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ON THE PARABOLIC MARTIN BOUNDARY OF THE ORNSTEIN–UHLENBECK OPERATOR ON WIENER SPACE

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We study the positive parabolic functions of the Ornstein–Uhlenbeck operator on an abstract Wiener space E using the approach developed by Dynkin. This involves first proving a characterization of the entrance space of the corresponding Ornstein–Uhlenbeck semigroup and deriving an integral representation for an arbitrary entrance law in terms of extreme ones. It is shown that the Cameron–Martin densities are extreme parabolic functions, but that if dim $E = \infty$, not every positive parabolic function has an integral representation in terms of those (which is in contrast to the finite-dimensional case). Furthermore, conditions for a parabolic function to be representable in terms of Cameron–Martin densities are proved.

1. Introduction. In recent years there has been considerable interest in the elliptic and parabolic Martin boundaries of the Ornstein–Uhlenbeck operator/process on $\mathbb{R}^d$ (cf. [30], [8] and [21]). More recently, Taylor [35], [36] (among other things) obtained some results for the corresponding infinite-dimensional situation, that is, for the Ornstein–Uhlenbeck process on Wiener space, and suggested looking at its harmonic and parabolic functions in further detail.

In this paper we take up Taylor's suggestion and concentrate on the square-integrable parabolic functions on an abstract Wiener space (E, H, μ). A family $p := (p(t, \cdot))_{t \in \mathbb{R}}$ of functions in (real) $L^2(E; \mu)$ is called (L-) *parabolic* if $p(t, \cdot) \in D(L)$ and

$$Lp(t, \cdot) = \frac{d}{dt} p(t, \cdot), \qquad t \in \mathbb{R} \tag{1.1}$$

(cf. Definition 3.1 below), where L is the *Ornstein–Uhlenbeck operator* on E, that is, the $L^2(E; \mu)$-generator of the transition function $(\pi_t)_{t \geq 0}$ of the Ornstein–Uhlenbeck process on E. Let $\mathscr{P}$ denote the convex set of all positive parabolic functions on E normalized so that $\int_E p(t, z)\mu(dz) = 1$, $t \in \mathbb{R}$. We study its extreme points and discuss integral representations for any $p \in \mathscr{P}$ in terms of them. Our approach to this problem is entirely in the spirit of Dynkin (cf. [9]–[12]; see also [14] for the case of random fields). It can be roughly described as follows: One considers the entrance law $\mu_t := p(t, \cdot)\mu$, $t \in \mathbb{R}$, for $(\pi_t)_{t \geq 0}$ given by $p \in \mathscr{P}$ and constructs a probability measure P on the space

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$\Omega := C(\mathbb{R}, E)$ of continuous functions from $\mathbb{R}$ to E such that the canonical coordinate process $X_t: \Omega \to E$, $t \in \mathbb{R}$, is Markovian, has entrance law $(\mu_t)_{t \in R}$ and transition function $(\pi_t)_{t \geq 0}$. If $\mathscr{M}$ denotes the convex set of all such probability measures on Ω and $\mathscr{M}_e$ its extreme points, a fundamental result by Dynkin (cf. [12]) states that an arbitrary $P \in \mathscr{M}$ can be represented as an integral of elements in $\mathscr{M}_e$. This induces a corresponding representation for all the entrance laws of $(\pi_t)_{t \geq 0}$ which in many cases gives a representation for the densities $p(t, \cdot\,)$, $t \in \mathbb{R}$.

A major step to implement the above program is to characterize $\mathscr{M}_e$. We prove in Theorem 2.7 that $\mathscr{M}_e$ and hence the extreme entrance laws are parametrized by E. The methods to derive this result are quite similar to those in [29], Section 4 (see also [28] and for the case of random fields [26], [27]), where such a characterization was proved for a much more difficult case, namely for the Ornstein–Uhlenbeck process associated with the free quantum field (cf. Remark 2.12(iii) below and also Remark 2.9 concerning the precise relation with the recent results in [13]). The tools that we develop to prove this fact are also used to prove Dynkin's representation theorem in this particular case (Theorem 2.10) and to obtain more precise information about the representing measures.

In Section 3 we then consider whether the resulting representation for the entrance laws (cf. Corollary 2.11 below) implies a representation for elements in $\mathscr{P}$. The main difference from the cases studied in [9]–[12] is that if $\dim E = +\infty$ there is no transition density, that is, π_t is not absolutely continuous with respect to a fixed reference measure for all $t > 0$. (Note that the same is true for the corresponding resolvent kernels so that [19] is also not applicable.) In fact, it turns out that here the situation is entirely different from the finite-dimensional case. On the one hand the extreme entrance laws parametrized by $h \in H(\subset E)$ are absolutely continuous with respect to μ and their Radon–Nikodym derivatives, the Cameron–Martin densities p^h, are extreme points of $\mathscr{P}$. In contrast to the finite-dimensional case, however, not every $p \in \mathscr{P}$ can be represented as an integral over p^h, $h \in H$ (Example 3.10) as $H \neq E$ in an abstract infinite-dimensional Wiener space. Let $\mathscr{P}_1$ be the convex set generated by p^h, $h \in H$. For this subcone of $\mathscr{P}$ a characterization and representation theorem is proved (Theorem 3.7). A rather mild condition on $\nabla p(t, \cdot\,)$ ensures that $p \in \mathscr{P}_1$ (Theorem 3.11) and furthermore we show that $p \in \mathscr{P}_1$, if $p \in \mathscr{P}$ such that $p(t, \cdot\,) \in L^\infty(E; \mu)$ for some $t \in \mathbb{R}$ (Theorem 3.14).

By using an abstract Wiener space we reprove the integral representation theorem for the finite-dimensional case. In the case of the classical Wiener space, a result about the exit space that corresponds to Theorem 2.7 was obtained independently by Föllmer [15].

2. Characterization of the entrance space and integral representation. Let (E, H, μ) be an abstract Wiener space (cf. [17] and also [18] and [37]), that is, E is a separable real Banach space, μ is a Gaussian mean zero measure on $(E, \mathscr{B}(E))$ with $\mu(U) > 0$ for each open nonempty subset U of E and $(H, \langle\ ,\ \rangle_H)$ is a real separable Hilbert space such that $H \subset E$ continuously and densely and for all $l_1, l_2 \in E'$,

$$\int_E {}_{E'}\langle l_1, z\rangle_{E\,E'}\langle l_2, z\rangle_E \mu(dz) = \langle l_1, l_2\rangle_H.$$

Here $\mathscr{B}(E)$ denotes the Borel σ-algebra on E and E' the (topological) dual of E. We have that $E' \subset H \subset E$ continuously and densely, and ${}_{E'}\langle\ ,\ \rangle_E$ restricted to $E' \times H$ coincides with $\langle\ ,\ \rangle_H$. "Gaussian" means that each $l \in E'$ has a Gaussian distribution in $\mathbb{R}$ under μ [i.e., is $N(0, \|l\|_H^2)$-distributed, where $\|\ \|_H := \langle\ ,\ \rangle_H^{1/2}$]. Besides the operator norm $\|\ \|_{E'}$, we will also consider the weak*-topology on E'. Recall that $B'_n := \{l \in E' \mid \|l\|_{E'} \le n\}$, $n \in]0, \infty[$, equipped with the weak*-topology is compact and metrizable, hence, in particular, separable. Let D_n be a countable weak*-dense subset of B'_n, $n \in \mathbb{N}$, such that $D_n \subset D_{n+1}$ for all $n \in \mathbb{N}$, and let $\tilde{D}_n$ be its $\mathbb{Q}$-linear span. Define

$$D_0 := \bigcup_{n \in \mathbb{N}} \tilde{D}_n. \tag{2.0}$$

Define the semigroup $(\pi_t)_{t \ge 0}$ on the nonnegative $\mathscr{B}(E)$-measurable functions u by Mehler's formula

$$\begin{aligned} \pi_t(z, u) &:= \int_E u(z')\pi_t(z, dz') \\ &= \int_E u\left(e^{-t}z + \sqrt{1 - e^{-2t}}\, z'\right)\mu(dz'), \qquad t \ge 0, z \in E. \end{aligned} \tag{2.1}$$

As usual we set $\Omega := C(\mathbb{R}, E)$, $X_t(\omega) := \omega(t)$, $\omega \in \Omega$, $t \in \mathbb{R}$,

$$\mathscr{F} := \sigma\{X_t | t \in \mathbb{R}\} \quad \text{and} \quad \mathscr{F}_t := \sigma\{X_s | s \le t\}, \qquad t \in \mathbb{R}.$$

Let $\mathscr{M}$ be the convex set of all probability measures P on $(\Omega, \mathscr{F})$ such that

$$E_P[X_t \in dz | \mathscr{F}_s] = \pi_{t-s}(X_s, dz) \quad \text{for all } t, s \in \mathbb{R}, t > s. \tag{2.2}$$

In this section we characterize

$$\mathscr{M}_e := \text{extreme points of } \mathscr{M} \tag{2.3}$$

and show that every $P \in \mathscr{M}$ has an integral representation in terms of them. We recall that a family $(\mu_t)_{t \in \mathbb{R}}$ of probability measures on $(E, \mathscr{B}(E))$ is called an *entrance law* (cf. [9]–[12]) if

$$\mu_t \pi_s = \mu_{t+s} \quad \text{for all } t \in \mathbb{R}, s \ge 0, \tag{2.4}$$

that is, $\int \pi_s(z, B)\mu_t(dz) = \mu_{t+s}(B)$ for all $B \in \mathscr{B}(E)$. By considering the corresponding Fourier transforms, we obtain

$$\mu \pi_t = \mu \quad \text{for all } t \ge 0, \tag{2.5}$$

that is, μ is an *invariant measure* for $(\pi_t)_{t \ge 0}$, in particular an entrance law.

