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# OCCUPATION TIME LARGE DEVIATIONS FOR THE SYMMETRIC SIMPLE EXCLUSION PROCESS

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We obtain the decay rate of the large deviation probabilities of occupation time for the symmetric simple exclusion process. Furthermore, in dimension $d \neq 2$, we prove a large deviation principle for the occupation time. To obtain these results, we prove hydrodynamical limits for the weakly asymmetric simple exclusion process and we prove a large deviation principle for the empirical density for the symmetric simple exclusion process.

**1. Introduction.** One of the most studied interacting particle systems is the symmetric simple exclusion process which can be informally described as follows. Let $\mathbb{Z}^d$ denote the $d$-dimensional integers. Consider transition probabilities $p(k, j)$ on $\mathbb{Z}^d$ such that $p(k, j) = p(j, k)$. Each site of $\mathbb{Z}^d$ is occupied by at most one particle. A particle at site $k$ waits a mean 1 exponential time and then chooses a site $j$ with probability $p(k, j)$. If the site $j$ is empty, the particle at $k$ jumps to $j$, otherwise it stays at $k$. Throughout this paper, we will denote this process by $(\eta_t) \in D([0, \infty[, \{0, 1\}^{\mathbb{Z}^d})$ and by $\eta(k)$ the number of particles at a site $k \in \mathbb{Z}^d$ for the configuration $\eta$.

In this paper, we investigate the large deviations for the additive functional $\int_0^t \eta_s(0)\, ds$ for two reasons.

On the one hand, we know from [13] that for some transition probabilities $p(k, j)$, the product Bernoulli measures $\nu_\rho$ ($\nu_\rho\{\eta;\ \eta(k) = 1\} = \rho$) are extremal invariant for the process. Therefore, $(1/t)\int_0^t \eta_s(0)\, ds \to \rho$, $P_{\nu_\rho}$-almost surely. Moreover, Kipnis [9] proved a central limit theorem for this functional. So, it is a natural question to consider the large deviations.

On the other hand, since in dimensions 1 and 2 random walks with finite second moments are recurrent (see [15]), it is natural to expect that in these dimensions the asymptotic decay rates of large deviations will be $o(t)$ as $t \to \infty$ (see [12] for the recurrent case and [8] and [11] for the transient). This is indeed the case. In fact, we will show that the decay rates for the large deviations are

$$a_t = \begin{cases} \sqrt{t}\,, & \text{if } d = 1, \\ t/\log t, & \text{if } d = 2, \\ t, & \text{if } d \geq 3. \end{cases} \tag{1.1}$$

In this way, this paper is a sequel to the study of occupation time large

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deviation probabilities for particle systems initiated by Cox and Griffeath [5] and continued by several authors ([2], [3] and [4]).

In dimension $d \neq 2$, we will prove a large deviation principle for the additive functional $\int_0^t \eta_s(0)\,ds$. The idea will be the same in dimension 1 and $d \geq 3$. First, we prove a large deviation principle at a "process" level and then apply a contraction principle directly ($d \geq 3$) or with some work ($d = 1$) to obtain the large deviations for the functional considered. The method to prove the large deviation principle at the process level heavily depends on the reversibility of the process in equilibrium.

This paper is divided in three independent parts. In Section 2, we obtain the large deviations for $\int_0^t \eta_s(0)\,ds$ in dimension $d \geq 3$ with the techniques developed by de Acosta [6] and Deuschel and Stroock [7].

In Section 3, we prove a large deviation principle for $\int_0^t \eta_s(0)\,ds$ in dimension 1. This is done by proving first a large deviation principle for the empirical density of the symmetric simple exclusion process on $\mathbb{Z}$. A large deviation principle of this kind was first proved by Kipnis, Olla and Varadhan [10]. The proof of this large deviation principle relies on a superexponential estimate. We give a new proof of this estimate which holds for $\mathbb{Z}$. With this result, we obtain the large deviations for $\int_0^t \eta_s(0)\,ds$.

We prove large deviation principles for the occupation time applying contraction principles. For this reason the rate functions obtained are given by variational formulas. Since we are not able to obtain explicit expressions for these variational formulas, we need to show that the rate functions are not degenerate. In Section 4, by comparing the symmetric simple exclusion process with another process where each particle moves independently from the others with the same dynamic as the ones which move according to simple exclusion, we prove that the rate functions obtained in Sections 2 and 3 are bounded below by the rate functions obtained by Cox and Griffeath [5] for the independent process. In particular, this proves that they are not degenerate. Moreover, this result together with results obtained by Arratia [2] show that the decay rate for the large deviation probabilities for the occupation time in dimension 2 is $a_t$ given by (1.1).

The method we use to prove large deviations in dimension $d \neq 2$ does not apply in dimension 2. Therefore, a result in this direction would be welcome. Another interesting question is to consider the large deviations in the asymmetric case, where the process in equilibrium is no longer reversible and where therefore our techniques do not apply.

**2. The case $d \geq 3$.** In this section, we will consider the case $d \geq 3$, where no surprise arises. Throughout this paper, to simplify the notation, we will denote $\{0,1\}^{\mathbb{Z}^d}$ by $\mathbb{X}_d$ and for $d = 1$, we will omit the subscript. Let $p(k, j)$ be symmetric transition probabilities on $\mathbb{Z}^d$:

$$(2.1)\qquad p(k,j) \geq 0, \qquad \sum_j p(k,j) = 1, \qquad p(k,j) = p(j,k).$$

In this section, we will consider the symmetric simple exclusion process associated with the transition probabilities (2.1). This is the Markov process on $\mathbb{X}_d$, whose generator acts on cylindrical functions as

$$(2.2) \qquad Lf(\eta) = \sum_{k, j \in \mathbb{Z}^d} p(k,j)\left[f(\eta^{k,j}) - f(\eta)\right],$$

where

$$(2.3) \qquad \eta^{k,j}(i) = \begin{cases} \eta(i), & \text{if } i \neq k, j, \\ \eta(k), & \text{if } i = j, \\ \eta(j), & \text{if } i = k. \end{cases}$$

In the nearest-neighbor case, where $p(k,j) = 1/2d$ if $|j-k| = 1$ and 0 otherwise, we will denote $L$ by $L_n$.

For $\rho \in [0,1]$, we know from [13] that the product measures $\nu_\rho$ on $\mathbb{X}_d$, with marginals given by $\nu_\rho\{\eta;\ \eta(k) = 1\} = \rho$ for every $k$ in $\mathbb{Z}^d$, are reversible for this process. We will denote by $P_\rho$ the corresponding probability measure on the path space $D([0,\infty[, \mathbb{X}_d)$.

To state the theorem, we will need to introduce some notation. Let $M$ be the space of probability measures on $\mathbb{X}_d$, endowed with the weak* topology which is metrizable on $M$, and $\mathscr{F}_+$ the space of positive real measurable functions defined on $\mathbb{X}_d$. For $f, g\colon \mathbb{X}_d \to \mathbb{R}$, let $\langle f, g\rangle_\rho = \int f(\eta) g(\eta) \nu_\rho(d\eta)$. Define the Dirichlet form $D$ on $\mathscr{F}_+$ by

$$(2.4) \qquad D(f) = \tfrac{1}{2}\sum_{k,j} p(k,j) \int \left[\sqrt{f(\eta^{k,j})} - \sqrt{f(\eta)}\right]^2 \nu_\rho(d\eta)$$

and $\tilde{I}_d\colon M \to [0,\infty]$ by

$$(2.5) \qquad \tilde{I}_d(\sigma) = \begin{cases} D\left(\dfrac{d\sigma}{d\nu_\rho}\right), & \text{if } \sigma \ll \nu_\rho, \\ \infty, & \text{otherwise.} \end{cases}$$

It is not hard to see that the function $\tilde{I}_d$ is convex and is *not* lower semicontinuous. Indeed, to see that $\tilde{I}_d$ is not lower semicontinuous, for a sufficiently large integer $N_0$, consider the product measures $(\sigma_N)$, $N_0 \le N \le \infty$ with marginals given by

$$\sigma_N\{\eta(k) = 1\} = \begin{cases} \rho + \dfrac{1}{k_1}, & \text{if } k_j = 0 \text{ for } 2 \le j \le d,\ k_1 \text{ even and } N_0 \le k_1 \le N, \\ \rho, & \text{otherwise.} \end{cases}$$

Then $\lim_N \sigma_N = \sigma_\infty$, $\limsup_N \tilde{I}_d(\sigma_N) < \infty$ and $\sigma_\infty$ is not absolutely continuous with respect to $\nu_\rho$. To overcome this difficulty, we introduce the function $I_d$: $M \to [0,\infty]$:

$$(2.6) \qquad I_d(\sigma) = \lim_{\varepsilon \to 0} \inf_{\nu \in B(\sigma,\varepsilon)} \tilde{I}_d(\nu),$$

where $B(\sigma, \varepsilon)$ is the ball centered at $\sigma$ with radius $\varepsilon$. Then $I_d$ is convex and lower semicontinuous.

Let $\mathscr{L}_t = (1/t)\int_0^t \delta_{\eta_s}\, ds \in M$, where for $\eta \in \mathbb{X}_d$, $\delta_\eta$ is the probability measure concentrated on the configuration $\eta$. We can now state the following theorem.

THEOREM 2.1. *For every closed subset $F$ and for every open subset $G$ of $M$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P_\rho[\mathscr{L}_t \in F] \le - \inf_{\sigma \in F} I_d(\sigma),$$

$$\liminf_{t\to\infty} \frac{1}{t} \log P_\rho[\mathscr{L}_t \in G] \ge - \inf_{\sigma \in G} I_d(\sigma).$$

PROOF OF THE UPPER BOUND. The upper bound will be obtained by a method introduced by de Acosta [6].

