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ERGODIC THEORY OF STOCHASTIC PETRI NETWORKS

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Stochastic Petri networks provide a general formalism for describing the dynamics of discrete event systems. The present paper focuses on a subclass of stochastic Petri networks called stochastic event graphs, under the assumption that the variables used for their "timing" form stationary and ergodic sequences of random variables. We show that such stochastic event graphs can be seen as a $(\max, +)$ linear system in a random, stationary and ergodic environment. We then analyze the associated Lyapounov exponents and construct the stationary and ergodic regime of the increments, by proving an Oseledec-type multiplicative ergodic theorem. Finally, we show how to construct the stationary marking process from these results.

1. Introduction. Timed Petri networks can be viewed as a general formalism for describing the dynamics of discrete event systems. It is beyond the scope of the present paper to review the domains of application of this formalism and the interested reader should refer to [4] and [6] for some entry points to the relevant literature. It is worthwhile mentioning that this formalism is powerful enough for allowing one to describe most of the existing models in queuing theory. In particular, the subclass of timed Petri networks considered here, namely, event graphs, contains several classical queuing models (e.g., single server queues, queues in tandem, closed cyclic networks, synchronized queuing networks, network of queues with blocking, etc.; see [2]). The main practical concerns of the paper are the construction of the stationary behavior of stochastic event graphs and the conditions under which such a stationary regime exists, when assuming that the sequences used in the *timing* of the Petri net are stationary and ergodic. The paper is structured as follows. Timed Petri networks and the subclass of timed event graphs are described together with some of their basic properties in Section 2. This section has no probability theory at all and is based on basic graph theoretic considerations. The main new result consists of showing that the choice of adequate state variables allows one to see a stochastic event graph as a linear system in a random environment, where the linearity is understood with respect to the semifield $(\mathbb{R}, \max, +)$. The probabilistic issues are addressed in Section 3, where the statistical assumptions are described. Section 3.2 is concerned with the determination of the maximal Lyapounov exponents of this

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type of linear system. These Lyapounov exponents, which are known as *cycle times* in the Petri net literature, allow one to characterize the linear growth rate of the state variables of the event graph. The construction of the stationary regime of the increments of the state variables is the object of Sections 3.3 and 3.4, and it can be seen as an Oseledec-type multiplicative ergodic theorem in $(\mathbb{R}, \max, +)$. In the *nonautonomous* case, there is a unique stationary regime that is reached in finite time regardless of the initial *lag times* in the event graph, provided the cycle time compares adequately with the asymptotic rate of the input. In the *autonomous* case, there is no uniqueness of the stationary regimes in general. We provide a simple condition for the stationary regime to be unique and to be reached with coupling. Finally, the construction of the stationary marking is addressed in Section 3.5.

The results on cycle times and on the construction of the stationary regime can be seen as stochastic extensions of known results on the periodic regimes of deterministic timed Petri networks [8], [3]. The observation that deterministic event graphs can be seen as $(\mathbb{R}, \max, +)$ linear systems was first made in [3], where the algebraic framework that is used in the present paper was also developed. The main probabilistic tool that is used to extend these results to the stochastic case is the theory of stochastic recursive sequences [2].

2. Timed Petri networks.

2.1. *Definition and notation.* The aim of this section is to sketch the formal definition of a Petri net. The reader is invited to consult [6] for more details and for examples. A Petri net is a pair $(\mathscr{G}, \mu)$. $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a bipartite graph with a finite set of nodes $\mathscr{V}$ which is partitioned into the disjoint sets $\mathscr{P}$ (the set of *places*) and $\mathscr{Q}$ (the set of *transitions*); the set of arcs $\mathscr{E}$ consists of pairs of the form (p_i, q_j) or (q_j, p_i), with $p_i \in \mathscr{P}$ and $q_j \in \mathscr{Q}$. We denote the sets of predecessors and the set of successors of a node t of $\mathscr{G}$ by $\pi(t)$ and $\sigma(t)$, respectively. If $p_i \in \pi(q_j)$, we also write $i \in \pi^q(j)$, $i = 1, \dots, |\mathscr{P}|$, $j = 1, \dots, |\mathscr{Q}|$; similarly, if $q_j \in \pi(p_i)$, we write $j \in \pi^p(i)$, and if $q_j \in \sigma(p_i)$, $j \in \sigma^p(i)$, etc. The graph $\mathscr{G}$ is also assumed to be connected. It is also assumed that, for all i, $\pi(p_i) \neq \varnothing$. The Petri net is *autonomous* provided $\pi(q_j) \neq \varnothing$, for all j.

The *initial marking* μ is a $|\mathscr{P}|$-vector with nonnegative integer entries. The integer μ_i is interpreted as the number of *tokens* initially in place p_i. Roughly speaking, places represent conditions and transitions represent events. A transition has a certain number of input and output places representing the preconditions. The presence of a token in a place is interpreted as the condition associated with that place being fulfilled. Petri networks can be seen as dynamical systems where changes occur according to the following rules:

1. A transition is said to be *enabled* if each upstream place contains at least one token.
2. The *firing* of an enabled transition removes one token from each of its upstream places and adds one token to each of its downstream places.

For q_j to be enabled in the initial marking, we need that $\mu_i \geq 1$, for all $p_i \in \pi(q_j)$. If the enabled transition q_j fires, then a new marking μ' is obtained with $\mu'_i = \mu_i - 1 \geq 0$, if $p_i \in \pi(q_j)$, $\mu'_i = \mu_i + 1$, if $p_i \in \sigma(q_j)$, and $\mu'_i = \mu_i$, otherwise. By firing a transition that is enabled for the marking μ', we get a new marking μ'' in the same way. The markings μ' and μ'' are said to be reachable (here in one and two steps, respectively) from μ.

A Petri net, with initial marking μ, is said to be *live* if, for all markings ν reachable from μ and for all transitions q, there exists a marking o reachable from ν where q is enabled. A Petri net is called an *event graph* if each place has exactly one upstream and one downstream transition. In the literature, event graphs are also referred to as *marked graphs* or as *decision free Petri networks*. Under the foregoing assumptions, an autonomous event graph with initial marking μ is live if and only if every circuit of $\mathscr{G}$ contains at least one place with $\mu_i > 0$ (see [6]).

2.2. *Timed event graphs.*

2.2.1. *Firing times and holding times.* Time can be introduced in two basic ways: by associating durations with either transition firings or with the sojourn of tokens in places. By definition, the *firing time* of a transition is the time that elapses between the starting and the completion of a firing of the transition. The tokens to be consumed by a transition remain in the preceding places during the firing time and are called *reserved tokens*. When a transition produces a token in a place, this token cannot immediately contribute to the enabling of the downstream transitions; it must first spend some *holding time* in that place.

Another general rule states that transitions fire as soon as they are enabled and that tokens start enabling the transition downstream as soon as they have completed their holding times.

REMARK. With our definitions, nothing prevents a transition from having several simultaneous firings. If we want to preclude such a phenomenon, we may add an extra place associated with this transition. This extra place should have the transition under consideration as unique predecessor and successor, and one token in the initial marking. The addition of this loop models a mechanism that is called a *recycling* of the transition. With this mechanism, the $(k+1)$st firing can only start after the completion of the kth.

2.2.2. *Initial condition.* Assume that we start looking at the system evolution at time $t = 0$ and that the piecewise constant function $N_i(t)$ describing the evolution of the number of tokens present in p_i, $i = 1, \ldots, |\mathscr{P}|$, at time $t \in \mathbb{R}$, is right continuous, and take $N_i(0) = \mu_i$. The general idea behind the initial condition is that the $N_i(0)$ $(= \mu_i)$ tokens visible in p_i at time $t = 0$ are assumed to have entered p_i before time 0 and to be either completing their holding times or ready to be consumed. The initial condition is defined through a vector of $\mathbb{R}$-valued initial *lag times*, where the lag time of a token of the initial marking of p_i is the epoch when this token starts contributing to enabling $\sigma(p_i)$.

However, these lag times should be compatible with the general rules that transitions fire as soon as they are enabled, and so on. For instance, if the lag time of an initial token exceeds its holding time, this token cannot have entered the place before time 0; similarly, if the lag times (which are possibly negative) are such that one of the transitions completes firing and consumes tokens of the initial marking before $t = 0$, these tokens cannot be part of the marking seen at time 0 since they should then have left before time 0. The initial condition of a timed event graph (i.e., the initial marking and a vector of lag times) is said to be compatible if (1) the lag time of each initial token does not exceed its holding time and (2) the first epoch when a transition completes firing is nonnegative.

2.2.3. FIFO *places and transitions*. A place p_i is first-in-first-out (FIFO) if the kth token to enter this place is also the kth to contribute enabling the transition $q_j = \sigma(p_i)$. For instance, a place with constant holding times is FIFO. A transition q_j is FIFO if the kth firing of q_j to start is also the kth to complete. For instance, a transition with constant firing times is always FIFO. If a transition is recycled, its $(k + 1)$st firing cannot start before the completion of the kth one, so that a recycled transition is necessarily FIFO, regardless of the firing times. An event graph is FIFO if all its places and transitions are FIFO. A typical example of FIFO timed event graph is that of a system with constant holding times and recycled transitions with possibly variable firing times. In the sequel, we assume that the event graph under consideration is FIFO and has all its transitions recycled.