PROPOSITION 2.1. *Let $P \in \mathscr{M}$ and set*

$$\mu_t := P \circ X_t^{-1}, \qquad t \in \mathbb{R} \tag{2.6}$$

(i.e., the one-dimensional marginals of P). Then $(\mu_t)_{t \in \mathbb{R}}$ is an entrance law.

Conversely, for each entrance law $(\mu_t)_{t \in \mathbb{R}}$ *on* $(E, \mathscr{B}(E))$ *there exists a unique* $P \in \mathscr{M}$ *satisfying* (2.6). *In particular, there exists* $P_\mu \in \mathscr{M}$ *with entrance law* μ.

PROOF. The first part is obvious. To prove the second part, note that by Kolmogorov's existence theorem there is a unique probability measure $\tilde{P}$ on $(E^{\mathbb{R}}, \mathscr{A})$ (where $\mathscr{A}$ is the σ-algebra generated by the canonical projections $\tilde{X}_t$: $E^{\mathbb{R}} \to E$) such that for $-\infty < t_0 < \cdots < t_n < \infty$ and $A_0, \dots, A_n \in \mathscr{B}(E)$,

$$\tilde{P}\left[\tilde{X}_{t_0} \in A_0, \dots, \tilde{X}_{t_n} \in A_n\right] = \int_{A_0}\int_{A_1} \cdots \int_{A_n} \pi_{t_n - t_{n-1}}(x_{n-1}, dx_n) \cdots \pi_{t_1 - t_0}(x_0, dx_1)\mu_{t_0}(dx_0). \tag{2.7}$$

We have to show the existence of a probability measure P on $(\Omega, \mathscr{F})$ having the same finite-dimensional distributions as $\tilde{P}$. We shall do this in two steps and consider first the special case where $\mu_t = \mu$ for all $t \in \mathbb{R}$. Then for $t > s$ and all $\alpha > 0$,

$$\begin{aligned}
\int \|\tilde{X}_s - \tilde{X}_t\|_E^\alpha \, d\tilde{P} &= \int \|y - z\|_E^\alpha \pi_{t-s}(y, dz)\mu(dy) \\
&= \int\int \left\|(1 - e^{-(t-s)})y - \sqrt{1 - e^{-2(t-s)}}\, z\right\|_E^\alpha \mu(dz)\mu(dy) \\
&\le 2^\alpha (1 - e^{-(t-s)})^{\alpha/2}\left[\int \left\|\sqrt{1 - e^{-(t-s)}}\, y\right\|_E^\alpha \mu(dy) \right.\\
&\qquad \left. + \int \left\|\sqrt{1 + e^{-(t-s)}}\, z\right\|_E^\alpha \mu(dz)\right] \\
&\le 2^{1 + 3\alpha/2}(t - s)^{\alpha/2} \int \|z\|_E^\alpha \mu(dz).
\end{aligned}$$

But $\int \|z\|_E^\alpha \mu(dz) < \infty$ by the Fernique–Skorohod theorem (cf. [34], Theorem (3.41)). Consequently, by the Kolmogorov–Prohorov continuity criterion, $\tilde{P}$ gives rise to a probability measure P_μ on $(\Omega, \mathscr{F})$ with the same finite-dimensional distributions. In the general case by ([6], 63.5, and the following remark), it suffices to prove that for each fixed $t_0 \in \mathbb{R}$ there is a probability measure P_{t_0} on $(\Omega, \mathscr{F})$ such that for all $t_0 < t_1 < \cdots < t_n < \infty$, $A_1, \dots, A_n \in \mathscr{B}(E)$, $P_{t_0}[X_{t_1} \in A_1, \dots, X_{t_n} \in A_n]$ is given by the right-hand side of (2.7) with $A_0 = E$.

Define for $z \in E$, Θ_z: $\Omega \to \Omega$ by

$$\Theta_z(\omega)(t) := \begin{cases} \omega(t) + e^{-(t-t_0)}(z - \omega(t_0)), & \text{if } t \ge t_0, \\ z, & \text{if } t < t_0, \end{cases}$$

for $\omega \in \Omega$, and let $P_z := P_\mu \circ \Theta_z^{-1}$ (where P_μ is as in step 1). Define

$$P_{t_0} := \int_E P_z \, \mu_{t_0}(dz).$$

Then P_{t_0} is a probability measure on $(\Omega, \mathscr{F})$ and considering the corresponding Fourier transforms we see that the finite-dimensional distributions are as required (cf. [29], proof of 2.4). □

REMARK 2.2. (i) Note that under P_μ, $(X_t)_{t\in\mathbb{R}}$ is just the *Ornstein–Uhlenbeck process* on E.

(ii) By the special form of $(\pi_t)_{t\geq 0}$ and the martingale convergence theorem, it is easy to see that $P \in \mathscr{M}$ if and only if

$$E_P\left[\exp(i_{E'}\langle l, X_t\rangle_E)|\mathscr{F}_s\right] = \pi_{t-s}\left(X_s, \exp\left(i_{E'}\langle l, \cdot\rangle_E\right)\right), \quad P\text{-a.s.},$$

for all $s, t \in \mathbb{Q}$, $s < t$, and all $l \in D_0$. Note that these are only countably many conditions (which will be important below in order to "control" zero sets).

(iii) Let $P \in \mathscr{M}$. Since $(\Omega, \mathscr{F})$ is standard Borel (cf. [23], Chapter V, Definition 2.2), there exists a regular conditional probability $\Pi^P(\omega, d\omega')$ given $\mathscr{F}_{-\infty}$. Since each $\mathscr{F}_s$ is countably generated, by (ii) we see that $\Pi^P(\omega, \cdot) \in \mathscr{M}$ for P-a.e. $\omega \in \Omega$. (Later it will turn out that in fact we can replace $\mathscr{M}$ by $\mathscr{M}_e$ here.)

Define for $\omega \in \Omega$, $T_\omega(\omega') := \omega' + \omega$, $\omega' \in \Omega$, and for $P \in \mathscr{M}$ denote its image measure under T_ω by $P \circ T_\omega^{-1}$. Correspondingly, for $z \in E$ define $T_z(z') := z' + z$, $z' \in E$, and $\nu \circ T_z^{-1}$ for a positive measure ν on $(E, \mathscr{B}(E))$. Furthermore, define for $z \in E$, $K_z \in \Omega$ by

$$K_z(t) := e^{-t}z, \qquad t \in \mathbb{R}. \tag{2.8}$$

PROPOSITION 2.3. (i) *Let $P \in \mathscr{M}$. Then $P \circ T_{K_z}^{-1} \in \mathscr{M}$ for each $z \in E$.*

(ii) *Let $\omega := (\omega(t))_{t\in\mathbb{R}} \in E^{\mathbb{R}}$. Then $(\mu \circ T_{\omega(t)}^{-1})_{t\in\mathbb{R}}$ is an entrance law for $(\pi_t)_{t\geq 0}$ if and only if $\omega(t) = e^{-t}z$ for all $t \in \mathbb{R}$ for some $z \in E$. In this case $P_\mu \circ T_{K_z}^{-1}$ is the corresponding element in $\mathscr{M}$ with entrance law $(\mu \circ T_{\omega(t)}^{-1})_{t\in\mathbb{R}}$.*

PROOF. (i) (2.1) implies that for nonnegative $\mathscr{B}(E)$-measurable u,

$$\pi_t(z, u(\cdot + z_0)) = \pi_t(z + e^t z_0, u), \qquad t > 0;\ z, z_0 \in E. \tag{2.9}$$

Now (i) is obvious.

(ii) We have for $\mu_t := \mu \circ T_{\omega(t)}^{-1}$, $t \in \mathbb{R}$, and all $l \in E'$ and $t \in \mathbb{R}$, $s > 0$,

$$\int_E \exp(i_{E'}\langle l, z\rangle_E)\mu_{t+s}(dz) = \exp\left[i_{E'}\langle l, \omega(t+s)\rangle_E - \tfrac{1}{2}\|l\|_H^2\right]$$

and by (2.9) and (2.5),

$$\begin{aligned}\int_E &\pi_s(z, \exp(i_{E'}\langle l, \cdot\rangle_E))\mu_t(dz)\\ &= \exp\left[i_{E'}\langle l, e^{-s}\omega(t)\rangle_E\right]\int_E \pi_s(z, \exp(i_{E'}\langle l, \cdot\rangle_E))\mu(dz)\\ &= \exp\left[i_{E'}\langle l, e^{-s}\omega(t)\rangle_E - \tfrac{1}{2}\|l\|_H^2\right].\end{aligned}$$

Defining $z := \omega(0)$, we obtain the first part of (ii). The second part follows from Proposition 2.1 and (i). □

For the reader's convenience the following folklore lemma is proved in the Appendix.