Let $V\colon \mathbb{X}_d \to \mathbb{R}$ be a continuous function. By the Feynman–Kac formula and the spectral theorem,

$$E_\rho\left[\exp\left(\int_0^t V(\eta_s)\, ds\right)\right] \le e^{t\lambda_V}, \tag{2.7}$$

where $\lambda_V$ is the greatest eigenvalue of the self-adjoint operator $L + V$. From the variational formula for the greatest eigenvalue, we have

$$\lambda_V = \sup_f \left\{ \int V(\eta) f(\eta) \nu_\rho(d\eta) - D(f) \right\}, \tag{2.8}$$

where the sup is taken over all the distributions with density $f$, that is, over all nonnegative functions $f$ such that

$$\int f(\eta) \nu_\rho(d\eta) = 1,$$

and $D$ is the Dirichlet form given by (2.4).

Since if $\sigma(d\eta) = f(\eta)\nu_\rho(d\eta)$, $\tilde{I}_d(\sigma) = D(f)$, we can write

$$\lambda_V = \sup_{\substack{\sigma \in M \\ \sigma << \nu_\rho}} \left\{ \int V(\eta)\sigma(d\eta) - \tilde{I}_d(\sigma) \right\}. \tag{2.9}$$

Since $\tilde{I}_d(\sigma) = \infty$ if $\sigma$ is not absolutely continuous relative to $\nu_\rho$ and since $V$ is bounded, we can omit the restriction $\sigma \ll \nu_\rho$ in the sup. From (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log E_\rho\left[\exp\left(\int_0^t V(\eta_s)\, ds\right)\right] &\le \sup_{\sigma \in M} \left\{ \int V(\eta)\sigma(d\eta) - \tilde{I}_d(\sigma) \right\} \\ &\le \sup_{\sigma \in M} \left\{ \int V(\eta)\sigma(d\eta) - I_d(\sigma) \right\},\end{aligned}$$

because $I_d(\sigma) \le \tilde{I}_d(\sigma)$.

Now, since $M$ is compact, applying a continuous version of Theorem 2.1 of [6], we obtain that for every closed subset $F$ of $M$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P_\rho[\mathscr{L}_t \in F] \le - \inf_{\sigma \in F} J(\sigma),$$

where

$$J(\sigma) = \sup_{V \in C(\mathbb{X}_d)} \left\{ \int V(\eta)\sigma(d\eta) - \sup_{\nu \in M} \left[ \int V(\eta)\nu(d\eta) - I_d(\nu) \right] \right\}.$$

Since $I_d$ is lower semicontinuous, convex and not identically equal to $\infty$ and since the dual of $M$ with the weak* topology is $C(\mathbb{X}_d)$, $J \equiv I_d$ completing the proof of the upper bound. □

PROOF OF THE LOWER BOUND. It is easy to show that the hypotheses of Theorem 5.3.10 of [7] are satisfied. Therefore, for every open set $G$ of $M$,

$$\begin{aligned} \liminf_{t \to \infty} \frac{1}{t} \log P_\rho[\mathscr{L}_t \in G] &\geq - \inf_{\sigma \in G} \tilde{I}_d(\sigma) \\ &= - \inf_{\sigma \in G} I_d(\sigma), \end{aligned}$$

where the equality follows from the definition of $I_d$. □

From Theorem 2.1, we will prove the large deviations for the occupation time.

Since $\Phi$: $M \to \mathbb{R}$ defined by $\Phi(\sigma) = \int \eta(0)\sigma(d\eta)$ is continuous and $I_d$ is a rate function in the terminology of [7], from the contraction principle we obtain the following corollary as an immediate consequence of the last theorem.

COROLLARY. *For every closed set $F$ and every open set $G$ of* $[0,1]$,

$$\limsup_{t \to \infty} \frac{1}{t} \log P_\rho \left[ \frac{1}{t} \int_0^t \eta_s(0)\, ds \in F \right] \leq - \inf_{\alpha \in F} \psi_d(\alpha),$$

$$\liminf_{t \to \infty} \frac{1}{t} \log P_\rho \left[ \frac{1}{t} \int_0^t \eta_s(0)\, ds \in G \right] \geq - \inf_{\alpha \in G} \psi_d(\alpha),$$

*where*

$$\psi_d(\alpha) = \inf_{\sigma;\, \int \eta(0)\sigma(d\eta) = \alpha} I_d(\sigma).$$

In the case where $p(k, j) = 1/2d$ if $|j - k| = 1$ and 0 otherwise, it is not hard to see that $\psi_d \equiv 0$ in dimensions 1 and 2. To show it, fix $0 < \alpha < 1$ and let $\lambda$: $\mathbb{R}^d \to \mathbb{R}_+$ be a twice differentiable function such that $\lambda|_{\mathbb{R}^d - [-1,1]^d} \equiv 1$ and $\lambda(0) = \alpha(1 - \rho)/\rho(1 - \alpha)$. For each integer $N$, consider the product measure

$$\sigma_N^\lambda(d\eta) = K_N \prod_{k \in \Lambda_N} \lambda\left(\frac{k}{N}\right)^{\eta(k)} \nu_\rho(d\eta),$$

where $K_N$ is a normalizing constant and $\Lambda_N = \{-N, \dots, N\}^d$. Then $\int \eta(0)\sigma_N^\lambda(d\eta) = \alpha$ for every $N$ and a simple computation shows that in dimensions 1 and 2 there exists a sequence $(\lambda_k)$ for which $\lim_k \lim_N I_d(\sigma_N^{\lambda_k}) = 0$.

Since for every $0 < \alpha < 1$, $0 \le \psi_d(\alpha) = \inf_{\sigma;\ \int \eta(0)\sigma(d\eta) = \alpha} I_d(\sigma) \le \lim_k \lim_N I_d(\sigma_N^{\lambda_k}) = 0$, this proves that in these dimensions $\psi_d \equiv 0$.

Therefore, in dimensions 1 and 2, the contraction principle yields no information on the real decay rate. On the other hand, in Section 4, we will prove that $\psi_d$ is not degenerate in dimension $d \ge 3$.

**3. The case $d = 1$.** In this section, we will consider the case $d = 1$. First, we show that to prove the occupation time large deviations, it is enough to consider the large deviations for the empirical density. This is the content of Lemma 3.1. Then, in the following two subsections, we prove a large deviation principle for the empirical density, following the ideas of [10]. Finally, in the fourth subsection, from the large deviation principle for the empirical density, we obtain the occupation time large deviations.

In this section, for each integer $N$, we consider the speeded up symmetric simple exclusion process on $\mathbb{X}$, which is the Markov process whose generator acts on cylindrical functions as

$$L_N f(\eta) = \frac{N^2}{2} \sum_{|j-k|=1} \left[ f(\eta^{j,k}) - f(\eta) \right], \tag{3.1}$$

where $\eta^{j,k}$ is given by (2.3).

For a metric space $\mathscr{S}$, we will denote, respectively, by $C(\mathscr{S})$, $C_K(\mathscr{S})$ and $C^j(\mathscr{S})$ the space of real continuous functions, real continuous functions with compact support and real functions with a $j$th continuous derivative.

Given a function $\gamma(x)$ on $\mathbb{R}$ such that $0 \le \gamma(x) \le 1$, we denote by $\nu_\gamma^N$ the product measure on $\mathbb{X}$ with marginals given by $\nu_\gamma^N(\eta;\ \eta(j) = 1) = \gamma(j/N)$, $j \in \mathbb{Z}$. We shall identify the constant $\rho \in [0,1]$ with the constant real function equal to $\rho$. We also define, for $\alpha \in [0,1]$ and $\phi \in C(\mathbb{X})$,

$$\tilde{\phi}(\alpha) = \nu_\alpha^N(\phi).$$

We denote by $P_N^\gamma$ the probability on the space $D([0,T], \mathbb{X})$ corresponding to the process with generator given by (3.1) and with initial measure $\nu_\gamma^N$. We will also have to consider, for real continuous functions $H$ in $C_K(\mathbb{R} \times [0,T])$, the weakly asymmetric simple exclusion process (WASEP) which is the strong Markov process whose generator acts on cylindrical functions $f$ as

$$\begin{aligned} L_{N,t}^H f(\eta) = \frac{N^2}{2} \sum_{|k-j|=1} \eta(k)[1 - \eta(j)] e^{H(j/N,t) - H(k/N,t)} \\ \times \left[ f(\eta^{k,j}) - f(\eta) \right], \end{aligned} \tag{3.2}$$

where $\eta^{k,j}$ is given by (2.3).

The probability on $D([0,T], \mathbb{X})$ corresponding to this process with initial measure $\nu_\gamma^N$ will be denoted by $P_N^{H,\gamma}$. We denote by $M_1$ the set of functions $\mu$ on $\mathbb{R}$ such that $0 \le \mu \le 1$ and by $M_1(\rho)$ the set of functions $\gamma$ in $M_1$ for which there exists $x_0 \in \mathbb{R}_+$ such that $\gamma(x) = \rho$ if $|x| \ge x_0$.

We shall use the notation

$$\langle \mu; G \rangle = \int_{\mathbb{R}} G(x)\mu(x)\,dx \quad \text{for } \mu \in M_1, G \in C_K(\mathbb{R}),$$

and consider on $M_1$ the topology induced by $C_K(\mathbb{R})$ with the duality $\langle \cdot, \cdot \rangle$. We shall observe that this topology, if we consider $M_1$ as a measure space, is the vague topology which is metrizable.

Given any path on $D([0,T], \mathbb{X})$, the empirical density is defined by

$$\mu^N(s, x; \eta) = \sum_{k \in \mathbb{Z}} \eta_s(k) 1_{[k/N, (k+1)/N[}(x),$$

so that $\mu^N(\cdot, \eta) \in D([0,T], M_1)$.