2.2.4. *Numbering of events*. The following way of numbering the tokens that traverse a place and the firings of a transition is adopted: by convention, the kth token, $k \geq 1$, of place p_i is the kth token to contribute enabling transition $\sigma(p_i)$ during the evolution of the event graph, including the tokens of the initial marking. The kth firing, $k \geq 1$, of transition q_i is the kth firing of q_i to be initiated, including the firings that consume initial tokens.

2.2.5. *Dynamics*. In the sequel, the sequences of holding times $\alpha_i(k)$, $i = 1, \ldots, |\mathscr{P}|$, $k \in \mathbb{Z}$, and of firing times $\beta_j(k)$, $j = 1, \ldots, |\mathscr{Q}|$, $k \in \mathbb{Z}$, are assumed to be given. The dynamics of the event graph are defined as follows:

1. The kth token of place p_i incurs the holding time $\alpha_i(k)$.
2. Once it is enabled, the kth firing of q_j takes the firing time $\beta_j(k)$.

We now state a few basic properties of the numbering in a FIFO event graph with a compatible initial condition. For i such that $\mu_i \geq 1$, denote by $w_i = (w_i(1), w_i(2), \ldots, w_i(\mu_i))$ the vector of the lag times of the initial tokens of place p_i ordered in a nondecreasing way.

Observe that if the initial condition is compatible and if the timed event graph is FIFO, then for all $1 \leq k \leq \mu_i$, such that $\mu_i \geq 1$, the initial token with lag time $w_i(k)$ is also the kth token of place p_i, according to the numbering convention. If this last property does not hold for some place p_i, then necessarily a token which does not belong to the initial marking of p_i and which enters p_i after time 0 contributes enabling $\sigma(p_i)$ before one of the tokens of the initial marking. Since the tokens of the initial marking enter p_i before time 0, this contradicts the assumption that p_i is FIFO.

Using this observation, it is easily checked that the firing of q_j that consumes the kth token of p_i [for all $p_i \in \pi(q_j)$] is also the kth firing of q_i, and that the completion of the kth firing of q_j, $k \geq 1$, produces the $(k + \mu_i)$th token of p_i, for all $p_i \in \sigma(q_j)$.

2.3. *The* $(\mathbb{R}^*, \max, +)$ *semifield*. Consider the set $\mathbb{R}^* \stackrel{\text{def}}{=} \mathbb{R} \cup \varepsilon$, where $\varepsilon \stackrel{\text{def}}{=} -\infty$, and the internal operations $\oplus = \max$ and $\otimes = +$. It is easy to check that $(\mathbb{R}^*, \oplus, \otimes)$ is a commutative and idempotent semifield: $\oplus$ is associative and commutative and has ε for neutral element; $\otimes$ is associative and commutative with neutral element $e = 0$; and $\otimes$ is both right and left distributive with respect to $\oplus$. In particular, we can define matrix multiplications in this semifield by the usual relation

$$(A \otimes B)_{ij} \stackrel{\text{def}}{=} \bigoplus_{k=1}^{J} A_{ik} \otimes B_{kj}.$$

Matrix multiplication is easily checked to be associative. As in the usual algebra, we often drop the $\otimes$ symbol (for instance, we also rewrite the last expression as $\oplus_{k=1 \cdots J} A_{ik} B_{kj}$). Observe that in this semifield, $x^k = x \otimes \cdots \otimes x$ is equal to k times x in the usual algebra. In the same vein, $x^{1/k}$ is simply x divided by k. For more on this structure, see [3].

2.4. *Autonomous* (max, +) *linear systems*.

2.4.1. *The basic recursive equations*. The state variable $x_j(k)$, $j = 1, \ldots, |\mathscr{Q}|$, $k \geq 1$, of the event graph is the epoch when transition q_j starts firing for the kth time, with the convention that, for all q_i, $x_i(k) = \infty$ if q_i fires less than k times. These state variables are continued to negative values of k by the relation $x_j(k) = \varepsilon$, for all $k \leq 0$. Let $M = \max_{i=1,\ldots,|\mathscr{P}|} \mu_i$. Define the $|\mathscr{Q}| \times |\mathscr{Q}|$ matrices $A(k,k), A(k,k-1), \ldots, A(k,k-M)$,

$$A_{jl}(k,k-m) \tag{2.1}$$
$$\stackrel{\text{def}}{=} \begin{cases} B_l(k-m) \otimes \bigoplus_{\{i \in \pi^q(j) | \pi^p(i) = l\}} \alpha_i(k), & \text{if } \mu_{\pi^q(j)} = m, \\ \varepsilon, & \text{otherwise}, \end{cases}$$

and the $|\mathscr{Q}|$-dimensional vector $v(k)$, $k = 1, \ldots, M$,

$$v_j(k) \stackrel{\text{def}}{=} \begin{cases} \bigoplus_{\{i \in \pi^q(j) | \mu_i \geq k\}} w_i(k), & \text{if } \{i \in \pi^q(j) | \mu_i \geq k\} \neq \varnothing, \\ \varepsilon, & \text{otherwise}. \end{cases} \tag{2.2}$$

In these definitions, we adopt the convention that the $\oplus$-sum over an empty set is ε.

THEOREM 1. *For a timed event graph with recycled transitions, the state vector $x(k) = (x_j(k))$ satisfies the recursive equations*

$$x(k) = A(k,k)x(k) \oplus A(k,k-1)x(k-1) \oplus \cdots \oplus A(k,k-M)x(k-M), \tag{2.3}$$

$k = M+1, M+2, \dots$, with the initial conditions

$$x(k) = A(k,k)x(k) \oplus \cdots \oplus A(k,k-M)x(k-M) \oplus v(k), \quad \text{for } k = 1,2,\dots,M,$$

where $x_j(k) \overset{\text{def}}{=} \varepsilon$ for all $k \le 0$.

PROOF. We first prove that the variables $x_j(k)$, $j = 1,\dots,|\mathscr{Q}|$, $k \ge 1$, satisfy the recursive equations

$$x_j(k) = \bigoplus_{\{i \in \pi^q(j) | k > \mu_i\}} \left(x_{\pi^p(i)}(k-\mu_i) \otimes \beta_{\pi^p(i)}(k-\mu_i) \otimes \alpha_i(k) \right) \oplus \bigoplus_{\{i \in \pi^q(j) | k \le \mu_i\}} w_i(k). \tag{2.4}$$

The kth firing, $k \ge 1$, of transition q_j starts as soon as, for all $i \in \pi^q(j)$, the kth token of p_i contributes to enabling q_j. In view of the FIFO assumptions, for $k > \mu_i$, this kth token is produced by the $(k-\mu_i)$th firing of the transition $\pi(p_i)$, so that the epoch when this token contributes enabling $\sigma(p_i)$ is $x_{\pi^p(i)}(k-\mu_i) \otimes \beta_{\pi^p(i)}(k-\mu_i) \otimes \alpha_i(k)$. For $k \le \mu_i$, this event takes place at time $w_i(k)$, in view of the FIFO assumptions, which completes the proof of (2.4). Now use the associativity and commutativity of $\oplus$, together with our convention on $\oplus$-sums over empty sets, to rewrite $x_j(k)$, $k > M$, as

$$\bigoplus_{\{m=0,\dots,M\}} \bigoplus_{\{l=1,\dots,|\mathscr{Q}|\}} \bigoplus_{\{i \in \pi^q(j) | \pi^p(i) = l,\, \mu_i = m\}} x_l(k-m) \otimes \beta_l(k-m) \otimes \alpha_i(k).$$

The distributivity of $\otimes$ with respect to $\oplus$ implies in turn that $x_j(k)$ is equal to

$$\bigoplus_{\{m=0,\dots,M\}} \bigoplus_{\{l=1,\dots,|\mathscr{Q}|\}} \left(x_l(k-m) \otimes \bigoplus_{\{i \in \pi^q(j) | \pi^p(i) = l,\, \mu_i = m\}} \beta_l(k-m) \otimes \alpha_i(k) \right),$$

which completes the proof of (2.3), in view of the definition of A. The proof of (2.3) for $k \le M$ follows the same lines. □

2.4.2. *Simplifications.* The recursive equations (2.4) are unchanged if we put all the firing times equal to e and if one replaces $\alpha_i(k)$ by $\beta_{\pi^p(i)}(k-\mu_i) \otimes \alpha_i(k)$. There is hence no loss of generality assuming $\beta \equiv e$, which will be done from now on.

