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BROWNIAN EXIT DISTRIBUTIONS FROM NORMAL BALLS IN $S^3 \times H^3$

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Let X_t be Brownian motion on a Riemannian manifold M started at m and let T be the first time X_t exits a normal ball about m. The first exit time T for $M = S^3 \times H^3$ has the same distribution as the first exit time for $M = \mathbf{R}^6$. For $M = S^3 \times H^3$, T and X_T are independent random variables.

1. Introduction. Let M be a Riemannian manifold. Let $B_m(\varepsilon)$ denote the image under the exponential map of the ball of radius ε about the origin in the tangent space $T_m M$. We say that $B_m(\varepsilon)$ is a normal ball if ε is small enough so that the exponential map is a diffeomorphism of the ball of radius ε. Let X be Brownian motion on M started at m. We examine the joint distribution of the first exit time from $B_m(\varepsilon)$,

$$T_\varepsilon = \inf\{t > 0 : d(m, X_t) = \varepsilon\},$$

and the first exit place, $X(T_\varepsilon)$. Here $d(m, p)$ denotes the geodesic distance between m and p. For simplicity, we will often write T for T_ε. The joint distribution of T and X_T has been studied through asymptotic expansions of $E[T^k f(X_T)]$ as $\varepsilon \to 0$.

For $M = \mathbf{R}^n$, the mean exit time $E[T_\varepsilon] = \varepsilon^2/n$. In Gray and Pinsky (1983) it is proved that if for all $m \in M$, $E_m[T_\varepsilon] = \varepsilon^2/n$ and dim $M = n < 6$, then M is flat. They also provided a class of nonflat manifolds, including the product of $S^3(k^2)$ (constant curvature k^2) and $H^3(-k^2)$ (constant curvature $-k^2$), for which the mean exit time agrees with that of $\mathbf{R}^6$ up to $O(\varepsilon^{10})$. This result was extended to $O(\varepsilon^{12})$ for $S^3 \times H^3$ in Hughes (1988). We will prove that this agreement is exact. For convenience, we consider only the case $k = 1$:

PROPOSITION 1. *For $\varepsilon < \pi$, the first exit times T_ε for $S^3 \times H^3$ and $\mathbf{R}^6$ have the same distribution.*

For $M = \mathbf{R}^n$, T and X_T are independent random variables. It is shown in Liao (1988a) and Kozaki and Ogura (1988) that if for all $m \in M$ and small ε, T and X_T are independent, then M has constant scalar curvature. In Hughes (1988) and Kozaki and Ogura (1988), additional curvature conditions are derived. It is shown in Liao (1988b) and Kozaki and Ogura (1988) that T and X_T are independent for normal balls in any harmonic space. We will show that

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$S^3 \times H^3$ is an example of a manifold which has this independence property but is not an Einstein manifold and thus is not harmonic:

PROPOSITION 2. *For $M = S^3 \times H^3$ and $\varepsilon < \pi$, T and X_T are independent random variables.*

2. Brownian motion on $S^3 \times H^3$. Let X be Brownian motion on $S^3 \times H^3$. Fix $m = (m^1, m^2) \in S^3 \times H^3$. The Laplace–Beltrami operator for $S^3 \times H^3$ can be expressed in terms of geodesic polar coordinates for each of S^3 and H^3:

$$\Delta = \frac{\partial^2}{(\partial r^1)^2} + 2\cot r^1 \frac{\partial}{\partial r^1} + \sin^{-2} r^1 \Delta_{\theta^1} + \frac{\partial^2}{(\partial r^2)^2} + 2\coth r^2 \frac{\partial}{\partial r^2} + \sinh^{-2} r^2 \Delta_{\theta^2},$$

where r^1 is the geodesic distance from m^1 in S^3, r^2 is the geodesic distance from m^2 in H^3 and Δ_{θ^1} and Δ_{θ^2} are two-dimensional spherical Laplacians, expressed only in terms of angular coordinates for S^3 and H^3, respectively.

