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THE SURVIVAL OF ONE-DIMENSIONAL CONTACT PROCESSES IN RANDOM ENVIRONMENTS[1]

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Consider the inhomogeneous contact process on Z^1 with recovery rate $\delta(k)$ at site k and infection rates $\lambda(k)$ and $\rho(k)$ at site k due to the presence of infected neighbors at $k-1$ and $k+1$ respectively. A special case of the main result in this paper is the following: Suppose that the environment is chosen in such a way that the $\delta(k)$'s, $\lambda(k)$'s and $\rho(k)$'s are all mutually independent, with the $\delta(k)$'s having a common distribution, and the $\lambda(k)$'s and $\rho(k)$'s having a common distribution. Then the process survives if

$$E\frac{\delta(\lambda+\rho+\delta)}{\lambda\rho} < 1,$$

while the right edge r_t of the process with initial configuration $\cdots 111000 \cdots$ satisfies

$$\limsup_{t\to\infty} r_t = +\infty$$

if

$$E\log\frac{\delta(\lambda+\rho+\delta)}{\lambda\rho} < 0.$$

If the environment is deterministic and periodic with period p, we prove survival if

$$\prod_{k=1}^{p}\frac{\delta(k)[\lambda(k)+\rho(k-1)+\delta(k)]}{\lambda(k)\rho(k-1)} < 1$$

and

$$\prod_{k=1}^{p}\frac{\delta(k-1)[\lambda(k)+\rho(k-1)+\delta(k-1)]}{\lambda(k)\rho(k-1)} < 1.$$

1. Introduction. A recent new direction in the development of the theory of interacting particle systems is the study of these systems in random environments. By now, a number of papers have been written analyzing the effect of the random environment on various types of systems. Several of these are listed in the references [Bramson, Durrett and Schonmann (1991), Chen and Liggett (1992), Ferreira (1990), Greven (1985), Greven (1990), Liggett (1990) and Liu (1991)].

In this paper we will consider the one-dimensional contact process in a random environment. By this we mean the Markov process with state space

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$\{0, 1\}^Z$, where Z is the set of integers, which has the following transitions:

$$1 \to 0 \quad \text{at site } k \text{ at rate } \delta(k)$$

and

$$0 \to 1 \quad \text{at site } k \text{ at rate } \rho(k)\eta(k+1) + \lambda(k)\eta(k-1),$$

where $\eta \in \{0, 1\}^Z$ is the configuration of the process, and $\{(\delta(k), \rho(k), \lambda(k)), k \in Z\}$ is the environment. We assume that the environment is a stationary ergodic process with strictly positive entries. The (infinite) process is said to die out if for almost every choice of environment, the only invariant measure is the pointmass on the configuration $\eta \equiv 0$. Otherwise, it is said to survive. The first major issue concerning the contact process in a random environment which should be settled is that of determining conditions on the distribution of the environment which imply survival or extinction of the process.

In Liggett (1990), we observed that the process dies out if

$$E \log \frac{\rho(k)}{\delta(k)} < 0 \quad \text{and} \quad E \log \frac{\lambda(k)}{\delta(k)} < 0. \tag{1.1}$$

We also gave a complicated and unintuitive sufficient condition for its survival, which we unfortunately were only able to verify in a very special case: $\delta(k) \equiv 1$, $\rho(2k) \equiv \lambda(2k+1) \equiv \lambda_1$, $\rho(2k+1) \equiv \lambda(2k) \equiv \lambda_2$, $\lambda_1\lambda_2 \geq 4$. (Note that a periodic environment can be thought of as a stationary ergodic environment, by choosing each of the translates of the environment with equal probability.) Based on these two results, one might be tempted to guess that a sufficient condition for survival is

$$E \log \frac{\rho(k)}{\delta(k)} \geq c \quad \text{and} \quad E \log \frac{\lambda(k)}{\delta(k)} \geq c \tag{1.2}$$

for some appropriately chosen constant c. This is in fact not the case, at least not for a general stationary ergodic environment. An easy modification of the proof of Theorem 2 in Liggett (1990) can be used to show that if the environment is periodic of period 2 and $\rho(k) = \lambda(k)$ for every k, then the process dies out whenever

$$\delta(1)\delta(2)[\lambda(1) + \lambda(2) + \delta(1) + \delta(2)] > 2\lambda(1)\lambda(2)[\delta(1) + \delta(2)]. \tag{1.3}$$

If a stationary ergodic environment is chosen by taking this periodic environment with probability 1/2 and its translate with probability 1/2, then (1.2) becomes

$$\frac{\lambda(1)\lambda(2)}{\delta(1)\delta(2)} \geq e^{2c}.$$

Taking $\delta(1) = \delta(2) = 1$, $\lambda(1) = \varepsilon$ and $\lambda(2) = \varepsilon^{-2}$ for sufficiently small ε, we see that there is no choice of c so that (1.2) implies survival. Note that the essential difference between the two periodic cases discussed above is that in the first case, infection rates were assigned to bonds, while in the second they were assigned to sites.

One survival result has been obtained recently by Bramson, Durrett and Schonmann (1991). They showed that if the environment satisfies $\lambda(k) \equiv \rho(k) \equiv 1$ and $\{\delta(k),\ k \in Z\}$ are independent and identically distributed with distribution $P[\delta(k) = \delta] = p$ and $P[\delta(k) = \Delta] = 1 - p$, then for every choice of $p \in (0, 1)$ and $\Delta > 0$, there exists a $\delta_c > 0$ so that the process survives whenever $\delta < \delta_c$. Estimates on the size of δ_c and of its dependence on p and Δ are not readily available in that paper.

In this paper, we are interested in obtaining sufficient conditions for survival which apply to larger classes of random environments, and which are easily verifiable. In order to state the main result of this paper, let A_t be the dual contact process, which has rates $\delta'(k) = \delta(k)$, $\rho'(k) = \lambda(k + 1)$ and $\lambda'(k) = \rho(k - 1)$. Let r_t be the position of the rightmost infected site for the process with initial configuration $\cdots 111000 \cdots$, and let l_t be the position of the leftmost infected site for the process with initial configuration $\cdots 000111 \cdots$. The behavior of these edges is related to the survival of the process in the following way: If

$$\lim_{t\to\infty} r_t = +\infty \quad \text{and} \quad \lim_{t\to\infty} l_t = -\infty,$$

then the original system survives. [See Theorem 2.2 of Chapter 6 of Liggett (1985).]

THEOREM 1.4. (i) *Suppose that the following two series converge*:

$$\sum_{j=0}^{\infty} E\frac{1}{\lambda(j+1)} \sum_{k=1}^{j} \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\lambda(k)\rho(k-1)},$$

$$\sum_{j=0}^{\infty} E\frac{1}{\rho(-1)} \prod_{k=1}^{j} \frac{\delta(k-1)[\lambda(k) + \rho(k-1) + \delta(k-1)]}{\lambda(k)\rho(k-1)}.$$

Then the contact process in a random environment survives.

(ii) *Suppose that*

$$E\log\lambda(k) > -\infty \quad \text{and} \quad E\log\rho(k) > -\infty,$$

$$E\log\frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\lambda(k)\rho(k-1)} < 0$$

and

$$E\log\frac{\delta(k-1)[\lambda(k) + \rho(k-1) + \delta(k-1)]}{\lambda(k)\rho(k-1)} < 0.$$

Then

$$\limsup_{t\to\infty} r_t = +\infty \quad \text{or} \quad \liminf_{t\to\infty} l_t = -\infty.$$

REMARKS. 1. We suspect that it is possible to build on the techniques of this paper to improve both parts of Theorem 1.4. Perhaps the right result is that the assumption of part (ii) implies survival, while the assumption of part (i) implies linear growth of the edges:

$$\lim_{t\to\infty}\frac{r_t}{t}-\lim_{t\to\infty}\frac{l_t}{t}>0.$$

This would represent a substantial improvement. For example, as it stands, Theorem 1.4 does not imply the Bramson, Durrett and Schonmann result which is stated above. On the other hand, the proposed improvement of part (ii) would give it as a special case, and in fact would provide a value for δ_c as the solution of

$$[\delta_c(2+\delta_c)]^p[\Delta(2+\Delta)]^{1-p}=1.$$

As another example, suppose $\delta(k)\equiv 1$ and $\lambda(k)=\rho(k-1)$ are i.i.d. and uniformly distributed on $[0,M]$. Then the assumption of part (i) is never satisfied, but the assumption of part (ii) is satisfied for $M>7.1$. The extinction criterion (1.1) is satisfied if $M<e$. Another reason for wanting to obtain the improvement is that the general sufficient condition for extinction (1.1) involves a logarithmic moment, so it would be more natural to prove survival under a logarithmic moment assumption.

2. If the distribution of the environment is spatially symmetric, then of course the "or" in the conclusion of part (ii) can be replaced by an "and."