LEMMA 2.4. *Let $P \in \mathscr{M}$; then $P \in \mathscr{M}_e$ if and only if $P(F) = 1$ or 0 for every $F \in \mathscr{F}_{-\infty}$, where*

$$\mathscr{F}_{-\infty} := \bigcap_{t \in \mathbb{R}} \mathscr{F}_t. \tag{2.10}$$

PROPOSITION 2.5. *$P_\mu(F) = 1$ or 0 for $F \in \mathscr{F}_{-\infty}$.*

PROOF. By Remark 2.2 (iii) (first half) and because of the one-to-one correspondence between entrance laws and elements in $\mathscr{M}$, it suffices to prove that P_μ – a.s.,

$$\begin{aligned} &E_{P_\mu}\left[\exp\left(i_{E'}\langle l, X_t\rangle_E\right) \mid \mathscr{F}_{-\infty}\right] \\ &\quad = \int \exp\left(i_{E'}\langle l, X_t\rangle_E\right) dP_\mu \quad \text{for all } t \in \mathbb{Q},\ l \in D_0. \end{aligned} \tag{2.11}$$

So, let $t \in \mathbb{Q}$, $l \in D_0$. By the reverse martingale convergence theorem, it follows since $P_\mu \in \mathscr{M}$, that P_μ – a.s.,

$$\begin{aligned} &E_{P_\mu}\left[\exp\left(i_{E'}\langle l, X_t\rangle_E\right) \mid \mathscr{F}_{-\infty}\right] \\ &\quad = \lim_{n\to\infty} E_{P_\mu}\left[\exp\left(i_{E'}\langle l, X_t\rangle_E\right) \mid \mathscr{F}_{-n}\right] \\ &\quad = \lim_{n\to\infty} \exp\left(ie^{-(t+n)}{}_{E'}\langle l, X_{-n}\rangle_E - \tfrac{1}{2}\left(1 - e^{-2(t+n)}\right)\|l\|_H^2\right) \\ &\quad = \exp\left(-\tfrac{1}{2}\|l\|_H^2\right) \\ &\quad = \int \exp\left(i_{E'}\langle l, X_t\rangle_E\right) dP_\mu, \end{aligned} \tag{2.12}$$

where in step 2 we used (2.1) and in step 3 we used the fact that $\lim_{n\to\infty} e^{-n} X_{-n} = 0$, P_μ – a.s. since $P_\mu \circ X_t^{-1} = \mu$ for all $t \in \mathbb{R}$ and hence

$$\sum_{n=1}^{\infty} \int \left(e^{-n}{}_{E'}\langle l, X_{-n}\rangle_E\right)^2 dP_\mu = \sum_{n=1}^{\infty} e^{-2n}\|l\|_H^2 < \infty. \tag{2.13}$$ □

Below we shall make use of the following well-known lemma whose proof is a simple application of some basic facts in functional analysis. For completeness we also include this proof in the Appendix.

LEMMA 2.6. *Let ν be a (not necessarily mean zero) Gaussian measure on $(E, \mathscr{B}(E))$ and $u \in L^p(E;\nu)$ for some $p \in]1,\infty[$. Then there exists $z_0 \in E$ such* *that*

$$\int {}_{E'}\langle l, z\rangle_E u(z)\nu(dz) = {}_{E'}\langle l, z_0\rangle_E \quad \textit{for all } l \in E'. \tag{2.14}$$

We now prove the characterization of $\mathscr{M}_e$.

THEOREM 2.7. $\mathscr{M}_e = \{P_\mu \circ T_{K_z}^{-1} | z \in E\}$.

PROOF. Let $z \in E$. By Proposition 2.3(i) we know that $P_\mu \circ T_{K_z}^{-1} \in \mathscr{M}$ and since $F \in \mathscr{F}_{-\infty}$ implies that $F - \omega := \{\omega' - \omega | \omega' \in F\} \in \mathscr{F}_{-\infty}$ for all $\omega \in \Omega$, we conclude that $P_\mu \circ T_{K_z}^{-1}$ is trivial on $\mathscr{F}_{-\infty}$ since so is P_μ (by Proposition 2.5). Hence the set on the right-hand side is contained in $\mathscr{M}_e$ by Lemma 2.4. To show the dual inclusion, let $P \in \mathscr{M}_e$. Fix $t \in \mathbb{R}$ and let L_0 be a countable dense subset of $L^1_{\mathbb{C}}(\mathbb{R}; ds)$ (= all *complex-valued* Lebesgue integrable functions on $\mathbb{R}$). Since P is trivial on $\mathscr{F}_{-\infty}$ and belongs to $\mathscr{M}$, the reverse martingale convergence theorem and Fubini's theorem imply that there exists $\omega \in \Omega$ such that for all $f \in L_0$ and $l \in D_0$,

$$\begin{aligned}
&\int f(s) \int \exp\big(is_{E'}\langle l, X_t\rangle_E\big)\, dP\, ds \\
&\quad = \lim_{n\to\infty} E_P\left[\int f(s)\exp\big(is_{E'}\langle l, X_t\rangle_E\big)\, ds | \mathscr{F}_{-n}\right](\omega) \\
&\quad = \lim_{n\to\infty} \int f(s)\pi_{t+n}\big(X_{-n}(\omega), \exp\big(is_{E'}\langle l, \cdot\,\rangle_E\big)\big)\, ds \\
&\quad = \lim_{n\to\infty} \int f(s)\exp\Big(ise^{-(t+n)}{}_{E'}\langle l, X_{-n}(\omega)\rangle_E \\
&\qquad\qquad - \tfrac{1}{2}s^2\big(1 - e^{-2(t+n)}\big)\|l\|_H^2\Big)\, ds.
\end{aligned} \tag{2.15}$$

By Lemma 2.8 below it follows that ${}_{E'}\langle l, X_t\rangle_E$ is Gaussian under P with variance $\|l\|_H^2$ for all $l \in D_0$. Hence (e.g., by [22], Lemma 1.5) the same holds for all $l \in E'$, that is, $\mu_t := P \circ X_t^{-1}$ is Gaussian on E with the same covariance as μ. Lemma 2.6 implies that its mean $\omega(t)$ belongs to E; thus $\mu_t = \mu \circ T_{\omega(t)}^{-1}$. Since $P \in \mathscr{M}$, $(\mu_t)_{t\in\mathbb{R}}$ is an entrance law for $(\pi_t)_{t\geq 0}$, consequently by Proposition 2.3(ii), $P = P_\mu \circ T_{K_z}^{-1}$ for some $z \in E$. It remains to prove Lemma 2.8 (cf. [26], Proposition 8.2).

LEMMA 2.8. *Let ρ_n, $n \in \mathbb{N}$, and ρ be probability measures on $\mathbb{R}$ and L_0 a dense subset of $L^1_{\mathbb{C}}(\mathbb{R}; ds)$. Assume that for the corresponding Fourier transforms $\hat\rho_n$, $n \in \mathbb{N}$, and $\hat\rho$,*

$$\lim_{n\to\infty} \int f\hat\rho_n\, ds = \int f\hat\rho\, ds \quad \textit{for each } f \in L_0. \tag{2.16}$$

Then $(\rho_n)_{n\in\mathbb{N}}$ converges weakly to ρ. If, in addition, all ρ_n are Gaussian with means m_n such that their variances $(\sigma_n)_{n\in\mathbb{N}}$ converge to σ, then ρ is Gaussian with variance σ and mean $m := \lim_{n\to\infty} m_n$.

PROOF. Since $|\hat{\rho}_n|, |\hat{\rho}| \le 1$ on $\mathbb{R}$, (2.16) holds for all $f \in L^1_{\mathbb{C}}(\mathbb{R}; ds)$. Now the proof of the first part is analogous to the converse of Lévy's continuity theorem (cf. [6], Proof of Theorem 48.7, for the version we have in mind). If all ρ_n are Gaussian and $\lim_{n\to\infty}\sigma_n = \sigma$, then by Lévy's continuity theorem $\hat{\rho}_n(s) \to_{n\to\infty} \hat{\rho}(s)$ and hence $\exp(ism_n) \to_{n\to\infty} \varphi(s)$ locally uniformly in $s \in \mathbb{R}$ for some continuous function φ on $\mathbb{R}$; consequently, the Dirac measures $(\varepsilon_{m_n})_{n\in\mathbb{N}}$ weakly converge. It follows that $m := \lim_{n\to\infty} m_n$ exists in $\mathbb{R}$ and hence $\hat{\rho}(s) = \exp(ism - \frac{1}{2}s^2\sigma^2)$. □

REMARK 2.9. In the case where E is a conuclear space instead of a Banach space there is a result by Dynkin (cf. [13], Theorem 5.1) related to Theorem 2.7 above, but instead of $\mathscr{M}$ only a subclass $\mathscr{M}^1$ is considered. It consists of all $P \in \mathscr{M}$ having the property that for $t \in \mathbb{R}$, $l \mapsto \int_{E'}\langle l, X_t\rangle_E\, dP$, $l \in E'$, is weak*-continuous. However, in the conuclear case, [13] also treats the time-inhomogeneous case.

Now, we shall prove the integral representation mentioned above. We emphasize that [10], Theorems 3.1 and 2.1, or [12], Theorem 9.1, are not directly applicable since in those references $\Omega(= C(\mathbb{R}, E))$ is replaced by $E^{\mathbb{R}}$. With some work, it is of course possible to reduce our situation to those general theorems. We prefer to give a direct proof which uses the Gaussian nature of our case to obtain additional specific information on how to obtain the representing measure corresponding to a given $P \in \mathscr{M}$.

THEOREM 2.10. *There exists an $\mathscr{F}_{-\infty}/\mathscr{F}$-measurable map $M: \Omega \to \Omega$ with $M(\Omega) = \{K_z | z \in E\}$ [cf. (2.8)] such that for every $P \in \mathscr{M}$,*

$$(2.17)\qquad P(F) = \int_E P_\mu(F - K_z)\, m_P(dz), \qquad F \in \mathscr{F},$$

where m_P is the probability measure on $(E, \mathscr{B}(E))$ defined by

$$(2.18)\qquad m_P(B) := P\{\omega \in \Omega | M(\omega)(0) \in B\}, \qquad B \in \mathscr{B}(E).$$

Moreover, (2.17) establishes a one-to-one correspondence between probability measures on $(E, \mathscr{B}(E))$ and elements in $\mathscr{M}$.

PROOF. Define the linear space

$$(2.19)\quad \Omega_0 := \Big\{\omega \in \Omega | z_l(\omega) := \lim_{n\to\infty} {}_{E'}\langle l, e^{-n}\omega(-n)\rangle_E \text{ exists for each } l \in D_0\Big\}.$$

Clearly, $\Omega_0 \in \mathscr{F}_{-\infty}$ and for $\omega \in \Omega_0$ the map $l \mapsto z_l(\omega)$ is $\mathbb{Q}$-linear on D_0. Now we define the linear space

$$\Omega_1 := \{\omega \in \Omega_0 | \text{there exists } \alpha(\omega) \in E \text{ such that } z_l(\omega) = {}_{E'}\langle l, \alpha(\omega)\rangle_E \text{ for all } l \in D_0\}. \tag{2.20}$$

Note that $\alpha(\omega)$ defined in (2.20) is unique. It is shown in the Appendix that $\Omega_1 \in \mathscr{F}_{-\infty}$. Define $M(\omega)$: $\mathbb{R} \to E$, $\omega \in \Omega$, by

$$M(\omega)(t) := \begin{cases} e^{-t}\alpha(\omega), & \text{if } \omega \in \Omega_1, \\ 0, & \text{if } \omega \in \Omega \setminus \Omega_1. \end{cases} \tag{2.21}$$

Then $M(\omega) \in \Omega$ [$= C(\mathbb{R}, E)$] and the map M: $\omega \mapsto M(\omega)$, $\omega \in \Omega$, is $\mathscr{F}_{-\infty}/\mathscr{F}$-measurable. Furthermore, we see that

$$M|_{\Omega_1} \text{ is linear, } K_z \in \Omega_1 \text{ and } M(K_z) = K_z \text{ for all } z \in E. \tag{2.22}$$

[Hence, $M(\Omega) \subset \Omega_1$ and $M^2 = M$ by (2.21).]

Now we define a probability kernel Π: $\Omega \times \mathscr{F} \to [0,1]$ by

$$\Pi(\omega, F) := P_\mu(F - M(\omega)), \qquad \omega \in \Omega,\ F \in \mathscr{F}.$$

Clearly, $\omega \mapsto \Pi(\omega, F)$, $\omega \in \Omega$, is $\mathscr{F}_{-\infty}$-measurable for every $F \in \mathscr{F}$ by (the first half of) Fubini's theorem. Furthermore, we know by (2.21) and Theorem 2.7 that $\Pi(\omega, \cdot\,) \in \mathscr{M}_e$ for all $\omega \in \Omega$.

Let $P \in \mathscr{M}$. Let Π^P: $\Omega \times \mathscr{F} \to [0,1]$ be a regular conditional probability of P given $\mathscr{F}_{-\infty}$ such that $\Pi^P(\omega, \cdot\,) \in \mathscr{M}$ for each $\omega \in \Omega$ [cf. Remark 2.2 (iii)]. Let L_0 be a countable dense subset of $L^1_{\mathbb{C}}(\mathbb{R}; ds)$. By the reverse martingale convergence theorem, there exists a set $\Omega_2 \in \mathscr{F}$ with $P(\Omega_2) = 1$ such that for every $\omega \in \Omega_2$,

$$\begin{aligned} &\int f(s) \int \exp(is_{E'}\langle l, X_t(\omega')\rangle_E)\Pi^P(\omega, d\omega')\, ds \\ &\quad = \lim_{n\to\infty} \int f(s) \int \exp\Big(ise^{-(t+n)}{}_{E'}\langle l, X_{-n}(\omega)\rangle_E \\ &\qquad\qquad - \tfrac{1}{2}s^2(1 - e^{-2(t+n)})\|l\|_H^2\Big)\, ds \end{aligned} \tag{2.23}$$

for all $f \in L_0$, $l \in D_0$ and $t \in \mathbb{Q}$ [cf. (2.15)]. Fix $\omega \in \Omega_2$. By Lemma 2.8 and [22], Lemma 1.5, it follows that $\Pi^P(\omega, \cdot\,) \circ X_t^{-1}$ is Gaussian on E with the same covariance as μ for every $t \in \mathbb{R}$. Lemma 2.6 implies that its mean $S(\omega)(t)$ belongs to E. Thus

$$\Pi^P(\omega, \cdot) \circ X_t^{-1} = \mu \circ T^{-1}_{S(\omega)(t)}, \qquad t \in \mathbb{R}.$$

Since $\Pi^P(\omega, \cdot\,) \in \mathscr{M}$, Proposition 2.3(ii) implies that $S(\omega) \in \Omega$, $S(\omega)(t) = e^{-t}S(\omega)(0)$, $t \in \mathbb{R}$, and that $\Pi^P(\omega, \cdot\,) = P_\mu \circ T^{-1}_{S(\omega)}$. Since by Lemma 2.8, ${}_{E'}\langle l, S(\omega)(0)\rangle_E = \lim_{n\to\infty} {}_{E'}\langle l, e^{-n}X_{-n}(\omega)\rangle_E$ for all $l \in D_0$, it follows that $\omega \in \Omega_1$ and that $S(\omega) = M(\omega)$. Hence we have shown that

$$\Pi^P(\omega, \cdot) = P_\mu \circ T^{-1}_{M(\omega)} \quad \text{for } P - \text{a.e. } \omega \in \Omega;$$

in particular, for each $F \in \mathscr{F}$,

$$\begin{aligned} \int F\,dP &= \int E_P[F|\mathscr{F}_{-\infty}]\,dP \\ &= \int P_\mu(F - M(\omega))\,P(d\omega) \\ &= \int P_\mu(F - K_{M(\omega)(0)})P(d\omega) \\ &= \int P_\mu(F - K_z)\,m_P(dz), \end{aligned} \tag{2.24}$$

where m_p is defined by (2.18), and the first half of the assertion is proved. To prove uniqueness, we note that (2.13) clearly implies that $P_\mu(\Omega_1) = 1$ and that

$$M(\omega) \equiv 0 \quad \text{for } P_\mu - \text{a.e. } \omega \in \Omega. \tag{2.25}$$

Hence if $\tilde{m}$ is another probability measure on $(E, \mathscr{B}(E))$ such that (2.17) holds, it follows by (2.22) that for each $B \in \mathscr{B}(E)$, if $F := \{\omega \in \Omega | M(\omega)(0) \in B\}$,

$$\begin{aligned} m_P(B) = P(F) &= \int\int 1_F(\omega + K_z)P_\mu(d\omega)\tilde{m}(dz) \\ &= \int\int 1_B(M(\omega)(0) + z)P_\mu(d\omega)\tilde{m}(dz) \\ &= \tilde{m}(B). \end{aligned}$$

Furthermore, let m be any probability measure on $(E, \mathscr{B}(E))$ and define

$$P(F) := \int P_\mu(F - K_z)\,m(dz), \qquad F \in \mathscr{F}.$$

To prove that $P \in \mathscr{M}$, let $t > s$ and let f be a bounded, $\mathscr{F}_s$-measurable function on Ω. Then for each bounded $\mathscr{B}(E)$-measurable function u on E,

$$\begin{aligned} \int fu(X_t)\,dP &= \int\int f(\omega + K_z)u(X_t(\omega) + e^{-t}z)P_\mu(d\omega)m(dz) \\ &= \int\int f(\omega + K_z)\pi_{t-s}(X_s(\omega) + e^{-s}z, u)P_\mu(d\omega)m(dz) \\ &= \int f\pi_{t-s}(X_s, u)\,dP, \end{aligned}$$

where in the second step we used that $f(\cdot + K_z)$ is also $\mathscr{F}_s$-measurable and (2.9). Hence $P \in \mathscr{M}$ and the proof is complete. □

As an immediate consequence of Theorem 2.10 and Proposition 2.1, we obtain a corresponding statement for the convex set of entrance laws.

COROLLARY 2.11. *Let $(\mu_t)_{t\in\mathbb{R}}$ be an entrance law for $(\pi_t)_{t\geq 0}$. Then there exists a unique probability measure m on $(E, \mathscr{B}(E))$ such that*

$$(2.26)\qquad \mu_t = \int \mu\circ T_{e^{-t}z}^{-1} m(dz), \qquad t\in\mathbb{R}.$$

In particular, $(\mu_t)_{t\in\mathbb{R}}$ is extreme if and only if $\mu_t = \mu\circ T_{e^{-t}z}^{-1}$, $t\in\mathbb{R}$, for some $z\in E$.

REMARK 2.12. (i) Theorems 2.7 and 2.10 say that E itself is the entrance space for $(\pi_t)_{t\geq 0}$ in the sense of [9], [10] and [12].

(ii) Obviously, each probability measure m on $(E, \mathscr{B}(E))$ defines an entrance law for $(\pi_t)_{t\geq 0}$ by (2.26). Corollary 2.11 implies, in particular, that μ is the unique invariant measure for $(\pi_t)_{t\geq 0}$.

(iii) For simplicity, here we have only presented the case where $(\pi_t)_{t\geq 0}$ is given by (2.1). The same methods work for transition semigroups of more general Ornstein–Uhlenbeck processes on E (e.g., those where the drifts are given by bounded symmetric linear operators A rather than the identity). We refer in particular to [28] and [29] where the case of $A = \sqrt{-\Delta + 1}$ was solved. This case is much more difficult since A is not bounded and does not have discrete spectrum.

3. Representation of positive parabolic functions. In this section we will apply Theorem 2.10 to obtain a representation of a class of positive parabolic functions on E in terms of extreme ones using Dynkin's method. However, as explained in Section 1, those final results in [9]–[12] and [19], which concern the representation of parabolic functions, are not applicable here. We need some preparations.

We denote the usual inner product in (real) $L^2(E;\mu)$ by $(\,,\,)_\mu$. We have for every π_t, $t\geq 0$, as in (2.1) that

$$(3.1)\qquad \int u\,\pi_t(\cdot, v)\,d\mu = \int \pi_t(\cdot, u)v\,d\mu$$

for all bounded $\mathscr{B}(E)$-measurable $u, v\colon E\to\mathbb{R}$. Hence for each $t\geq 0$,

$$(3.2)\qquad T_t u := \pi_t(\cdot, u), \qquad u\ \mathscr{B}(E)\text{-measurable, bounded},$$

defines an operator on $L^2(E;\mu)$ which extends uniquely to a symmetric contraction T_t on $L^2(E;\mu)$ (cf. [16], Section 1.4) such that $(T_t)_{t\geq 0}$ is a semigroup of Markovian operators on $L^2(E;\mu)$ (i.e., $0\leq T_t u\leq 1$, μ-a.e. if $0\leq u\leq 1$, μ-a.e.). $(T_t)_{t\geq 0}$ is *analytic* on $L^p(E;\mu)$ for all $1<p<\infty$ (cf., e.g., [25], Theorem X.55) and hypercontractive (cf. [25], Theorem X.61). Let L be the associated generator. L is the well-known *Ornstein–Uhlenbeck operator* on Wiener space (cf. [20] and [37]). Set

$$(3.3)\qquad \begin{aligned}\mathscr{F}C^\infty := \{u\colon E\to\mathbb{R} \mid u = f(l_1,\dots,l_m) &\text{ for some } m\in\mathbb{N},\\ f\in C^\infty(\mathbb{R}^m) &\text{ and } l_1,\dots,l_m\in E'\}\end{aligned}$$

and define $\mathscr{F}C_b^\infty$ (resp. Pol) as those subsets of $\mathscr{F}C^\infty$ for which f [in (3.3)] and all its partial derivatives are bounded (resp. f is a polynomial in m variables).

For $u \in \mathscr{F}C^\infty$ and $k \in E$ we define

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z+sk)\bigg|_{s=0}, \qquad z \in E. \tag{3.4}$$

Then for $z \in E$ we define $\nabla u(z)$ to be the unique element in H representing the continuous linear map $k \mapsto (\partial u/\partial k)(z)$, $k \in H$. It is well known that $\mathscr{F}C_b^\infty \cup \text{Pol} \subset D(L)$ and that if $u = f(l_1, \dots, l_m) \in \mathscr{F}C_b^\infty \cup \text{Pol}$ and $k_1, \dots, k_N \in E'$ form an orthonormal system in H having $l_1, \dots, l_m$ in their linear span, then

$$Lu = \sum_{n=1}^{N} \frac{\partial}{\partial k_n}\left[\frac{\partial u}{\partial k_n}\right] - \sum_{n=1}^{N} {}_{E'}\langle k_n, \cdot\,\rangle_E \frac{\partial u}{\partial k_n}. \tag{3.5}$$

Furthermore, both $\mathscr{F}C_b^\infty$ and Pol are (operator) cores for L; hence L is uniquely determined by (3.5) (cf., e.g., [3], Section 7, Part I, resp. [37], for details). The quadratic form $(\mathscr{E}, D(\mathscr{E}))$ associated to L on $L^2(E;\mu)$ [i.e., $\mathscr{E}(u,v) := (\sqrt{-L}\,u, \sqrt{-L}\,v)_\mu$ with domain $D(\mathscr{E}) := D(\sqrt{-L}\,)$] is the closure of

$$\mathscr{E}(u,v) = \int \langle \nabla u, \nabla v\rangle_H \, d\mu, \qquad u,v \in \mathscr{F}C_b^\infty$$

(cf. [3], Section 7, Part I); it is therefore a classical Dirichlet form in the sense of [1–5]. Below we also denote the closure of the operator ∇: $\mathscr{F}C_b^\infty \to L^2(E \to H;\mu)$ by ∇ and hence

$$\mathscr{E}(u,v) = \int \langle \nabla u, \nabla v\rangle_H \, d\mu \quad \text{for all } u,v \in D(\mathscr{E}). \tag{3.6}$$

Definition 3.1. Let $p := (p(t,\cdot\,))_{t\in\mathbb{R}}$ be a family of functions on E. p is called *L-parabolic* (on E) if $p(t,\cdot\,) \in D(L)$ for each $t \in \mathbb{R}$, $t \mapsto p(t,\cdot\,)$ is differentiable as a map from $\mathbb{R}$ to $L^2(E;\mu)$ and

$$Lp(t,\cdot) = \frac{d}{dt}p(t,\cdot), \qquad t \in \mathbb{R}. \tag{3.7}$$

Remark 3.2. In [9] a positive function p on $\mathbb{R} \times E$ is called *P_μ-harmonic* if $(p(t,X_t), \mathscr{F}_t, P_\mu)_{t\in\mathbb{R}}$ is a martingale. There is clearly a one-to-one correspondence between P_μ-harmonic functions such that $\int p(t,z)\mu(dz) = 1$ for all $t \in \mathbb{R}$, and Radon–Nikodym derivatives $d\mu_t/d\mu =: p(t,\cdot\,)$, $t \in \mathbb{R}$, of entrance laws $(\mu_t)_{t\in\mathbb{R}}$ for $(\pi_t)_{t\ge 0}$. By the following proposition it follows that (up to a constant) our L-parabolic functions are exactly those P_μ-harmonic functions of Dynkin which are in $L^2(E;\mu)$ for each fixed $t \in \mathbb{R}$.

PROPOSITION 3.3. *Let $p := (p(t,\cdot))_{t\in\mathbb{R}}$ be a family of positive functions on E. Then the following are equivalent*:

(i) *p is L-parabolic.*
(ii) *There exists a constant $c_p > 0$ such that*

$$\mu_t := c_p^{-1} p(t,\cdot)\mu, \qquad t\in\mathbb{R}, \tag{3.8}$$

is an entrance law for $(\pi_t)_{t\geq 0}$, and there exists arbitrarily small $t\in\mathbb{R}$ such that $p(t,\cdot)\in L^q(E;\mu)$ for one (resp. all) $1<q<\infty$.

In this case $\int p(t,z)\mu(dz)(=c_p)$ is independent of $t\in\mathbb{R}$.

PROOF. Assume (i). By the product rule we have for all $s\in\mathbb{R}$, $t\geq 0$,

$$\frac{d}{dt}(T_t p(s-t,\cdot)) = T_t(Lp(s-t,\cdot)) + T_t\left(\frac{d}{dt}(p(s-t,\cdot))\right) = 0.$$

Since T_0 is the identity on $L^2(E;\mu)$, it follows that $T_t p(s-t,\cdot) = p(s,\cdot)$ for all $s\in\mathbb{R}$, $t\geq 0$. Hence by hypercontractivity $p(s,\cdot)\in L^q(E;\mu)$ for all $1<q<\infty$. By (3.2) and symmetry, (ii) now follows with $c_p := \int p(t,z)\mu(dz)$, which is independent of t since $T_t 1 = 1$, $t\geq 0$.

Assume (ii) for some $q>1$. Then by (3.2), $T_t p(s,\cdot) = p(s+t,\cdot)$ for all $s\in\mathbb{R}$, $t\geq 0$; thus by hypercontractivity $p(t,\cdot)\in L^2(E;\mu)$ for all $t\in\mathbb{R}$. Since $(T_t)_{t\geq 0}$ is analytic, hence $T_t u\in D(L)$ for all $u\in L^2(E;\mu)$, $t>0$, it follows that $p(t,\cdot)\in D(L)$, that $t\mapsto p(t,\cdot)$ is differentiable and that $(d/dt)p(t,\cdot) = Lp(t,\cdot)$, $t\in\mathbb{R}$. □

DEFINITION 3.4. A positive L-parabolic function $p = (p(t,\cdot))_{t\in\mathbb{R}}$ is called *normalized* if $\int p(t,z)\mu(dz) = 1$ for all $t\in\mathbb{R}$ (cf. Proposition 3.3). Let $\mathscr{P}$ denote the convex set of all normalized positive L-parabolic functions on E.

Examples of normalized positive L-parabolic functions are given by the *Cameron–Martin densities* defined for $t\in\mathbb{R}$, $h\in H$ by

$$p^h(t,z) := \frac{d(\mu\circ T_{e^{-t}h}^{-1})}{d\mu}(z) = \exp\left(e^{-t}X_h(z) - \tfrac{1}{2}e^{-2t}\|h\|_H^2\right), \qquad z\in E \tag{3.9}$$

(cf. Lemma A.2). Here $X_h(\cdot) := \lim_{n\to\infty E'}\langle k_n,\cdot\rangle_E$ in $L^2(E;\mu)$ for any sequence $(k_n)_{n\in\mathbb{N}}$ in E' converging to h in H.

LEMMA 3.5. *$p^h := (p^h(t,\cdot))_{t\in\mathbb{R}}\in\mathscr{P}$ for each $h\in H$.*

PROOF. Note that $p^h(t,\cdot)\in L^2(E;\mu)$ for all $t\in\mathbb{R}$. Using (2.1), we have that for each $t\in\mathbb{R}$, $T_t(\exp X_h) = \exp(e^{-t}X_h)\exp[\frac{1}{2}(1-e^{-2t})\|h\|_H^2]$. Now $t\mapsto \exp(e^{-t}X_h)\in L^2(E;\mu)$ is differentiable and

$$d/dt\,\exp(e^{-t}X_h) = -e^{-t}X_h\exp(e^{-t}X_h).$$

Consequently, $p^h(t,\cdot) \in D(L)$ and

$$Lp^h(t,\cdot) = \left(e^{-2t}\|h\|_H^2 - e^{-t}X_h\right)p^h(t,\cdot) = \frac{d}{dt}p^h(t,\cdot), \qquad t \in \mathbb{R}. \qquad \square$$

REMARK 3.6. (i) Note that by [11], Lemma 4.1, for $t \in R$ we can choose μ-versions of $p^h(t,\cdot)$, hence of X_h, so that $(h,z) \mapsto p^h(t,z)$ and hence $(h,z) \mapsto X_h(z)$ is $\mathscr{B}(H) \otimes \mathscr{B}(E)$-measurable. Here $\mathscr{B}(H)$ denotes the Borel σ-algebra on H. We will always use these particular versions below without further notice.

(ii) Let $h \in H$. Since the entrance law $\mu_t^h := p^h(t,\cdot)\mu$, $t \in \mathbb{R}$, given by p^h according to Proposition 3.3, is extreme by Corollary 2.11, p^h is extreme in $\mathscr{P}$.

(iii) It is well known that $\mu \circ T_z^{-1}$, $z \in E$, is absolutely continuous w.r.t. μ if and only if $z \in H$ (cf. Lemma A.2 below). Therefore, the entrance laws $\mu \circ T_{e^{-t}z}^{-1}$, $z \in E \setminus H$, do not have densities w.r.t. μ and hence do not give rise to extreme L-parabolic functions.

We now prove the following representation theorem for a subclass of $\mathscr{P}$.

THEOREM 3.7. *Let $p = (p(t,\cdot))_{t\in\mathbb{R}} \in \mathscr{P}$ and let m_p be the unique probability measure on $(E, \mathscr{B}(E))$ representing the corresponding entrance law $\mu_t := p(t,\cdot)\mu$, $t \in \mathbb{R}$. Suppose that $m_p(H) = 1$, then*

$$(3.10) \qquad p(t,z) = \int_H p^h(t,z)\, m_p(dh), \qquad t \in \mathbb{R}, \textit{ for } \mu\text{-a.e. } z \in E.$$

$(p^h(t,\cdot))_{t\in\mathbb{R}}$, $h \in H$, are exactly the extreme functions of the convex set $\mathscr{P}_1$ consisting of all $p \in \mathscr{P}$ with $m_p(H) = 1$.

PROOF. By Corollary 2.11 and (3.9),

$$p(t,z)\mu(dz) = \int p^h(t,z)\mu(dz)\, m_p(dh);$$

hence by Remark 3.6 (i) and Fubini's theorem, (3.10) follows. The rest of the assertion is obvious. $\square$

REMARK 3.8. (i) Note that $\mathscr{P}_1$ is a so-called *face* of $\mathscr{P}$.

(ii) If $\dim E < \infty$, that is, $E = \mathbb{R}^d$, then $E = H$; hence $m_p(H) = m_p(E) = 1$ and (3.10) holds for each $p \in \mathscr{P}$. Thus we have reproved the finite-dimensional case (cf. [30] and [8]). Here, in particular, by (2.26) and Remark 3.6 (iii), each entrance law for $(\pi_t)_{t\geq 0}$ is absolutely continuous w.r.t. μ. Hence by Proposition 3.3 the corresponding family of densities is in $\mathscr{P}$ if they are in $L^q(E;\mu)$ for some $q > 1$.

One can also use (3.10) to construct L-parabolic functions since we have the following proposition.

PROPOSITION 3.9. *Let m be a probability measure on $(E, \mathscr{B}(E))$ such that $m(H) = 1$ and let $p := (p(t, \cdot))_{t \in \mathbb{R}}$ be defined by*

$$p(t,z) := \int p^h(t,z)m(dh), \qquad t \in \mathbb{R}, z \in E. \tag{3.11}$$

Then the following assertions are equivalent:

(i) *p is L-parabolic.*
(ii) $\int_H \int_H \exp(e^{-2t}\langle h, h'\rangle_H) m(dh)m(dh') < \infty$ *for all $t \in \mathbb{R}$.*

PROOF. Since each $(p^h(t, \cdot)\mu)_{t \in \mathbb{R}}$, $h \in H$, is an entrance law, it follows by Remark 3.6(i) and Fubini's theorem that $(p(t, \cdot)\mu)_{t\in\mathbb{R}}$ is an entrance law for $(\pi_t)_{t \ge 0}$. Hence by Proposition 3.3, (i) is equivalent with $p(t, \cdot) \in L^2(E; \mu)$ for all $t \in \mathbb{R}$. But

$$\int p(t,z)^2 \mu(dz) = \int\int\int p^h(t,z)p^{h'}(t,z)\mu(dz)m(dh)m(dh')$$

$$= \int\int \exp(e^{-2t}\langle h, h'\rangle_H)m(dh)m(dh'),$$

where again we used Fubini's theorem and the formula for the Laplace transform of a Gaussian measure on $\mathbb{R}$. Hence the equivalence of (i) and (ii) follows. □

Unfortunately, the assumption $m_p(H) = 1$ in Theorem 3.7, which is crucial for the representation (3.10), is not always fulfilled for any $p \in \mathscr{P}$. We shall construct a function $p \in \mathscr{P}$ such that even $m_p(H) = 0$. In the case of the classical Wiener space an example of an entrance law $\mu_t := p(t, \cdot)\mu$, $t \in \mathbb{R}$, such that $m_p(H) = 0$ has also been constructed by Föllmer using a slightly different technique based on a theorem by Kakutani (cf. [15]). We believe that also in Föllmer's example $(p(t, \cdot))_{t\in\mathbb{R}}$ is in fact L-parabolic.

EXAMPLE 3.10. Suppose that E (or equivalently H) is infinite dimensional. Let $\lambda_n \in]0, \infty[$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty}\lambda_n = +\infty$ and $\sum_{n=1}^{\infty}\lambda_n^2 < \infty$. Let $\{k_n | n \in \mathbb{N}\} \subset E'$ be an orthonormal basis of H and define an operator A on H by

$$Ah := \sum_{n=1}^{\infty} \lambda_n \langle k_n, h\rangle_H k_n, \qquad h \in H.$$

A is a compact, self-adjoint and nonnegative definite. By the converse of Gross' fundamental theorem (cf. [17] and [7]), E is the completion of H w.r.t. some μ-measurable norm on H. Since A is a bounded operator on H, it follows by [18], Lemma 4.3, and [38], Theorems 3.1 and 4.1, that there exists a unique probability measure m on $(E, \mathscr{B}(E))$ such that

$$\int \exp(i_{E'}\langle k, z\rangle_E)m(dz) = \exp(-\tfrac{1}{2}\langle k, Ak\rangle_H), \qquad k \in E'.$$

Since m is Gaussian and A is not nuclear, it follows by the Minlos–Sazonov theorem that $m(H) = 0$. Define for $t \in \mathbb{R}$ the measure μ_t on $(E, \mathscr{B}(E))$ by

$$\mu_t := \int \mu \circ T^{-1}_{e^{-t}z} m(dz), \qquad t \in \mathbb{R}.$$

Then by Remark 2.12(ii), $(\mu_t)_{t \in \mathbb{R}}$ is an entrance law for $(\pi_t)_{t \geq 0}$ and

$$\int \exp(i_{E'}\langle k, z \rangle_E)\mu_t(dz)$$
$$= \exp\left[-\tfrac{1}{2}\left(\|k\|_H^2 + e^{-2t}\langle k, Ak\rangle_H\right)\right], \qquad k \in E', t \in \mathbb{R}.$$

Hence μ_t is the unique mean zero Gaussian measure on $(E, \mathscr{B}(E))$ with covariance operator $B_t := Id_H + e^{-2t}A$, $t \in \mathbb{R}$. But $B_t - Id_H$ is Hilbert–Schmidt; hence by [32] μ_t is absolutely continuous w.r.t. μ (cf. also [33], Theorem I.23). Indeed, let us fix $t \in \mathbb{R}$ and define

$$C_{t,n} := \left(1 + e^{-2t}\lambda_n\right)^{-1}, \qquad n \in \mathbb{N},$$

$$f_n := \sqrt{C_{t,n}} \exp\left(-\tfrac{1}{2}(C_{t,n} - 1)_{E'}\langle k_n, z\rangle_E^2\right), \qquad n \in \mathbb{N},$$

and

$$F_N := \prod_{n=1}^{N} f_n, \qquad N \in \mathbb{N}.$$

Then as a consequence of a lemma by Segal (cf. [31] and also [33], Lemma I.24)

$$p(t, \cdot) := \lim_{N \to \infty} F_N$$

exists as a limit in $L^1(E; \mu)$ and in fact $p(t, \cdot\,) \in L^2(E; \mu)$. It is easy to check that for all $k \in E'$,

$$\int \exp(i_{E'}\langle k, z\rangle_E)p(t, z)\mu(dz) = \exp\left[-\tfrac{1}{2}\left(\|k\|_H^2 + e^{-2t}\langle k, Ak\rangle_H\right)\right].$$

Hence $\mu_t = p(t, \cdot\,)\mu$. It follows by Proposition 3.3 that $p := (p(t, \cdot\,))_{t \in \mathbb{R}} \in \mathscr{P}$. But m_p (defined as in Theorem 3.7) is the Gaussian measure m constructed above; hence $m_p(H) = m(H) = 0$.