Define a pair of processes by $(R_t^1, R_t^2) = (r^1(X_t), r^2(X_t))$. Then R^1 is a radial process on S^3 centered at m^1, R^2 is a radial process on H^3 centered at m^2. We also have the angular processes Θ^1 and Θ^2 which are independent Brownian motions on S^2 run with clocks $\int_0^t \sin^{-2} R^1(s)\, ds$ and $\int_0^t \sinh^{-2} R^2(s)\, ds$, respectively. The pair (R^1, R^2) is a diffusion on $\mathbf{R}_+^2 = \{(x^1, x^2)\colon x^1, x^2 \geq 0\}$ with diffusion measures $\{P_y\}$ generated by

$$A = \tfrac{1}{2}\left[(\partial_1)^2 + (\partial_2)^2\right] + \cot x^1\, \partial_1 + \coth x^2\, \partial_2$$

with domain

$$\mathscr{D}(A) = \left\{ f \in C_b^2(\mathbf{R}_+^2)\colon \partial_1 f|_{x^1=0} = 0 = \partial_2 f|_{x^2=0} \right\}.$$

Let T be the first time the Brownian motion X_t on $S^3 \times H^3$ exits $B_m(\varepsilon)$. Then T is also the first time (R_t^1, R_t^2) exits $B_0(\varepsilon)$.

3. Transformation of drift. Let $\hat{A}$ be the infinitesimal generator of a pair of Bessel processes of index 3 ($\mathscr{D}(\hat{A}) = \mathscr{D}(A)$). Let $\{\hat{P}_y\}$ be the associated diffusion measures. Then the operators A and $\hat{A}$ differ by a drift vector field:

$$A - \hat{A} = \left(\cot x^1 - \frac{1}{x^1}\right)\partial_1 + \left(\coth x^2 - \frac{1}{x^2}\right)\partial_2.$$

We expect that the method of transformation of drift can be used to express P in terms of $\hat{P}$.

We examine a more general situation. Let $\hat{R} = (\hat{R}^1, \hat{R}^2, \dots, \hat{R}^k)$ be constructed from k independent Bessel processes with indices $n_1, n_2, \dots, n_k$, respectively. Let $\mathbf{R}_+^k = \{x \in \mathbf{R}^k\colon x^1, x^2, \dots, x^k \geq 0\}$. Then $\hat{R}$ is a diffusion on

$\mathbf{R}_+^k$ with diffusion measures $\{\hat{P}_y\}$ generated by

$$\hat{A} = \frac{1}{2}\left[\Delta + \sum_{i=1}^{k} \frac{n_i - 1}{x^i}\partial_i\right].$$

Let $h \in \mathscr{D}(\hat{A})$ and let $\{P_y\}$ be the diffusion measures generated by

$$A = \hat{A} + \sum_{i=1}^{k} (\partial_i h)\partial_i.$$

Then we have:

PROPOSITION 3. *Suppose $\hat{A}h + \frac{1}{2}\|\nabla h\|^2 = c$, where c is a constant. Let $\hat{A}$, A, $\hat{P}_y$ and P_y be defined as above. Let $M_t = \exp\{h(\hat{R}_t) - h(y) - ct\}$. Then M_t is an exponential martingale and P_y has density M with respect to $\hat{P}_y$ (i.e., $P_y = M \cdot \hat{P}_y$).*

PROOF. Define $g(t, x) = \exp\{h(x) - h(y) - ct\}$. It is easy to verify that $(\hat{A} + \partial/\partial t)g = 0$. It follows that $M_t = g(t, \hat{R}_t)$ is a martingale.