It is now shown that the initial condition of (2.3) can be further simplified whenever the lag times satisfy certain additional constraints. For all i such that $\mu_i > 0$, denote by $y_i(k)$, $k \leq 0$, the *entrance time* function associated with place p_i, defined by the relation

$$y_i(k - \mu_i) \stackrel{\text{def}}{=} \begin{cases} w_i(k) - \alpha_i(k), & \text{for } 1 \leq k \leq \mu_i, \\ \varepsilon, & \text{for } k > \mu_i. \end{cases} \tag{2.5}$$

The initial condition is said to be *strongly compatible* if, for any pair of places p_i and p_j which follow the same transition, the entrance times $y_i(k)$ and $y_j(k)$ coincide provided $k \geq \min(\mu_i, \mu_j)$, namely, if there exist functions $z_j(k)$, $j = 1, \dots, |\mathcal{Q}|$, $k \leq 0$, such that

$$y_i(k) = z_{\pi^p(i)}(k), \quad \forall\, i, k \text{ such that } -\mu_i + 1 \leq k \leq 0. \tag{2.6}$$

Observe that the function $z_j(k)$ is only defined through (2.6) for $-M_j < k \leq 0$, provided $M_j \stackrel{\text{def}}{=} \max_{i \in \sigma^q(j)}(\mu_i) \geq 1$. For other values of k, or if $M_j = 0$, take $z_j(k) = \varepsilon$.

COROLLARY 1. *For a FIFO timed event graph with a strongly compatible initial condition, the state vector $x(k) = (x_j(k))$ satisfies the recursive equations*

$$\begin{aligned} x(k) = A(k,k)x(k) \oplus A(k, k-1)x(k-1) \\ \oplus \cdots \oplus A(k, k-M)x(k-M), \end{aligned} \tag{2.7}$$

$k = 1, 2, \dots,$ *provided the continuation of $x(k)$ to negative values of k is defined by $x_j(k) = z_j(k)$, $\forall\, k \leq 0$, $j = 1, \dots, |\mathcal{Q}|$.*

PROOF. By successively using (2.5) and (2.6) we get

$$\bigoplus_{\{i \in \pi^q(j) | k \leq \mu_i\}} w_i(k) = \bigoplus_{\{i \in \pi^q(j) | k \leq \mu_i\}} \left(z_{\pi^p(i)}(k - \mu_i) \otimes \alpha_i(k) \right),$$

for all $k = 1, 2, \dots,$ so that we can rewrite (2.4) as indicated when using the proper continuation of x. □

2.4.3. *Constructiveness of the recursive equations.*

LEMMA 1. *The event graph is live if and only if there exists a permutation matrix P for which the matrix $P'A(k,k)P$ is strictly lower triangular.*

PROOF. If the matrix $P'A(k,k)P$ is strictly lower triangular for some permutation P, then there is no cycle with zero initial marking, in view of the definition of $A(k,k)$ [see (2.1)]. Conversely, if the event graph is live, the matrix $A(k,k)$ has no circuit, and there exists a permutation of the coordinates that makes A strictly lower triangular. □

If the matrix $A(k,k)$ is strictly lower triangular, $(A(k,k))^n = \varepsilon$ for $n \geq |\mathcal{Q}|$, and the matrix

$$A^*(k,k) \stackrel{\text{def}}{=} e \oplus A(k,k) \oplus A^2(k,k) \oplus \cdots$$

(where e denotes here the identity matrix with scalar e on the diagonal and ε elsewhere) is finite. Let

$$\bar{A}(k, k-l) \stackrel{\text{def}}{=} A^*(k,k)A(k,k-l), \qquad l = 1, \dots, M,$$

$$\bar{v}(k) \stackrel{\text{def}}{=} A^*(k,k)v(k), \qquad k \in \mathbb{Z},$$

with $v_j(k) \stackrel{\text{def}}{=} \varepsilon$ for $k \leq 0$ or $k > M$.

THEOREM 2. *If the event graph is live, the recursive equations* (2.3) *can be rewritten as*

$$x(k) = \bar{A}(k, k-1)x(k-1) \oplus \cdots \oplus \bar{A}(k, k-M)x(k-M) \oplus \bar{v}(k), \tag{2.8}$$

$k = 1, 2, \dots, M$, *where* $x_j(k) \stackrel{\text{def}}{=} \varepsilon$, *for all* $k \leq 0$.

PROOF. From (2.3) we get by induction on n that, for all $k = 1, 2, \dots,$

$$x(k) = A^{n+1}(k,k)x(k) \oplus \left(\bigoplus_{m=0,\dots,n} A^m(k,k) \right) \left(\bigoplus_{l=1,\dots,M} A(k,k-l)x(k-l) \oplus v(k) \right).$$

Equation (2.8) follows from the last relation by letting n go to ∞. □

As a direct consequence of the preceding theorem, we get that if the holding times and the lag times are all finite, so are the state variables $x_j(k)$, $j = 1, \dots, |\mathcal{Q}|$, $k \geq 1$.

2.4.4. *Standard autonomous equations.* It may be desirable to replace the initial recurrence (2.8), which is of order M, by an equivalent recurrence of order 1. This is done by using the standard technique which consists in extending the state vector. Let $\tilde{x}(k)$ and $\tilde{v}(k)$ be the $|\mathcal{Q}| \times M$-vectors

$$\tilde{x}(k) \stackrel{\text{def}}{=} \begin{pmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k+1-M) \end{pmatrix}, \qquad \tilde{v}(k) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{v}(k+1) \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix},$$

where ε represents here the $|\mathcal{Q}|$-dimensional null vector. Let $\tilde{A}(k)$, $k \in \mathbb{Z}$, be the $(|\mathcal{Q}| \times M) \times (|\mathcal{Q}| \times M)$ matrix defined by the relation

$$\tilde{A}(k) = \begin{pmatrix} \bar{A}(k+1,k) & \bar{A}(k+1,k-1) & \cdots & \cdots & \bar{A}(k+1,k+1-M) \\ e & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & e & \ddots & \vdots & \varepsilon \\ \vdots & \ddots & e & \varepsilon & \varepsilon \\ \varepsilon & \cdots & \varepsilon & e & \varepsilon \end{pmatrix},$$

where e and ε denote the $|\mathcal{Q}| \times |\mathcal{Q}|$ identity and null matrices, respectively. If we adopt the convention that $x_j(k)$ and $v_j(k)$ are equal to ε for $k \le 0$, it should be clear that (2.9) is a mere rewriting of (2.8).

COROLLARY 2. *The extended state space vector* $\tilde{x}(k)$ *satisfies the* $M \times |\mathscr{Q}|$-*dimensional recurrence of order* 1:

$$\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) \oplus \tilde{v}(k), \qquad k = 1,2,\dots . \tag{2.9}$$

Equation (2.9) will be referred to as the *standard form* of the recursive equations (2.8). In the particular case of a strongly compatible initial condition, these equations read $\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k)$, $k = 1,2,\dots,$ provided the continuation that is taken for $x_j(k)$, $k \le 0$ is adequate. One can associate a timed event graph with (2.9) by associating one transition with each coordinate and one place with each entry of the matrix $\tilde{A}(k)$ that is not ε. One interesting property of this event graph is that its initial marking is such that each μ_i is 1.

2.5. *Nonautonomous* (max, +) *linear systems*. In a nonautonomous event graph, transitions without predecessors (called input transitions) are allowed. This set of transitions is denoted $\mathscr{I}$. With each input transition $q_j \in \mathscr{I}$ is associated an input sequence, that is, an increasing sequence $u_j(k)$, $k \in \mathbb{Z}$, where $u_j(k)$, $k \ge 1$, gives the epoch when q_j fires for the kth time after the origin of time. This sequence is compatible if $u_j(1) \ge 0$ and $u_j(0) < 0$, for all $q_j \in \mathscr{I}$ [for instance, take $u_j(1) = 0$, $q_j \in \mathscr{I}$]. Accordingly, at time $u_j(k)$, $k \ge 1$, of the input sequence, the $(k + \mu_i)$th token of p_i enters this place, for all $p_i \in \sigma(q_j)$. As in the autonomous case, we can prove that the firing times can be taken equal to zero without loss of generality, which will be assumed throughout this section. The input sequence $u_j(k)$, $j \in \mathscr{I}$, is strongly compatible if it is compatible and if, for all $p_i \in \sigma(q_j)$ with $\mu_i \ge 1$, $w_i(k) = u_j(k - \mu_i) \otimes \alpha_i(k)$, for all $1 \le k \le \mu_i$.

2.5.1. *Basic nonautonomous equations*. Define the $|\mathscr{Q}| \times |\mathscr{I}|$ matrices $B(k,k), B(k,k-1),\dots, B(k,k-M)$ by

$$B_{jl}(k,k-l) \overset{\text{def}}{=} \begin{cases} \displaystyle\bigoplus_{\{i \in \pi^q(j) | \pi^p(i) = l\}} \alpha_i(k), & \text{if } q_l \in \mathscr{I},\ q_j \in \mathscr{Q},\ \mu_{\pi^q(j)} = l, \\ \varepsilon, & \text{otherwise,} \end{cases} \tag{2.10}$$

and the $|\mathscr{I}|$-dimensional vector $u(k) \overset{\text{def}}{=} (u_1(k),\dots,u_{|\mathscr{I}|}(k))$, $k = 1,2,\dots$. Using the same arguments as in Theorem 1 we get the following theorem.