3. If the environment is periodic of period p, the assumptions of parts (i) and (ii) both reduce to

$$\prod_{k=1}^{p}\frac{\delta(k)[\lambda(k)+\rho(k-1)+\delta(k)]}{\lambda(k)\rho(k-1)}<1$$

and

$$\prod_{k=1}^{p}\frac{\delta(k-1)[\lambda(k)+\rho(k-1)+\delta(k-1)]}{\lambda(k)\rho(k-1)}<1.$$

Theorem 1.4 is proved in Section 4. Sections 2 and 3 develop some preliminary results. To give an idea of the approach used in these proofs, consider for a moment the homogeneous contact process, in which $\delta(k)\equiv\delta$, $\lambda(k)\equiv\lambda$ and $\rho(k)\equiv\rho$. In this case, it was proved in Holley and Liggett (1978) that the process survives if $\lambda+\rho\ge 4\delta$. An overly simplified version of that proof goes as follows: Suppose the process survives, and let v be its upper invariant measure. Define

$$\alpha=v\{\eta:\eta(k)=1\}\quad\text{and}\quad\phi(n)=v\{\eta:\eta(k)=\eta(k+n)=1,\ \eta(j)=0\text{ for }k<j<k+n\}.$$

If we pretend as a first approximation that v is a stationary renewal measure (which is of course not the case), then the invariance of v implies that these quantities satisfy the convolution equations

$$(\lambda + \rho + 2\delta)\phi(1) = (\lambda + \rho)[\phi(2) + \alpha]$$

and

$$(\lambda + \rho + 2\delta)\phi(n) = (\lambda + \rho)\phi(n+1) + \frac{\delta}{\alpha}\sum_{k=1}^{n-1}\phi(k)\phi(n-k)$$

for $n \geq 2$. It is not hard to solve these equations explicitly. In the spatially inhomogeneous case, on the other hand, it is not in general possible to solve the analogous system of equations. [See (2.4).] Since our objective in both cases is to keep α away from 0, we would like to say something about α without having to solve for the $\phi(n)$'s. To do so, multiply the second equation by $n(n-1)/2$ and sum for $n \geq 2$, using the fact that

$$\alpha = \sum_{n=1}^{\infty}\phi(n) \quad \text{and} \quad 1 = \sum_{n=1}^{\infty} n\phi(n)$$

and assuming that

$$\sum_{n=1}^{\infty} n^2\phi(n) < \infty$$

to obtain the following quadratic equation for α:

$$(\lambda + \rho)\alpha^2 - (\lambda + \rho)\alpha + \delta = 0. \tag{1.5}$$

Note that this has a real solution if and only if $\lambda + \rho \geq 4\delta$, which is the Holley–Liggett condition.

In Section 2 we prove an analog of (1.5) for periodic systems, which is given in Theorem 2.27. In this case, $\alpha(k)$ becomes a (periodic) function of one variable, and $\phi(k, l)$ becomes a function of two variables. The analog of (1.5) is one identity involving the α's, but not the ϕ's. Guided by this result, we study in Section 3 the invariant measure for the inhomogeneous contact process (i.e., with a deterministic environment) on a finite interval with boundary conditions. This can be regarded as a periodic system in which $\delta(0) = 0$. The main result in that section, which is the key tool in the proof of Theorem 1.4, is the following.

THEOREM 1.6. *Let v be the invariant measure for the inhomogeneous contact process with fixed ones at* 0 *and $p > 1$. Then there exists a positive function α on $\{0, 1, \dots, p\}$ which satisfies $\alpha(0) = \alpha(p) = 1$,*

$$\lambda(k)\alpha(k-1) \leq [\lambda(k) + \delta(k)]\alpha(k), \tag{1.7}$$

$$\rho(k)\alpha(k+1) \leq [\rho(k) + \delta(k)]\alpha(k) \tag{1.8}$$

and

$$v\{\eta: \eta(k) = 1\} \geq \alpha(k) \tag{1.9}$$

for $1 \le k < p$, *and the following identity*:

$$\begin{aligned}
&\sum_{i=1}^{p-1}\sum_{j=i}^{p-1}\frac{1}{\lambda(j+1)\rho(i-1)}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i}^{j}\frac{\delta(l)}{\alpha(l)}\\
&\qquad+\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i+1}^{j-1}\frac{\delta(l)}{\alpha(l)}\\
&\quad=\sum_{i=0}^{p-1}\sum_{j=i}^{p-1}\frac{1}{\lambda(j+1)}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i+1}^{j}\frac{\delta(l)}{\alpha(l)}\\
&\qquad+\sum_{i=1}^{p}\sum_{j=i}^{p}\frac{1}{\rho(i-1)}\sum_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i}^{j-1}\frac{\delta(l)}{\alpha(l)}.
\end{aligned}\tag{1.10}$$

In order to see why (1.10) is likely to be useful, recall that one way to prove survival of systems such as the contact process is to show that the invariant measures for the systems on finite sets with boundary condition $= 1$ do not converge to the pointmass on the zero configuration as the finite set increases to the whole set of sites. The difficulty has always been that it is practically impossible in nonreversible situations to say anything very precise about these invariant measures. Thus one wants to find lower bounds of the type given in (1.9). If the $\alpha(k)$'s could be computed explicitly, one would get very good results. While we cannot compute them, equality (1.10) is nearly as useful. To see this, note by counting the number of $\alpha(k)$'s which occur in each of the four expressions in (1.10) that, roughly speaking, one can write that identity in the form

$$O(\alpha^{-1}) + O(\alpha) = O(1).$$

If an α satisfies such an identity, it cannot be too small. Inequalities (1.7) and (1.8) are needed to show that expressions of the form

$$\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\frac{\delta(k)}{\alpha(k)}$$

are bounded above and below by known quantities (i.e., quantities which do not depend on the α's).

2. The periodic system. Throughout this section, we assume that the transition rates are periodic of period p: $\lambda(k+p) = \lambda(k)$, $\rho(k+p) = \rho(k)$ and $\delta(k+p) = \delta(k)$ for all k. Our starting point is the following two results which were proved in Liggett (1990) and Liggett (1991) respectively. The α's and ϕ's which appear in the statements of these results have the following interpretation: The contact process is to be run with an initial distribution which is the inhomogeneous renewal measure μ on $\{0,1\}^Z$ determined by

$$\alpha(k) = \mu\{\eta\colon \eta(k) = 1\} \quad \text{and} \quad \phi(k,l) = \mu\{\eta\colon \eta(k) = \eta(l) = 1,\ \eta(j) = 0 \text{ for } k < j < l\}.$$

THEOREM 2.1. *Suppose that there exist* $\alpha(k+p) = \alpha(k) > 0$ *for all* k *and* $\phi(k+p, l+p) = \phi(k,l) \geq 0$ *for all* $k < l$, *which satisfy*

$$\sum_{k \leq i \leq l} \phi(k,l) = 1 \tag{2.2}$$

and

$$\alpha(i) = \sum_{k>i} \phi(i,k) = \sum_{k<i} \phi(k,i) \tag{2.3}$$

for all i, *and*

$$\begin{aligned} 0 = {} & \lambda(k)\phi(k-1,l) + \rho(l)\phi(k,l+1) + \sum_{k<j<l} \delta(j)\frac{\phi(k,j)\phi(j,l)}{\alpha(j)} \\ & + [\lambda(k+1)\alpha(k) + \rho(k)\alpha(k+1)]1_{\{l=k+1\}} \\ & - [\delta(k) + \delta(l) + \lambda(k+1) + \rho(l-1)]\phi(k,l) \end{aligned} \tag{2.4}$$

and

$$\phi(k,l)\phi(k-1,l+1) \geq \phi(k-1,l)\phi(k,l+1) \tag{2.5}$$

for all $k < l$. *Then the contact process survives, and the upper invariant measure* v *satisfies*

$$v\{\eta\colon \eta(k) = 1\} \geq \alpha(k) \tag{2.6}$$

for all k.

REMARK. The periodicity assumptions are not required in Liggett (1990). The inequality (2.6) was not explicitly stated in Theorem 3.15 of that paper, but it is a special case of the final sentence in its proof.

THEOREM 2.7. *Suppose* α *and* ϕ *satisfy* (2.2)–(2.4) *and*

$$\sum_{k=1}^{p} \sum_{l=k+1}^{\infty} (l-k)^2 \phi(k,l) < \infty. \tag{2.8}$$

For $1 \leq k, l \leq p$, *define*

$$F(k,l) = \sum_{i>k,\, i \equiv l} \phi(k,i) = \sum_{i<l,\, i\equiv k} \phi(i,l),$$

where $\equiv$ *denotes equality modulo* p. *For* $1 \leq k \leq p$, *let* $M(k)$, $N(k)$, $G(k)$ *and*