Let $f \in \mathscr{D}(A) = \mathscr{D}(\hat{A})$. Then it is easy to check that

$$\left(\hat{A} + \frac{\partial}{\partial t}\right)(gf)(t,x) = g(t,x)(Af)(x).$$

Let $I_t = \int_0^t (Af)(\hat{R}_s)\,ds$. Then by Itô's formula,

$$\begin{aligned} M_t f(\hat{R}_t) - M_t I_t &= M_t f(\hat{R}_t) - \int_0^t M_s (Af)(\hat{R}_s)\,ds - \int_0^t I_s\,dM_s \\ &= (gf)(t, \hat{R}_t) - \int_0^t \left(\hat{A} + \frac{\partial}{\partial t}\right)(gf)(s, \hat{R}_s)\,ds - \int_0^t I_s\,dM_s \end{aligned}$$

is a martingale (with respect to $\hat{P}_y$). Therefore $M \cdot \hat{P}_y$ is the diffusion measure generated by A. □

Let $\hat{T}$ be the first exit time of $\hat{R}$ from $B_0(\varepsilon)$. Note that each $\hat{R}^i$ can be constructed from n_i independent one-dimensional Brownian motions by $\hat{R}^i = \|(B^1, B^2, \dots, B^{n_i})\|$. In this way, $\hat{R}$ can be constructed from Brownian motion on $\mathbf{R}^n$, where $n = \sum_{i=1}^k n_i$. The first exit times from $B_0(\varepsilon)$ are identical for the n-dimensional Brownian motion and for the process $\hat{R}$ derived from it. It also follows that when $\hat{R}$ is started at the origin, $\hat{T}$ and $\hat{R}_{\hat{T}}$ are independent since the first exit times and places are independent for the n-dimensional Brownian motion started at the origin.

Let R be the diffusion on $\mathbf{R}_+^k$ started at the origin and generated by A given as in Proposition 3 above and let T be the first exit time of R from $B_0(\varepsilon)$. We have the following:

COROLLARY 1. *If $c = 0$, T and $\hat{T}$ have the same distributions.*

PROOF. Letting E_0 and $\hat{E}_0$ stand for expectation with respect to P_0 and $\hat{P}_0$, respectively, we have

$$E_0[e^{-\alpha T}] = \hat{E}_0\left[e^{-\alpha \hat{T}} M_{\hat{T}}\right] = \hat{E}_0[e^{-\alpha \hat{T}}]\hat{E}_0[M_{\hat{T}}] = \hat{E}_0[e^{-\alpha \hat{T}}],$$

since $M_{\hat{T}} = \exp\{h(\hat{R}_{\hat{T}}) - h(0)\}$, $\hat{T}$ and $\hat{R}_{\hat{T}}$ are independent and $\hat{E}_0[M_{\hat{T}}] = 1$. □

COROLLARY 2. *The exit time T and the exit place R_T are independent random variables.*

PROOF. We proceed in a way similar to the proof of Corollary 1. For bounded functions ϕ and ψ,

$$\begin{aligned} E_0[\phi(T)\psi(R_T)] &= \hat{E}_0\left[\phi(\hat{T})\psi(\hat{R}_{\hat{T}})M_{\hat{T}}\right] \\ &= \hat{E}_0\left[\phi(\hat{T})e^{-c\hat{T}}\right]\hat{E}_0\left[\psi(\hat{R}_{\hat{T}})\exp\left\{h(\hat{R}_{\hat{T}}) - h(0)\right\}\right] \\ &= \hat{E}_0\left[\phi(\hat{T})e^{-c\hat{T}}\right]\hat{E}_0[M_{\hat{T}}]\hat{E}_0\left[\psi(\hat{R}_{\hat{T}})\exp\left\{h(\hat{R}_{\hat{T}}) - h(0)\right\}\right] \\ &= \hat{E}_0\left[\phi(\hat{T})M_{\hat{T}}\right]\hat{E}_0\left[\psi(\hat{R}_{\hat{T}})M_{\hat{T}}\right] \\ &= E_0[\phi(T)]E_0[\psi(R_T)]. \end{aligned}$$

Therefore T and R_T are independent. □

4. