d}{ds} \exp\{-m(r(t), a(\infty))s\}|_{s=0} = -m(r(t), a(\infty)),$$

with $M(0) = 1$. So

$$\mathbb{P}[\tau > t] \le \exp - \left\{\int_0^{\beta(t)} m(r(s), a(\infty))\, ds\right\}.$$

But $m(r(s), a(\infty)) = 2r(s) m(\frac{1}{2}, a(\infty)(2r(s))^{1/2})$ by Lemma 2.2.1, so

$$\begin{aligned}
\mathbb{P}[\tau > t] &\le \exp - \left\{\int_0^{\beta(t)} 2r(s) m\left(\tfrac{1}{2}, a(\infty)(2r(s))^{1/2}\right) ds\right\} \\
&= \exp - \left\{\int_1^t \frac{a^2(\infty) m\left(\frac{1}{2}, a(\infty)\left(2r(\beta(u))^{1/2}\right) 2r(\beta(u))\right)}{u a^2(u)}\, du\right\}.
\end{aligned}$$

This last step follows the change of variable, $u = \alpha(s)$. Now $r(s) = g(u)g'(u)$ and

$$g'(t) = \frac{1}{2t^{1/2}}\frac{a(t)}{a(\infty)} + \frac{t^{1/2}a'(t)}{a(\infty)}.$$

So

$$g(u)g'(u) = \frac{1}{2}\frac{a^2(u)}{a^2(\infty)} + \frac{ua'(u)a(u)}{a^2(\infty)}$$

and

$$-\log \mathbb{P}[\tau > t] \geq \int_1^t \frac{m\left(\frac{1}{2}, a(u)\left[1 + \frac{2ua'(u)}{a(u)}\right]^{1/2}\right)}{u} h(u)\,du,$$

where

$$h(u) = 1 + \frac{2ua'(u)}{a(u)}.$$

Thus

$$-\log \mathbb{P}[\tau > t] \geq \int_1^t \frac{m\left(\frac{1}{2}, a(u)\right) - (k'2ua'(u))/(a(u))}{u}\,du,$$

where k' is a local Lipschitz constant for the function $m(\frac{1}{2}, \cdot\,)$ around $a(\infty)$, which exists by Lemma 2.2.5.

But $\int_1^t (2k'a'(u))/(a(u))\,du$ converges, so

$$-\log \mathbb{P}[\tau > t] \geq \text{constant} + \int_1^t \frac{m(\frac{1}{2}, a(u))}{u}\,du.$$

Here we have used that a' is nonnegative. Therefore,

$$\mathbb{P}[\tau > t] \leq k \exp\left\{-\int_1^t \frac{m(\frac{1}{2}, a(u))}{u}\,du\right\}$$

for some constant k.

Now it remains to show that the result holds for all initial distributions, but this is clear from the arguments in the converse, earlier in this proof, which show that all starting points yield the same asymptotic behaviour for the hitting time, thus completing the proof. □

Note that condition (A2) is only needed for the second part of the proof. That is, the inequality

$$\mathbb{P}[{}_f\tau > t] \geq \exp\left\{-\int_1^t \frac{m(\frac{1}{2}, a(s))}{s}\,ds\right\} \tag{3.21}$$

has been proved without (A2). This is important for considering cases such as

$$a'(t) \le O\left(\frac{1}{t^2}\right) \quad \text{as } t \to \infty,$$

where (A2) is never satisfied. However, in this and similar cases, Taksar's approximation [Taksar (1982)] turns out to be equivalent up to a multiplicative constant and so gives the converse result. Specifically, we have the following theorem.

Here and later in the article, we will abbreviate $m(\frac{1}{2}, \cdot\,)$ by $m(\cdot)$.

THEOREM 3.3. *Suppose*

$$\left|\int_1^\infty \log t\, dm(a(t))\right| < \infty \tag{3.22}$$

for some approximate square-root boundary satisfying (A1). *Then*

$$\mathbb{P}[{}_f\tau > t] = O\left(\exp\left\{-\int_1^t \frac{m(a(s))}{s}\, ds\right\}\right).$$

PROOF. Taksar's approximation is the following:

$$\mathbb{P}[\tau > t] \le O(t^{-m(a(t))}),$$

so that

$$\int_1^t -\frac{m(a(s))}{s}\, ds = -(\log t) m(a(t)) + \int_1^t \log s\, dm(a(s)),$$

and

$$\exp\left\{-\int_1^t \frac{m(a(s))}{s}\, ds\right\} = t^{-m(a(t))} \exp\left\{\int_1^t \log s\, dm(a(s))\right\}.$$

So clearly if the integral $\int_1^\infty \log t\, dm(a(t))$ is finite, then this expression is bounded by multiples of $t^{-m(a(t))}$, which is Taksar's approximation. □

4. Lower-case boundaries. An obvious way to proceed in this case, since the rate of increase of the boundary function is "relatively small", is to look at the approximations obtained by approximating the distribution of the conditioned Brownian motion itself. This has the added advantage that no normalizing time change is necessary. However, we can obtain marginally better results by the natural scaling and its appropriate normalizing time change, which converts the problem to a constant boundary hitting problem.

DEFINITION 4.1. f is a simple lower-case boundary if $f(t)f'(t)$ is asymptotically nonincreasing to 0.

Note that this implies that $a(t) \to 0$ as $t \to \infty$, where $a(t) = f(t)t^{-1/2}$.

THEOREM 4.1. *Suppose f is a simple lower-case boundary and α is the solution of*

$$\alpha'(t) = f^2(\alpha(t)) \tag{4.1}$$

such that $\alpha(0) = 1$. Let

$$\tau = \inf\{t \geq 1; |B_t| \geq af(t)\};$$

then

$$\mathbb{P}[\tau > t] = \exp\left\{-\frac{\beta(t)\pi^2}{8a^2}\right\}h(t),$$

where $\beta(\cdot)$ is the inverse function of $\alpha(\cdot)$ and

$$\exp\left\{k_1\int_0^{\beta(t)} r(s)\,ds\right\} \leq h(t) \leq k_2 t^p,$$

where $r(t) = f(\alpha(t))f'(\alpha(t))$ and k_1, k_2, p are positive constants.

PROOF. Consider the first inequality first. Let

$$X_t = \frac{B_{\alpha(t)}}{f(\alpha(t))}, \tag{4.2}$$

then

$$dX_t = dB'_t - X_t r(t)\,dt,$$

where B' is another Brownian motion.