$H(k)$ be the (unique) solutions of the following systems of equations:

$$(2.9)\quad [\lambda(k+1)+\delta(k)]M(k) = M(k-1)\lambda(k) + \sum_{i=1}^{p}\frac{\delta(i)}{\alpha(i)}M(i)F(k,i), \qquad \sum_{i=1}^{p} M(i)=1,$$

$$(2.10)\quad [\rho(k-1)+\delta(k)]N(k) = N(k+1)\rho(k) + \sum_{i=1}^{p}\frac{\delta(i)}{\alpha(i)}N(i)F(i,k), \qquad \sum_{i=1}^{p} N(i)=1,$$

$$(2.11)\quad \begin{aligned} G(k-1)\lambda(k) &+ \sum_{i=1}^{p}\frac{\delta(i)}{\alpha(i)}G(i)F(k,i) - G(k)[\lambda(k+1)+\delta(k)] \\ &= \lambda(k)\alpha(k-1) - \lambda(k+1)\alpha(k) - \rho(k)\alpha(k+1) \\ &\quad + \sum_{i=1}^{p}\rho(i-1)F(k,i), \qquad \sum_{i=1}^{p} G(i)=0, \end{aligned}$$

$$(2.12)\quad \begin{aligned} H(k+1)\rho(k) &+ \sum_{i=1}^{p}\frac{\delta(i)}{\alpha(i)}H(i)F(i,k) - H(k)[\rho(k-1)+\delta(k)] \\ &= \rho(k)\alpha(k+1) - \rho(k-1)\alpha(k) - \lambda(k)\alpha(k-1) \\ &\quad + \sum_{i=1}^{p}\lambda(i+1)F(i,k), \qquad \sum_{i=1}^{p} H(i)=0. \end{aligned}$$

Then

$$(2.13)\quad \begin{aligned} 0 = p^2\sum_{k=1}^{p}\frac{\delta(k)}{\alpha(k)}M(k)N(k) &+ \sum_{k=1}^{p}\alpha(k)[\lambda(k+1)+\rho(k-1)] \\ + p\sum_{k=1}^{p}\Bigg\{\frac{\delta(k)}{\alpha(k)}&[M(k)H(k)+N(k)G(k)] \\ &-\lambda(k+1)M(k) - \rho(k-1)N(k)\Bigg\} \\ + \sum_{k=1}^{p}\Bigg\{\frac{\delta(k)}{\alpha(k)}&G(k)H(k) - \lambda(k+1)G(k) - \rho(k-1)H(k)\Bigg\}. \end{aligned}$$

REMARKS. 1. This result can be found as (2.12) in Liggett (1991), noting that as a consequence of (2.2), the S which occurs there equals p.

2. The uniqueness of the solutions to (2.9)–(2.12) is most easily seen by considering the irreducible continuous-time Markov chains X_t and Y_t on

$\{1, \dots, p\}$ with the following transition rates:

> X_t goes from k to l at rate $\delta(k)F(l,k)/\alpha(k)$, and from k to $k+1$ at (an additional) rate $\lambda(k+1)$;
>
> Y_t goes from k to l at rate $\delta(k)F(k,l)/\alpha(k)$, and from k to $k-1$ at (an additional) rate $\rho(k-1)$.

Then (2.9) and (2.10), for example are simply the statements that M and N are the stationary distributions for X_t and Y_t respectively. These chains will appear again later.

3. The remainder of this section is devoted to simplifying (2.13), and reexpressing it so that it does not depend on the ϕ's. Identity (2.13) appears later in various forms in (2.18), (2.20), the conclusion of Theorem 2.27 and (2.34). A limiting form of it, as the recovery rate $\delta(p)$ at p tends to 0, can be found in the statement of Theorem 2.35, which is just identity (1.10).

As it stands, (2.13) does not appear to be very useful, since $M(k)$, $N(k)$, $G(k)$ and $H(k)$ are very complicated functions of $F(k,l)$, which are not known explicitly. However, as is observed in Liggett (1991), $F(k,l)$ satisfy the following identities as a consequence of (2.3) and (2.4):

$$\alpha(k) = \sum_{i=1}^{p} F(k,i) = \sum_{i=1}^{p} F(i,k), \tag{2.14}$$

$$\begin{aligned} 0 = {} & \lambda(k)F(k-1,l) + \rho(l)F(k,l+1) + \sum_{i=1}^{p} \delta(i)\frac{F(k,i)F(i,l)}{\alpha(i)} \\ & + \{\delta(k)\alpha(k) - \lambda(k)\alpha(k-1) - \rho(k)\alpha(k+1)\}1_{\{l=k\}} \\ & + [\lambda(k+1)\alpha(k) + \rho(k)\alpha(k+1)]1_{\{l=k+1\}} \\ & - [\delta(k) + \delta(l) + \lambda(k+1) + \rho(l-1)]F(k,l). \end{aligned} \tag{2.15}$$

In (2.15), we use the periodicity conventions $F(0,l) = F(p,l)$ and $F(k,p+1) = F(k,1)$.

We will now see that as a consequence of (2.14), (2.15) and a lot of cancellation, the combinations of the $M(k)$'s, $N(k)$'s, $G(k)$'s and $H(k)$'s which occur in (2.13) can be rewritten in such a way that they depend explicitly on the $\alpha(k)$'s, but are independent of the $\phi(k,l)$'s and $F(k,l)$'s. This leads to a useful identity among the $\alpha(k)$'s (which appears as the conclusion in Theorem 2.27, and in a limiting form in Theorem 2.35). In order to simplify (2.9)–(2.13), let

$$g(k) = G(k) + pM(k) - \frac{\rho(k-1)\alpha(k)}{\delta(k)}$$

and

$$h(k) = H(k) + pN(k) - \frac{\lambda(k+1)\alpha(k)}{\delta(k)}.$$

Also, set

$$u(k) = \lambda(k)\alpha(k-1) + \rho(k-1)\alpha(k)$$

and

$$v(k) = \frac{\lambda(k+1)\rho(k-1)\alpha(k)}{\delta(k)},$$

since these combinations will occur often. By (2.9)–(2.12), these satisfy

$$(2.16)\qquad \begin{aligned} g(k-1)\lambda(k) + \sum_{i=1}^{p} \frac{\delta(i)}{\alpha(i)} g(i)F(k,i) - g(k)[\lambda(k+1) + \delta(k)] \\ = u(k) - u(k+1) - v(k-1) + v(k), \\ \sum_{i=1}^{p} g(i) = p - \sum_{i=1}^{p} \frac{\rho(i-1)\alpha(i)}{\delta(i)} \end{aligned}$$

and

$$(2.17)\qquad \begin{aligned} h(k+1)\rho(k) + \sum_{i=1}^{p} \frac{\delta(i)}{\alpha(i)} h(i)F(i,k) - h(k)[\rho(k-1) + \delta(k)] \\ = u(k+1) - u(k) - v(k+1) + v(k), \\ \sum_{i=1}^{p} h(i) = p - \sum_{i=1}^{p} \frac{\lambda(i+1)\alpha(i)}{\delta(i)}. \end{aligned}$$

Identity (2.13) becomes

$$(2.18)\qquad \sum_{i=1}^{p} \frac{\delta(i)}{\alpha(i)} g(i)h(i) + \sum_{i=1}^{p} \{u(i) - v(i)\} = 0.$$

REMARK. From the definitions of $M(k)$, $N(k)$, $G(k)$ and $H(k)$ in Liggett (1991), one can see that $g(k)$ and $h(k)$ are given by

$$g(k) = \sum_{i=k+1}^{\infty} (i-k)\phi(k,i) - \frac{\rho(k-1)\alpha(k)}{\delta(k)}$$

and

$$h(k) = \sum_{i=-\infty}^{k-1} (k-i)\phi(i,k) - \frac{\lambda(k+1)\alpha(k)}{\delta(k)}.$$

The first part of (2.16) comes from multiplying (2.4) by $l-k$ and then summing on all $l > k$ for fixed k. The second part of (2.16) is a consequence of (2.2). Similarly, the first part of (2.17) comes from multiplying (2.4) by $l-k$ and then summing on all $k < l$ for fixed l, while the second part is a consequence of (2.2). Equation (2.18) is obtained by multiplying (2.4) by $(l-k)(l-k-1)/2$ and summing on all k and l satisfying $1 \le k \le p$ and $l > k$. Thus the reader can check the basic equations (2.16)–(2.18) without referring to Liggett (1991).

The next step is to write (2.16) and (2.17) using matrix notation. In order to do so, let D be the diagonal $p \times p$ matrix whose ith entry is $\delta(i)/\alpha(i)$, and $A = (a_{i,j})$ and $B = (b_{i,j})$ be the $p \times p$ matrices with entries

$$a_{i,j} = \begin{cases} \dfrac{\delta(j)}{\alpha(j)} F(i,j) + \lambda(i) 1_{\{j=i-1\}} - [\lambda(i+1) + \delta(i)] 1_{\{j=i\}}, & \text{if } 1 < i \le p, \\ 1, & \text{if } i = 1, \end{cases}$$

and

$$b_{i,j} = \begin{cases} \dfrac{\delta(i)}{\alpha(i)} F(i,j) + \rho(j) 1_{\{j=i-1\}} - [\rho(i-1) + \delta(i)] 1_{\{j=i\}}, & \text{if } 1 \le j < p, \\ 1, & \text{if } j = p. \end{cases}$$

Then (2.16) and (2.17) become

$$Ag = x \quad \text{and} \quad hB = y, \tag{2.19}$$

where x and y are the vectors with entries

$$x(k) = \begin{cases} u(k) - u(k+1) - v(k-1) + v(k), & \text{if } 1 < k \le p, \\ p - \displaystyle\sum_{i=1}^{p} \frac{\rho(i-1)\alpha(i)}{\delta(i)}, & \text{if } k = 1, \end{cases}$$

and

$$y(k) = \begin{cases} u(k+1) - u(k) - v(k+1) + v(k), & \text{if } 1 \le k < p, \\ p - \displaystyle\sum_{i=1}^{p} \frac{\lambda(i+1)\alpha(i)}{\delta(i)}, & \text{if } k = p. \end{cases}$$

Next, let $W = (w_{i,j})$ be the matrix $AD^{-1}B$, and compute its entries using (2.14) and (2.15) as follows:

$$w_{1,p} = \sum_{k=1}^{p} \frac{\alpha(k)}{\delta(k)},$$

$$w_{1,j} = \frac{\rho(j)\alpha(j+1)}{\delta(j=1)} - \frac{\rho(j-1)\alpha(j)}{\delta(j)}, \quad \text{for } 1 \le j < p,$$

$$w_{i,p} = \frac{\lambda(i)\alpha(i-1)}{\delta(i-1)} - \frac{\lambda(i+1)\alpha(i)}{\delta(i)}, \quad \text{for } 1 < i \le p,$$

and for $1 < i \le p$ and $1 \le j < p$,

$$\begin{aligned} w_{i,j} &= \sum_{k=1}^{p} \frac{\alpha(k)}{\delta(k)} \left\{ \frac{\delta(k)}{\alpha(k)} F(i,k) + \lambda(i) 1_{\{k=i-1\}} - [\lambda(i+1) + \delta(i)] 1_{\{k=i\}} \right\} \\ &\quad \times \left\{ \frac{\delta(k)}{\alpha(k)} F(k,j) + \rho(j) 1_{\{k=j+1\}} - [\rho(j-1) + \delta(j)] 1_{\{k=j\}} \right\} \\ &= \sum_{k=1}^{p} \frac{\delta(k)}{\alpha(k)} F(i,k) F(k,j) + \rho(j) F(i,j+1) + \lambda(i) F(i-1,j) \\ &\quad - [\delta(i) + \delta(j) + \lambda(i+1) + \rho(j-1)] F(i,j) \\ &\quad + v(i-1) 1_{\{i-1=j+1\}} + \frac{\alpha(i)}{\delta(i)} [\lambda(i+1) + \delta(i)][\rho(i-1) + \delta(i)] 1_{\{i=j\}} \\ &\quad - [v(i) + v(j) + u(i)] 1_{\{i=j+1\}} \\ &= \begin{cases} v(i-1), & \text{if } j = i-2, \\ -[u(i) + v(i) + v(i-1)], & \text{if } j = i-1, \\ u(i) + u(i+1) + v(i), & \text{if } j = i, \\ -u(i+1), & \text{if } j \equiv i+1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the matrices A and B are invertible, since for example, $\mu B = 0$ is just the statement that μ is an invariant (signed) measure for the irreducible Markov chain Y_t which has total mass 0, and hence $\mu = 0$. By (2.19), $g = A^{-1}x$ and $h = yB^{-1}$. It follows from the definition of W that W is invertible as well and $W^{-1} = B^{-1}DA^{-1}$. Therefore, $yW^{-1}x = hDg$, which is the first sum in (2.18). Thus we conclude that

$$yW^{-1}x = \sum_{i=1}^{p} \{v(i) - u(i)\}. \tag{2.20}$$

Note that as promised, none of the terms appearing in (2.20) depend explicitly on the $F(i,j)$'s. It remains to express the left side of (2.20) in a more convenient form.

In order to do so, we define $p \times p$ matrices $Q = (q_{i,j})$, $R = (r_{i,j})$ and $S = (s_{i,j})$ as follows:

$$q_{i,j} = \begin{cases} c(j), & \text{if } i = 1, \\ 1, & \text{if } i = j > 1, \\ -1, & \text{if } i = j+1, \end{cases}$$

$$r_{i,j} = \begin{cases} v(i), & \text{if } i = j, \\ -u(j), & \text{if } j = i+1, \\ -u(1), & \text{if } i = p, j = 1, \end{cases}$$

and

$$s_{i,j} = \begin{cases} e(i), & \text{if } j = p, \\ 1, & \text{if } i = j < p, \\ -1, & \text{if } i = j + 1. \end{cases}$$

In the above, $c = (c(1), \dots, c(p))$ and $e = (e(1), \dots, e(p))$ are vectors to be determined later. [In formulas appearing below, these vectors are defined for other integer arguments by periodicity: $c(i) = c(i + p)$ and $e(i) = e(i + p)$.] Easy verifications show that these matrices have determinants

$$\det Q = \sum_{k=1}^{p} c(k),$$

$$\det S = \sum_{k=1}^{p} e(k), \tag{2.21}$$

$$\det R = \prod_{k=1}^{p} v(k) - \prod_{k=1}^{p} u(k),$$

and adjoints with entries

$$(\operatorname{Adj} Q)_{i,j} = \begin{cases} 1, & \text{if } j = 1, \\ -\sum_{k=j}^{p} c(k), & \text{if } i < j, \\ \sum_{k=1}^{j-1} c(k), & \text{if } 1 < j \le i, \end{cases}$$

$$(\operatorname{Adj} R)_{i,j} = \begin{cases} \prod_{k=i=1}^{j} u(k) \prod_{k=j+1}^{i+p-1} v(k), & \text{if } i \le j, \\ \prod_{k=i+1}^{j+p} u(k) \prod_{k=j+1}^{i-1} v(k), & \text{if } i > j, \end{cases} \tag{2.22}$$

$$(\operatorname{Adj} S)_{i,j} = \begin{cases} 1, & \text{if } i = p, \\ -\sum_{k=1}^{i} e(k), & \text{if } i < j, \\ \sum_{k=i+1}^{p} e(k), & \text{if } j \le i < p. \end{cases}$$

The connection between these matrices and W is given in the following lemma.

LEMMA 2.23. *Assume that* α *and* ϕ *satisfy* (2.2)–(2.4) *and* (2.8). *Suppose that the vectors* c *and* e *satisfy*

$$c(k)v(k) - c(k-1)u(k) + \frac{v(k)}{\lambda(k+1)} = \sigma \quad \text{for some constant } \sigma, \tag{2.24}$$

$$e(k)v(k) - e(k+1)u(k+1) + \frac{v(k)}{\rho(k-1)} = \tau \quad \text{for some constant } \tau \tag{2.25}$$

and

$$\sum_{k=1}^{p} \left\{ u(k)e(k)c(k-1) - v(k)e(k)c(k) + \frac{\alpha(k)}{\delta(k)} \right\} = 0. \tag{2.26}$$

Then $W = QRS$.

PROOF. The $(1, p)$ entry in QRS is

$$\sum_{k=1}^{p} c(k)v(k)e(k) - \sum_{k=1}^{p} c(k)u(k+1)e(k+1).$$

So by (2.26), it agrees with the $(1, p)$ entry in W. For $1 \le j < p$, the $(1, j)$ entry in QRS is

$$c(j)v(j) - c(j+1)v(j+1) - c(j-1)u(j) + c(j)u(j+1),$$

which agrees with the $(1, j)$ entry in W by (2.24). The same argument applies to the equality of the (i, p) entries for $1 < i \le p$, using (2.25). For $1 < i \le p$ and $1 \le j < p$, the (i, j) entry in QRS is

$$\sum_{k=1}^{p} q_{i,k} v(k) s_{k,j} - \sum_{k=1}^{p} q_{i,k} u(k+1) s_{k+1,j},$$

which agrees with the corresponding entry in W. □

Next, we come to the main result in this section, which gives a useful identity which is satisfied by the vector α.

THEOREM 2.27. *Assume that* α *and* ϕ *satisfy* (2.2)–(2.4) *and* (2.8). *If*

$$\prod_{k=1}^{p} v(k) \ne \prod_{k=1}^{p} u(k), \tag{2.28}$$

then there exist constants σ *and* τ, *and vectors* c *and* e *which satisfy* (2.24)–(2.26). *Furthermore*,

$$\sum_{i=1}^{p} \sum_{j=i}^{p+i-1} \left[1 - \frac{v(i)}{\lambda(i+1)}\right]\left[1 - \frac{v(j)}{\rho(j-1)}\right] \prod_{k=i+1}^{j} u(k) \sum_{l=j+1}^{i+p-1} v(l)$$
$$= \left[\prod_{k=1}^{p} v(k) - \prod_{k=1}^{p} u(k)\right] \sum_{k=1}^{p} \frac{v(k)}{\lambda(k+1)\rho(k-1)}.$$

PROOF. Define vectors z_1, z_2 and z_3 by

$$z_1(k) = \frac{v(k)}{\lambda(k+1)} = \frac{\rho(k-1)\alpha(k)}{\delta(k)},$$

$$z_2(k) = \frac{v(k)}{\rho(k-1)} = \frac{\lambda(k+1)\alpha(k)}{\delta(k)}$$

and

$$z_3(k) = \frac{v(k)}{\lambda(k+1)\rho(k-1)} = \frac{\alpha(k)}{\delta(k)}.$$

Also, let 1 be the vector all of whose entries are 1. (We do not distinguish between row and column vectors—the choice is dictated by which side of a matrix they appear on.) By (2.21) and (2.28), R is invertible. Note that (2.24) can be written as $cR = \sigma 1 - z_1$, and (2.25) can be written as $Re = \tau 1 - z_2$. Therefore, for given σ and τ, these equations can be solved for c and e respectively, giving

$$c = \sigma 1 R^{-1} - z_1 R^{-1} \quad \text{and} \quad e = \tau R^{-1} 1 - R^{-1} z_2. \tag{2.29}$$

Equation (2.26) can be written as $cRe = 1z_3$, so using (2.29), it becomes

$$\tau(\sigma 1 R^{-1} - z_1 R^{-1})1 = \sigma(1R^{-1}z_2) - z_1 R^{-1} z_2 + 1z_3. \tag{2.30}$$

Choose σ and τ so that (2.30) is satisfied. The conditions of Lemma 2.23 are now satisfied, and hence $W = QRS$. Since W is invertible, Q and S are as well. By (2.21) and (2.29),

$$\det Q = c1 = \sigma(1R^{-1}1) - (z_1 R^{-1} 1)$$

and

$$\det S = 1e = \tau(1R^{-1}1) - (1R^{-1}z_2).$$

In order to evaluate the left side of (2.20), we need to compute yS^{-1} and $Q^{-1}x$, which we now proceed to do using (2.21) and (2.22):

$$\begin{aligned}(yS^{-1})(j) &= \frac{1}{\det S}\sum_{i=1}^{p} y(i)(\operatorname{Adj} S)_{i,j} \\ &= -\frac{1}{\det S}\sum_{1\le k\le i<j}[u(i+1)-u(i)-v(i+1)+v(i)]e(k) \\ &\quad + \frac{1}{\det S}\sum_{j\le i<k\le p}[u(i+1)-u(i)-v(i+1)+v(i)]e(k) \\ &\quad + \frac{p-1z_2}{\det S} \\ &= \frac{1}{\det S}\sum_{k=1}^{p}[u(k)-u(j)-v(k)+v(j)]e(k) + \frac{p^{-1}z_2}{\det S} \\ &= \frac{p(1-\tau)+[v(j)-u(j)]1e}{\det S} = v(j)-u(j)+\frac{p(1-\tau)}{\det S},\end{aligned} \tag{2.31}$$

since $1e = \det S$ as observed above, and

$$\sum_{k=1}^{p} [u(k) - v(k)]e(k) = 1z_2 - \tau p$$

by (2.25). Similarly,

$$Q^{-1}x(i) = v(i) - u(i+1) + \frac{p(1-\sigma)}{\det Q}. \tag{2.32}$$

Therefore, (2.20) becomes

$$\begin{aligned} &\sum_{i,j=1}^{p} \left[v(i) - u(i) + \frac{p(1-\tau)}{\det S} \right] \\ &\qquad \times (\operatorname{Adj} R)_{i,j} \left[v(j) - u(j+1) + \frac{p(1-\sigma)}{\det Q} \right] \\ &= \det R \sum_{k=1}^{p} [v(k) - u(k)]. \end{aligned} \tag{2.33}$$

From the definition of R,

$$(1R)(i) = v(i) - u(i) \quad \text{and} \quad (R1)(j) = v(j) - u(j+1).$$

Therefore, the left side of (2.33) can be written as

$$\det R \left\{ 1R1 + \frac{p^2(1-\tau)}{\det S} + \frac{p^2(1-\sigma)}{\det Q} + \frac{p^2(1-\tau)(1-\sigma)(1R^{-1}1)}{(\det S)(\det Q)} \right\}.$$

The right side of (2.33) is just $(\det R)(1R1)$. After some cancellation, (2.33) then becomes

$$(1-\tau)\det Q + (1-\sigma)\det S + (1-\tau)(1-\sigma)(1R^{-1}1) = 0. \tag{2.34}$$

Using (2.30) and the expressions for $\det Q$ and $\det S$ which occur just after it, (2.34) becomes

$$\left(z_1 R^{-1} 1\right) + \left(1R^{-1}z_2\right) + (1z_3) = (1R^{-1}1) + \left(z_1 R^{-1} z_2\right).$$

But this is the identity in the statement of the theorem. □

The final result in this section is intended to motivate the approach taken in Section 3 to the proof of Theorem 1.6.

THEOREM 2.35. *Fix values of* $\lambda(k)$, $1 \le k \le p$, $\rho(k)$, $1 \le k \le p$, *and* $\delta(k)$, $1 \le k \le p-1$. *Suppose there exists an* $\varepsilon > 0$ *so that if* $0 < \delta(p) < \varepsilon$, *then there are solutions* α *and* ϕ *of* (2.2)–(2.4) *which satisfy* (2.8). *Suppose also that* $\alpha(k)$ *has a nonzero limit as* $\delta(p) \to 0$ *for each* $1 \le k \le p$. *Then this limit* *satisfies the following identity*:

$$\sum_{i=1}^{p-1}\sum_{j=i}^{p-1}\frac{1}{\lambda(j+1)\rho(i-1)}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i}^{j}\frac{\delta(l)}{\alpha(l)}$$
$$+\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i+1}^{j-1}\frac{\delta(l)}{\alpha(l)}$$
$$=\sum_{i=0}^{p-1}\sum_{j=i}^{p-1}\frac{1}{\lambda(j+1)}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i+1}^{j}\frac{\delta(l)}{\alpha(l)}$$
$$+\sum_{i=1}^{p}\sum_{j=i}^{p}\frac{1}{\rho(i-1)}\prod_{k=i+1}^{j}\left[\frac{\alpha(k-1)}{\rho(k-1)}+\frac{\alpha(k)}{\lambda(k)}\right]\prod_{l=i}^{j-1}\frac{\delta(l)}{\alpha(l)}.$$

PROOF. We will pass to the limit in the identity given in Theorem 2.27 as $\delta(p)\to 0$. All of the quantities in the identity except $v(p)$ have nonzero finite limits; $v(p)$, on the other hand, tends to ∞. Therefore, (2.28) is eventually satisfied. The dominant term on each side of the identity is

$$\frac{v(p)}{\lambda(1)\rho(p-1)}\prod_{k=1}^{p}v(k),$$

which is quadratic in $v(p)$. After subtracting this term from each side of the identity, the dominant terms are those which contain the first power of $v(p)$. Dividing by $v(p)$ and passing to the limit in the identity yields the following:

$$-\left[\frac{1}{\lambda(1)}+\frac{1}{\rho(p-1)}\right]\prod_{k=1}^{p-1}v(k)+\sum_{i=1}^{p-1}\sum_{j=i}^{p-1}\left[1-\frac{v(i)}{\lambda(i+1)}\right]\left[1-\frac{v(j)}{\rho(j-1)}\right]$$
$$\times\prod_{k=i+1}^{j}u(k)\prod_{l=1}^{i-1}v(l)\prod_{l=j+1}^{p-1}v(l)$$
$$(2.36)\qquad -\sum_{j=1}^{p-1}\frac{1}{\lambda(1)}\left[1-\frac{v(j)}{\rho(j-1)}\right]\prod_{k=1}^{j}u(k)\prod_{l=j+1}^{p-1}v(l)$$
$$-\sum_{i=1}^{p-1}\frac{1}{\rho(p-1)}\left[1-\frac{v(i)}{\lambda(i+1)}\right]\prod_{k=i+1}^{p}u(k)\prod_{l=1}^{i-1}v(l)$$
$$=\left[\prod_{k=1}^{p-1}v(k)\right]\sum_{k=1}^{p-1}\frac{v(k)}{\lambda(k+1)\rho(k-1)}-\frac{1}{\lambda(1)\rho(p-1)}\prod_{k=1}^{p}u(k).$$

The first expression on the left above comes from the term $i=j=p$ in the identity in Theorem 2.27, the second from the terms with $i\neq p$ and $j\neq p$, the third from those with $i=p$ and $j\neq p$, and the fourth from those with $i\neq p$ and $j=p$.

Dividing (2.36) by

$$\prod_{k=1}^{p-1} v(k)$$

and combining terms with the same λ and/or ρ factor leads to

$$\begin{aligned} 0 = {} & \sum_{i=1}^{p-1} \sum_{j=i}^{p-1} \frac{1}{v(i)} \prod_{k=i+1}^{j} \frac{u(k)}{v(k)} \\ & + \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} \frac{v(j)}{\lambda(i+1)\rho(j-1)} \prod_{k=i+1}^{j} \frac{u(k)}{v(k)} \\ & - \sum_{i=0}^{p-1} \sum_{j=i}^{p-1} \frac{1}{\lambda(i+1)} \prod_{k=i+1}^{j} \frac{u(k)}{v(k)} - \sum_{i=1}^{p} \sum_{j=i}^{p} \frac{v(j)}{\rho(j-1)v(i)} \prod_{k=i+1}^{j} \frac{u(k)}{v(k)}. \end{aligned} \tag{2.37}$$

Using the definitions of $u(k)$ and $v(k)$ in (2.37) gives the required identity. □

3. The system on $\{1,\ldots,p-1\}$. The difficulty with the results in Section 2 is of course that they all assume that the system (2.2)–(2.4) has a solution. This begs the original question, since if we knew that, Theorem 2.1 could be applied directly to conclude that the contact process survives, at least in the case of a periodic environment. We will rectify this situation in this section by studying the system on $\{1,\ldots,p-1\}$ with fixed boundary ones at 0 and at p. This will be regarded as a periodic system in which $\delta(p)=0$. We cannot simply set $\delta(p)=0$ in the results of the previous section, since $\delta(p)$ appears in the denominator of various expressions. More serious is the observation that in Theorem 2.35 we saw that there is no information in the leading terms in the identity from Theorem 2.27—the useful information is in the second-order terms. Therefore, for most of the quantities appearing in Section 2, we will need to keep track of what would correspond to both the first-order and second-order terms in $\delta(p)$. We will not be using any of the results proved in Section 2 here—they will merely be a guide in developing identities involving these quantities. We will use the suggestive notation g', h', and so forth, to denote the versions of these quantities corresponding to the second-order terms. In this section, we will take periodic rates $\lambda(k+p)=\lambda(k)$, $\rho(k+p)=\rho(k)$ and $\delta(k+p)=\delta(k)$ for all k. These are all strictly positive, except that $\delta(k)=0$ for $k\equiv p$ (recall that $\equiv$ means equality modulo p). The purpose of this section is to prove Theorem 1.6. We begin with the following easy result.

THEOREM 3.1. *There exist $\alpha(k)>0$ for all k and $\phi(k,l)\ge 0$ for all $k<l$ with the following properties*:

(i) $\alpha(k+p)=\alpha(k)$ *for all k, and* $\phi(k+p,l+p)=\phi(k,l)$ *for all $k<l$.*
(ii) $\alpha(p)=1$, *and* $\phi(k,l)=0$ *if $k<m<l$ for some $m\equiv p$.*
(iii) $\alpha(k)$ *and* $\phi(k,l)$ *satisfy* (2.2)–(2.5).
(iv) $\lambda(k)\alpha(k-1)\le[\lambda(k)+\delta(k)]\alpha(k)$ *and*
$\rho(k)\alpha(k+1)\le[\rho(k)+\delta(k)]\alpha(k)$ *for all k.*

PROOF. We may assume without loss of generality that the rates $\lambda(k)$, $\rho(k)$ and $\delta(k)$ are all less than $1/4$. Let Φ be the set of all $\alpha \geq 0$ and $\phi \geq 0$ which satisfy (i), (ii), (2.2) and (2.3). As a consequence of part (ii), the sums occurring in (2.2) and (2.3) contain at most p^2 nonzero terms. Therefore, Φ is compact. It is also convex. Define a mapping Γ on Φ by $(\beta, \psi) = \Gamma(\alpha, \phi)$, where

$$\begin{aligned}\psi(k,l) = {} & \phi(k,l) + \lambda(k)\phi(k-1,l) + \rho(l)\phi(k,l+1) \\ & + \sum_{k<j<l} \delta(j)\frac{\phi(k,j)\phi(j,l)}{\alpha(j)} \\ & + [\lambda(k+1)\alpha(k) + \rho(k)\alpha(k+1)]1_{\{l=k+1\}} \\ & - [\delta(k) + \delta(l) + \lambda(k+1) + \rho(l-1)]\phi(k,l)\end{aligned}$$

[if $\alpha(j) = 0$ for some j, the corresponding term in the sum above is taken to be 0] and

$$\beta(i) = \sum_{k>i} \psi(i,k) = \sum_{k<i} \psi(k,i). \tag{3.2}$$

It is easy to check that Γ maps Φ into itself, and that the second equality in (3.2) holds, so that this equation can be used to define $\beta(i)$. The assumption that the rates are less than $1/4$ guarantees that $\psi(k,l) \geq 0$. Now sum the identity above which defines $\psi(k,l)$ on $l > k$ for fixed k. Noting that

$$\begin{aligned}\sum_{l=k=1}^{\infty} \sum_{k<j<l} \delta(j)\frac{\phi(k,j)\phi(j,l)}{\alpha(j)} &= \sum_{j=k+1}^{\infty} \delta(j)\frac{\phi(k,j)}{\alpha(j)} \sum_{l=j+1}^{\infty} \phi(j,l) \\ &= \sum_{j=k+1}^{\infty} \delta(j)\phi(k,j)\end{aligned}$$

since (α, ϕ) satisfies (2.3), the result is

$$\begin{aligned}\beta(k) = {} & [1-\delta(k)]\alpha(k) + \lambda(k)[\alpha(k-1) - \phi(k-1,k)] \\ & + \rho(k)[\alpha(k+1) - \phi(k,k+1)].\end{aligned} \tag{3.3}$$

Note that Γ is continuous, since $\phi(k,l) \leq \min\{\alpha(k), \alpha(l)\}$ on Φ. Therefore, Γ has a fixed point, which we call (α, ϕ). This solves (2.2)–(2.4). By (3.3),

$$\begin{aligned}\delta(k)\alpha(k) = {} & \lambda(k)[\alpha(k-1) - \phi(k-1,k)] \\ & + \rho(k)[\alpha(k+1) - \phi(k,k+1)].\end{aligned} \tag{3.4}$$

Using $\phi(k-1, k) \leq \min\{\alpha(k-1), \alpha(k)\}$ and $\phi(k, k+1) \leq \min\{\alpha(k), \alpha(k+1)\}$, we see that

$$\lambda(k)\alpha(k-1) \leq [\lambda(k) + \delta(k)]\alpha(k)$$

and

$$\rho(k)\alpha(k+1) \leq [\rho(k) + \delta(k)]\alpha(k).$$

Since $\alpha(p) = 1$, it follows that $\alpha(k) > 0$ for all k. The fact that ϕ satisfies (2.5) is a consequence of Proposition 3.17 of Liggett (1990). The hypotheses of that proposition are very easy to verify, since $\delta(p) = 0$ and ϕ satisfies part (ii). Thus (α, ϕ) has all the required properties. □

From now on, we will fix a choice of (α, ϕ) with the properties given in Theorem 3.1. Guided by the corresponding definitions in the previous section, let g, g', h, h', x, x', y and y' be the p-dimensional vectors with entries

$$g(k) = \begin{cases} -\rho(p-1), & \text{if } k = p, \\ 0, & \text{if } 1 \le k < p, \end{cases}$$

$$g'(k) = \sum_{i>k} (i-k)\phi(k,i) - \frac{\rho(k-1)\alpha(k)}{\delta(k)} 1_{\{1 \le k < p\}},$$

$$h(k) = \begin{cases} -\lambda(1), & \text{if } k = p, \\ 0, & \text{if } 1 \le k < p, \end{cases}$$

$$h'(k) = \sum_{i<k} (k-i)\phi(i,k) - \frac{\lambda(k+1)\alpha(k)}{\delta(k)} 1_{\{1 \le k < p\}},$$

$$x(k) = \begin{cases} -\rho(p-1), & \text{if } k = 1, \\ 0, & \text{if } 1 < k < p, \\ \lambda(1)\rho(p-1), & \text{if } k = p, \end{cases}$$

$$x'(k) = \begin{cases} p - \sum_{i=1}^{p-1} \dfrac{\rho(i-1)\alpha(i)}{\delta(i)}, & \text{if } k = 1, \\ u(k) - u(k+1) - v(k-1) + v(k), & \text{if } 1 < k < p, \\ u(p) - u(1) - v(p-1), & \text{if } k = p, \end{cases}$$

$$y(k) = \begin{cases} 0, & \text{if } 1 \le k < p-1, \\ -\lambda(1)\rho(p-1), & \text{if } k = p-1, \\ -\lambda(1), & \text{if } k = p, \end{cases}$$

and

$$y'(k) = \begin{cases} u(k+1) - u(k) - v(k+1) + v(k), & \text{if } 1 \le k < p-1, \\ u(p) - u(p-1) + v(p-1), & \text{if } k = p-1, \\ p - \sum_{i=1}^{p-1} \dfrac{\lambda(i+1)\alpha(i)}{\delta(i)}, & \text{if } k = p. \end{cases}$$

Also, let A, A', B and B' be the p-dimensional matrices with entries

$$a_{i,j} = \begin{cases} \dfrac{\delta(j)}{\alpha(j)} F(i,j) + \lambda(i) 1_{\{j=i-1\}} - [\lambda(i+1) + \delta(i)] 1_{\{j=i\}}, & \text{if } 1 < i \le p, \\ 1, & \text{if } i = 1, \end{cases}$$

$$a'_{i,j} = F(i,p) 1_{\{1 < i \le p,\, j=p\}} - 1_{\{i=j=p\}},$$

$$b_{i,j} = \begin{cases} \dfrac{\delta(i)}{\alpha(i)} F(i,j) + \rho(j) 1_{\{j=i-1\}} - [\rho(i-1) + \delta(i)] 1_{\{j=i\}}, & \text{if } 1 \le j < p, \\ 1, & \text{if } j = p, \end{cases}$$

and

$$b'_{i,j} = F(0,j) 1_{\{1 \le j < p,\, i=p\}}.$$

Here $F(i, j)$ is defined as in the previous section. The analogs of (2.18) and (2.19) are given next.

PROPOSITION 3.5. *The vectors and matrices defined above satisfy the following identities*:

$$Ag = x \quad \text{and} \quad hB = y, \tag{3.6}$$

$$Ag' + A'g = x' \quad \text{and} \quad h'B + hB' = y', \tag{3.7}$$

$$\sum_{i=1}^{p-1} \frac{\delta(i)}{\alpha(i)} g'(i) h'(i) + \sum_{i=1}^{p} u(i) - \sum_{i=1}^{p-1} v(i) = \lambda(1) g'(p) + \rho(p-1) h'(p). \tag{3.8}$$

PROOF. We prove the first parts of (3.6) and (3.7) only, since the second parts are similar. For the first part of (3.6), write

$$\begin{aligned}(Ag)(i) &= \sum_{j=1}^{p} a_{i,j} g(j) = -\rho(p-1) a_{i,p} \\ &= \rho(p-1)\lambda(1) 1_{\{i=p\}} - \rho(p-1) 1_{\{i=1\}} = x(i).\end{aligned}$$

To prove the equality of the first coordinates of the two sides of the first part of (3.7), write

$$(Ag')(1) + (A'g)(1) = \sum_{j=1}^{p} a_{1,j} g'(j) = p - \sum_{j=1}^{p-1} \frac{\rho(j-1)\alpha(j)}{\delta(j)} = x'(1),$$

where the middle equality follows from (2.2). For the other coordinates of the first part of (3.7), take $1 < k \le p$, multiply (2.4) by $l - k$ and sum on $l > k$ as follows:

$$
\begin{aligned}
0 &= \sum_{l \ge k} [(l-k+1)-1][\lambda(k)\phi(k-1,l) + \rho(l)\phi(k,l+1)] \\
&\quad + \lambda(k+1)\alpha(k) + \rho(k)\alpha(k+1) \\
&\quad + \sum_{j>k} \sum_{l>j} [(l-j)+(j-k)]\delta(j)\frac{\phi(k,j)\phi(j,l)}{\alpha(j)} \\
&\quad - \sum_{l>k} (l-k)[\delta(k)+\delta(l)+\lambda(k+1)+\rho(l-1)]\phi(k,l) \\
&= \lambda(k)\left\{g'(k-1) + \frac{\rho(k-2)\alpha(k-1)}{\delta(k-1)}\right\} \\
&\quad - [\lambda(k+1)+\delta(k)]\left\{g'(k) + \frac{\rho(k-1)\alpha(k)}{\delta(k)}1_{\{k \ne p\}}\right\} \\
&\quad + u(k+1) - \lambda(k)\alpha(k-1) - \sum_{l \ge k,\, l \equiv p-1} \rho(l)\phi(k,l+1) \\
&\quad + \sum_{j>k} \delta(j)\frac{\phi(k,j)}{\alpha(j)} g'(j),
\end{aligned}
\tag{3.9}
$$

where a substantial amount of cancellation has occurred in order to obtain the final expression. On the other hand, for $1 < k \le p$,

$$
\begin{aligned}
&(Ag' + A'g - x')(k) \\
&\quad = \sum_{j=1}^{p} a_{k,j} g'(j) + \sum_{j=1}^{p} a'_{k,j} g(j) - x'(k) \\
&\quad = \sum_{j=1}^{p-1} \frac{\delta(j)}{\alpha(j)} F(k,j) g'(j) + \lambda(k) g'(k-1) \\
&\qquad - [\lambda(k+1)+\delta(k)]g'(k) - \rho(p-1)\left[F(k,p) - 1_{\{k=p\}}\right] \\
&\qquad - u(k) + u(k+1) + v(k-1) - v(k)1_{\{k \ne p\}}.
\end{aligned}
\tag{3.10}
$$

Using the definitions of $u(k)$, $v(k)$ and $F(k,j)$, it is easy to check that the right sides of (3.9) and (3.10) agree, and hence the first part of (3.7) is proved for $1 < k \le p$. To prove (3.8), multiply (2.4) by $(l-k)(l-k-1)/2$ and sum on all k and l satisfying $1 \le k \le p$ and $l > k$ to obtain

$$
\begin{aligned}
0 = {} & \sum_{k=1}^{p} \sum_{l>k} \left\{ \frac{(l-k+1)(l-k)}{2} - (l-k+1) + 1 \right\} \\
& \qquad \times [\lambda(k)\phi(k-1,l) + \rho(l)\phi(k,l+1)] \\
& + \sum_{k=1}^{p} \sum_{j>k} \sum_{l>j} \left\{ \frac{(l-j)(l-j-1)}{2} + \frac{(j-k)(j-k-1)}{2} \right. \\
& \qquad \left. + (l-j)(j-k) \right\} \delta(j) \frac{\phi(k,j)\phi(j,l)}{\alpha(j)} \qquad (3.11) \\
& - \sum_{k=1}^{p} \sum_{l>k} \frac{(l-k)(l-k-1)}{2} [\delta(k) + \delta(l) + \lambda(k+1) \\
& \qquad + \rho(l-1)] \phi(k,l) \\
= {} & - \sum_{k=1}^{p} \lambda(k) \left\{ g'(k-1) + \frac{\rho(k-2)\alpha(k-1)}{\delta(k-1)} 1_{\{k \ne 1\}} - \alpha(k-1) \right\} \\
& - \sum_{l=1}^{p} \rho(l) \left\{ h'(l+1) + \frac{\lambda(l+2)\alpha(l+1)}{\delta(l+1)} 1_{\{l \ne p-1\}} - \alpha(l+1) \right\} \\
& + \sum_{j=1}^{p-1} \frac{\delta(j)}{\alpha(j)} \left\{ g'(j) + \frac{\rho(j-1)\alpha(j)}{\delta(j)} \right\} \left\{ h'(j) + \frac{\lambda(j+1)\alpha(j)}{\delta(j)} \right\}.
\end{aligned}
$$

The key fact which leads to the last equality is the cancellation of the terms which involve $(l-k+1)(l-k)$, $(l-j)(l-j-1)$, $(j-k)(j-k-1)$ and $(l-k)(l-k-1)$. Further cancellation of terms on the right side of (3.11) leads to (3.8). □

Now fix $\varepsilon > 0$ and define the diagonal $p \times p$ matrix D by setting $d_{i,i} = \delta(i)/\alpha(i)$ for $1 \le i < p$ and $d_{p,p} = \varepsilon$. D is invertible, so we can define the matrix W by

$$W = (A + \varepsilon A')D^{-1}(B + \varepsilon B').$$

Compute the entries of this matrix using (2.14) and (2.15) as follows:

$$w_{i,j} = \sum_{k=1}^{p-1} a_{i,k} \frac{\alpha(k)}{\delta(k)} b_{k,j} + \varepsilon^{-1}[a_{i,p} + \varepsilon a'_{i,p}][b_{p,j} + \varepsilon b'_{p,j}],$$

so that

$$w_{1,p} = \sum_{k=1}^{p-1} \frac{\alpha(k)}{\delta(k)} + \varepsilon^{-1},$$

$$w_{1,j} = \frac{\rho(j)\alpha(j+1)}{\delta(j+1)} 1_{\{j \ne p-1\}} + \varepsilon^{-1}\rho(j) 1_{\{j=p-1\}} - \frac{\rho(j-1)\alpha(j)}{\delta(j)}, \qquad \text{for } 1 \le j < p,$$

$$w_{i,p} = \frac{\lambda(i)\alpha(i-1)}{\delta(i-1)} - \varepsilon^{-1}\lambda(1) 1_{\{i=p\}} - \frac{\lambda(i+1)\alpha(i)}{\delta(i)} 1_{\{i \ne p\}}, \qquad \text{for } 1 < i \le p,$$

and for $1 < i \le p$ and $1 \le j < p$,

$$\begin{aligned} w_{i,j} = \sum_{k=1}^{p-1} \frac{\alpha(k)}{\delta(k)} &\left\{ \frac{\delta(k)}{\alpha(k)} F(i,k) + \lambda(i) 1_{\{k=i-1\}} - [\lambda(i+1) + \delta(i)] 1_{\{k=i\}} \right\} \\ &\times \left\{ \frac{\delta(k)}{\alpha(k)} F(k,j) + \rho(j) 1_{\{k=j+1\}} - [\rho(j-1) + \delta(j)] 1_{\{k=j\}} \right\} \\ &+ \varepsilon^{-1}\left[-\lambda(1) 1_{\{i=p\}} + \varepsilon\{F(i,p) - 1_{\{i=p\}}\}\right] \\ &\quad \times \left[\rho(p-1) 1_{\{j=p-1\}} + \varepsilon F(0,j)\right] \\ = \varepsilon^{-1}\lambda(1)\rho(p-1) &1_{\{i=p, j=p-1\}} + \varepsilon F(0,j)\left[F(i,p) - 1_{\{i=p\}}\right] \\ &+ \begin{cases} v(i-1), & \text{if } j = i-2, \\ -\left[u(i) + v(i) 1_{\{i \ne p\}} + v(i-1)\right], & \text{if } j = i-1, \\ u(i) + u(i+1) + v(i), & \text{if } j = i, \\ -u(i+1), & \text{if } j \equiv i+1. \end{cases} \end{aligned}$$

We are now in a position to carry out the remainder of the proof of Theorem 1.6.