THEOREM 3. *Under the foregoing assumptions, the state vector* $x(k) = (x_1(k),\dots,x_{|\mathscr{Q}|}(k))$ *satisfies the recursive equations*

$$\begin{aligned} x(k) &= A(k,k)x(k) \oplus A(k,k-1)x(k-1) \\ &\quad \oplus \cdots \oplus A(k,k-M)x(k-M) \\ &\quad \oplus B(k,k)u(k) \oplus \cdots \oplus B(k,k-M)u(k-M) \oplus v(k), \end{aligned} \tag{2.11}$$

$k = 1, 2, \ldots,$ *where* $x_j(k) \stackrel{\text{def}}{=} \varepsilon$ *for all* $k \le 0$, $v_j(k)$ *is defined in* (2.2) *for* $1 \le k \le M$ *and is* ε *otherwise.*

If the event graph is live, then the recursive equations are constructive. To see this, define

$$\overline{B}(k, k-l) \stackrel{\text{def}}{=} A^*(k,k)B(k,k-l), \qquad k \in \mathbb{Z},\, l = 0, \ldots, M,$$

and use the following theorem (which is proved like Theorem 2).

THEOREM 4. *If the event graph is live, the recursive equations* (2.11) *can be rewritten as*

$$\begin{aligned} x(k) &= \overline{A}(k,k-1)x(k-1) \\ &\quad \oplus \cdots \oplus \overline{A}(k,k-M)x(k-M) \oplus v(k) \\ &\quad \oplus \overline{B}(k,k)u(k) \oplus \cdots \oplus \overline{B}(k,k-M)u(k-M), \end{aligned} \tag{2.12}$$

$k = 1, 2, \ldots, M$, *with the same simplification as in Corollary* 1, *provided the initial lag times are strongly compatible.*

2.5.2. *Standard nonautonomous equations.* Define the $M \times |\mathscr{I}|$-dimensional vector $\tilde{u}(k) \stackrel{\text{def}}{=} (u(k+1), u(k), \ldots, u(k+2-M))$, and define $\tilde{B}(k)$, $k \in \mathbb{Z}$, to be the $(|\mathscr{I}| \times M) \times (|\mathscr{Q}| \times M)$ matrix

$$\tilde{B}(k) = \begin{pmatrix} \overline{B}(k+1,k+1) & \overline{B}(k+1,k) & \cdots & \overline{B}(k+1,k+2-M) \\ \varepsilon & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \cdots & \varepsilon \end{pmatrix}.$$

COROLLARY 3. *The extended state space vector* $\tilde{x}(k)$ *satisfies the* $M \times |\mathscr{Q}|$-*dimensional recurrence of order* 1:

$$\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) \oplus \tilde{B}(k)\tilde{u}(k) \oplus \tilde{v}(k), \qquad k = 1, 2, \ldots. \tag{2.13}$$

If the lag times are strongly compatible, we can rewrite (2.13) as

$$\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) \oplus \tilde{B}(k)\tilde{u}(k), \qquad k = 1, 2, \ldots. \tag{2.14}$$

If we define $\hat{x}(k) = (\tilde{u}(k), \tilde{x}(k))$ and if $\hat{A}(k)$ is the matrix

$$\hat{A}(k) = \begin{pmatrix} \tilde{U}(k) & \varepsilon \\ \tilde{B}(k) & \tilde{A}(k) \end{pmatrix}, \tag{2.15}$$

where $\tilde{U}(k)$ is the diagonal matrix with entries $\tilde{U}_{jj}(k) = \tilde{u}_j(k+1) - \tilde{u}_j(k)$, then it is immediate that (2.14) can also be rewritten as

$$\hat{x}(k+1) = \hat{A}(k)\hat{x}(k), \qquad k = 1, 2, \ldots. \tag{2.16}$$

3. Stochastic event graphs. This section is based on the standard forms of the recursive equations which were established in Corollaries 2 and 3. We apply some of the results that were obtained on this class of recursive equations in [2] and rephrase them within the linear system theoretic formalism. Finally, we come back to the stochastic event graph itself and we show how to construct the stationary marking process.

3.1. *Statistical assumptions.* A timed event graph is a stochastic event graph if the holding times, the firing times and the lag times are all random variables defined on a common probability space. In the autonomous case, it is assumed that the holding times are θ-stationary, while in the nonautonomous case, the assumption is that the holding times and the increments $U_{jj}(k) = u_j(k+1) - u_j(k)$ are jointly θ-stationary. More precisely, the random variables $\alpha_j(k)$, $j = 1, \dots, J$ [resp., $\alpha_i(k)$] and $U_{jj}(k)$, $q_j \in \mathscr{I}$, are assumed to be defined on a common probability space $(\Omega, \mathbb{F}, P)$, endowed with a shift θ which leaves P invariant and is ergodic. Let $\alpha_i = \alpha_i(0)$ and $U_{jj} = U_{jj}(0)$. We have hence $\alpha_i(k) = \alpha_i \circ \theta^k$, $p_i \in \mathscr{P}$, $U_{jj}(k) = U_{jj} \circ \theta^k$, $q_j \in \mathscr{I}$, $k \in \mathbb{Z}$. It is also assumed that the variables α_i and U_{jj} are integrable. In view of these assumptions, if we denote by $\tilde{A}$ the matrix $\tilde{A}(0)$ in (2.9), we have hence $\tilde{A}(k) = \tilde{A} \circ \theta^k$. In addition, the entries of $\tilde{A}$ that are not identically equal to ε are integrable. The same properties hold for $\hat{A}(k)$ in (2.16).

3.2. *Rate theorems.* The basic datum of this section is equation (3.1):

$$x(k+1) = A(k)x(k), \qquad k = 1, 2, \dots, \tag{3.1}$$

where the initial condition is $x(0) = z$. The matrices $A(k)$ are such that $A(k) = A \circ \theta^k$, $k \in \mathbb{Z}$; each entry of A not a.s. equal to ε is nonnegative and integrable; each diagonal element of A is nonnegative. When needed, we stress the dependence on the initial condition in (3.1) by writing $x(k; z)$. In view of our remarks at the end of Sections 2.4.4 and 2.5, and above, this framework covers both the autonomous and nonautonomous models considered in the previous section.

3.2.1. *Maximal Lyapounov exponent.* For $x \in \mathbb{R}^J$ and $A \in \mathbb{R}^{J \times J}$, let $\|x\| = \oplus_{i=1}^J x_i$ and $\|A\| = \oplus_{i,j=1}^J a_{i,j}$. For $x \in \mathbb{R}^J$, we also use the notation $/x/ \overset{\text{def}}{=} \min_{j=1,\dots,J} x_j$.

THEOREM 5. *There exists a constant* $0 \le \mathbf{a} < \infty$ *such that, for all finite initial conditions* $z = x(0)$,

$$\lim_{k \to \infty} \|x(k; z)\|^{1/k} = \lim_{k \to \infty} \|A \circ \theta^{k-1} \cdots A \circ \theta A z\|^{1/k} = \mathbf{a} \quad a.s. \tag{3.2}$$

If the initial condition is integrable, the former limits also hold in mean.

PROOF. The proof is essentially that of Theorem 2.5 in [2], a matrix product version of which is given below. One first gets by induction $e \le$

$E[\|x(k;e)\|] < \infty$, for all $k \geq 0$. Let $X_{m,m+k} = \|x(k;e)\| \circ \theta^m$, $m \in \mathbb{Z}$, $k \geq 0$. We have

$$\|x(k,e)\| = \|A \circ \theta^{k-1} \cdots Ae\| = \|A \circ \theta^{k-1} \cdots A\|.$$

For all pairs of matrices A and B, $\|A \otimes B\| \leq \|A\| \otimes \|B\|$. Therefore, for all $k \geq 1$ and all $0 \leq p \leq k$,

$$\begin{aligned}&\|A \circ \theta^{k-1} \cdots A \circ \theta^p A \circ \theta^{p-1} \cdots A\| \circ \theta^m \\ &\quad \leq \|A \circ \theta^{k-1} \cdots A \circ \theta^p\| \circ \theta^m \|A \circ \theta^{p-1} \cdots A\| \circ \theta^m,\end{aligned}$$

that is, $X_{m,m+k} \leq X_{m,m+p} + X_{m+p,m+k}$, so that the result for $\|x(k,e)\|$ follows immediately from Kingman's theorem on subadditive ergodic processes.

The property (3.2) for arbitrary z follows from the relation

$$\|x(k;e)\|^{1/k}/z/^{1/k} \leq \|x(k;z)\|^{1/k} \leq \|x(k;e)\|^{1/k}\|z\|^{1/k}, \qquad k \geq 0.$$

If z is integrable, the convergence in mean follows immediately from this and the convergence in mean for $\|x(k,e)\|^{1/k}$. □

It is enough to have a circuit of the communication graph with an entry of A with a positive mean value to have $\mathbf{a} > 0$.

3.2.2. *Decomposition into strongly connected components.* The *communication graph* associated with A in (3.1) is the graph on $\mathscr{Q} = \{1, \ldots, J\}$, with arcs the set of pairs (i,j) such that $A_{ji} \neq \varepsilon$ a.s. This graph is assumed to be connected. Let $\mathscr{G}_1 = (\mathscr{Q}_1, \mathscr{E}_1), \ldots, \mathscr{G}_N = (\mathscr{Q}_N, \mathscr{E}_N)$, be the set of *maximal strongly connected subgraphs* of the communication graph (a directed graph is strongly connected if there exists a path from any node to any other node). Due to our assumptions, $\mathscr{Q}_1 \cup \cdots \cup \mathscr{Q}_N = \mathscr{Q}$.