The general idea is to approximate μ_t, the distribution of $[X_t | \beta(\tau) > t]$, by the stationary distribution for the O.U. $(r(t), 1)$ process with boundaries $-a, a$ and initial distribution $\delta_\infty(r(0), a)$: $\delta_\infty(r(t), a)$.

$$\frac{\mathbb{P}[\beta(\tau) > t]}{\mathbb{P}[\beta(\tau) > t - \varepsilon]} = \mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon],$$

and assuming $X_0 \sim \delta_\infty(r(0), a)$,

$$\begin{aligned}
&\mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon, X_0 \sim \delta_\infty(r(0), a)] \\
&\quad = \mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon, X_{t-\varepsilon} \sim \mu_{t-\varepsilon}] \\
&\quad \geq \mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon, X_{t-\varepsilon} \sim \delta_\infty(r(t - \varepsilon), a)],
\end{aligned}$$

since as in the proof of Corollary 2.1.5 we have $\bar{\delta}_\infty(r(0), a) \overset{st.}{\leq} \delta_\infty(r(t - \varepsilon), a)$, so by Theorem 2.1.4, $\bar{\delta}_\infty(r(t - \varepsilon), a) \overset{st.}{\leq} \bar{\mu}_{t-\varepsilon}$. Now if X'_s is a process satisfying

$$X'_s = X_s, \ \leq t - \varepsilon,$$

$$dX'_s = dB_s - r(t)X'_s\,ds, \qquad t - \varepsilon < s \leq t,$$

then $\mathrm{Law}(|X'_s|) \overset{st.}{\geq} \mathrm{Law}(|X_s|)$, $0 \leq s \leq t$; see, for example, Roberts (1988). Therefore,

$$\mathbb{P}[\beta(\tau) > t | \beta(\tau) > t - \varepsilon,\ X_{t-\varepsilon} \sim \delta_\infty(r(t-\varepsilon), a)]$$
$$\geq \mathbb{P}[\gamma > t | \gamma > t - \varepsilon,\ X'_{t-\varepsilon} \sim \delta_\infty(r(t-\varepsilon), a)],$$

where $\gamma = \inf\{t \geq 0;\ |X'_t| > a\}$.

Now let μ'_t be the distribution of $[X'_t | \gamma > t]$, then $\bar{\mu}'_t \overset{st.}{\leq} \bar{\delta}_\infty(r(t), a)$ since

$$\bar{\delta}_\infty(r(t - \varepsilon), a) \overset{st.}{\leq} \bar{\delta}_\infty(r(t), a),$$

by Theorem 2.1.4. So

$$\mathbb{P}[\beta(\tau') > t | \beta(\tau') > t - \varepsilon,\ X'_{t-\varepsilon} \sim \delta_\infty(r(t-\varepsilon), a)]$$
$$\geq \mathbb{P}[\beta(\tau') > t | \beta(\tau') > t - \varepsilon,\ X'_{t-\varepsilon} \sim \delta_\infty(r(t), a)]$$
$$= \exp\{-\varepsilon m(r(t), a)\},$$

and so as in the proof of Theorem 3.2,

$$\frac{\mathbb{P}[\beta(\tau) > t]}{\mathbb{P}[\beta(\tau) > t - \varepsilon]} \geq \exp\{-\varepsilon m(r(t), a)\}.$$

So $\mathbb{P}[\beta(\tau) > t] \geq M(t)$, where

$$\frac{M(t)}{M(t - \varepsilon)} = \exp\{-\varepsilon m(r(t), a)\},$$

and $M(1) = 0$. Taking limits as $\varepsilon \downarrow 0$,

$$\frac{M'(t)}{M(t)} = -m(r(t), a),$$

$$M(t) = \exp\left\{-\int_1^t m(r(s), a)\, ds\right\}.$$

However, for small $r(s)$,

$$m(r(s), a) = \frac{\pi^2}{8a^2} - O(r(s)).$$

This follows using a simple argument and an expansion for the confluent hypergeometric function; see Abramowitz and Stegun (1972), page 510. So

$$\mathbb{P}[\beta(\tau) > t] \geq M(t) \geq \exp\left\{\frac{-\pi^2 t}{8a^2}\right\} \exp\left\{k_1 \int_1^t r(s)\, ds\right\}$$

for some positive constant k. Therefore,

$$\mathbb{P}[\tau > t] \geq \exp\left\{-\frac{\pi^2 \beta(t)}{8a^2}\right\} \exp\left\{k_1 \int_1^{\beta(t)} r(s)\, ds\right\}$$

gives a lower inequality for $h(t)$, assuming $X_0 \sim \delta_\infty(r(1), a)$.

We prove the reverse inequality by using the usual exponential transformation, $X_t = e^{-t/2}X_{e^t}$. Then if μ_t is the distribution of $[X_t | \tau > e^t]$, an application of Lemma 2.1.2 gives

$$\bar{\mu}_t \overset{st.}{\geq} \bar{\delta}_\infty\left(\tfrac{1}{2}, a(e^t)\right),$$

and so we can use a similar argument to that for the previous inequality to show that

$$\mathbb{P}[\tau > e^t] \leq M(t),$$

where

$$\frac{M'(t)}{M(t)} = -m\left(\frac{1}{2}, a(t)\right),$$
$$a(t) = t^{-1/2}f(t)$$

and

$$M(t) = \exp\left\{-\int_1^t m\left(\frac{1}{2}, a(e^s)\right) ds\right\},$$

and for small $a(s)$, from Abramowitz and Stegun (1972),

$$M(t) = \exp\left\{-\int_1^t \left(\frac{\pi^2}{8a^2(e^s)} - O(1)\right) ds\right\}.$$

So

$$\begin{aligned}
\mathbb{P}[\tau > t| &\leq \exp\left\{-\int_0^{\log t}\left(\frac{\pi^2}{8a^2(e^s)} - O(1)\right) ds\right\} \\
&\leq kt^p \exp\left\{-\int_1^t \frac{\pi^2}{8a^2(u)u}\, du\right\} \\
&= kt^p \exp\left\{-\int_1^t \frac{\pi^2}{8a^2 f^2(u)}\, du\right\} \\
&= kt^p \exp\left\{-\frac{\pi^2}{8a^2}\beta(t)\right\}.
\end{aligned}$$

We can now apply identical arguments to those of Theorem 3.2 to show that these distributional inequalities hold for all initial distributions. □

5. Upper-case boundaries.

LEMMA 5.1. *Recall that $m(x) = m(\frac{1}{2}, x)$, given in Section 2.2. Then*

$$m(x) = \frac{xe^{-x^2/2}}{2\sqrt{2}\,\Gamma(\frac{3}{2})}\left(1 + O\left(\frac{1}{x^2}\right)\right)$$

as $x \to \infty$.

PROOF. $m(x)\downarrow 0$ as $x \to \infty$, so we consider $M(-b, \frac{1}{2}, z)$ for small positive b.

$$M(-b, \tfrac{1}{2}, z) = 1 - 2bF(b, z),$$

where

$$\frac{d}{dz}F(b, z) = M(1 - b, \tfrac{3}{2}, z)$$

and

$$F(b, 0) = 0.$$

Thus

$$m(a) = \frac{1}{2F(1 - m(a), a^2/2)}.$$

Now we are interested in the behaviour of $m(a)$ for large a, so we use an approximation for M given in Abramowitz and Stegun (1972):

$$M(\alpha, \beta, z) = \frac{e^z z^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)}\left[1 + O(z^{-1})\right],$$

and so

$$\frac{d}{dz}F(b, z) = \frac{e^z z^{-b-1/2}\Gamma(\frac{3}{2})}{\Gamma(1 - b)}\left[1 + O(z^{-1})\right].$$

It follows by a simple integration by parts argument that

$$F(b, z) = \frac{e^z z^{-b-1/2}\Gamma(\frac{3}{2})}{\Gamma(1 - b)}\left[1 + O(z^{-1})\right],$$

and, furthermore, the $O(z^{-1})$ term is uniformly bounded for b in some neighbourhood of 0, so

$$\begin{aligned} m(a) &= \frac{1}{2}e^{-a^2/2}\left(\frac{a^2}{2}\right)^{1/2+m(a)}\frac{\Gamma(1 - m(a))}{\Gamma(\frac{3}{2})}(1 + O(a^{-2})) \\ &= \frac{e^{-a^2/2}\Gamma(1 - m(a))}{2\sqrt{2}\,\Gamma(\frac{3}{2})}a^{2m(a)+1}(1 + O(a^{-2})). \end{aligned}$$

But $m(a)$ is bounded for $a \in [A, \infty)$ say, so $m(a) = o(e^{-a})$ and $a^{2m(a)} - 1 = o(a^{-2})$. Also, this implies that

$$m(a) = O(ae^{-a^2/2}),$$

so that $\Gamma(1 - m(a)) - 1 = o(a^{-2})$, and

$$m(a) = \frac{ae^{-a^2/2}}{2\sqrt{2}\,\Gamma(\frac{3}{2})}(1 + O(a^{-2}))$$

as required. □

DEFINITION 5.1. A function f is called a simple upper-case boundary if $f \in C^2$, $f(t) = a(t)t^{1/2}$ and the following conditions are satisfied:

(B1) a is asymptotically nondecreasing to ∞.
(B2) If $\alpha(\cdot)$ is the solution of

$$\alpha'(t) = f^2(\alpha(t)),$$

with $\alpha(0) = 1$ and

$$r(s) = f(\alpha(s))\,f'(\alpha(s)) = \frac{1}{2}\frac{\alpha''(s)}{\alpha'(s)},$$

then $r(s)$ is asymptotically nondecreasing to ∞.
(B3) Let

$$p(u) = a(u)\left[1 + \frac{2ua'(u)}{a(u)}\right]^{1/2},$$

then $p(\cdot)$ is asymptotically increasing.

(B4) $$\int^{\infty} \frac{a(u)e^{-a^2(u)/2}}{u}\left(\frac{a'(u)}{p'(p^{-1}(a(u)))} - 1\right) du < \infty.$$

THEOREM 5.2. *Suppose f is a simple upper-case boundary and $\{B_t,\ t \geq 1\}$ is a Brownian motion with hitting time*

$$\tau_f(B) = \inf\{t \geq 1; |B_t| \geq f(t)\}.$$

Then

$$\mathbb{P}\big[\tau_f(B) > t\big] = O\left(\exp\left\{-\int_1^t \frac{m(a(s))}{s}\,ds\right\}\right)$$

as $t \to \infty$.

PROOF. The result is identical in form to that for the simple approximate square-root boundary and we proceed in a similar fashion.

Let $X_t = e^{-t/2}B_{e^t}$. As usual, we assume initially that $X_0 \sim \delta_\infty(a(0))$ and $a(\cdot)$ is nondecreasing $\forall\ t \geq 0$. Under these assumptions,

$$\bar{\mu}_t \overset{st.}{\leq} \mathrm{Law}\big[X_t | \tau_g(X) > t\big],$$

where $g(s) = a(t)$, $s \leq t$, and

$$\mathrm{Law}\big[|X_t|\ |\tau_g(X) > t\big] \overset{st.}{\leq} \mathrm{Law}\big[|X_t|\ |\tau_g(X) > t,\ X_0 \sim \delta_\infty(a(t))\big].$$

These results follow from Lemmas 2.1.2 and 2.1.1 and Corollary 2.1.5. But

$$\mathrm{Law}\big[X_t | \tau_g(X) > t,\ X_0 \sim \delta_\infty(a(t))\big] = \delta_\infty(a(t))$$

by Theorem 2.1.4, so $\bar{\mu}_t \overset{st.}{\leq} \bar{\delta}_\infty(a(t))$, and

$$\mathbb{P}[\tau_a(X) > t] \geq k \exp - \left\{ \int_0^t \frac{m(a(s))}{s} \, ds \right\}$$

for some constant k, by using similar arguments to those in the proof of Theorem 3.2 for general initial distributions.

For the other inequality, let $Y_t = B_{\alpha(t)}/f(\alpha(t))$, where α is defined as in Definition 4.1. We also let β equal the inverse function of α. Y satisfies

$$dY_t = dB_t^* - r(t)\, dt,$$

where B^* is an associated Brownian motion and $r(t) = f(\alpha(t)) f'(\alpha(t))$. We assume in the usual way that r is actually nondecreasing everywhere. Denoting by μ_t the law of $[Y_t | \tau_f(Y) > t]$, we can follow the proof of Theorem 3.2 to obtain

$$\bar{\mu}_t \overset{st.}{\geq} \bar{\delta}_\infty(r(t), 1),$$

$$\mathbb{P}\left[\tau_f(Y) > t\right] \leq \exp - \left\{ \int_0^t m(r(s), 1)\, ds \right\},$$

$$\mathbb{P}\left[\tau_f(B) > t\right] \leq \exp - \left\{ \int_0^{\beta(t)} m(r(s), 1)\, ds \right\},$$

$$-\log \mathbb{P}\left[\tau_f(B) > t\right] \geq \int_0^t \frac{m\left(\frac{1}{2}, p(u)\right)}{u}\, du.$$

Using the transformation $a(v) = p(u)$ [note that this is certainly always possible by restriction (B3)], we obtain

$$-\log \mathbb{P}\left[\tau_f > t\right] \geq \int_0^{a^{-1}(p(t))} \frac{m\left(\frac{1}{2}, a(v)\right) a'(v)}{p^{-1}(a(v)) p'(p^{-1}(a(v)))}\, dv$$

$$\geq \int_0^t \frac{m(a(v))}{p^{-1}(a(v))} \frac{a'(v)}{p'(p^{-1}(a(v)))}\, dv,$$

since the integrand is positive and $(a^{-1}p)(t) \geq t$. Also, $p^{-1}a(v) \leq v$, $\forall\, v$, so

$$-\log \mathbb{P}[\tau > t] \geq \int_0^t \frac{m(a(v))}{v} \frac{a'(v)}{p'(p^{-1}(a(v)))}\, dv$$

$$\geq k + \int_0^t \frac{m(a(v))}{v}\, dv$$

for some constant k, by (B4) and Lemma 5.1.

The final result for general μ_0 follows in the usual way. Note that the two inequalities have been proved here using the same initial distribution. This simplifies the argument for general μ_0. □

REMARKS.

(i) The result is most interesting in the case when the boundary is attained with probability 1, that is, roughly speaking boundaries a such that

$$a(t) \le \left((2\log\log t)^{1/2}\right)$$

asymptotically.