A natural problem that arises now is to find necessary and sufficient conditions for a $p \in \mathscr{P}$ to have the property $m_p(H) = 1$. Theorems 3.11 and 3.14 below give rather mild sufficient conditions for this to hold. Let $\|\ \|_q$ denote the usual norm on $L^q(E; \mu)$, $q > 1$.

THEOREM 3.11. *Let $p = (p(t, \cdot\,))_{t \in \mathbb{R}} \in \mathscr{P}$ and let m_p be as in Theorem 3.7. Let $\{k_n | n \in \mathbb{N}\} \subset E'$ be an orthonormal basis of H separating the points of E. Assume that for some $t \in \mathbb{R}$, $q > 1$,*

$$\sum_{n=1}^{\infty} \|\langle \nabla p(t, \cdot), k_n\rangle_H\|_q < \infty. \tag{3.12}$$

Then $m_p(H) = 1$. *In particular, p has the unique representation*

$$p(t,z) = \int_H \exp\big(e^{-t}X_h(z) - \tfrac{1}{2}e^{-2t}\|h\|_H^2\big)m_p(dh) \quad \text{for } \mu\text{-a.e. } z \in E. \tag{3.13}$$

For the proof of Theorem 3.11 we need the following lemma.

LEMMA 3.12. *Let* p, m_p *be as in Theorem 3.11. Let* $t \in \mathbb{R}$ *and let* $k_1, \dots, k_N \in E'$ *form an orthonormal system of H. Let* $u(z) := f({}_{E'}\langle k_1, z\rangle_E, \dots, {}_{E'}\langle k_N, z\rangle_E)$, $z \in E$, *for some* $f \in \mathbb{C}_b^\infty(\mathbb{R}^m)$ *or f a polynomial in m variables. Then*

$$\begin{aligned}&\int_E Lu(z)p(t,z)\mu(dz)\\ &\quad = -\sum_{n=1}^N \int_E {}_{E'}\langle k_n, e^{-t}z'\rangle_E \int_E \frac{\partial u}{\partial k_n}(z + e^{-t}z')\mu(dz)m_p(dz').\end{aligned} \tag{3.14}$$

PROOF. It follows by the first part of Theorem 3.7, (2.26) and Fubini's theorem that for all $v: E \to \mathbb{R}$, $\mathscr{B}(E)$-measurable, bounded,

$$\int_E v(z)p(t,z)\mu(dz) = \int_E\int_E v(z + e^{-t}z')\mu(dz)m_p(dz').$$

Consequently, since $u \in D(L)$ and by (3.5) we obtain

$$\begin{aligned}&\int_E Lu(z)p(t,z)\mu(dz)\\ &\quad = \int_E\int_E Lu(z + e^{-t}z')\mu(dz)m_p(dz')\\ &\quad = \int_E\int_E L\big(u(\cdot + e^{-t}z')\big)(z)\mu(dz)m_p(dz')\\ &\qquad - \sum_{n=1}^N \int_E\int_E {}_{E'}\langle k_n, e^{-t}z'\rangle_E \frac{\partial u}{\partial k_n}(z + e^{-t}z')\mu(dz)m_p(dz').\end{aligned}$$

But for all $v \in D(L)$ we have that $\int Lv\,d\mu = \int L1v\,d\mu = 0$, since $L1 = 0$. Hence the assertion follows. □

PROOF OF THEOREM 3.11. For $N \in \mathbb{N}$ let $u_N(z) := \sum_{n=1}^N {}_{E'}\langle k_n, z\rangle_E^2$, then by Lemma 3.12,

$$\begin{aligned}&\int_E \langle \nabla p(t,\cdot), \nabla u_N\rangle_H\,d\mu\\ &\quad = -\int_E Lu_N p(t,\cdot)\,d\mu\\ &\quad = 2\sum_{n=1}^N \int_E {}_{E'}\langle k_n, e^{-t}z'\rangle_E \int_E {}_{E'}\langle k_n, z + e^{-t}z'\rangle_E \mu(dz)m_p(dz').\end{aligned}$$

Hence

$$\sum_{n=1}^{N} \int_E \langle \nabla p(t,z), k_n \rangle_H \, {}_{E'}\langle k_n, z\rangle_E \mu(dz) = e^{-2t} \sum_{n=1}^{N} \int_E {}_{E'}\langle k_n, z'\rangle_E^2 m_p(dz').$$

Since ${}_{E'}\langle k_n, \cdot\, \rangle_E$ is $N(0,1)$-distributed, we have for $1 \le q' < \infty$ that $\|{}_{E'}\langle k_n, \cdot\, \rangle_E\|_{q'} \le c$ for all $n \in \mathbb{N}$ for some constant c depending only on q'. Hence by Hölder's inequality,

$$\sum_{n=1}^{N} \int_E {}_{E'}\langle k_n, z'\rangle_E^2 m_p(dz') \le c e^{2t} \sum_{n=1}^{N} \|\langle \nabla p(t, \cdot), k_n \rangle_H\|_q.$$

Consequently, if $t \in \mathbb{R}$, $q > 1$, is as in (3.12), then

$$\sum_{n=1}^{\infty} \int_E {}_{E'}\langle k_n, z'\rangle_E^2 m_p(dz') < \infty.$$

Hence $\sum_{n=1}^{\infty} {}_{E'}\langle k_n, z\rangle_E^2 < \infty$ for m_p-a.e. $z \in E$. But if this sum is finite for a $z \in E$, then $h := \sum_{n=1}^{\infty} {}_{E'}\langle k_n, z\rangle_E k_n \in H(\subset E)$ and ${}_{E'}\langle k_n, h\rangle_E = \langle k_n, h\rangle_H = {}_{E'}\langle k_n, z\rangle_E$ for all $n \in \mathbb{N}$. Since $\{k_n | n \in \mathbb{N}\}$ separates the points of E, it follows that $h = z$. Therefore, we can now conclude that $m_p(H) = 1$. The rest of the assertion follows from Theorem 3.7. □

REMARK 3.13. (i) By [3], Lemma 5.6, there always exist $k_n \in E'$, $n \in \mathbb{N}$, separating the points of E which form an orthonormal basis of H. So, in order to satisfy the condition in Theorem 3.11, one needs to find such a basis so that in addition (3.12) holds for some $t \in \mathbb{R}$.

(ii) Note that each $p \in \mathscr{P}$ always satisfies

$$\sum_{n=1}^{\infty} \|\langle \nabla p(t, \cdot), k_n \rangle_H\|_2^2 < \infty$$

for all $t \in \mathbb{R}$ and any orthonormal basis $\{k_n | n \in \mathbb{N}\} \subset E'$ of H, since each $p(t, \cdot\,) \in D(\mathscr{E})$ [cf. (3.6)].

(iii) Let $p \in \mathscr{P}$ be such that for some $t \in \mathbb{R}$, $p(t, \cdot\,) \in \mathscr{F}C_b^\infty \cup \mathrm{Pol}$. Obviously, one can then choose an orthonormal basis $\{k_n | n \in \mathbb{N}\} \in E'$ of H in such a way that the sum in (3.12) is in fact finite. Hence the representation (3.13) for p holds in this case.

THEOREM 3.14. *Let $p = (p(t, \cdot\,))_{t \in \mathbb{R}} \in \mathscr{P}$ and let m_p be as in Theorem* 3.7. *If $p(t, \cdot\,) \in L^\infty(E; \mu)$ for some $t \in \mathbb{R}$, then $m_p(H) = 1$ and* (3.13) *holds.*

PROOF. Let $\lambda \in]-1, \infty[$ and let $\{k_n | n \in \mathbb{N}\} \subset E'$ be an orthonormal basis of H separating the points of E. By the first part of Theorem 3.7, (2.26) and

Fubini's theorem we have that for each $N \in \mathbb{N}$,

$$\begin{aligned}\int_E \exp\left[-\frac{\lambda}{2}\sum_{n=1}^{N} {}_{E'}\langle k_n, z\rangle_E^2\right] p(t,z)\mu(dz) &= \int_E\int_E \exp\left[-\frac{\lambda}{2}\sum_{n=1}^{N} {}_{E'}\langle k_n, z + e^{-t}z'\rangle_E^2\right]\mu(dz)m_p(dz') \\ &= \int_E \exp\left[-\frac{\lambda}{2}e^{-2t}\sum_{n=1}^{N} {}_{E'}\langle k_n, z'\rangle_E^2\right] g(z')m_p(dz'),\end{aligned}$$

where

$$\begin{aligned}g(z') &= \prod_{n=1}^{N}\int_E \exp\left[-\frac{\lambda}{2}\left({}_{E'}\langle k_n, z\rangle_E^2 + 2e^{-t}{}_{E'}\langle k_n, z'\rangle_E\,{}_{E'}\langle k_n, z\rangle_E\right)\right]\mu(dz) \\ &= (\lambda+1)^{-N/2}\exp\left[\frac{\lambda^2}{2(\lambda+1)}e^{-2t}\sum_{n=1}^{N} {}_{E'}\langle k_n, z'\rangle_E^2\right], \qquad z' \in E.\end{aligned}$$