Let $\mu \in D([0,T], M_1)$. Consider in $C_K^{2,1}(\mathbb{R} \times [0,T])$ the scalar product

$$[H, G] = \int_0^T dt \left\langle \mu_t, \frac{\partial H}{\partial x}\frac{\partial G}{\partial x}(\cdot, t) \right\rangle.$$

Consider in $C_K^{2,1}(\mathbb{R} \times [0,T])$ the equivalence relation $G \sim H$ if $[H - G, H - G] = 0$. Let $H^1(\mu)$ be the Hilbert space defined as the completion of $C_K^{2,1}(\mathbb{R} \times [0,T])|_\sim$ with respect to this scalar product.

3.1. *Reduction to the empirical density.* First, we will state a lemma which establishes that to prove the occupation time large deviations, it will be sufficient to consider the large deviations for the empirical density. This lemma was indicated by S. R. S. Varadhan.

LEMMA 3.1. *For every* $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P_N^\rho \left[ \left| \int_0^t \left( \eta_s(0) - \frac{1}{2N\varepsilon + 1} \sum_{|j| \le N\varepsilon} \eta_s(j) \right) ds \right| \ge \delta \right] = -\infty. \tag{3.3}$$

PROOF. It is enough to prove the lemma without the absolute value. In this case, the probability in (3.3) is less than or equal to

$$e^{-a\delta N} E_N^\rho \left[ \exp\left( aN \int_0^t V_{N,\varepsilon}(\eta_s)\, ds \right) \right] \quad \text{for every } a > 0, \tag{3.4}$$

if

$$V_{N,\varepsilon}(\eta) = \eta(0) - \frac{1}{2N\varepsilon + 1} \sum_{|j| \le N\varepsilon} \eta(j).$$

As in Section 2, by the Feynman–Kac formula, the spectral theorem and the classical variational formula for the largest eigenvalue of a self-adjoint operator, this term is less than or equal to

$$
\begin{aligned}
&\exp\left\{-a\delta N + t\sup_f\left[aN\int V_{N,\varepsilon}(\eta) f(\eta)\nu_\rho(d\eta) - N^2D(f)\right]\right\} \\
(3.5)\qquad &= \exp\left\{N\left(-a\delta + t\sup_f\left[a\int V_{N,\varepsilon}(\eta) f(\eta)\nu_\rho(d\eta) - ND(f)\right]\right)\right\},
\end{aligned}
$$

where the sup is taken over all distributions on $\mathbb{X}$ with density $f$ and $D(f)$ is the Dirichlet form given by

$$D(f) = \frac{1}{4}\sum_{|k-j|=1}\int\left[\sqrt{f(\eta^{k,j})} - \sqrt{f(\eta)}\right]^2\nu_\rho(d\eta).$$

In view of (3.4) and (3.5), in order to prove

$$\text{(3.6)}\qquad \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N^\rho\left[\int_0^t V_{N,\varepsilon}(\eta_s)\,ds \ge \delta\right] \le -a\delta,$$

it is enough to show that for every $a > 0$,

$$\text{(3.7)}\qquad \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_f\left\{a\int V_{N,\varepsilon}(\eta) f(\eta)\nu_\rho(d\eta) - ND(f)\right\} \le 0.$$

Letting $a \uparrow \infty$ in (3.6), we shall obtain the lemma. To prove (3.7), we develop the integral which appears in the left-hand side. We have

$$a\int V_{N,\varepsilon}(\eta) f(\eta)\nu_\rho(d\eta) = \frac{a}{2N\varepsilon + 1}\sum_{|j|\le N\varepsilon}\int(\eta(0) - \eta(j)) f(\eta)\nu_\rho(d\eta).$$

Changing in the integral the variable $\eta$ to $\eta^{0,j}$, we obtain that the last expression is equal to

$$\frac{a}{2[2N\varepsilon + 1]}\sum_{|j|\le N\varepsilon}\int[\eta(0) - \eta(j)][f(\eta) - f(\eta^{0,j})]\nu_\rho(d\eta).$$

Since

$$|f(\eta) - f(\eta^{0,j})| = \left|\sqrt{f(\eta)} - \sqrt{f(\eta^{0,j})}\right|\left|\sqrt{f(\eta)} + \sqrt{f(\eta^{0,j})}\right|,$$

by the Schwarz inequality and remembering that $f$ is a distribution function, we obtain that this last expression is less than

$$\frac{a}{2N\varepsilon + 1}\sum_{|j|\le N\varepsilon}\left\{\int\left[\sqrt{f(\eta)} - \sqrt{f(\eta^{0,j})}\right]^2\nu_\rho(d\eta)\right\}^{1/2}.$$

Since

$$\eta^{0,j} = \left(\cdots\left(\left(\left(\cdots\left((\eta^{0,1})^{1,2}\right)\cdots\right)^{j-1,j}\right)^{j-2,j-1}\right)\cdots\right)^{1,0},$$

by the inequality

$$\left(\sum_{j=1}^{n} a_j\right)^2 \le n \sum_{j=1}^{n} a_j^2$$

and by the definition of the Dirichlet form, we obtain that this last expressio is less than

$$\sqrt{2}\, a \sqrt{(2N\varepsilon + 1) D(f)}\,.$$

Therefore, the left-hand side of (3.7) is bounded above by

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \left\{ \sup_{x \ge 0} \left\{ \sqrt{2}\, a \sqrt{(2N\varepsilon + 1) x} - Nx \right\} \right\}$$
$$= \lim_{\varepsilon \to 0} \limsup_{N \to \infty} \left\{ 2a^2 \frac{2N\varepsilon + 1}{4N} \right\},$$

which proves the assertion (3.7). □

3.2. *A superexponential estimate.* Now, we prove a large deviation principle for the empirical density. The proof relies on a superexponential estimate which first appeared in [10]. We give another proof of this estimate which has the advantage of holding for $\mathbb{Z}^1$ instead of the torus considered in that paper.

LEMMA 3.2. *Let $H \in C_K(\mathbb{R} \times \mathbb{R}_+)$, $\rho \in [0,1]$ and $\phi$ be a cylindrical function. Let*

$$V_{N,\varepsilon}^{H,\phi}(s,\eta) = \frac{1}{N} \sum_{i \in \mathbb{Z}} H\left(\frac{i}{N}, s\right) \left[ \tau_i \phi(\eta) - \tilde{\phi}\left( \frac{1}{2N\varepsilon + 1} \sum_{|j-i| \le N\varepsilon} \eta(j) \right) \right],$$

*where $\tau_i$ is the space shift on $\mathbb{X}$. Then, for every $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P_N^{\rho} \left[ \left| \int_0^t V_{N,\varepsilon}^{H,\phi}(s, \eta_s)\, ds \right| \ge \delta \right] = -\infty. \tag{3.8}$$

PROOF. It is easy to see that it is enough to prove the lemma for functions $H$ for which there exist $n$ in $\mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n = t$ such that

$$H(x,t) = H(x,t_j) \equiv H_j(x) \quad \text{for } t_j \le t < t_{j+1}, 0 \le j < n,$$

with each function $H_j$ continuous and with compact support. Since the measure $\nu_\rho$ is invariant for the process, to prove the lemma for such functions, it is enough to consider continuous functions with compact support $H$ which do not depend on time. Observe that in this case, $V_{N,\varepsilon}^{H,\phi}(s,\eta)$ is a function of $\eta$ only.

On the other hand, it is also sufficient to prove the lemma for cylindrical functions of the type

$$\phi(\eta) = \prod_{i \in A} \eta(i), \tag{3.9}$$

for finite subsets $A$ of $\mathbb{Z}^1$, because every cylindrical function is a finite linear combination of such product functions. To keep the notation simple, we will consider the case where the set $A$ in (3.9) is $\{0, 1\}$. The general case is proved in the same way.

It is sufficient to consider (3.8) without the absolute value. As in the proof of Lemma 3.1, to prove the lemma, it is enough to show

$$(3.10)\quad \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_f\left\{a\int V_{N,\varepsilon}^{H,\phi}(\eta)\,f(\eta)\nu_\rho(d\eta) - ND(f)\right\} \le 0.$$

To do so, we develop the integral which appears in the left-hand side. We have

$$\begin{aligned} a\int & V_{N,\varepsilon}^{H,\phi}(\eta)\,f(\eta)\nu_\rho(d\eta) \\ &= \frac{a}{N}\sum_i H\left(\frac{i}{N}\right)\int\left[\eta(i)\eta(i+1) - \left(\frac{1}{2N\varepsilon+1}\sum_{|j-i|\le N\varepsilon}\eta(j)\right)^2\right] \\ &\quad \times f(\eta)\nu_\rho(d\eta). \end{aligned}$$

Denoting the empty set of $\mathbb{Z}$ by $A_0$, $\{0\}$ by $A_1$ and $\{0, 1\}$ by $A_2$, we see that this expression can be rewritten, introducing intermediary terms, as

$$\begin{aligned} \frac{a}{N}\sum_i H\left(\frac{i}{N}\right)\sum_{k=0}^{1}\int&\left\{\left(\prod_{j\in A_{2-k}}\eta(i+j)\right)\left(\frac{1}{2N\varepsilon+1}\sum_{|l-i|\le N\varepsilon}\eta(l)\right)^k\right. \\ &\left. -\left(\prod_{j\in A_{1-k}}\eta(i+j)\right)\left(\frac{1}{2N\varepsilon+1}\sum_{|l-i|\le N\varepsilon}\eta(l)\right)^{k+1}\right\}f(\eta)\nu_\rho(d\eta). \end{aligned}$$

We will consider only the term $k = 0$. For the other one we would proceed in the same way. This term is equal to

$$\frac{a}{N[2N\varepsilon+1]}\sum_i\sum_{|l|\le N\varepsilon}H\left(\frac{i}{N}\right)\int\eta(i)[\eta(i+1)-\eta(i+l)]\,f(\eta)\nu_\rho(d\eta).$$

For $l \ne 0$, changing in the integral the variable $\eta$ to $\eta^{i+1,i+l}$, we obtain that the last expression is bounded above by

$$\begin{aligned} \frac{a}{2N[2N\varepsilon+1]}&\sum_i\sum_{\substack{|l|\le N\varepsilon\\ l\ne 0}}H\left(\frac{i}{N}\right)\int\eta(i)[\eta(i+1)-\eta(i+l)] \\ &\times\left[f(\eta)-f(\eta^{i+1,i+l})\right]\nu_\rho(d\eta) + O\left(\frac{1}{N}\right), \end{aligned}$$

where $O(1/N)$ appeared to take care of the case $l = 0$.

Now, we can proceed as in the last lemma to obtain (3.10). □

3.3. *Hydrodynamical limits and large deviations.* In this subsection, we will prove hydrodynamical limits for the weakly asymmetric process and a large deviation principle for the empirical density for the speeded up symmetric simple exclusion process. Almost all proofs will be omitted since they are similar to those in [10]. We should remark that, as in [10], to prove a large deviation principle for the empirical density, we need to consider WASEP processes whose perturbations depend on time. Indeed, we will see below in (3.12), that to define the rate function $I_\gamma$ we have to consider WASEP processes whose perturbations depend on space and time.

HYDRODYNAMICAL LIMITS. First, we observe that the superexponential estimate (3.8) also holds for $P_N^{H,\gamma}$ if $H \in C_K^{2,1}(\mathbb{R} \times [0,T])$ and $\gamma \in M_1(\rho)$. The proof of this result is the same as the one of Theorem 3.2 in [10]. Then, just as in Section 3 of [10], we can prove the following theorem.

THEOREM 3.1. *As $N \to \infty$, the empirical density $\mu^N(s,x;\eta)$ converges in $P_N^{H,\gamma}$-probability to the unique weak solution $\rho(t,x)$ of the equation*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}\left[\frac{\partial H}{\partial x}\rho(1-\rho)\right], \qquad \rho(0,x) = \gamma(x). \tag{3.11}$$

The uniqueness of weak solutions of (3.11) in the class of bounded functions can be established with a similar argument to the one used to prove Propositions 3.4 and 3.5 of [14].

LARGE DEVIATIONS; UPPER BOUND. For any $\mu \in D([0,T], M_1)$ and $\gamma \in M_1(\rho)$, define the linear functional on $C_K^{2,1}(\mathbb{R} \times [0,T])$:

$$l(\mu;G) = \langle \mu_T; G(\cdot,T)\rangle - \langle \mu_0; G(\cdot,0)\rangle - \int_0^T \left\langle \mu_t; \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) G(\cdot,t)\right\rangle dt$$

and introduce the following rate functions:

$$I_0(\mu) = \sup_{G \in C_K^{2,1}(\mathbb{R}\times[0,T])} \left\{ l(\mu;G) - \frac{1}{2}\int_0^T \left\langle \mu_s[1-\mu_s]; \left[\frac{\partial G}{\partial x}(\cdot,s)\right]^2 \right\rangle ds \right\},$$

$$\begin{aligned} h(\mu_0;\gamma) = \sup_{\phi_0,\phi_1 \in C_K(\mathbb{R})} \Big\{ &\langle \mu_0;\phi_0\rangle + \langle 1-\mu_0;\phi_1\rangle \\ &-\big\langle 1; \log\big[\gamma e^{\phi_0} + (1-\gamma)e^{\phi_1}\big]\big\rangle \Big\}, \end{aligned} \tag{3.12}$$

$$I_\gamma(\mu) = I_0(\mu) + h(\mu_0;\gamma).$$

As in Section 4 of [10], we prove the following theorem.

THEOREM 3.2. *Let $\gamma \in M_1(\rho)$. For any closed set $C \subset D([0,T], M_1)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^{\gamma}\left[\mu^N \in C\right] \le - \inf_{\mu \in C} I_{\gamma}(\mu).$$

LARGE DEVIATIONS; LOWER BOUND. As in Section 5 of [10], we first observe that if $\gamma$ and $\mu_0$ are such that $h(\mu_0;\gamma) < \infty$, we have

$$h(\mu_0;\gamma) = \left\langle \mu_0; \log \frac{\mu_0}{\gamma} \right\rangle + \left\langle 1 - \mu_0; \log \frac{1-\mu_0}{1-\gamma} \right\rangle.$$

Furthermore, if $I_0(\mu) < \infty$, there exists $H \in H^1(\mu(1-\mu))$ such that

$$I_0(\mu) = \frac{1}{2}\int_0^T \left\langle \mu(s,\cdot)[1-\mu(s,\cdot)]; \left[\frac{\partial H}{\partial x}(\cdot, s)\right]^2 \right\rangle ds, \tag{3.13}$$

and $\mu$ satisfies in the weak sense the equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}\left[\frac{\partial H}{\partial x}\rho(1-\rho)\right]. \tag{3.14}$$

We omit the proof of this result since it is the same as the one of Lemma 5.1 of [10].

THEOREM 3.3. *Let $\gamma \in M_1(\rho) \cap C(\mathbb{R})$ such that $0 < \gamma < 1$. For any open neighborhood $A$ of a given $\mu$ in $D([0,T], M_1)$,*

$$\liminf_{N\to\infty} \frac{1}{N} \log P_N^{\gamma}\left[\mu^N \in A\right] \ge -I_{\gamma}(\mu). \tag{3.15}$$

PROOF. We will just point out the new arguments needed in our case. Let $\mu \in A$. Of course, we only have to consider the case where $I_\gamma(\mu) < \infty$. Fix $\mu \in D([0,T], M_1)$ such that $I_\gamma(\mu) < \infty$. For $0 < a < 1$, let

$$\mu^a(t,x) = \rho + a[\mu(t,x) - \rho].$$

It is easy to show that $\mu^a \to \mu$ in $D([0,T], M_1)$ as $a \uparrow 1$. On the other hand, since $I_0$ is convex, lower semicontinuous and invariant under spatial translations and since $I_0(\rho) = 0$, $\lim_{a\to 1} I_0(\mu^a) = I_0(\mu)$. With our assumptions on $\gamma$, it is easy to see that $\lim_{a\to 1} h(\mu_0^a, \gamma) = h(\mu_0, \gamma)$, so that

$$\lim_{a\to 1} I_\gamma(\mu^a) = I_\gamma(\mu).$$

Since for every $0 < a < 1$, there exists $\delta(a) > 0$ such that

$$\delta \le \mu(t,x) < 1 - \delta, \tag{3.16}$$

we only have to prove (3.15) for paths $\mu$ with this property. Fix such $\mu \in D([0,T], M_1)$.

For $\varepsilon > 0$, let $\sigma_\varepsilon \in C_K^\infty(\mathbb{R})$ be smooth approximations of the identity in $L^1(\mathbb{R})$: $\sigma_\varepsilon \ge 0$, $\int_{\mathbb{R}} \sigma_\varepsilon(x)\,dx = 1$, $\sigma_\varepsilon(x) = 0$ if $|x| \ge \varepsilon$ and for every $f \in L^1(\mathbb{R})$,

$\sigma_\varepsilon * f \to f$ in $L^1(\mathbb{R})$ as $\varepsilon \to 0$, where $*$ denotes the convolution in $\mathbb{R}$. Define $\mu^\varepsilon$ as

$$\mu^\varepsilon(t,x) + \int_{\mathbb{R}} \mu(t, x-y)\sigma_\varepsilon(y)\,dy. \tag{3.17}$$

With the same arguments used before, we can prove that $\mu^\varepsilon \to \mu$ in $D([0,T], M_1)$ as $\varepsilon \downarrow 0$ and that

$$\lim_{\varepsilon \to 0} I_\gamma(\mu^\varepsilon) = I_\gamma(\mu).$$

Therefore, to prove the theorem, it is enough to prove (3.15) for spatial convolutions $\mu^\varepsilon$ in $D([0,T], M_1)$ with property (3.16). Fix $\varepsilon > 0$. Since $I_\gamma(\mu^\varepsilon) < \infty$, by (3.13), there exists $H_\varepsilon \in H^1(\mu^\varepsilon(1-\mu^\varepsilon))$ such that

$$I_0(\mu^\varepsilon) = \frac{1}{2}\int_0^T \left\langle \mu_t^\varepsilon[1-\mu_t^\varepsilon], \left[\frac{\partial H_\varepsilon}{\partial x}(\cdot, t)\right]^2 \right\rangle dt. \tag{3.18}$$

It follows from properties (3.16) and (3.17) of $\mu^\varepsilon$ that $H_\varepsilon$ is regular and that

$$\left\| \frac{\partial H_\varepsilon}{\partial x} \right\|_2 < \infty, \qquad \left\| \frac{\partial H_\varepsilon}{\partial x} \right\|_\infty < \infty. \tag{3.19}$$

Hence, $\partial H_\varepsilon / \partial x \in L^p(\mathbb{R} \times [0,T])$ for every $p \geq 2$.

The problem is that we only have proved the hydrodynamical limit for WASEP processes [defined in (3.2)] whose function $H$ is in $C_K^{2,1}(\mathbb{R} \times [0,T])$. There is no reason for the function $H_\varepsilon$ obtained above to be in $C_K^{2,1}(\mathbb{R} \times [0,T])$. Therefore, we will have to approximate this function $H_\varepsilon$ by $C_K^{2,1}(\mathbb{R} \times [0,T])$ functions $H_n$ in such a way that the weak solutions $\mu_n$ of (3.14) associated to $H_n$ converge in the Skorohod topology to $\mu^\varepsilon$ and that $I_\gamma(\mu_n)$ converge to $I_\gamma(\mu^\varepsilon)$.

Let $(H_n)$ be a sequence of functions in $C_K^{2,1}(\mathbb{R} \times [0,T])$ such that

(3.20) (i) $$\frac{\partial H_n}{\partial x} \to \frac{\partial H_\varepsilon}{\partial x} \quad \text{in } L^2(\mathbb{R} \times [0,T]),$$

(3.20) (ii) $$\left\| \frac{\partial H_n}{\partial x} \right\|_\infty \leq \left\| \frac{\partial H_\varepsilon}{\partial x} \right\|_\infty,$$

(3.20)(iii) $$\left| \frac{\partial H_n}{\partial x} \right| \leq \left| \frac{\partial H_\varepsilon}{\partial x} \right| + f_n \quad \text{where } \int_0^T dt \int_{\mathbb{R}} dx\, f_n^2(x,t) \leq \frac{1}{n}.$$

Let $m_n \in M_1(\rho) \cap C(\mathbb{R})$ such that

$$m_n|_{[-n,n]} \equiv \mu^\varepsilon(0, \cdot)|_{[-n,n]}. \tag{3.21}$$

Let $\mu_n$ be the unique bounded solution of

$$\begin{cases} \dfrac{\partial \mu}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \mu}{\partial x^2} - \dfrac{\partial}{\partial x}\left[\dfrac{\partial H_n}{\partial x}\mu(1-\mu)\right], \\ \mu(0,\cdot) = m_n(\cdot). \end{cases}$$

With similar methods to those developed in the proof of Proposition 3.5 of [14], from the propeties (3.19) of $H_\varepsilon$, we can prove that $\mu_n \to \mu^\varepsilon$ uniformly over the compact sets of $\mathbb{R} \times [0, T]$. In particular, $\mu_n \to \mu^\varepsilon$ in $D([0, T], M_1)$. From the definition (3.21) of $\mu_n(0, \cdot\,)$ it is easy to see that $h(\mu_n(\cdot, 0), \gamma) \to h(\mu_0^\varepsilon, \gamma)$. Finally from the representation (3.18) and since, from (3.20), $\partial H_n/\partial x \to \partial H_\varepsilon/\partial x$ in $L^2(\mathbb{R} \times [0, T])$, it follows that $I_0(\mu_n) \to I_0(\mu^\varepsilon)$.

Thus, to prove the theorem, we only have to show (3.15) for $\mu \in D([0, T], M_1)$ with $I_\gamma(\mu) < \infty$ and such that $\mu(0, \cdot\,) \in C(\mathbb{R}) \cap M_1(\rho)$ and for which the associated function $H$ is in $C_K^{2,1}(\mathbb{R} \times [0, T])$. This is done just as in the proof of Theorem 5.2 of [10]. □

3.4. *Occupation time large deviations.* In this subsection, from the large deviation principle for the empirical density established in the last section, we shall obtain a large deviation principle for the occupation time of a site in the symmetric simple exclusion process.

If we define

$$U = \{\mu \in D([0, T], M_1);\ \mu \text{ is continuous in space, uniformly in time}\},$$

let, for $\alpha$ in $[0, 1]$ and $T \in \mathbb{R}_+$,

$$\tilde{\psi}_{1,T}(\alpha) = \inf_{\substack{\mu;\,(1/T)\int_0^T \mu(s,0)\,ds = \alpha \\ \mu \in U}} I_\gamma(\mu),$$

where $I_\gamma(\mu)$ was defined in the last subsection. Since $I_\gamma$ and $U$ are convex, it follows that $\tilde{\psi}_{1,T}$ is also convex. Let

$$\psi_{1,T}(\alpha) = \lim_{\varepsilon \to 0} \inf_{\substack{|\beta - \alpha| < \varepsilon \\ \beta \in [0,1]}} \tilde{\psi}_{1,T}(\beta).$$

Then, since $\tilde{\psi}_{1,T}$ is convex, $\psi_{1,T}$ is convex. We also have that $\psi_{1,T}$ is lower semicontinuous and $\psi_{1,T}(\alpha) \le \tilde{\psi}_{1,T}(\alpha)$ for any $\alpha \in [0, 1]$. Now, we will prove that $\psi_{1,T}$ is the rate function of the large deviation principle for the occupation time of a site in the symmetric simple exclusion process.

THEOREM 3.4. *Let $\gamma \in M_1(\rho) \cap C(\mathbb{R})$ such that $0 < \gamma < 1$. For every closed subset $F$ and every open subset $G$ of* $[0, 1]$,

$$\begin{aligned} &\limsup_{N \to \infty} \frac{1}{N} \log P_N^\gamma\left[\frac{1}{T}\int_0^T \eta_s(0)\,ds \in F\right] \le -\inf_{\alpha \in F} \psi_{1,T}(\alpha), \\ &\liminf_{N \to \infty} \frac{1}{N} \log P_N^\gamma\left[\frac{1}{T}\int_0^T \eta_s(0)\,ds \in G\right] \ge -\inf_{\alpha \in G} \psi_{1,T}(\alpha). \end{aligned} \tag{3.22}$$

PROOF OF THE UPPER BOUND. Let $F$ be a closed subset of $[0, 1]$ and for $\delta > 0$ define

$$F^\delta = \left\{\alpha \in [0, 1];\ \inf_{\beta \in F} |\alpha - \beta| \le \delta\right\}.$$

For every $\delta > 0$, we have

$$P_N^\gamma\left[\frac{1}{T}\int_0^T \eta_s(0)\,ds \in F\right] \le P_N^\gamma\left[\frac{1}{T}\left|\int_0^T\left(\eta_s(0) - \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\right)ds\right| > \delta\right]$$
$$+ P_N^\gamma\left[\frac{1}{T}\int_0^T \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\,ds \in F^\delta\right].$$

From Lemma 3.1, we know that for every $\delta > 0$,

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N^\gamma\left[\frac{1}{T}\left|\int_0^T\left(\eta_s(0) - \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\right)ds\right| \ge \delta\right]$$
$$= -\infty.$$

Therefore, the left-hand side of (3.22) is less than or equal to

$$\inf_{\delta>0}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N^\gamma\left[\frac{1}{T}\int_0^T\frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\,ds \in F^\delta\right].$$

Since the set $\{\mu \in D([0,T], M_1);\ (1/2T\varepsilon)\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds \in F^\delta\}$ is closed in $D([0,T], M_1)$, by the large deviation principle proved in the last subsection the last expression is bounded above by

$$(3.23)\qquad \begin{aligned} &-\sup_{\delta>0}\liminf_{\varepsilon\to 0}\inf\left\{I_\gamma(\mu);\ \frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds \in F^\delta\right\}\\ &\quad= -\sup_{\delta>0}\liminf_{\varepsilon\to 0}\inf\left\{I_\gamma(\mu);\ \frac{1}{T}\int_0^T\mu^\varepsilon(s,0)\,ds \in F^\delta\right\},\end{aligned}$$

if, for $\mu \in D([0,T], M_1)$ and for $\varepsilon > 0$, we denote by $\mu^\varepsilon$ the function such that

$$\mu^\varepsilon(s,x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}\mu(s,y)\,dy \quad \text{for every } (s,x) \in [0,T]\times\mathbb{R}.$$

Since $I_0$ is convex, lower semicontinuous and translation invariant, $I_0(\mu^\varepsilon) \le I_0(\mu)$. On the other hand, since $\gamma \in M_1(\rho) \cap C(\mathbb{R})$, it is easy to show that $h(\mu^\varepsilon, \gamma) \le h(\mu, \gamma) + O(\varepsilon)$, where $\lim_{\varepsilon\to 0} O(\varepsilon) = 0$ uniformly in $\mu \in D([0,T], M_1)$. Therefore, (3.23) is less than or equal to

$$-\sup_{\delta>0}\liminf_{\varepsilon\to 0}\inf\left\{I_\gamma(\mu^\varepsilon);\ \frac{1}{T}\int_0^T\mu^\varepsilon(s,0)\,ds \in F^\delta\right\}.$$

Recalling the definition of $U$, we see that for every $\varepsilon > 0$, $\mu^\varepsilon \in U$. Thus this last expression is less than or equal to

$$-\sup_{\delta>0}\inf\left\{I_\gamma(\mu);\ \frac{1}{T}\int_0^T\mu(s,0)\,ds \in F^\delta \text{ and } \mu \in U\right\},$$

which, by the definition of $\tilde{\psi}_{1,T}$, is equal to

$$-\sup_{\delta>0}\inf_{\alpha\in F^\delta}\tilde{\psi}_{1,T}(\alpha).$$

Since $\psi_{1,T}(\alpha) \le \tilde{\psi}_{1,T}(\alpha)$ for every $\alpha \in [0,1]$, this last term is less than or equal to

$$-\sup_{\delta>0} \inf_{\alpha\in F^{\delta}} \psi_{1,T}(\alpha).$$

Finally, since $\psi_{1,T}$ is lower semicontinuous and $[0,1]$ is compact, this is equal to

$$-\inf_{\alpha\in F} \psi_{1,T}(\alpha),$$

which proves the upper bound. □

Proof of the lower bound. Let $\alpha \in G$ and $\delta > 0$ such that

$$\inf_{\beta\in G^c} |\alpha-\beta| \ge 4\delta.$$

Therefore,

$$P_N^{\gamma}\left[\frac{1}{T}\int_0^T \eta_s(0)\,ds \in G\right] \ge P_N^{\gamma}\left[\left|\frac{1}{T}\int_0^T \eta_s(0)\,ds - \alpha\right| < 4\delta\right].$$

Since by Lemma 3.1, for every $\chi > 0$,

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N^{\gamma}\left[\frac{1}{T}\left|\int_0^T\left(\eta_s(0) - \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\right)ds\right| > \chi\right]$$
$$= -\infty,$$

$$\liminf_{N\to\infty}\frac{1}{N}\log P_N^{\gamma}\left[\left|\frac{1}{T}\int_0^T \eta_s(0)\,ds - \alpha\right| < 4\delta\right]$$
$$= \lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\left\{P_N^{\gamma}\left[\left|\frac{1}{T}\int_0^T \eta_s(0)\,ds - \alpha\right| < 4\delta\right]\right.$$
$$\left. + P_N^{\gamma}\left[\frac{1}{T}\left|\int_0^T\left(\eta_s(0) - \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\right)ds\right| > 2\delta\right]\right\}.$$

This last expression is greater than or equal to

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log P_N^{\gamma}\left[\left|\frac{1}{T}\int_0^T \frac{1}{2N\varepsilon+1}\sum_{|j|\le N\varepsilon}\eta_s(j)\,ds - \alpha\right| < 2\delta\right].$$

Since

$$\left\{\mu \in D([0,T], M_1);\left|\frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds - \alpha\right| < 2\delta\right\}$$

is open in $D([0,T], M_1)$, by the large deviation principle proved in the last subsection this last term is greater than or equal to

$$(3.24)\qquad -\lim_{\varepsilon\to 0}\inf\left\{I_\gamma(\mu);\left|\frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds-\alpha\right|<2\delta\right\}.$$

Fix $\tilde{\mu}\in U$ such that $|(1/T)\int_0^T\tilde{\mu}(s,0)\,ds-\alpha|<\delta$. Since $\tilde{\mu}\in U$, there exists $\varepsilon_0>0$ such that for $\varepsilon<\varepsilon_0$,

$$\left|\frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\tilde{\mu}(s,x)\,dx\,ds-\alpha\right|<2\delta.$$

Therefore,

$$I_\gamma(\tilde{\mu})\geq\inf\left\{I_\gamma(\mu);\left|\frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds-\alpha\right|<2\delta\right\}\quad\text{for every }\varepsilon<\varepsilon_0.$$

Then it follows that

$$I_\gamma(\tilde{\mu})\geq\lim_{\varepsilon\to 0}\inf\left\{I_\gamma(\mu);\left|\frac{1}{2T\varepsilon}\int_0^T\int_{-\varepsilon}^{\varepsilon}\mu(s,x)\,dx\,ds-\alpha\right|<2\delta\right\}.$$

Thus (3.24) is greater than or equal to

$$-\inf\left\{I_\gamma(\mu);\ \mu\in U\text{ and }\left|\frac{1}{T}\int_0^T\mu(s,0)\,ds-\alpha\right|<\delta\right\},$$

which, by the definition of $\tilde{\psi}_{1,T}$, is equal to

$$-\inf_{|\alpha-\beta|<\delta}\tilde{\psi}_{1,T}(\beta),$$

which is equal to

$$-\inf_{|\alpha-\beta|<\delta}\psi_{1,T}(\beta),$$

which proves the theorem. □

Returning to the symmetric simple exclusion process with generator $L_n$ given by (2.2), let $\psi_1=\psi_{1,1}$. We have the following immediate corollary.

COROLLARY. *For any closed subset $F$ and open subset $G$ of* $[0,1]$,

$$\limsup_{T\to\infty}\frac{1}{\sqrt{T}}\log P_\rho\left[\frac{1}{T}\int_0^T\eta_s(0)\,ds\in F\right]\leq-\inf_{\alpha\in F}\psi_1(\alpha),$$

$$\liminf_{T\to\infty}\frac{1}{\sqrt{T}}\log P_\rho\left[\frac{1}{T}\int_0^T\eta_s(0)\,ds\in G\right]\geq-\inf_{\alpha\in G}\psi_1(\alpha).$$

We do not have an explicit formula for the rate function $\psi_1$. Nevertheless, in the next section we will show that $\psi_1$ is not degenerate.

**4. The case $d=2$.** In this section, we do not establish a large deviation principle for the occupation time for the symmetric simple exclusion process in dimension 2, but we determine the order of magnitude of the decay rate of the large deviation probabilities. Furthermore, Theorem 4.1 together with Theorem 1 of [2] prove that the rate functions $\psi_d$, $d \neq 2$, obtained in Sections 2 and 3 are not degenerate.

In this section, we consider the symmetric simple exclusion process on $\mathbb{Z}^d$ with transition probabilities given by

$$p(x,y) = \frac{1}{2d} \quad \text{if } |x-y| = 1. \tag{4.1}$$

This is the Markov process on $\mathbb{X}_d$ with generator $L_n$ given by (2.2).

For $x$ in $\mathbb{Z}^d$, we will denote by $(X_\cdot^x)$ independent random walks on $\mathbb{Z}^d$ starting from $x$ with mean 1 holding time and transition probabilities given by (4.1). Let

$$p_s(x,y) = P[X_s^x = y] \quad \text{and} \quad G(x,y) = \int_0^\infty p_s(x,y)\,ds.$$

To state the main result of this section, we will have to introduce some notation. Let

$$\lambda_d = \begin{cases} \infty, & \text{if } d = 1,\\ \pi, & \text{if } d = 2,\\ 1/G(0,0), & \text{if } d \geq 3, \end{cases} \tag{4.2}$$

and

$$\psi(\lambda) = \begin{cases} 2\rho\left[\dfrac{e^{\lambda^2/2}-1}{\lambda} + \sqrt{\dfrac{2}{\pi}}\,\dfrac{e^{\lambda^2/2}\int_0^\lambda e^{-s^2/2}\,ds - \lambda}{\lambda}\right], & d = 1, \lambda \neq 0,\\ 0, & d = 1, \lambda = 0,\\ \dfrac{\pi\rho\lambda}{\pi-\lambda}, & \text{for } \lambda < \lambda_2 \text{ if } d = 2,\\ \dfrac{\rho\lambda}{1-\lambda G(0,0)}, & \text{for } \lambda < \lambda_d \text{ if } d \geq 3. \end{cases} \tag{4.3}$$

For $\alpha$ in [0, 1], let

$$I(\alpha) = \sup_{0<\lambda<\lambda_d} \{\lambda\alpha - \psi(\lambda)\}. \tag{4.4}$$

It is easy to see that in every dimension $I(\alpha) > 0$ for $\alpha > \rho$ and that

$$I(\alpha) = \begin{cases} \pi\left(\sqrt{\alpha} - \sqrt{\rho}\right)^2, & \text{if } d = 2,\\ \dfrac{1}{G(0,0)}\left(\sqrt{\alpha} - \sqrt{\rho}\right)^2, & \text{if } d \geq 3. \end{cases}$$

Now, we can state the main theorem of this section.

THEOREM 4.1. *For any $\alpha > \rho$,*

$$\limsup_{t\to\infty} \frac{1}{a_t} \log P_\rho\left[\frac{1}{t}\int_0^t \eta_s(0)\,ds \ge \alpha\right] \le -I(\alpha),$$

*with $a_t$ defined in* (1.1).

Theorem 4.1 together with Theorem 1 of [2] prove that the $a_t$ are the asymptotic decay rates of occupation time large deviation probabilities for the symmetric simple exclusion process (2.2) with generator $L_n$.

The main argument in the proof of Theorem 4.1 is an inequality comparing the simple exclusion process and independent random walks which holds in a larger context. To state this result, we consider a countable space $S$ and transition probabilities $p(x, y)$ on $S$ such that

$$p(x,y) = p(y,x) \ge 0 \quad\text{and}\quad \sum_y p(x,y) = 1. \tag{4.5}$$

Since no confusion can arise, until the proof of Theorem 4.1 we will also denote by $(\eta_t)$ the symmetric simple exclusion process on $S$ and by $(\xi_t)$ the process of independent random walks associated with these transition probabilities. Formally, $(\eta_t)$ is the Markov process on $\{0,1\}^S$ whose generator acts on cylindrical functions as (2.2), with $S$ instead of $\mathbb{Z}^d$ and $(\xi_t)$ is the Markov process on $\mathbb{N}^S$ whose generator acts on cylindrical functions as

$$Lf(\xi) = \sum_{x,y\in S} p(x,y)\xi(x)[f(\xi^{x,y}) - f(\xi)],$$

where

$$\xi^{x,y}(z) = \begin{cases} \xi(z), & \text{if } z \ne x,y, \\ \xi(x) - 1, & \text{if } z = x, \\ \xi(y) + 1, & \text{if } z = y, \end{cases}$$

and where $\mathbb{N}$ denotes the nonnegative integers.

For $A \subset S$, the expectation of the process starting from the configuration $\eta_0$ $(\xi_0)$ such that

$$\eta_0(x)[\xi_0(x)] = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases}$$

will be denoted by $E^A$. Choose a point in $S$ and denote it by 0. Now, we can state the following lemma.

LEMMA 4.1. *For every subset $A$ of $S$, $r$ in $\mathbb{R}$, $n$ in $\mathbb{N}$ and $0 \le s_1 < s_2 < \cdots < s_n < \infty$,*

$$E^A\left[\exp\left(r\sum_{j=1}^n \eta_{s_j}(0)\right)\right] \le E^A\left[\exp\left(r\sum_{j=1}^n \xi_{s_j}(0)\right)\right].$$

PROOF. Assume that $A$ is finite; the case of infinite $A$ will follow by taking limits. Fix a finite subset $A$ of $S$, $r \in \mathbb{R}$, $n \in \mathbb{N}$ and $0 \leq s_1 < \cdots < s_n < \infty$. Let $m = |A|$ and $\mathscr{M}$ be the set of matrices $m \times n$ with entries equal to 0 or 1. The elements of $\mathscr{M}$ will be denoted by $a$.