(ii) For such boundaries, (B3) will be satisfied unless a exhibits some oscillatory behaviour. Any attempt to give more explicit expressions for (B1)–(B4) would only lead to a weakening of the result. However, almost all cases of interest can be either solved directly or by means of an approximation scheme.

EXAMPLE. Consider the boundary

$$a(t) = \sqrt{2l_3(t)}\,,$$

where $l_i(\cdot) \equiv \log l_{i-1}(\cdot)$, $\log_1(\cdot) \equiv \log(\cdot)$. It is easy to check that (B1), (B2) and (B3) are satisfied. For (B4), we must consider

$$\int_0^t \frac{\sqrt{2l_3(u)}}{ul_2(u)}\left(\frac{a'(u)}{p'(p^{-1}(a(u)))} - 1\right) du.$$

After much algebra we obtain

$$\frac{a'(u)}{p'(p^{-1}(a(u)))} \approx \frac{w(l(w))(l_2(w))(l_3(w))^{1/2}}{u(l(u))(l_2(u))(l_3(u))^{1/2}},$$

where

$$u = w\left(1 + \frac{1}{2w^2e^{w^2}e^{e^{w^2}}}\right).$$

And so

$$\frac{a'(u)}{p'(p^{-1}(a(u)))} - 1 \le O\left(e^{-e^{u^2/2}}e^{-u^2/2}u^{-2}\right).$$

So (B4) is clearly satisfied, and

$$\mathbb{P}[\tau > t] = O\left(\exp\left\{\int_0^t \frac{(2l_3(u))^{1/2}\,du}{ul_2(u)}\right\}\right).$$

Now consider a similar curve

$$a(t) = \sqrt{2l_3(t)(1 + \sin t)}\,.$$

In this case (B1), (B2) and (B3) are all contravened while (B4) holds [this follows from the calculation for $\sqrt{2l_3(t)}$.] However, if we define the function

$$a_\theta(t) = \sqrt{2l_3(t)(1 + e^{-\theta t}(1 + \sin t))}\,.$$

then a_θ satisfies (B1), (B2) and (B4), and we can therefore consider $a(\cdot)$ as $\lim_{\theta\downarrow 0} a_\theta(\cdot)$ and it is not too difficult to prove

$$\mathbb{P}[\tau_a > t] = \lim_{\theta\downarrow 0}\mathbb{P}\left[\tau_{a_\theta} > t\right].$$

A simple consequence of Theorem 5.1 is the following well-known result, a generalisation of the law of the iterated logarithm.

COROLLARY 5.3 (Kolmogorov, Erdös, Feller and Petrowski). *Under the conditions* (B1)–(B4),

$$\tau_f < \infty \text{ a.s.} \Leftrightarrow \int_0^\infty \frac{a(s)\exp\{-a^2(s)/2\}}{s}\,ds = \infty.$$

PROOF. This follows immediately from Lemma 5.1 and Theorem 5.2. □

6. One-sided boundary hitting problems.

6.1. In this section we consider the behaviour of stopping times τ, such as

$$\tau = \inf\{t \geq 0;\, X_t \geq f(t)\},$$

where X is a diffusion process. We attempt to extend the ideas of Jacka and Roberts (1987) and Section 2 to the one-sided boundary case. The main results, giving the asymptotic behaviour of one-sided boundaries for Brownian motion, are stated in Section 6.3. We give first-order expansions for the distribution function of the hitting times by considering the four distinct cases determined by the behaviour of $a(t) = t^{-1/2}f(t)$:

(i) Approximate square-root boundaries, that is, boundaries such that $a(t) \to a(\infty)$, a finite nonzero limit.
(ii) Lower-case boundaries, that is, $a(t) \to 0$.
(iii) Positive upper-case boundaries, $a(t) \to \infty$.
(iv) Negative upper-case boundaries, $a(t) \to \infty$.

In Section 6.2 we consider the time-homogeneous problem, that is, we look at hitting times of the form:

$$\tau = \inf\{t \geq 0\colon X_t \geq b\},$$

where X is time homogeneous. For the applications in Section 6.3 we are only interested in the case where X is an Ornstein–Uhlenbeck process. In this particular case, we are able to prove analogous results to those of Jacka and Roberts (1987). In fact, we can carry out a self-adjoint analysis of the infinitesimal generator $\mathscr{L}$ of X, but in general this is not possible.

In Jacka and Roberts (1987), the finiteness of the interval on which the functions in the domain of $\mathscr{L}$ are defined simplifies the problem for three main reasons:

1. First, the stationary behavior exhibited by X with respect to the hitting time $\tau = \inf\{t \geq 0;\, X_t = a \text{ or } b\}$ in the finite interval case does not necessar-

ily follow in the semi-infinite case. More precisely, if

$$\mu_t = \mathrm{Law}[X_t | \tau > t],$$

then in the finite case it is always true that μ_t has a limit which we call δ_∞, but this is not always the case when $a = -\infty$.

2. Second, in the finite interval case, at least for well-behaved $\mathscr{L}$, the spectrum of $\mathscr{L}$ is purely discrete on the space S, where

$$S = \{C^2 \text{ functions } f \text{ on } (a,b) \text{ such that } f(a) = f(b) = 0\}.$$

This allows us to take eigenfunction expansions in terms of an infinite sum, and, furthermore, the eigenfunctions are always in S.

3. Third, suppose we form the following subspace of S by imposing the integrability conditions on function in S:

$$L_i = \{f \in S;\ f^i(x) m'(x) \in L(a,b)\},$$

where m' is the speed measure of X. The nature of the problem compels us to work in the space L_1 since the inherent probabilistic restriction on δ_∞ is that it integrates to unity, whereas in L_2 we are able to define an inner product to make $\mathscr{L}$ self-adjoint and the eigenfunction expansions are easier to handle as well as having been more widely studied. Of course, in the finite interval case, trivially we have $S = L_1 = L_2$, and so we can work with whatever structure we choose. However, in the semi-infinite case, $L_1 \neq L_2$, and so in general it is not possible to adopt the self-adjoint approach.

In Section 6.2 we look at the problem of one-sided boundary hitting problems for Brownian motion. Some of the proofs are similar to those of the previous sections and so to avoid repetition, parts of these proofs are merely sketched. Also, the theorems of this section are by no means a definitive collection of results that can be proved by the methods which are developed in Section 2. Extensions to higher dimensions and to other diffusions are obvious examples of areas where the methods can be applied, but also the consideration of different classes of boundaries can yield similar inequalities. An example of this is Theorem 6.3.1, where we consider a class of functions of the form $f(t) = a(t)t^{1/2}$, where $a(t)$ is asymptotically increasing to a finite limit $a(\infty)$. An analogous theorem for the case $a(t)$ asymptotically decreasing to $a(\infty)$ is easily derived with all the inequalities running the other way. So in a sense, the results of this section should be viewed as illustrative of the power of the techniques used.