The associated reduced graph is the directed graph on $\{1, 2, \ldots, N\}$ with the set of arcs

$$\{(m,n) | m,n = 1, \ldots, N, \exists\, i,j \in \mathscr{Q}, i \in \mathscr{G}_m, j \in \mathscr{G}_n, A_{ji} \neq \varepsilon\}.$$

This graph is acyclic and connected. The notation $\pi(n)$ is used to represent the set of nodes that are direct predecessors of n in the reduced graph. Without loss of generality, the numbering of the nodes is assumed to be compatible with the graph in the sense that if (m,n) is an arc, then $m < n$. In particular, the source subgraphs (the subgraphs without predecessors) are numbered $\{1, \ldots, N^0\}$. We also denote $\pi^*[n]$ [resp., $\pi^*(n)$] the set of possibly indirect predecessors of n including (resp., excluding) node n.

3.2.3. *Individual growth rates in the strongly connected case.*

COROLLARY 4. *If the matrix A has a strongly connected communication graph, then, for any finite initial condition $z = x(0)$ and for any transition $j = 1, \ldots, J$,*

$$\lim_{k \to \infty} \left(x_j(k;z)\right)^{1/k} = \mathbf{a}, \quad a.s., \tag{3.3}$$

where $\mathbf{a}$ *is the maximal Lyapounov exponent of Theorem* 5. *If the initial condition is integrable, we also have convergence in mean.*

PROOF. From the strong connectedness assumption, for all $i, j = 1, \dots, J$, there exists a path from i to j of length less than J. This plus the assumption on the diagonal elements imply that $x_j(k; z) \geq x_i(k - J; z)$ for all $i, j = 1, \dots, J$ and $k > J$. The property (3.3) follows then from the bounds $\|x(k - J; z)\| \leq x_j(k; z) \leq \|x(k; z)\|$, $j = 1, \dots, J$, and from Theorem 5. □

3.2.4. *General case*. Consider the case of an event graph with N strongly connected subgraphs. For all $1 \leq n \leq N$, and for any vector $x \in \mathbb{R}^J$ (resp., matrix $A \in \mathbb{R}^{J \times J}$) let $x^{\{n\}}(k)$ [resp., $A^{\{n\}}(k)$] be the restriction of $x(k)$ [resp., $A(k)$] on the subspace of $\mathbb{R}^J$ corresponding to $\mathcal{Q}_n$, and let $\mathbf{a}_{\{n\}}$ be the maximal Lyapounov exponent of $A^{\{n\}}(k)$; similarly, let $x^{[n]}(k)$ [resp., $A^{[n]}(k)$] be the restriction of $x(k)$ [resp., $A(k)$] on the subspace corresponding to $\bigcup_{m \in \pi^*[n]} \mathcal{Q}_m$, and let $\mathbf{a}_{[n]}$ be the maximal Lyapounov exponent of $A^{[n]}(k)$.

THEOREM 6. *For all finite initial conditions,*

$$\lim_{k \to \infty} \left\| x^{\{n\}}(k) \right\|^{1/k} = \mathbf{a}_{[n]} = \bigoplus_{m \in \pi^*[n]} \mathbf{a}_{\{m\}} \quad a.s. \tag{3.4}$$

Similarly, for all $j \in \mathcal{Q}_n$, $\lim_{k \to \infty} x_j(k)^{1/k} = \mathbf{a}_{[n]}$, *a.s., with the usual addition concerning the convergence in mean, provided the initial condition is integrable.*

PROOF. It is obvious from the definition that $\|x^{\{n\}}(k)\| \leq \|x^{[n]}(k)\|$, so that $\liminf_k \|x^{\{n\}}(k)\|^{1/k} \leq \mathbf{a}_{[n]}$. Using the fact that for all $j \in \mathcal{Q}_n$ and $h \in \bigcup \mathcal{Q}_m$, $m \in \pi^*(n)$, there is a path of length less than J from h to j, we get the following bound from (3.1):

$$x_j(k+1) \geq \bigoplus_{\{h,\, h \in \mathcal{Q}_m,\, m \in \pi^*(n)\}} x_h(k - J), \qquad \forall j \in \mathcal{Q}_n,$$

provided $k \geq J$. Therefore, $\|x^{\{n\}}(k+1)\| \geq \|x^{[n]}(k - J)\|$, for $k \geq J$, so that $\limsup_k \|x^{\{n\}}(k)\|^{1/k} \geq \mathbf{a}_{[n]}$ a.s.. The proof of the individual a.s. limits follows the same lines as in Corollary 4. The proof for the convergence of the expectations in the integrable case is immediate. A proof of the second equality in (3.4) can be found in Theorem 2.7 and Corollary 2.8 in [2]. □

We can draw the following practical conclusions for stochastic event graphs: All the transitions in a strongly connected component of the graph have the same asymptotic firing rate (if there is only one such component, the constant $\mathbf{a}$ is also called the *cycle time* of the event graph); the firing rates of the components are obtained from the constants $\mathbf{a}_{\{n\}}$ (namely, the cycle times of the strongly connected components "in isolation") by (3.4).

3.3. *Multiplicative ergodic theorem—Nonautonomous case.* Consider a nonautonomous stochastic event graph with associated equation

$$x(k+1) = A(k)x(k) \oplus B(k)u(k), \qquad k = 1, 2, \dots, \tag{3.5}$$

and with initial condition $x(0) = z$. The statistical assumptions are the following: the $J \times J$ matrices $A(k)$ and the $J \times I$ matrices $B(k)$ satisfy the relations $A(k) = A \circ \theta^k$ and $B(k) = B \circ \theta^k$, where the entries of A and B that are not a.s. equal to ε are positive and integrable; the increments $U_{ij}(k) \overset{\text{def}}{=} u_i(k+1) - u_j(k)$, $i, j = 1, \dots, I$, $k \in \mathbb{Z}$, are such that $U_{ij}(k) = U_{ij} \circ \theta^k$, where U_{ij} is integrable and

$$\lim_{k \to \infty} \|u(k)\|^{1/k} = \lim_{k \to \infty} /u(k)/^{1/k} = \mathbf{u} \quad \text{a.s.}, \tag{3.6}$$

so that $E[U_{ii}] = \mathbf{u}$ for all $i = 1, \dots, |\mathscr{I}|$. The following notation is used for A: $\mathbf{a}$ denotes the maximum Lyapounov exponent, and $\mathbf{a}_{\{n\}}$, $\mathbf{a}_{[n]}$, $n = 1, \dots, N$, the constants defined in Section 3.2.4, so that $\mathbf{a} = \oplus_n \mathbf{a}_{\{n\}}$. In view of the results of Section 2.7, this framework is that of a nonautonomous stochastic event graph with strongly compatible initial condition. In addition to this, the following nondegeneracy of the input will be assumed: for all $n - 1, 1, \dots, N^0$, there exists a $j \in \mathscr{Q}_n$ such that $(B(k)u(k))_j \neq \varepsilon$.

3.3.1. *Construction of stationary increments.* The basic process of interest is the increment process

$$\delta_{ji}(k; z) \overset{\text{def}}{=} x_j(k+1, z) - u_i(k), \qquad j = 1, \dots, J, i = 1, \dots, I.$$

THEOREM 7. *If $\mathbf{a} < \mathbf{u}$, there exists a unique finite random matrix $\delta = (\delta_{ij})$ such that the increments $\delta_{ij}(k)$ couple in finite time with the stationary and ergodic process $\delta_{ij} \circ \theta^k$, regardless of the initial condition. If $\mathbf{a} > \mathbf{u}$, let n_0 be the first $n = 1, \dots, N$ such that $\mathbf{a}_{\{n\}} > \mathbf{u}$. Then all increments of the form $\delta_{ji}(k)$, $j \in \mathscr{Q}_{n_0}$, tend to ∞ a.s., for all finite initial conditions.*

PROOF. Let $s(k+1) = B(k)u(k)$, $k \in \mathbb{Z}$. It may happen that, for some $j = 1, \dots, J$, $s_j(k)$ is identically equal to ε. In a first step, we assume that this is not the case. Let us show that under this assumption, the increments $s_j(k+1) - s_i(k)$ are such that $s_j(k+1) - s_i(k) = S_{ji}(k) = S_{ji} \circ \theta^k$, where S_{ji} is integrable. For all i, j, we get the relation

$$\begin{aligned} s_i(k+1) - s_j(k) &= \bigoplus_{\{l \mid B_{il} \neq \varepsilon\}} B_{il}(k+1)u_l(k+1) - \bigoplus_{\{m \mid B_{jm} \neq \varepsilon\}} B_{jm}(k)u_m(k) \\ &= \bigoplus_{\{l \mid B_{il} \neq \varepsilon\}} \min_{\{m \mid B_{jm} \neq \varepsilon\}} \left\{ U_{lm}(k) \otimes B_{il}(k+1) - B_{jm}(k) \right\}, \end{aligned}$$

from which the θ-stationarity and the integrability of $s_i(k+1) - s_j(k)$ are easily deduced. By letting k go to ∞ in the relation

$$s_i(k)^{1/k} = \bigoplus_{\{l \mid B_{il} \neq \varepsilon\}} B_{il}(k)^{1/k} u_l(k)^{1/k},$$

and by using the assumption that $E[U_{ii}] = \mathbf{u}$, for all i, and the integrability of the nonnegative entries of B, we get immediately that $\lim_k s_j(k)^{1/k} = \mathbf{u}$ a.s., so that necessarily $E[S_{jj}] = \mathbf{u}$. Let

$$(3.7) \qquad \delta_j(k) \stackrel{\text{def}}{=} x_j(k) - s_j(k), \qquad j = 1, \dots, J,\ k \geq 0.$$