PROOF OF THEOREM 1.6. Define $v(p) = \varepsilon^{-1}$, and let g^*, h^*, x^*, y^*, A^*, B^*, W^*, Q^*, R^* and S^* be defined as the corresponding "unstarred" vectors and matrices in Section 2, with $\delta(p)$ replaced by ε. Take ε sufficiently small that (2.28) holds, and let σ, τ, c, and e be defined as in the statement and proof of Theorem 2.27. Note that

$$g^* = g' + \varepsilon^{-1} g, \qquad h^* = h' + \varepsilon^{-1} h, \qquad x^* = x' + \varepsilon^{-1} x \qquad y^* = y' + \varepsilon^{-1} y,$$
$$A^* = A + \varepsilon A', \qquad B^* = B + \varepsilon B' \quad \text{and} \quad W^* = W + \varepsilon W',$$

where W' is the matrix with entries

$$w'_{i,j} = F(0,j)\left[F(i,p) - 1_{\{i=p\}}\right] 1_{\{i \ne 1, j \ne p\}}.$$

The conclusions of Proposition 3.5 can then be restated as

$$A^* g^* = x^* + O(\varepsilon), \qquad h^* B^* = y^* + O(\varepsilon) \tag{3.12}$$

and

$$h^* D g^* = \sum_{k=1}^{p} [v(k) - u(k)] + O(\varepsilon), \tag{3.13}$$

where $O(\varepsilon)$ refers to the behavior as $\varepsilon \downarrow 0$. By examining the expressions in (2.21) and (2.22), we see that $\det R^* = O(\varepsilon^{-1})$, $(R^*)^{-1} = O(1)$, and the entries in $(R^*)^{-1}$ in the pth row and column are $O(\varepsilon)$. Since $\det W^* = O(\varepsilon^{-1})$ and $W^* = Q^* R^* S^*$ by Lemma 2.23, it follows that $\det Q^* \det S^* = O(1)$. Combining these observations with (2.29) and (2.30), we see that the choice of σ and τ can be made so that they, as well as Q^*, S^* and their adjoints are $O(1)$ as $\varepsilon \downarrow 0$. Therefore, $(W^*)^{-1} = O(1)$ as well. As observed in Section 2, A^* and B^*

are invertible, so that we can rewrite (3.12) as

$$g^* = (A^*)^{-1}[x^* + O(\varepsilon)] \quad \text{and} \quad h^* = [y^* + O(\varepsilon)](B^*)^{-1}.$$

Using this in (3.13) and recalling that $W = A^*D^{-1}B^*$, it follows that

$$(3.14) \quad [y^* + O(\varepsilon)][W^* - \varepsilon W']^{-1}[x^* + O(\varepsilon)] = \sum_{k=1}^{p} [v(k) - u(k)] + O(\varepsilon).$$

Since $(W^*)^{-1} = O(1)$, we can combine the $O(\varepsilon)$ terms to conclude that (2.20) holds for the starred quantities, provided that we add a $O(\varepsilon)$ term to the right-hand side. It now remains to repeat the proofs of Theorems 2.27 and 2.35, keeping track of the order of the additional error terms which are carried along. Equation (2.33) is modified by the addition of an $O(1)$ term since $\det R^* = O(\varepsilon^{-1})$. Equation (2.34) is modified by adding an $O(\varepsilon)$ term since $\det Q^* \det S^* = O(1)$. The identity in the statement of Theorem 2.27 is modified by adding an $O(1)$ term. Identity (1.10) follows from this as in the proof of Theorem 2.35. Inequalities (1.7) and (1.8) were already stated as part (iv) of Theorem 3.1. Inequality (1.9) follows from Theorems 2.1 and 3.1. □

4. The general system. In this section, we will prove Theorem 1.4. For $m < n$, let $v_{m,n}$ be the invariant measure for the contact process with fixed ones at m and n. For $m \le k \le n$, put

$$\sigma_{m,n}(k) = v_{m,n}\{\eta: \eta(k) = 1\}.$$

By attractiveness, this quantity decreases as $n \uparrow$ and as $m \downarrow$ for fixed k.

Apply identity (1.10) with 0 and p replaced by m and n respectively. Discard all terms in the second expression on the left side of that identity, and discard all but the terms with $i = j$ in the first expression on the left side. Use (1.9) to eliminate the α from the remaining expression on the left side. Next, eliminate the α's from both expressions on the right of the identity, using (1.7) and (1.8) in the form

$$\frac{\rho(k-1)}{\rho(k-1) + \delta(k-1)} \le \frac{\alpha(k-1)}{\alpha(k)} \le \frac{\lambda(k) + \delta(k)}{\lambda(k)}.$$

The result is

$$(4.1) \qquad \sum_{i=m+1}^{n-1} \frac{\delta(i)}{\lambda(i+1)\rho(i-1)} \frac{1}{\sigma_{m,n}(i)} \le \sum_{i=m}^{n} W_i,$$

where

$$W_i = \sum_{j=i}^{\infty} \frac{1}{\lambda(j+1)} \prod_{k=i+1}^{j} \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\rho(k-1)\lambda(k)} + \sum_{j=-\infty}^{i} \frac{1}{\rho(j-1)} \prod_{k=j+1}^{i} \frac{\delta(k-1)[\lambda(k) + \rho(k-1) + \delta(k-1)]}{\rho(k-1)\lambda(k)}.$$

PROOF OF THEOREM 1.4(i). Using the monotonicity of $\sigma_{m,n}(i)$ in m and n, we can take expected values in (4.1), divide by $n - m$ and pass to the limit as $n \uparrow \infty$ and as $m \downarrow -\infty$ (since the sequences of random variables which occur are stationary), to obtain

$$E \frac{\delta(i)}{\lambda(i+1)\rho(i-1)} \frac{1}{v\{\eta: \eta(i) = 1\}} \le EW_i,$$

where v is the upper invariant measure of the contact process. The assumption of part (i) of Theorem 1.4 is just the statement that $EW_i < \infty$. Therefore,

$$v\{\eta: \eta(i) = 1\} > 0$$

for almost every environment, and hence the process survives. □

PROOF OF THEOREM 1.4(ii). Let A_t be the dual contact process, which has rates $\delta'(k) = \delta(k)$, $\rho'(k) = \lambda(k+1)$ and $\lambda'(k) = \rho(k-1)$. Then the duality relation [see Theorem 1.10 of Chapter 6 of Liggett (1985), for example] gives

$$\sigma_{m,n}(k) = P^{\{k\}}[m \in A_t \text{ or } n \in A_t \text{ for some } t]$$

for $m \le k \le n$. By attractiveness, we have

$$\sigma_{m,n}(k) \le P^{\{k, k+1, \dots\}}[m \in A_t \text{ for some } t] + P^{\{\dots, k-1, k\}}[n \in A_t \text{ for some } t]. \tag{4.2}$$

Let

$$U_k = P^{\{k, k+1, \dots\}}[k - 1 \in A_t \text{ for some } t]$$

and

$$V_k = P^{\{\dots, k-1, k\}}[k + 1 \in A_t \text{ for some } t].$$

Using attractiveness again and (4.2), it follows that

$$\sigma_{m,n}(k) \le U_{m+1}U_{m+2} \cdots U_k + V_k V_{k+1} \cdots V_{n-1}.$$

Using this in (4.1) gives

$$\sum_{i=m+1}^{n-1} \frac{\delta(i)}{\lambda(i+1)\rho(i-1)} \frac{1}{\prod_{k=m+1}^{i} U_k + \prod_{k=i}^{n-1} V_k} \le \sum_{i=m}^{n} W_i. \tag{4.3}$$

By the assumption of part (ii) of Theorem 1.4,

$$\gamma \equiv \exp\left\{E \log \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\lambda(k)\rho(k-1)}\right\} < 1.$$

By the ergodic theorem, if $\gamma_1 < \gamma < \gamma_2 < 1$, there exists a random integer N so that for $j \ge N$,

$$\gamma_1^j \le \prod_{k=0}^{j} \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\rho(k-1)\lambda(k)} \le \gamma_2^j$$

and

$$\gamma_1^j \le \prod_{k=-j}^{0} \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\rho(k-1)\lambda(k)} \le \gamma_2^j.$$

Since

$$E \log \lambda(k) > -\infty,$$

if $\sigma < 1$, this N can be taken so large that $\lambda(j) \ge \sigma^j$ and $\lambda(-j) \ge \sigma^j$ for $j \ge N$ as well. Using these estimates, one sees that

$$\sup_i \left(\frac{\gamma_1 \sigma}{\gamma_2}\right)^{|i|} \sum_{j=i}^{\infty} \frac{1}{\lambda(j+1)} \prod_{k=i+1}^{j} \frac{\delta(k)[\lambda(k) + \rho(k-1) + \delta(k)]}{\rho(k-1)\lambda(k)} < \infty.$$

The expression in parentheses can be made arbitrarily close to 1. Applying the same argument with $\delta(k)$ replaced by $\delta(k-1)$ and $\lambda(k)$ replaced by $\rho(k)$ and then using the definition of W_i yields the conclusion that the right side of (4.3) grows more slowly than exponentially in $n - m$. Applying the ergodic theorem to the nonpositive stationary ergodic sequences $\{\log U_k\}$ and $\{\log V_k\}$, we see from (4.3) that either $U_k \equiv 1$ or $V_k \equiv 1$. A similar argument shows that for each positive integer m, either

$$P^{\{k, k+1, \dots\}}[k - m \in A_t \text{ for some } t] \equiv 1$$

or

$$P^{\{\dots, k-1, k\}}[k + m \in A_t \text{ for some } t] \equiv 1.$$

This gives the needed conclusion. □

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