Fix $c > 0$, which will converge to $\infty$. The idea of the proof is to consider two auxiliary processes $(z_1(t); a(t))$ and $(z_2(t); a(t))$. In these processes, we will have two types of particles called first- and second-class particles. The first-class particles will evolve on $S$, while the second-class particles will not move and lie on $\{1, \dots, m\} \times \{1, \dots, n\}$. In the first process, in each point of $A$, we put a particle, label them from 1 to $m$ and call them first-class particles. We start without second-class particles $[a(0) = 0 \in \mathscr{M}]$. The first-class particles evolve on $S$ according to a symmetric simple exclusion process with mean 1 holding times and transition probabilities given by (4.5). These particles create second-class particles in the following way. When the $k$th particle is at 0 between times $s_j$ and $s_j + 1/c$, it creates a second-class particle in $a$ at the position $(k, j)$ at rate $c^2$. Each position $(k, j)$ is occupied by at most one second-class particle. Therefore, if a first-class particle tries to create a second-class particle at a position already occupied, nothing happens. The second process is the same, but with the first-class particles evolving as independent symmetric random walks on $S$.

When $c$ is large, if there is a particle at site 0 between times $s_j$ and $s_j + 1/c$, with high probability a second-class particle is created. In this way, knowing the state of the process at time $t \geq s_n$, we will be able to guess the number of particles on 0 at times $s_j$, $1 \leq j \leq n$. Comparing these two processes by a method introduced by Liggett (see Proposition 8.1.7 of [13]), we will prove the lemma. See [2] for a similar construction.

Let $T^m = \{\alpha \in S^m; \alpha_i \neq \alpha_j \text{ for every } i \neq j\}$ and fix $c > 0$. As usual, we will identify $z \in T^m$ with a subset $A$ of $S$ by

$$A = \{z_j; 1 \leq j \leq m\}.$$

The entries of an element $a$ of $\mathscr{M}$ will be denoted by $a_k^j$ for $1 \leq k \leq m$ and $1 \leq j \leq n$. Until the end of the proof, we will omit $c$ when no confusion can arise for the sake of simplicity. Consider the Markov processes on $S^m \times \mathscr{M}$ whose generators are given by

$$L_1^* f(z; a) = \sum_{k=1}^{m} \sum_{y \in S} p(z_k, y)\left[ f(z_1, \dots, z_{k-1}, y, z_{k+1}, \dots, z_m; a) - f(z; a)\right] \times 1_{[y \neq z_i, 1 \leq i \leq m]},$$

$$L_2^* f(z; a) = \sum_{k=1}^{m} \sum_{y \in S} p(z_k, y)\left[ f(z_1, \dots, z_{k-1}, y, z_{k+1}, \dots, z_m; a) - f(z; a)\right],$$

$$L_3^* f(z; a) = \sum_{j=1}^{n} \sum_{k=1}^{m} c_t^j 1_{[z_k = 0]}\left[ f\left(z; a_1^1, \dots, a_k^{j-1}, 1, a_k^{j+1}, \dots, a_m^n\right) - f(z; a)\right],$$

where

$$c_t^j = \begin{cases} c^2, & \text{if } t \in \left[s_j, s_j + \dfrac{1}{c}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$L_j = L_j^* + L_3^*, \qquad j = 1, 2,$$

$$f(z;a) = \exp\left(r \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^j\right), \qquad (z;a) \in S^m \times \mathscr{M}.$$

Although the rates are not homogeneous in time, to keep the notation simple, we denote by

$(U_t^j)_{t \geq 0}$ the semigroup associated with the generator $L_j$, $j = 1, 2$.

Since in the process with generator $L_2$, the first-class particles evolve and create second-class particles independently,

$$\begin{aligned} U_t^2 f(z;0) &= E_2^{(z,0)}\left[\exp\left(r \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^j(t)\right)\right] \\ &= \prod_{k=1}^{m} E_2^{(z_k,0)}\left[\exp\left(r \sum_{j=1}^{n} a_k^j(t)\right)\right], \end{aligned} \tag{4.6}$$

where we denoted by $E_j^{(z;a)}$ the expectation relative to the process with generator $L_j$ for $j = 1, 2$, starting from the configuration $(z;a)$. With this observation, it is easy to see that the function $U_t^2 f_0(z) = U_t^2 f(z;0)$ is bounded, symmetric and positive definite in the terminology of [13]. Then, with an integration by parts formula and an induction to take care of the inhomogeneous particles' creation rate, just as in the proof of Proposition 8.1.7 of [13],

$$U_t^2 f(z;0) - U_t^1 f(z;0) \geq 0 \quad \text{for every } (z;a) \in T^m \times \mathscr{M}. \tag{4.7}$$

On the other hand, we have by the bounded convergence theorem that

$$\begin{aligned} \lim_{c\to\infty} E_1^{(A,0)}\left[\exp\left(r \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^j(t)\right)\right] &= E^A\left[\exp\left(r \sum_{j=1}^{n} \eta_{s_j}(0)\right)\right], \\ \lim_{c\to\infty} E_2^{(A,0)}\left[\exp\left(r \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^j(t)\right)\right] &= E^A\left[\exp\left(r \sum_{j=1}^{n} \xi_{s_j}(0)\right)\right]. \end{aligned} \tag{4.8}$$

Therefore, since

$$E_j^{(A,0)}\left[\exp\left(r \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^j(t)\right)\right] = U_t^j f(A;0) \quad \text{for } j = 1, 2,$$

(4.7) and (4.8) prove the lemma. □

REMARK 4.1. E. Andjel gave us another proof of Lemma 4.1 based on correlation inequalities for the symmetric simple exclusion process (see [1]). His proof is obtained by induction on $n$. For $n = 0$ it is clear. Fix $n \geq 1$, $A \subset S$ such that $|A| = k < \infty$ and for $s_1 < \cdots < s_n$ let $\tilde{s} = (s_1, \ldots, s_n)$, $f(A, \tilde{s}) = E^A[\exp(r\sum_{j=1}^n \eta_{s_j}(0))]$, $g(A, \tilde{s}) = E^A[\exp(r\sum_{j=1}^n \xi_{s_j}(0))]$ and $F(r) = \sup_{s_1 \leq r} \sup_{|A|=k}[f(A, \tilde{s}) - g(A, \tilde{s})]$. Following the proof of Theorem 2.1 in [1], for fixed $A$ and $\tilde{s}$ we bound above the expression $f(A, \tilde{s}) - g(A, \tilde{s})$. In order to do it, we use the induction hypothesis and we observe from (4.6) that for $A$ such that $A \cap \{x, y\} = \emptyset$, $2g(A^{x,y}, \tilde{s}) \leq g(A^{x,x}, \tilde{s}) + g(A^{y,y}, \tilde{s})$ if $A^{x,y}$ is the configuration of $S^{\mathbb{N}}$ with one particle on each site of $A \cup \{x, y\}$ (two particles on $x$ if $x = y$). Proceeding as in [1], we obtain that for every $r < \infty$, $F(r) \leq \int_0^r ke^{-ku}\, du\, F(r)$ and this shows that $F(r) \leq 0$.

The proof of the following proposition relies on a simple trick. This result will enable us to prove Theorem 4.1.

PROPOSITION 4.1. *For every subset $A$ of $S$, $\lambda$ in $\mathbb{R}$ and $t \geq 0$,*

$$E^A\left[\exp\left(\lambda \int_0^t \eta_s(0)\, ds\right)\right] \leq E^A\left[\exp\left(\lambda \int_0^t \xi_s(0)\, ds\right)\right].$$

PROOF. Just as in the proof of the lemma, we assume first that $A$ is finite and prove the general case by taking limits. Since $\eta(0) \in \{0, 1\}$, we have

$$\begin{aligned}
&E^A\left[\exp\left(\lambda \int_0^t \eta_s(0)\, ds\right)\right] \\
&\quad = E^A\left[\exp\left(\lambda \int_0^t \frac{e^{r\eta_s(0)} - 1}{e^r - 1}\, ds\right)\right] \quad \text{for every } r > 0 \\
&\quad = e^{-\lambda t/(e^r - 1)} \sum_{n \geq 0} \left(\frac{\lambda}{e^r - 1}\right)^n \frac{1}{n!} \int_0^t ds_1 \cdots \int_0^t ds_n\, E^A\left[\exp\left(r \sum_{j=1}^n \eta_{s_j}(0)\right)\right] \\
&\quad \leq e^{-\lambda t/(e^r - 1)} \sum_{n \geq 0} \left(\frac{\lambda}{e^r - 1}\right)^n \frac{1}{n!} \int_0^t ds_1 \cdots \int_0^t ds_n\, E^A\left[\exp\left(r \sum_{j=1}^n \xi_{s_j}(0)\right)\right] \\
&\quad = E^A\left[\exp\left(\lambda \int_0^t \frac{e^{r\xi_s(0)} - 1}{e^r - 1}\, ds\right)\right],
\end{aligned}$$

where the inequality follows from Lemma 4.1. Letting $r \downarrow 0$, we obtain the result by the bounded convergence theorem. □

We now return to the proof of Theorem 4.1 and consider the symmetric simple exclusion process $(\eta_t)$ on $\mathbb{Z}^d$ with transition probabilities given by (4.1).

PROOF OF THEOREM 4.1. Fix $\alpha > \rho$. For $x$ in $\mathbb{Z}^d$, remember that $(X_s^x)$ denote independent random walks starting from $x$ with mean 1 holding time and transition probability given by (4.1).