When we subtract $s_j(k+1) \neq \varepsilon$ from the jth line of the matrix relation (3.7), we get

$$\delta_j(k+1) = \left(\bigoplus_i A_{ji}(k) x_i(k) - s_j(k+1) \right) \oplus e = \left(\bigoplus_i \Xi_{ji}(k) \delta_i(k) \right) \oplus e,$$

where $\Xi_{ji}(k) \stackrel{\text{def}}{=} A_{ji}(k) - S_{ji}(k)$, $i, j = 1, \dots, J$, $k \geq 0$. Therefore, the state variables $\delta(k)$ satisfy the recursion

$$(3.8) \qquad \delta(k+1) = \Xi(k)\delta(k) \oplus e, \qquad k \geq 0,$$

with the initial condition $\delta_j(0) \stackrel{\text{def}}{=} x_j(0) - s_j(0)$. This type of equation is in the class considered in Sections 2 and 3 of [2], and the statements of the theorem are direct consequences of the properties stated in Theorems 2.7, 3.1 and 3.3 thereof.

If the vector $s(k)$ has some of its coordinates equal to ε, we can consider the sequence $x'(k) \stackrel{\text{def}}{=} x(k+J)$, $k \geq 1$, which satisfies the equation

$$x'(k+1) = A'(k)x'(k) \oplus s(k),$$

where $A'(k) \stackrel{\text{def}}{=} A(k+J)$ and $s'(k) \stackrel{\text{def}}{=} s(k+J)$, and with the initial condition $x'(0) \stackrel{\text{def}}{=} x(J)$. It is easy to check that under the foregoing assumptions on the input, for all j, there exists $p(j)$ such that $s_{p(j)}(k) \neq \varepsilon$ and $x'_j(k) \geq s'_{p(j)}(k-J)$ for all $k \geq 1$. Therefore, if $s''_j(k) = s_{j'}(k) \oplus s_{p(j)'}(k-J) > \varepsilon$, one can replace $s'(k)$ by $s''(k)$ in the last equation without altering the result of the recursive equations. In other words, we can replace the initial equation by another one that satisfies the same type of statistical properties and where all entries of $s(k)$ are different from ε. □

It is an immediate consequence of Theorem 7 that, under the conditions stated there, other increments like $x_j(k+1) - x_i(k)$, $k = 1, \dots, J$, do also couple with a stationary and ergodic process.

Remark. It was shown in Section 2.5.2 that the nonautonomous equation (3.5) can be rewritten as $\hat{x}(k+1) = \hat{A} \circ \theta^k \hat{x}(k)$, where

$$(3.9) \qquad \hat{A} = \begin{pmatrix} U & \varepsilon \\ B & A \end{pmatrix}, \qquad \hat{x}(k) = \begin{pmatrix} u(k) \\ x(k) \end{pmatrix},$$

and where U is the diagonal matrix with entries $U_{ii} = u_i(1) - u_i(0)$. We know from Theorem 7 that, under the stability condition, (3.5) has a solution with stationary increments. Since there is no loss of generality in assuming that $u_1(0) = e$, we get from (3.5) that this solution satisfies the equation

$$u_1(1)(x(1) - u_1(1)\eta) = A(x(0) - u_1(0)\eta) \oplus B(u(0) - u_1(0)\eta),$$

where $\eta = (e, \dots, e)$. Similarly, we have the obvious relation

$$u_1(1)(u(1) - u_1(1)\eta) = U(u(0) - u_1(0)\eta).$$

Define the $(I + J)$-dimensional vector $\hat{X} = (u(0) - u_1(0)\eta, x(0) - u_1(0)\eta)$ and the random variable $\hat{\lambda} = u_1(1)$. Due to the stationarity of the increments, the preceding relations imply that $(\hat{\lambda}, \hat{X})$ satisfies the eigenpair property $\hat{A}\hat{X} = \hat{\lambda}\hat{X} \circ \theta$. It can be shown that $(\hat{\lambda}, \hat{X})$ is the unique solution eigenpair such that $X_1 = e$. Theorem 7 can hence be seen as an Oseledec-type multiplicative ergodic theorem in $(\mathbb{R}^*, \oplus, \otimes)$ for matrices of the form (3.9).

In conclusion, under appropriate conditions bearing on the rates, each pair of transitions q_i, q_j of a nonautonomous event graph is such that the process $x_j(k + 1) - x_i(k)$ couples in finite time with a uniquely defined stationary and ergodic sequence. Observe, however, that the assumption that $U_{ij}(k)$ is θ-stationary may not be consistent with certain types of compatibility assumptions on the input [like, for instance, the assumption that $u_i(1) = 0$ for all i]; nevertheless, it can be shown that Theorem 7 remains true if we replace the assumption that $U_{ij}(k)$ is θ-stationary by the weaker assumption that $U_{ij}(k)$ couples in finite time with a θ-stationary sequence, so that the actual value of $u(k)$ for k small has no influence on the asymptotic behavior of the increments.

3.4. *Multiplicative ergodic theorem—Autonomous case.* Consider a stochastic event graph with associated equation

$$x(k + 1) = A(k)x(k), \qquad k = 0, 1, 2, \dots, \tag{3.10}$$

where $x(0)$ is some initial condition, with the same assumptions on A as before.

3.4.1. *The strongly connected case.* We assume that the communication graph associated with the matrix A is strongly connected. The basic process of interest is the increment process

$$\delta_{ji}(k; z) = x_j(k + 1, z) - x_i(k; z), \qquad i, j = 1, \dots, J,$$

where $z = x(0)$. Consider the random matrices

$$C(k) \stackrel{\text{def}}{=} A(k - J + 1)A(k - J) \cdots A(k + 1)A(k), \qquad k \in \mathbb{Z}. \tag{3.11}$$

We know that $C_{ij}(k) \geq e$ for all pairs (i, j) because of the strong connectedness assumption. Let $X(k) = x(Jk)$, $k \geq 0$. It is easily checked from (3.10) that the state variables $X(k)$ satisfy the relation

$$X(k+1) = C(k)X(k), \qquad k \geq 0,\ X(0) = x(0) \overset{\text{def}}{=} z. \tag{3.12}$$

Let $\mathscr{B}$ be the event

$$\mathscr{B} = \{\exists\, h_* | C_{jh_*} \circ \Theta \geq C_{jh} \circ \Theta + C_{hi} - C_{h_* i}, \forall\, h, i, j = 1, \dots, J\}, \tag{3.13}$$

where $\Theta = \theta^J$. The following theorem summarizes results proved on the solutions of equations of the type (3.12) in [2], Section 4.

THEOREM 8. *If $\Theta = \theta^J$ is ergodic, the condition $P[\mathscr{B}] > 0$ implies that there exist a finite initial condition $X(0) = Y \in \mathbb{R}^J$ and uniquely defined integrable random variables Δ_{ij}, $i, j = 1, \dots, J$, such that the increments $\Delta_{ji}(k, Y) = X_j(k+1, Y) - X_i(k, Y)$, $k \geq 0$, satisfy the condition $\Delta_{ji}(k, Y) = \Delta_{ji} \circ \Theta^k$, for all $i, j = 1, \dots, J$, $k \geq 0$. In addition, whatever the initial condition z, the increment process $\Delta_{ji}(k, z)$ couples in finite time with the stationary sequence $\Delta_{ji} \circ \Theta^k$. Conversely, if $\Delta_{ji}(k)$, $k \geq 0$, couples in finite time with the stationary process $\Delta'_{ji} \circ \Theta^k$, for all $i, j = 1, \dots, J$, then $\Delta' = \Delta$.*

The results of Theorem 8 yield the following multiplicative ergodic theorem, which can be understood as (part of) an Oseledec theorem for positive matrices in $(\mathbb{R}^*, \oplus, \otimes)$.

COROLLARY 5. *If $C(k) = C \circ \Theta^k$ is a sequence of matrices with nonnegative and integrable entries, where Θ is an ergodic shift, and if $P[\mathscr{B}] > 0$, then there exists a unique finite eigenpair λ, X, with $X_1 = e$ and such that*

$$CX = \lambda X \circ \Theta. \tag{3.14}$$

This eigenpair is integrable, and $E[\lambda] = \mathbf{c}$, where $\mathbf{c}$ is the maximal Lyapounov exponent of C.

PROOF. Let $X(k)$ be the solution of (3.12) with stationary increments. It is easily checked that $X = X(0) - X_1(0)\eta$ and $\lambda = X_1(1) - X_1(0)$ satisfy the properties stated in the theorem. In particular, the property that $E[\lambda] = \mathbf{c}$ follows immediately from the rate property $X_1(k)^{1/k} \to \mathbf{c}$ a.s.

As for the uniqueness, it is immediate that if (λ, X) is an eigenpair as defined above, then $X(k) = \lambda \cdots \lambda \circ \Theta^{k-1} X \circ \Theta^k$, $k \geq 0$, satisfies the equation $X(k+1) = C(k)X(k)$, $k \geq 0$. From the very definition of $X(k)$, it is immediate that the increments $X_j(k+1) - X_i(k)$ are Θ-stationary for all pairs $i, j = 1, \dots, J$. Therefore, for the last property of Theorem 8, we get the relation $X_j(1) - X_i(0) = \Delta_{ji}$. This implies that $\lambda = X_j(1) - X_1(0)$ is then equal to Δ_{11} and that $X_j \circ \Theta = \Delta_{j1} - \Delta_{11}$, so that X and λ are uniquely defined from Δ. □

The results of Theorem 8 can be continued to the increments $\delta(k)$ defined previously.

COROLLARY 6. *Under the conditions of Theorem* 5, *the increments* $\delta(k)$ *also admit a* θ*-stationary regime in the sense that there exists an initial condition for which the relation* $\delta(k) = \delta \circ \theta^k$ *holds for all* $k \geq 0$. *This stationary regime is unique and integrable, and, whatever the initial condition,* $\delta(k)$ *couples with it in finite time.*

PROOF. In the proof of Theorem 5, we established that X is the unique initial condition such that $X_1 = e$ and such that the increments $\Delta_{ji}(k; X) = x_j((k+1)J, X) - x_i(kJ; X)$ are stationary in k, and more precisely such that $\Delta_{ji}(k; X) = \Delta_{ji} \circ \Theta^k$, $k \geq 0$. This implies that the increments $(x_j((k+1)J + 1; X) - x_i(kJ + 1; X))$, $i, j = 1, \dots, J$, are stationary in k, as can be seen when writing them as

$$\bigoplus_{\{h,\, A_{jh} \neq \varepsilon\}} \min_{\{l,\, A_{il} \neq \varepsilon\}} \left\{ \Delta_{hl}(k; X) \otimes A_{jh}((k+1)J) - A_{il}(kJ) \right\}$$

and when using the stationarity of $\Delta(k; X)$. But this increment process is the one generated by the event graph when taking $A(k) \circ \theta$, $k \geq 0$, as timing sequence and Z as initial condition, where $Z_j = x_j(1; X) - x_1(1; X)$, $j = 1, \dots, J$. In view of the uniqueness property mentioned in Theorem 5, we get immediately from this that $x_j(J+1; X) - x_i(1; X) = \Delta_{ji} \circ \theta$, $i, j = 1, \dots, J$. Since $Z_1 = e$, this in turn entails that $Z = X \circ \theta$, in view of the uniqueness property mentioned above.

We show that the increment process $\delta(k; X)$ satisfies the desired property. We have

$$(3.15)\qquad \begin{aligned} \delta_{ji}(1; X) &= \bigoplus_{\{h,\, A_{jh} \neq \varepsilon\}} \left\{ x_h(1; X) \otimes A_{jh}(1) - x_i(1; X) \right\} \\ &= \left\{ X_h \otimes A_{jh}(0) - X_i \right\} \circ \theta, \end{aligned}$$

so that $\delta(k; X) = \delta(0, X) \circ \theta^k$ for $k = 1$. In addition, $\delta(k)$ satisfies the recursive equations $\delta(k+1) = f(\delta(k), A(k+1), A(k))$, where

$$f_{ji}(\delta(k), A(k+1), A(k)) = \bigoplus_{\{h,\, A_{jh} \neq \varepsilon\}} \min_{\{l,\, A_{il} \neq \varepsilon\}} \left\{ \delta_{hl}(k) A_{jh}(k+1) - A_{il}(k) \right\},$$

$i, j = 1, \dots, J$. Using this relation, we prove by an immediate induction that $\delta(k)$ satisfies the preceding relation for all $k \geq 0$.

One proves in the same way that the coupling of the Δ increments with a uniquely defined stationary process implies the same property for δ. The integrability property follows from the integrability of X and the relation

$$\delta_{ji}(0; X) = \bigoplus_{\{h,\, A_{jh} \neq \varepsilon\}} \left\{ X_h \otimes A_{jh}(0) - X_i \right\}. \qquad \square$$

3.4.2. *The general autonomous case.* Consider first the case where the reduced graph has a single source, namely $N^0 = 1$, and assume that it satisfies the assumption of Theorem 5. Then, we get from Theorem 5 that the increments of the state variables $x_j(k)$, $j \in \mathscr{Q}_1$, couple in finite time with a stationary, ergodic and integrable process which satisfies the property $E[x_i(k+1) - x_i(k)] = \mathbf{a}_{\{1\}}$, where $\mathbf{a}_{\{1\}}$ is the maximal Lyapounov exponent associated with $A^{\{1\}}$. The techniques of [2] that were used for the analysis of the general nonautonomous case yield the following theorem.

THEOREM 9. *Under the assumptions of Theorem 5 concerning $A^{\{1\}}$, if $\bigoplus_{n=2,\dots,N} \mathbf{a}_{\{n\}} < \mathbf{a}_{\{1\}}$, then there exists a unique finite random matrix $\delta = (\delta_{ij})$ such that the increments $\delta_{ji}(k)$, $i, j \in \mathscr{Q}$, couple in finite time with a stationary and ergodic process $\delta_{ij} \circ \theta^k$, regardless of the initial condition. If $\bigoplus_{n=2,\dots,N} \mathbf{a}_{\{n\}} > \mathbf{a}_{\{1\}}$, let n_0 be the first $n = 2, \dots, N$ such that $\mathbf{a}_{\{n\}} > \mathbf{a}_{\{1\}}$. Then all increments of the form $\delta_{ji}(k)$, $j \in \mathscr{Q}_{n_0}$, $i \in \mathscr{Q}_m$, $m < n_0$, tend to ∞ a.s., for all initial conditions.*

REMARK. The increments of the variables $x_j(k)$, $j \in \mathscr{Q}_1$, are always integrable (see Corollary 6). However, it is not always true that all the increments of the variables $x_j(k)$, $j \in \mathscr{Q}$, are integrable for $n > 1$. The general law is as follows: Increments of the form $x_j(k+1) - x_i(k)$, $i, j \in \mathscr{Q}_n$, are always integrable, while increments of the form $x_j(k+1) - x_i(k)$, $i \in \mathscr{Q}_m$, $j \in \mathscr{Q}_n$, $m \neq n$, may be finite and nonintegrable.

Consider now the case where the reduced graph has several sources, namely, $N_0 > 1$. If the sources have different cycle times, it is clear that some of the increments of the processes $x_j(k)$, $j \in \mathscr{Q}_n$, $n = 1, \dots, N^0$, can neither be made stationary nor couple with a stationary sequence. Even if all these subgraphs have the same cycle times, nothing general (namely, not depending on more elaborate statistics) can be said on the stationarity of the variables $x_j(k+1) - x_i(k)$ for $j \in \mathscr{Q}_n$, $i \in \mathscr{Q}_m$, $m, n = 1, \dots, N^0$, $m \neq n$, as exemplified in the following simple situation.

Consider a timed event graph with three recycled transitions (q_1, q_2 and t) and five places (p_1, p_2, p_1', p_2' and r); p_i (resp., r) is the place associated with the recycling of q_i, $i = 1, 2$ (resp., t) and p_i' is the place connecting q_i to t. We have two sources $\mathscr{G}_i$, with $\mathscr{Q}_i = \{q_i\}$ and $\mathscr{E}_i = (q_i, q_i)$, $i = 1, 2$, and one non-source subgraph G_3, with $\mathscr{Q}_3 = \{t\}$ and $\mathscr{E}_3 = (t, t)$. Assume the holding times in r, p_1' and p_2' are zero and where the holding times in p_1 and p_2 are independent i.i.d. sequences $\alpha_1(k)$ and $\alpha_2(k)$, with common mean λ. If the variables $\alpha_1(k)$ and $\alpha_2(k)$ are deterministic, the increments $x_1(k) - x_2(k)$ (with obvious notations) are stationary and finite whatever the initial condition. However, if the two sequences are made of exponentially distributed random variables with parameter λ, these increments form a null recurrent

Markov chain on $\mathbb{R}$ that admits no invariant measure with finite mass, so that they cannot be made stationary.

Hence, under the appropriate rate conditions and provided the condition $P[\mathscr{B}] > 0$ holds, the same conclusions hold as in the nonautonomous case. Concerning the practical meaning of the condition $P[\mathscr{B}] > 0$, consider the particular case where the random variables $\alpha_j(k)$ are mutually independent. It is easily checked that a sufficient condition for $P[\mathscr{B}] > 0$ is that there exist one transition in $\mathscr{Q}$ such that all the places that follow it have holding times with infinite support. If this condition is not satisfied, there may be several stationary regimes, and only a lower bound to all possible stationary regimes is known (see [2], Section 4).

REMARK. The notions of initial condition of an equation like (3.5) or (3.10) should not be mixed with the notion of initial condition of the event graph. For instance, the assertions that are made on the existence of initial conditions that make the increments of (3.10) stationary does not imply that this process can be obtained by choosing appropriately the lag times of the initial tokens.