6.2. *Time-homogeneous problem.* Let $X_t(\alpha)$ be an Ornstein–Uhlenbeck $(\alpha, 1)$ process with $\alpha > 0$, that is,

$$dX_t(\alpha) = dB_t - \alpha X_t\, dt,$$

and define the stopping time

$$\tau = \inf\{t \geq 0;\ X_t(\alpha) \geq b\},$$

and its distribution function

$$\phi(t, x) = \mathbb{P}[\tau > t | X_0(\alpha) = x].$$

Then the backward equation for ϕ is

$$\mathscr{L}(\alpha)\phi = \frac{\partial \phi}{\partial t},$$

where $\mathscr{L}(\alpha)$ is the infinitesimal generator of $X(\alpha)$.

The method of solution which naturally suggests itself, analogous to the approach of Jacka and Roberts (1987), is taking an appropriate eigenfunction expansion in a space where $\mathscr{L}$ is a self-adjoint operator. Proceeding in this fashion, we define the pre-Hilbert space $H = \{\tilde{L}_2(b, \alpha), \langle \cdot , \cdot \rangle\}$ as follows:

$$\tilde{L}_2(b, \alpha) = \left\{ \begin{matrix} C^2\text{-functions } f \text{ such that } \int_{-\infty}^{b} f^2(x) e^{-\alpha x^2}\, dx < \infty \text{ and} \\ f(b) = 0 \end{matrix} \right\}$$

and

$$\langle f, g \rangle = \int_{-\infty}^{b} f(x) g(x) e^{-\alpha x^2}\, dx.$$

It is clear that $\phi(t, \cdot) \in \tilde{L}_2$ since $|\phi(t, \cdot)| \le 1$ and it is easily checked that $\mathscr{L}(\alpha)$ is self-adjoint on H. We will need the following results about the spectrum of $\mathscr{L}(\alpha)$.

LEMMA 6.2.1. *The spectrum of $\mathscr{L}(\alpha)$ in H is purely discrete.*

PROOF. Define the pre-Hilbert space $H^* = \{L_2^*, \langle \cdot , \cdot \rangle_*\}$ as follows:

$$L_2^* = \left\{ \begin{matrix} C^2\text{-functions } f \text{ on } (-\infty, b) \text{ such that } \int_{-\infty}^{b} f^2(x)\, dx < \infty \text{ and} \\ f(b) = 0 \end{matrix} \right\}$$

and

$$\langle f, g \rangle_* = \int_{-\infty}^{b} fg\, dx.$$

Molchanov (1953) showed that the spectrum of an operator $G = \frac{1}{2} d^2/dx^2 - a(x)$ in H^* is purely discrete if and only if

$$\lim_{A \to -\infty} \int_{A}^{A+\varepsilon} a(x)\, dx = \infty,$$

for arbitrary positive ε and where $a(x)$ is bounded below. Consider the operator

$$G = \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{\alpha}{2} - \frac{\alpha^2 x^2}{2} \right), \qquad x \le b.$$

Clearly G satisfies Molchanov's conditions and so has a purely discrete spectrum.

Now f is an eigenfunction of G on H^* with corresponding eigenvalue λ if and only if $f(x)\exp\{\alpha x^2/2\}$ is an eigenfunction of $\mathscr{L}$ in H with eigenvalue λ. That is the spectra of G on H^* and $\mathscr{L}(\alpha)$ on H coincide and therefore the spectrum of $\mathscr{L}(\alpha)$ on H is discrete. □

LEMMA 6.2.2. *The eigenvalues of $\mathscr{L}(\alpha)$ on H are negative.*

PROOF. From the proof of Lemma 6.2.1 the spectrum of $\mathscr{L}(\alpha)$ in H coincides with the spectrum of G in H^*. However, suppose λ is a nonnegative eigenvalue of G in H^*. By successive applications of the maximum principle on intervals $[-R, b]$ as $R \to \infty$, we see that f is either nonincreasing or nondecreasing. But $f(b) = 0$ and $f \in L^2(-\infty, b)$, so $f \equiv 0$, giving a contradiction and completing the proof. □

Now by the spectral theorem, the eigenfunctions of $\mathscr{L}(\alpha)$ form an orthonormal basis for H. These results allow us to carry out a self-adjoint analysis in the manner of that of Jacka and Roberts (1987). We will denote by $-n(\alpha, b)$ the largest eigenvalue of $\mathscr{L}(\alpha)$ on H and let $\exp\{-\alpha x^2/2\}e_1(x)$ be its corresponding normalised eigenfunction, that is, e_1 is the corresponding eigenfunction for G on H^*. The existence of $-n(\alpha, b)$ is clear from Lemmas 6.2.1 and 6.2.2. In this context we can adopt the results of Sections 2 and3, which are summarised below.

THEOREM A. *A process $\tilde{X} = \lim_{T\to\infty}[X|\tau > T]$ exists as a weak limit and satisfies the S.D.E.*

$$d\tilde{X}_t = dB_t + dt\left(\frac{e_1'(\tilde{X}_t)}{e_1(\tilde{X}_t)} - \alpha \tilde{X}_t\right)$$

for a suitable Brownian motion B.

THEOREM B. *The process $\tilde{X}$ has a distributional limit given by*

$$v_\infty(dx) = \lim_{t\to\infty} \text{Law}\, \tilde{X}_t = e^{-\alpha x^2} e_1^2(x)\, dx.$$

THEOREM C. *Let μ_t denote the distribution of $[X_t|\tau > t]$. Then the following limit exists:*

$$\delta_\infty(dx) = \lim_{t\to\infty} \mu_t = e^{-\alpha x^2}\frac{e_1(x)}{\langle 1, e_1\rangle}\, dx,$$

and has the following properties:

(i)

$$\int_{-\infty}^{b} \phi(t, x)\delta_\infty(dx) = e^{-n(\alpha, b)t}, \qquad \forall\, t \geq 0.$$

(ii) *If $\mu_0 \overset{st.}{\leq} \delta_\infty$, then $\mu_t \overset{st.}{\leq} \delta_\infty$, and if $\mu_0 \overset{st.}{\geq} \delta_\infty$, then $\mu_t \overset{st.}{\geq} \delta_\infty$.*

THEOREM D. *If δ_∞^b denotes the limit distribution corresponding to the hitting time of b, then*

$$\delta_\infty^b \overset{st.}{\leq} \delta_\infty^{b+\varepsilon} \quad \textit{for } \varepsilon > 0.$$

In our particular problem, solutions of

$$\mathscr{L}(\alpha)\phi = \lambda\phi \tag{6.2.1}$$

are parabolic cylinder functions and we shall see that the only $\tilde{L}_2$ solution of (6.2.1) can be expressed in terms of a Whittaker function $D_\lambda(\cdot)$. First though, for notational simplicity, we reduce the problem to the case $\alpha = \frac{1}{2}$.

LEMMA 6.2.3.

$$n(\alpha, b) = 2\alpha n\left(\tfrac{1}{2}, b(2\alpha)^{1/2}\right).$$

PROOF. This is proved in a similar manner to Lemma 2.2.1. □

For the rest of this section, we shall assume that $\alpha = \frac{1}{2}$ and we abbreviate $n(\frac{1}{2}, b)$ by $n(b)$.

LEMMA 6.2.4. *$n(b)$ is a nonincreasing function of b.*

PROOF. The proof of this result revolves around the intuitive idea that since $n(b)$ is the exponential decay rate of the hitting time of b, then it must be greater for smaller b.

Consider $b_1 < b_2$ and let $\phi(t, x; b_i) = \mathbb{P}[\tau(b_i) > t | X_0 = x]$, $i = 1, 2$, where $\tau(b_i)$ is the first hitting time of the level b_i. Furthermore, denote by $\delta_\infty^{b_i}$ the limit distribution corresponding to $\tau(b_i)$, $i = 1, 2$. Clearly

$$\phi(t, x; b_1) \leq \phi(t, x; b_2),$$

and by Theorem D,

$$\delta_\infty^{b_1} \overset{st.}{\leq} \delta_\infty^{b_2}.$$

Using this, together with the fact that ϕ is a nonincreasing function of x, we see that

$$\begin{aligned} e^{-n(b_1)t} &= \int \phi(t, x; b_1)\, d\delta_\infty^{b_1}(x) \\ &\leq \int \phi(t, x; b_2)\, d\delta_\infty^{b_1}(x) \\ &\leq \int \phi(t, x; b_2)\, d\delta_\infty^{b_2}(x) = e^{-n(b_2)t}, \end{aligned} \tag{6.2.2}$$

since it is clear by Lemma 2.1.3 that $\delta_\infty^{b_2} \overset{st.}{\geq} \delta_\infty^{b_1}$, and $\phi(t, x; b)$ is a nonincreasing function of x (which can be seen, for example, by a simple coupling argument). So

$$n(b_1) \geq n(b_2), \qquad \forall\, b_1 < b_2.$$

Furthermore, since $d\delta_\infty^{b_1}$ is absolutely continuous with respect to Lebesgue measure, equality in (6.2.2) is only possible when $\phi(t, x; b_1) = \phi(t, x; b_2)$ for almost all x for each $t \geq 0$. But

$$\phi(t, x; b_2) - \phi(t, x; b_1) = \mathbb{P}\Big[\max_{s \geq t} X_s \in (b_1, b_2)\Big] > 0.$$

This implies that $n(b_1) > n(b_2)$, completing the proof. □

To continue our investigation of n, we need to study the corresponding eigenfunctions and thereby find an implicit characterization of n.

Eigenfunctions of G on H^* are solutions of

$$\frac{1}{2}\frac{d^2 e_i(x)}{dx^2} + \left(\frac{1}{4} - \frac{x^2}{8} - \lambda_i\right) e_i(x) = 0,$$

such that $e_i(b) = 0$. These are Whittaker functions, $D_\lambda(\cdot)$; see Abromowitz and Stegun (1972). So

$$e_i(x) = kD_{2\lambda_i}(-x),$$

where k is an L_2^* normalising constant, and λ_i is such that $D_{2\lambda_i}(b) = 0$.

Since $e_1(x)$ is positive for $x \in (-\infty, b)$ this leads to the following characterisation of n. Let $z(\lambda)$ be the smallest zero of the equation

$$D_{2\lambda}(-x) = 0.$$

Then $-n$ is the inverse function of z. See Abramowitz and Stegun (1972) for a summary of the properties of the Whittaker function that we have used.

We now extend our notation for the dominating eigenfunction and write e_b for the eigenfunction for G corresponding to the eigenvalue $-n(b)$, on the interval $(-\infty, b]$. We shall also denote by $\tilde{L}_2(b)$ the relevant $\tilde{L}_2$ space, and by $\langle \cdot , \cdot \rangle_b$ the corresponding inner product.

LEMMA 6.2.5.

$$[n(b+\varepsilon) - n(b)]\langle e_b, e_{b+\varepsilon}\rangle_{b+\varepsilon} = -\tfrac{1}{2}e'_{b+\varepsilon}(b+\varepsilon)e_b(b+\varepsilon),$$

where e' denotes the derivative with respect to s, the natural scale of $\mathscr{L}$.

PROOF. Now $e_b \notin L_2(b+\varepsilon)$, and taking its analytic continuation to the interval $(-\infty, b+\varepsilon]$ and using the inner product on $(-\infty, b+\varepsilon]$,

$$\begin{aligned}\langle \mathscr{L}e_b, e_{b+\varepsilon}\rangle_{b+\varepsilon} &= \int_{-\infty}^{b+\varepsilon} \frac{1}{2}\frac{d^2 e_b}{ds^2} e_{b+\varepsilon}\, ds \\ &= \int_{-\infty}^{b+\varepsilon} \frac{1}{2}\frac{d^2 e_{b+\varepsilon}}{ds^2} e_b\, ds - \frac{1}{2}[e'_{b+\varepsilon}e_b]_{-\infty}^{b+\varepsilon} \\ &= \langle \mathscr{L}e_{b+\varepsilon}, e_b\rangle_{b+\varepsilon} - \frac{1}{2}e'_{b+\varepsilon}(b+\varepsilon)e_a(b+\varepsilon).\end{aligned}$$

The second line follows from two integrations by parts. So

$$[n(b+\varepsilon)-n(b)]\langle e_b, e_{b+\varepsilon}\rangle_{b+\varepsilon} = -\tfrac{1}{2}e'_{b+\varepsilon}(b+\varepsilon)e_b(b+\varepsilon). \qquad \square$$

We wish to establish continuity results about $D_\mu(x)$ as a function of μ in order to take the limit as $\varepsilon \downarrow 0$ in Lemma 6.2.5. Now $D_\mu(x)$ can be written as

$$D_\mu(x) = A(\mu)e^{-x^2/4}M\left(-\frac{\mu}{2}, \frac{1}{2}, \frac{x^2}{2}\right) + B(\mu)xe^{-x^2/4}M\left(-\frac{\mu}{2} + \frac{1}{2}, \frac{3}{2}, \frac{x^2}{2}\right),$$

where M is the Kummer function and A and B are given by

$$A(\mu) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\mu/2+\frac{1}{2})}{2^{-\mu/2}}\cos\left(\frac{-\mu\pi}{2}\right),$$

$$B(\mu) = -\frac{1}{\sqrt{\pi}}\frac{\Gamma(\mu/2+1)}{2^{-\mu/2-1/2}}\sin\left(\frac{-\mu\pi}{2}\right).$$

Now since $n(\cdot)$ is decreasing by Lemma 6.2.4, and clearly $A(\lambda)$ and $B(\lambda)$ are C^∞ functions of λ for positive λ, we can directly apply Lemma 2.2.3 to give the necessary continuity properties. Furthermore, since the derivative of the Whittaker function is given by

$$D'_\mu(x) = -\tfrac{1}{2}xD_\mu(x) - \mu D_{\mu-1}(x),$$

we can derive similar results for $D'_\mu(x)$ also. We summarise these results in the following lemma, which then enables us to state the main result of this section.

LEMMA 6.2.6.

(i) *$D_\mu(x)$ is a continuous function of μ for $\mu < 0$ and this continuity is uniform in x for x on compact sets.*

(ii) *$D_{-2n(b)}(x)$ is a bounded function for b and x on compact sets.*

(iii) *$D'_\mu(x)$ is a continuous function of μ and this continuity is uniform in x for x on compact sets.*

(iv) *$D'_{-2n(b)}(x)$ is a bounded function for b and x on compact sets.*

THEOREM 6.2.7. *$n(b)$ is a continuously differentiable function such that*

$$n'(b) = \frac{1}{2}(e'_b(b))^2 = -\frac{1}{2}\frac{\left(D'_{-2n(b)}(-b)\right)^2}{\|D_{-2n(b)}(-x)\|^2}.$$

PROOF. Lemma 6.2.5 can be rewritten as

$$(6.2.3)\quad n(b+\varepsilon) - n(b) = -\frac{D'_{-2n(b+\varepsilon)}(-b-\varepsilon)D_{-2n(b)}(-b-\varepsilon)}{2\langle D_{-2n(b)}, D_{-2n(b+\varepsilon)}\rangle_{b+\varepsilon}}.$$

We wish to consider the limit as $\varepsilon \downarrow 0$ in (6.2.3). First, we look at the denominator.

We know that $n^+(b+\varepsilon) = \lim_{\varepsilon \downarrow 0} n(b+\varepsilon)$ exists since $n(b)$ is decreasing. It is also clear that $D_{-2n^+(b)}(-x) \geq 0$ for $x \in (-\infty, b)$ and is not identically 0 (since no parabolic cylinder function is). So $D_{-2n^+(b)}(-x)$ is a nonnegative eigenfunction of $\mathscr{L}$ on $(-\infty, b)$. This implies that $n(b) = n^+(b)$ since if not then $D_{-2n(b)}$ and $D_{-2n^+(b)}$ are orthogonal eigenfunctions on H, and this is contradicted by taking their inner product since $D_{-2n(b)}$ is a positive eigenfunction on $(-\infty, b)$. Thus we have proved that n is a right-continuous function. Now Lemma 6.2.6 ensures that the integrals converge to the expected limits,

$$\langle D_{-2n(b)}, D_{-2n(b+\varepsilon)} \rangle_{b+\varepsilon} \to \|D_{-2n(b)}\|^2 > 0 \quad \text{as } \varepsilon \downarrow 0.$$

Now we can strengthen Lemma 6.2.6 to say that $D_{-2n(a)}(-x)$ is a continuous function of both b and x and we can thus take limits in (6.2.3) to obtain the required result, since $D_{-2n(b)}(-b) = 0$. □

6.3. Time-dependent boundaries for Brownian motion.

THEOREM 6.3.1. *Let $\{\Omega, \mathscr{F}, \mathscr{F}_t, t \geq 1, \mathbb{P}\}$ be a filtered probability space on which is defined a Brownian motion $\{B_t, t \geq 1\}$. Let $f(t)$ be a one-sided boundary of the form*

$$f(t) = a - b(t)t^{1/2}, \qquad t \geq 1, \qquad a - b(1) > 0,$$

where $b(t) - at^{-1/2}$ is asymptotically nonincreasing to a limit $b(\infty)$ and $f(t)f'(t)$ is asymptotically nonincreasing to $\frac{1}{2}b(\infty)^2$. Let

$$\tau = \inf\{t \geq 1;\ B_t \geq f(t)\},$$

then

$$\mathbb{P}[\tau > t] = O\left(\exp\left\{-\int_1^t \frac{n(-b(s))}{s}\,ds\right\}\right) \quad \text{as } t \to \infty.$$

PROOF. Let

$$X_t = e^{-t/2}B_{e^t}.$$

Then $\{\tau = t\} = \{\tau' = \log t\}$, where

$$\tau' = \inf\{t \geq 0;\ X_t \geq ae^{-t/2} - b(e^t)\}.$$

Let $g(t) = ae^{-t/2} - b(e^t)$, and fix T such that $g(t)$ is nondecreasing for $t \geq T$. Adopting the notation introduced in Section 2, we let $\mu_t = \text{Law}\{X_t | \tau' > t\}$, and we assume

$$\mu_T \sim \delta_\infty^{g(T)}.$$

Then Lemmas 2.1.1 and 2.1.2 and Corollary 2.1.5 show us that

$$\mu_t \overset{st.}{\leq} \delta_\infty^{g(t)}, \qquad t \geq T.$$

Now we define the stopping time τ^* as follows:

$$\tau^* = \inf\{s \geq 0;\ B_s \geq f^*(s)\},$$

where

$$f^*(s) = \begin{cases} f(s), & s \leq t, \\ f(t), & s \geq t, \end{cases}$$

for some fixed $t \geq T$. Then

$$\begin{aligned} \mathbb{P}\big[\tau' > t + \varepsilon \,|\, \tau' > t,\ \mu_T \sim \delta_\infty^{g(T)}\big] &\geq \mathbb{P}\big[\tau' > t + \varepsilon \,|\, \tau' > t,\ \mu_t \sim \delta_\infty^{g(t)}\big] \\ &\geq \mathbb{P}\big[\tau^* > t + \varepsilon \,|\, \tau^* > t,\ \mu_t \sim \delta_\infty^{g(t)}\big] \\ &= e^{-\varepsilon n(g(t))}, \qquad t \geq T. \end{aligned}$$

So

$$\mathbb{P}\big[\tau' > t \,|\, \tau' > T,\ \mu_T \sim \delta_\infty^{g(T)}\big] \geq \exp\left[-\varepsilon \sum_{i=1}^{\operatorname{int}\{(t-T)/\varepsilon\}} n(g(T + i\varepsilon))\right].$$

But this is true for arbitrarily small ε, so by taking the limit as $\varepsilon \to 0$, in a similar manner to that of the proof of Theorem 3.2,

$$\mathbb{P}\big[\tau' > t \,|\, \tau' > T,\ \mu_T \sim \delta_\infty^{g(T)}\big] \geq \exp\left\{\int_T^t - n(g(s))\, ds\right\}, \qquad t \geq T.$$

Also using the standard argument involving the recurrence of Brownian motion and thus showing that all starting distributions have the same asymptotic behaviour for the hitting time (see the proof of Theorem 3.2), we can say

$$\mathbb{P}[\tau' > t \,|\, \tau' > T] \geq \exp\left\{k_1 - \int_T^t n(g(s))\, ds\right\}, \qquad t \geq T,$$

for some constant k_1, and so

$$\mathbb{P}[\tau' > t] \geq k_2 \exp\left\{\int_0^t - n(g(s))\, ds\right\}.$$

Now by the transformation $s = \log u$:

$$\mathbb{P}[\tau' > t] \geq k_2 \exp\left\{\int_1^{e^t} \frac{-n(au^{-1/2} - b(u))}{u}\, du\right\}$$

and

$$\mathbb{P}[\tau > t] \geq k_2 \exp\left\{\int_1^t \frac{-n(au^{-1/2} - b(u))}{u}\, du\right\}.$$

We have now given a lower bound for the distribution function of τ. The proof of the upper bound follows in a similar way to that of Theorem 3.1.