By the Chebyshev inequality, for any $\lambda > 0$,

$$P_\rho\left[\frac{1}{t}\int_0^t \eta_s(0)\,ds \ge \alpha\right] \le e^{-\lambda\alpha a_t} E_\rho\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t \eta_s(0)\,ds\right)\right]. \tag{4.9}$$

Now, we shall develop the expectation to compare it with independent random walks.

$$\begin{aligned}
&E_\rho\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t \eta_s(0)\,ds\right)\right]\\
&\quad = \int E^A\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t \eta_s(0)\,ds\right)\right]\nu_\rho(dA)\\
&\quad \le \int E^A\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t \xi_s(0)\,ds\right)\right]\nu_\rho(dA)\\
&\quad = \int E\left[\exp\left(\lambda\frac{a_t}{t}\sum_{x\in A}\int_0^t X_s^x(0)\,ds\right)\right]\nu_\rho(dA)\\
&\quad = \int \prod_{x\in\mathbb{Z}^d}\left(1_{\{A\not\ni x\}} + 1_{\{A\ni x\}}E\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t X_s^0(x)\,ds\right)\right]\right)\nu_\rho(dA),
\end{aligned}$$

where the inequality follows from Proposition 4.1 and the last equality from the independence of the random walks. Since $\nu_\rho$ is a product measure, we can compute this last expression and get that it is equal to

$$\prod_{x\in\mathbb{Z}^d}\left\{1 + \rho\left(E\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t X_s^0(x)\,ds\right)\right] - 1\right)\right\},$$

which is bounded above by

$$\exp\left\{\rho\sum_{x\in\mathbb{Z}^d}\left(E\left[\exp\left(\lambda\frac{a_t}{t}\int_0^t X_s^0(x)\,ds\right)\right] - 1\right)\right\}.$$

We can develop the expectation term to obtain that it is equal to

$$\rho\sum_{x\in\mathbb{Z}^d}\sum_{n\ge 1}\left(\frac{\lambda a_t}{t}\right)^n\frac{1}{n!}\int_0^t\cdots\int_0^t ds_1\cdots ds_n\,E\left[\prod_{1\le j\le n}X_{s_j}^0(x)\right]$$

$$\begin{aligned}
&= \rho\sum_{n\ge 1}\left(\frac{\lambda a_t}{t}\right)^n\int\cdots\int_{0\le s_1\le\cdots\le s_n\le t} ds_1\cdots ds_n\\
&\qquad \times\sum_{x\in\mathbb{Z}^d} p_{s_1}(0,x)p_{s_2-s_1}(0,0)\cdots p_{s_n-s_{n-1}}(0,0)
\end{aligned}\tag{4.10}$$

$$= \rho\sum_{n\ge 1}\left(\frac{\lambda a_t}{t}\right)^n\varphi_n(t),$$

if we define $\varphi_n(t)$ by

$$\int \cdots \int_{0 \le s_1 \le \cdots \le s_n \le t} ds_1 \cdots ds_n \, p_{s_2 - s_1}(0,0) \cdots p_{s_n - s_{n-1}}(0,0).$$

Cox and Griffeath [5] showed (proof of Theorem 1, page 548) that

$$\lim_{t\to\infty} \frac{\rho}{a_t} \sum_{n \ge 1} \left( \frac{\lambda a_t}{t} \right)^n \varphi_n(t) = \psi(\lambda) \quad \text{for } \lambda < \lambda_d, \tag{4.11}$$

where $\psi$ and $\lambda_d$ are given by (4.3). Therefore, by (4.9), (4.10) and (4.11),

$$\limsup_{t\to\infty} \frac{1}{a_t} \log P_\rho\left[ \frac{1}{t} \int_0^t \eta_s(0)\, ds \ge \alpha \right] \le -\lambda\alpha + \psi(\lambda) \quad \text{for every } 0 < \lambda < \lambda_d.$$

Minimizing in $\lambda$, we prove the theorem. □

We list in the next proposition the properties of the rate function $\psi_d$ obtained in Sections 2 and 3.

PROPOSITION 4.2. *For $d \ne 2$, $\psi_d$ is a continuous, bounded, convex function. Moreover, $I \le \psi_d$, where $I$ is given by (4.4). In particular, $\psi_d(\alpha) = 0$ if and only if $\alpha = \rho$.*

PROOF. *Boundedness*. For $d \ge 3$, from the definition of $\psi_d$, we obtain that $\psi_d \le \tilde{I}_d(\sigma)$ for every $\sigma \in M$ such that $\int \eta(0)\sigma(d\eta) = \alpha$, where $\tilde{I}_d$ is given by (2.5). Let

$$\sigma_\alpha(d\eta) = \left( \frac{\alpha}{\rho} \right)^{\eta(0)} \left( \frac{1-\alpha}{1-\rho} \right)^{1-\eta(0)} \nu_\rho(d\eta).$$

A simple computation shows that $I_d(\sigma_\alpha) \le \rho \vee (1-\rho)$. Thus

$$\sup_{0 \le \alpha \le 1} \psi_d(\alpha) \le \rho \vee (1-\rho).$$

In the same way, in dimension 1, we have that $\psi_1(\alpha) \le I_\rho(\mu)$ for every $\mu \in U$ such that $\int_0^1 \mu(s,0)\, ds = \alpha$, where $I_\rho$ is given by (3.12). Consider a smooth function $f \in M_1(\rho) \cap C^2(\mathbb{R})$ for which $f(0) = \alpha$. Let $\mu \in D([0,1], M_1)$ such that $\mu(t, \cdot) \equiv f$ for every $t \in [0,1]$. (3.14) enables us to compute $I_\rho(\mu)$. We obtain

$$\begin{aligned} I_\rho(\mu) = \int_{\mathbb{R}} \Bigg\{ & f(x) \log \frac{f(x)}{\rho} + (1 - f(x)) \log \frac{(1-f(x))}{(1-\rho)} \\ & + \frac{1}{8} \frac{[f'(x)]^2}{f(x)[1-f(x)]} \Bigg\}\, dx \\ = K(f). & \end{aligned}$$

Therefore, we have

$$\psi_1(\alpha) \le \inf_{\substack{f \in M_1(\rho) \cap C^2(\mathbb{R}) \\ f(0)=\alpha}} K(f) \le K(f_0),$$

where

$$f_0(x) = \begin{cases} \alpha + (\rho - \alpha)x^2, & \text{if } |x| < 1, \\ \rho, & \text{otherwise}. \end{cases}$$

A simple computation shows that $K(f_0)$ is bounded above by

$$\log \frac{1}{\rho(1-\rho)} + \frac{1}{\rho} + \frac{1}{(1-\rho)}.$$

*Convexity*. We know (see [7]) that $\tilde{I}_d$ given by (2.5) is convex. It is easy to show that this property is inherited by $I_d$ given by (2.6) and by $\psi_d$. On the other hand, we saw in Section 3.4 that $\psi_1$ is a convex function.

*Continuity*. Since $\psi_d$ is a convex and bounded function, $\psi_d$ is continuous in $(0, 1)$. To see that $\psi_d$ is continuous in $[0, 1]$, we just have to prove that $\psi_d$ is lower semicontinuous. By construction, $\psi_1$ is lower semicontinuous. On the other hand, since $M_1$ is compact and $I_d$ given by (2.6) is a lower semicontinuous function, $\psi_d$ is also lower semicontinuous.

$I \le \psi_d$. For $A \subset [0, 1]$, denote by $A^0$ the interior of $A$ and by $\bar{A}$ its closure. From the large deviation principle proved in Sections 2 and 3, we have that if $\inf_{\alpha \in A^0} \psi_d(\alpha) = \inf_{\alpha \in \bar{A}} \psi_d(\alpha)$, then

$$\lim_{t\to\infty} \frac{1}{a_t} \log P_\rho\left[\frac{1}{t}\int_0^t \eta_s(0)\, ds \in A\right] = -\inf_{\alpha \in A} \psi_d(\alpha).$$

Fix $\alpha > \rho$. The proof for $\alpha < \rho$ is similar. Since $\psi_d$ is nonnegative, convex and $\psi_d(\rho) = 0$, $\psi_d$ is nonincreasing on $[0, \rho]$ and nondecreasing on $[\rho, 1]$. Since $\psi_d$ is continuous, with the notation of Proposition 4.1,

$$\begin{aligned}
-\psi_d(\alpha) &= -\inf_{\beta \ge \alpha} \psi_d(\beta) \\
&= \lim_{t\to\infty} \frac{1}{a_t} \log P_\rho\left[\frac{1}{t}\int_0^t \eta_s(0)\, ds \ge \alpha\right] \\
&\le -\lambda\alpha + \liminf_{t\to\infty} \frac{1}{a_t} \log E_\rho\left[\exp\left(\lambda \frac{a_t}{t}\int_0^t \eta_s(0)\, ds\right)\right] \\
&\le -\lambda\alpha + \liminf_{t\to\infty} \frac{1}{a_t} \log E_\rho\left[\exp\left(\lambda \frac{a_t}{t}\int_0^t \xi_s(0)\, ds\right)\right] \\
&= -\lambda\alpha + \psi(\lambda),
\end{aligned}$$

where we applied the Chebyshev inequality to obtain the first inequality, Proposition 4.1 to obtain the second inequality and (4.11) to obtain the last equality. Minimizing in $\lambda$, we conclude the proof of Proposition 4.2. □

REMARK 4.2. Notice that we only used the special form of $p(x, y)$ to compute the limit in (4.11). But, from Section 5 of [5], we know that this limit can be evaluated for a larger class of transition probabilities. In this way, Theorem 4.1 extends immediately to this larger class.

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