3.5. *Construction of the stationary marking.* In this section, we assume for the sake of simplicity that all transitions are recycled and with positive holding times in the recycling. A place of the event graph is said to be stable if the number of tokens in this place at time t (the marking at time t), converges weakly to a finite random variable when time goes to ∞. The event graph is said to be stable if all the places are stable.

Choose some place p_i in $\mathscr{P}$, and let $q_j = \pi(p_i)$ and $q_l = \sigma(p_i)$. Assume there exists an initial condition X such that $x_j(0, X) = 0$ and such that the increments of the stochastic processes $x_h(k, X)$, $q_h \in \mathscr{Q}$, are stationary and ergodic. Then the sequences $\xi_h(k) \stackrel{\text{def}}{=} x_h(k, X) - x_j(k, X)$, $q_h \in \mathscr{Q}$, $k \geq 0$, can be continued to bi-infinite stationary and ergodic sequences by the relation $\xi_h(0) \circ \theta^k \stackrel{\text{def}}{=} \xi_h \circ \theta^k$, $k \in \mathbb{Z}$, where $\xi_h = \xi_h(0)$. A similar continuation also holds for the sequence $\chi(k) \stackrel{\text{def}}{=} x_j(k+1, X) - x_j(k, X) \stackrel{\text{def}}{=} \chi \circ \theta^k$, $k \geq 0$. Let $\mathscr{N}$ be the marked point process on $(\Omega, \mathbb{F}, P, \theta)$ with interevent times sequence $\chi(k)$, $k \in \mathbb{Z}$, and with the $\mathbb{R}^{|\mathscr{Q}|}$-valued mark sequence $\xi_h(k)$, $q_h \in \mathscr{Q}$, $k \in \mathbb{Z}$. Namely, the kth point of $\mathscr{N}$ is $t(k) \stackrel{\text{def}}{=} x_j(k, X)$, for $k \geq 0$ and $t(k) = \sum_{h=k}^{-1} - \chi(k)$, for $k < 0$, and its mark is $\{\xi_h(k), q_h \in \mathscr{Q}\}$. The interarrival times and the marks being θ-stationary, this point process is stationary (in its Palm version). In view of our assumptions, $\mathscr{N}$ has a finite intensity and no double points.

Let $T(k) = (T_1(k), \ldots, T_J(k))$ be the sequence $T_h(k) \stackrel{\text{def}}{=} t(k) + \xi_h(k)$, $q_h \in \mathscr{Q}$, $k \in \mathbb{Z}$, and let N_i^o be the random variable

$$N_i^o = \sum_{k \leq 0} 1_{\{T_l(k+\mu_i) > 0\}}, \tag{3.16}$$

where $q_l = \sigma(p_i)$. This variable is a.s. finite. Indeed, $T_l(k)$ satisfies the relations $\lim_{k \to \infty} T_l(k)/k = c > 0$, where c is a positive constant. Therefore $T_l(k)$ is an increasing sequence such that $\lim_{k \to -\infty} T_l(k) = -\infty$ a.s. Hence there exists a finite integer-valued random variable H such that $T_l(k) \le 0$ for all $k \le -H$.

THEOREM 10. *Under the assumptions of Theorem* 7 (*resp.*, 9), *if* $\mathbf{a}_{\{n\}} < \mathbf{u}$, *for all* $n = 1, \dots, N$ (*resp.*, $\mathbf{a}_{\{n\}} < \mathbf{a}_{\{1\}}$, *for all* $n = 2, \dots, N$), *then the event graph is stable, whatever the initial condition, and the marking in place* p_i *at arrival epochs converges weakly to the random variable* N_i^o. *Conversely, if* n_0 *is the first* $n = 1, \dots, N$ *such that* $\mathbf{a}_{\{n\}} > \mathbf{u}$ (*resp., the first* $n = 2, \dots, N$ *such that* $\mathbf{a}_{\{n\}} > \mathbf{a}_{\{n\}}$), *then the places connecting the transitions of* $\mathscr{Q}_m \cup \mathscr{I}$ (*resp.*, $\mathscr{Q}_m$), $m < n_0$, *to transitions of* $\mathscr{Q}_{n_0}$ *are all unstable whatever the initial condition.*

PROOF. Let $N_i^o(k)$ be the number of tokens in p_i just after time $x_j(k)$, $k \ge 1$, where $q_j = \pi(p_i)$. A token is in p_i at this time if and only if its index with respect to this place is h, with $1 \le h \le k + \mu_i$, and $x_l(h) > x_j(k)$, where $q_l = \sigma(p_i)$. Therefore,

$$(3.17) \quad N_i^o(k) = \sum_{1 \le h \le k + \mu_i} 1_{\{x_l(h) > x_j(k)\}} = \sum_{0 \le h < k + \mu_i} 1_{\{x_l(k + \mu_i - h) > x_j(k)\}}.$$

We first prove the last assertion of the theorem. Assume that p_i is a place connecting a transition of $\mathscr{Q}_m \cup \mathscr{I}$ (resp., $\mathscr{Q}_m$) to a transition of $\mathscr{Q}_{n_0}$. Due to the property that $\lim_k (x_l(k) - x_j(k))/k = \mathbf{a}_{\{n_0\}} - \mathbf{u} > 0$ [resp., $\lim_k (x_l(k) - x_j(k))/k = \mathbf{a}_{\{n_0\}} - \mathbf{a}_{\{1\}} > 0$], and to the increasingness of the sequences $x_l(k)$ and $x_j(k)$, we get that for all H, $\exists\ K$ such that $\forall\ k \ge K$, $h = 1, \dots, H$, $x_l(k - h) - x_j(k) \ge 0$. It follows immediately from this that $N_i^o(k) \ge H$ for $k \ge K$. Therefore, $N_i^o(k)$ tends to ∞ a.s.

We now prove the first part. We know that the increments of $x_h(k)$, $q_h \in \mathscr{Q}$, couple with their stationary regime in a finite random time K. This implies that, for all fixed h, the sequence $x_l(k + \mu_i - h) - x_j(k)$ couples with a stationary process. More precisely, for all $k \ge K + h$ and $h > \mu_i$, $x_l(k + \mu_i - h) - x_j(k) = -\rho_l(\mu_i - h) \circ \theta^k$, where

$$\rho_l(k) \stackrel{\text{def}}{=} \sum_{n=-1}^{k} \chi \circ \theta^n - \xi_l \circ \theta^{-k} = T_j(0) - T_l(k), \qquad k < 0,$$

in view of the uniqueness of the stationary regimes of the increments. Define

$$H = \inf\{k \ge K | x_l(h) - x_j(k) < 0, \forall\ h = 1, \dots, K\}.$$

H is a.s. finite since K is finite and $x_j(k)$ tends to ∞ a.s. Therefore,

$$(3.18) \quad N_i^o(k) = \sum_{1 \le h \le k - K} 1_{\{x_l(k + \mu_i - h) - x_j(k) > 0\}} = \sum_{1 \le h \le k - K} 1_{\{-\rho_l(\mu_i - h) \circ \theta^k > 0\}},$$

for all $k \ge H$. On the other hand,

$$(3.19) \qquad N_i^o \circ \theta^k = \sum_{0 \le h} 1_{\{T_l(k + \mu_i - h) - T_j(k) > 0\}} = \sum_{0 \le h} 1_{\{-\rho_l(\mu_i - h) \circ \theta^k > 0\}}.$$

Since $T_j(k)$ tends to ∞ as k goes to ∞, we get that there exists L such that

$\sum_{k-h \leq K} 1_{\{T_l(k+\mu_i-h)-T_j(k)>0\}} = 0$, for all $k \geq L$. Therefore, $N_i^o(k) = N_i^o \circ \theta^k$, for $k \geq \max(H, L)$, and the stationary regime of the marking process is reached with coupling, regardless of the initial condition. □

The preceding construction gives the Palm probability of the number of tokens in p_i at arrival epochs. The continuous time distribution of this variable is then directly obtained via the Palm inversion formula (see [1], page 17).

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REFERENCES

[1] BACCELLI, F. and BRÉMAUD, P. (1987). *Palm Probabilities and Stationary Queues. Lecture Notes in Statist.* **41**. Springer, New York.

[2] BACCELLI, F. and LIU, Z. (1992). On a class of stochastic recursive sequences arising in queueing theory. *Ann. Probab.* **20** 350–374.

[3] COHEN, G., DUBOIS, D., QUADRAT, J. P. and VIOT, M. (1985). A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE Trans. Automat. Control* **30** 210–220.

[4] HO, L. (1989). *Dynamics of Discrete Event Systems*. IEEE, New York.

[5] KINGMAN, J. F. C. (1976). Subadditive processes. *Ecole d'Eté de Probabilité de Saint-Flour V. Lecture Notes in Math.* **539** 165–223. Springer, New York.

[6] MURATA, T. (1989). Petri nets: Analysis and applications. *Proc. IEEE* **77** 541–580.

[7] OSELEDEC, V. I. (1968). A multiplicative ergodic theorem. Lyapounov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19** 197–231.

[8] RAMAMOORTHY, C. and HO, G. (1980). Performance evaluation of asynchronous concurrent systems using Petri nets. *IEEE Trans. Software Engrg.* **6** 440–449.

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