Hence for all $N \in \mathbb{N}$,

$$\begin{aligned}&\int_E \exp\left[-\frac{\lambda e^{-2t}}{2(\lambda+1)}\sum_{n=1}^{N} {}_{E'}\langle k_n, z'\rangle_E^2\right] m_p(dz') \\ (3.15)\qquad &\le \operatorname{ess\,sup} p(t,\cdot)\,(\lambda+1)^{N/2}\int_E \exp\left[-\frac{\lambda}{2}\sum_{n=1}^{N} {}_{E'}\langle k_n, z\rangle_E^2\right]\mu(dz) \\ &= \operatorname{ess\,sup} p(t,\cdot).\end{aligned}$$

If $\lambda \in]-1, 0[$, then $-\lambda/(\lambda+1) > 0$; hence letting $N \to \infty$ in (3.15) we obtain

$$\sum_{n=1}^{\infty}\int_E {}_{E'}\langle k_n, z'\rangle_E^2 m_p(dz') < \infty.$$

Consequently, $m_p(H) = 1$ (cf. the proof of Theorem 3.11). The rest of the assertion follows by Theorem 3.7. □

REMARK 3.15. (i) The proof of Theorem 3.14 shows that we can weaken the assumption on p as follows. Instead of $p(t,\cdot) \in L^\infty(E;\mu)$ for some $t \in \mathbb{R}$, it is enough to assume that there exists an orthonormal basis $\{k_n | n \in \mathbb{R}\}$ of H contained in E' and separating the points of E, $t \in \mathbb{R}$ and $\lambda \in]-1, 0[$ such that

$$(3.16)\qquad \limsup_{N\to\infty}(\lambda+1)^{N/2}\int_E \exp\left[-\frac{\lambda}{2}\sum_{n=1}^{N} {}_{E'}\langle k_n, z\rangle_E^2\right] p(t,z)\mu(dz) < \infty.$$

(ii) Observe that in both Theorems 3.11 and 3.14 we actually proved not only that $m_p(H) = 1$, but moreover that $\int_H \|h\|_H^2 m_p(dh) < \infty$, which is only implied by $m_p(H) = 1$ if m_p is Gaussian (cf. [34], Theorem (3.41)).

APPENDIX

PROOF OF LEMMA 2.4. Let $P \in \mathscr{M}$ and $F \in \mathscr{F}_{-\infty}$ with $0 < P(F) < 1$. Then $P = P(F)P_1 + P(\Omega \setminus F)P_2$, where $P_1 := 1_F P(F)^{-1}P$ and $P_2 := 1_{\Omega \setminus F} P(\Omega \setminus F)^{-1}P$. Clearly, $P_1, P_2 \in \mathscr{M}$, so if P is not trivial on $\mathscr{F}_{-\infty}$, then $P \notin \mathscr{M}_e$. Conversely, if P is trivial on $\mathscr{F}_{-\infty}$, then by the reverse martingale convergence theorem

$$(A.1) \quad P[X_t \in B] = P[X_t \in B | \mathscr{F}_{-\infty}] = \lim_{s \to \infty} \pi_{t+s}(X_{-s}, B), \quad P\text{-a.s.},$$

for all $B \in \mathscr{B}(E)$. The set Ω_P of all $\omega \in \Omega$ for which (A.1) holds is in $\mathscr{F}_{-\infty}$ and if $P' \in \mathscr{M}$ is trivial on $\mathscr{F}_{-\infty}$, then $\Omega_P \cap \Omega_{P'} \neq \emptyset$ implies $P = P'$. But, if $P = \alpha P_1 + (1 - \alpha)P_2$, $\alpha \in]0, 1[$, then $P_1, P_2 \in \mathscr{M}$ and both are trivial on $\mathscr{F}_{-\infty}$. Since they cannot be supported by disjoint sets in $\mathscr{F}_{-\infty}$, it follows that $P_1 = P_2$. □

PROOF OF LEMMA 2.6. We have to show that the linear functional $l \mapsto \int l(z)u(z)\nu(dz)$, $l \in E'$, is continuous in the weak*-topology. By a consequence of the Krein–Smulian theorem (cf. eg., [24], 2.5.11, Corollary), it suffices to prove that this functional restricted to the unit ball B_1' in $(E', \| \ \|_{E'})$ is weak*-continuous. Since the restriction of the weak*-topology to B_1' is metrizable, it suffices to prove sequential continuity. But if $l, l_n \in B_1'$, $n \in \mathbb{N}$, so that $l_n(z) \to_{n \to \infty} l(z)$ for every $z \in E$, it follows (since $l, l_n \in \mathbb{N}$ are jointly Gaussian) by [22], Lemma 1.5, that $l_n \to_{n \to \infty} l$ in $L^q(E; \nu)$ for all $q \in [1, \infty[$. Hence the assertion is proved. □

LEMMA A.1. *Let Ω_1 be defined as in* (2.20). *Then $\Omega_1 \in \mathscr{F}_{-\infty}$.*

PROOF. The weak*-topology restricted to each ball B_n', $n \in \mathbb{N}$, of radius n in $(E', \| \ \|_{E'})$ is defined by some metric d_n. Define

$$D_{n,m} := \left\{ (k, k') \in D_n \times D_n | d_n(k, k') < \frac{1}{m} \right\}$$

and

$$\Omega_3 := \bigcap_{j, n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{(l,k) \in D_{n,m}} \left\{ \omega \in \Omega_0 | \, |z_l(\omega) - z_k(\omega)| \le \frac{1}{j} \right\}.$$

Then $\Omega_3 \in \mathscr{F}_{-\infty}$ and for each $\omega \in \Omega_3$ the map $l \mapsto z_l(\omega)$, $l \in D_n$, extends to a weak*-continuous map on (B_n', d_n) for every $n \in \mathbb{N}$ in a compatible way to define a linear functional on E'. Using [24], 2.5.11, Corollary, again, this functional is weak*-continuous; hence there exists a unique $\alpha(\omega) \in E$ such that $z_l(\omega) = {}_{E'}\langle l, \alpha(\omega) \rangle_E$ for all $l \in D_0$. Consequently, $\Omega_3 \subset \Omega_1$. But obviously, $\Omega_1 \subset \Omega_3$ and the assertion follows. □

We have used the following well-known fact. Following the advice of the referee, we enclose a proof here.

LEMMA A.2. *Let $h \in E$. Then $\mu \circ T_h^{-1}$ is absolutely continuous w.r.t. μ if and only if $h \in H$. In this case*

$$\frac{d\mu \circ T_h^{-1}}{d\mu}(z) = \exp\left[X_h(z) - \frac{1}{2}\|h\|_H^2\right], \qquad z \in E.$$

PROOF. Assume $h \in H \setminus \{0\}$ and let $l \in E'(\subset H \subset E)$. Define

$$\alpha := \langle l, h\rangle_H / \langle h, h\rangle_H.$$

Then $\langle l - \alpha h, h\rangle_H = 0$. Hence since μ is Gaussian with covariance $\langle\ ,\ \rangle_H$,

$$\begin{aligned}
&\int_E \exp[i_{E'}\langle l, z\rangle_E]\exp\left[X_h(z) - \tfrac{1}{2}\|h\|_H^2\right]\mu(dz)\\
&\quad= \exp\left[-\tfrac{1}{2}\|h\|_H^2\right]\int_E \exp[iX_{l-\alpha h} + (1 + i\alpha)X_h]\,d\mu\\
&\quad= \exp\left[-\tfrac{1}{2}\|h\|_H^2\right]\int_E \exp[iX_{l-\alpha h}]\,d\mu \int_E \exp[(1 + i\alpha)X_h]\,d\mu\\
&\quad= \exp\left[-\tfrac{1}{2}\|l\|_H^2 + i\langle l, h\rangle_H\right]\\
&\quad= \int \exp[i_{E'}\langle l, z\rangle_E](\mu \circ T_h^{-1})(dz).
\end{aligned}$$

Conversely, suppose $\mu \circ T_h^{-1} = \rho_h \mu$ for some nonnegative Borel-measurable ρ_h on E. In order to show that $h \in H$ we have to show that $l \mapsto {}_{E'}\langle l, h\rangle_E$, $l \in E'$, is continuous w.r.t. $\|\ \|_H$. But if $l_n \in E'$, $n \in \mathbb{N}$, such that $l_n \to_{n\to\infty} 0$ w.r.t. $\|\ \|_H$, then ${}_{E'}\langle l_n, \cdot\ \rangle_E \to_{n\to\infty} 0$ in $L^2(E; \mu)$, in particular in μ-measure, hence in $(\rho_h\mu)$-measure. Since $\rho_h\mu$ is Gaussian, it follows that ${}_{E'}\langle l_n, \cdot\ \rangle_E \to_{n\to\infty} 0$ in $L^1(E; \rho_h\mu)$; hence

$$\limsup_{n\to\infty} |{}_{E'}\langle l_n, h\rangle_E| = \limsup_{n\to\infty}\left|\int {}_{E'}\langle l_n, z\rangle_E \rho_h(z)\mu(dz)\right| = 0. \qquad \square$$

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