Define the following transformation:

$$Y_t = \frac{B_{\alpha(t)}}{f(\alpha(t))},$$

where α is a time change given by

$$\alpha'(t) = f^2(\alpha(t)).$$

As in Section 3, we obtain the inequality

$$\mathbb{P}[\tau > t] \le k_3 \exp\left\{ \int_1^t \frac{-n(au^{-1/2} - b(u))}{u}\, du \right\}$$

for some constant k_3. Note that this uses crucially the existence of a local Lipschitz constant of $n(\cdot)$.

It remains to show that $\int_1^t n(au^{-1/2} - b(u))/u\, du$ is asymptotically equivalent to $\int_1^t n(-b(u))/u\, du$. However, again by the local Lipschitz property of $n(\cdot)$ around $b(\infty)$, noting that $n(\cdot)$ is decreasing:

$$n(-b(u)) - k_4 au^{-1/2} \le n(au^{-1/2} - b(u)) \le n(-b(u))$$

for some constant k_4. But, $u^{-3/2} \in L^1(1, \infty)$, and so

$$\exp\left\{ \int_1^t \frac{-n(-b(u))}{u}\, du \right\} \le \exp\left\{ \int_1^t \frac{-n(au^{-1/2} - b(u))}{u}\, du \right\}$$
$$\le k_5 \exp\left\{ \int_1^t \frac{-n(-b(u))}{u}\, du \right\},$$

thus completing the proof. □

For our result on lower-case boundaries, we need to define a well-behaved class of functions.

DEFINITION. A function $f = t^{1/2}a(t) \in C^+$ if it satisfies:

(i) $a(t) \to 0$ as $t \to \infty$.
(ii) $f(t)f'(t)$ is asymptotically decreasing to 0.

THEOREM 6.3.2. *Suppose $f \in C^+$; then for $a(t) = f(t)t^{-1/2}$,*

$$\mathbb{P}[\tau > t] = O\left(\exp\left\{ -\int_1^t \frac{n(a(s))}{s}\, ds \right\} \right).$$

PROOF. In the usual way we derive the inequalities

$$\mathbb{P}[\tau > t] \le k_1 \exp\left\{ -\int_1^t \frac{n(a(s))}{s}\, ds \right\}$$

and

$$\mathbb{P}[\tau > t] \geq k_2 \exp\left\{ -\int_{\beta(1)}^{\beta(t)} n(r(s), 1)\, ds \right\},$$

where $\beta(t) = \int_0^t 1/f^2(y)\,dy$, α is the inverse of β and $r(s) = f(\alpha(s))f'(\alpha(s))$, k_1 and k_2 are positive constants. Also

$$\begin{aligned}
\int_{\beta(1)}^{\beta(t)} n(r(s), 1)\, ds &= \int_{\beta(1)}^{\beta(t)} 2r(s) n\left((2r(s))^{1/2}\right) ds \\
&= \int_1^t n\left((a^2(s) + 2sa(s)a'(s))^{1/2}\right)\left(\frac{1}{s} + \frac{2a'(s)}{a(s)}\right) ds \\
&\leq \int_1^t \frac{n\left((a^2(s) + 2sa(s)a'(s))^{1/2}\right)}{s}\, ds \\
&\leq \int_1^t \frac{n(a(s)) + k_3 sa(s)a'(s)}{s}\, ds \\
&\leq k_3(a^2(t) - a^2(1)) + \int_1^t \frac{n(a(s))}{a}\, ds \\
&\leq k_4 + \int_1^t \frac{n(a(s))}{s}\, ds.
\end{aligned} \tag{6.3.1}$$

Here the equality (6.3.1) follows from Lemma 6.2.3 and the subsequent inequalities follow from the facts that n is a decreasing function and is continuously differentiable, and this leads to the existence of positive constants k_3 and k_4. So

$$\mathbb{P}[\tau > t] \geq k_5 \exp\left\{ -\int_1^t \frac{n(a(s))}{s}\, ds \right\}$$

for some constant k_5, completing the proof. □

DEFINITION. f is a simple positive upper-case function if $f(t)f'(t)$ is asymptotically nondecreasing to ∞.

THEOREM 6.3.3. *Suppose $f(t) = a(t)t^{1/2}$ is a simple positive upper-case function, then*

$$\mathbb{P}[\tau > t] = O\left(\exp\left\{ -\int_1^t \frac{n(a(s))}{s}\, ds \right\} \right).$$

PROOF. Since f is a simple positive upper-case function, we derive the usual inequalities

$$\mathbb{P}[\tau > t] \geq k_1 \exp\left\{ -\int_1^t \frac{n(a(s))}{s}\, ds \right\}$$

and

$$\mathbb{P}[\tau > t] \le k_2 \exp\left\{ - \int_{\beta(1)}^{\beta(t)} n(r(s), 1)\, ds \right\},$$

where $r(s) = f(\alpha(s)) f'(\alpha(s))$, $\alpha'(s) = f^2(\alpha(s))$ and β is the inverse of α. Again k_1 and k_2 are positive constants. But

$$\begin{aligned} \int_{\beta(1)}^{\beta(t)} n(r(s), 1)\, ds &\ge \int_1^t \frac{n\left(a(s)(1 + (2sa'(s))/a(s))^{1/2}\right)}{s}\, ds \\ &\ge \int_1^t \frac{n(a(s) + sa'(s))}{s}\, ds, \end{aligned}$$

and $n'(b)$ is bounded for large b. This can be seen, for example, from Theorem 6.2.7 and the facts that n maps $[b, \infty)$ onto a subset of the compact region $[0, n(b)]$ and $e_b(x) = k_3 D_{-2n(b)}$ for some positive constant k_3. So, let the lower bound on $n'(b)$ be $-k_4$:

$$\begin{aligned} \int_1^t \frac{n(a(s) + sa'(s))}{s}\, ds &\ge \int_1^t \frac{n(a(s))}{s}\, ds - k_4 \int_1^t \frac{n'(a(s)) sa'(s)}{s}\, ds \\ &= \int_1^t \frac{n(a(s))}{s}\, ds - k_4(n(a(t)) - n(a(1))). \end{aligned}$$

So clearly,

$$\mathbb{P}[\tau > t] \le k_5 \exp\left\{ - \int_1^t \frac{n(a(s))}{s} \right\}$$

for some constant k_5, as required. □

Unfortunately, for the negative upper-case boundary, the hitting time becomes "too fast" and conditional distributions change too quickly to allow our methods to give such good estimates. We can, however, derive the following theorem which will be stated without proof since the methodology does not involve any new ideas.

THEOREM 6.3.4. *Suppose f is a simple negative upper-case boundary, that is, a negative square-root boundary such that $a(t)a'(t)$ is asymptotically increasing to ∞. Then its hitting time τ satisfies*

$$\mathbb{P}[\tau > t] \le O\left(\exp\left\{ \int_1^t \frac{-n(a(s))}{s}\, ds \right\} \right)$$

and

$$\mathbb{P}[\tau > t] \ge O\left(\exp\left\{ \int_0^{\beta(t)} - n(r(s), 1)\, ds \right\} \right).$$

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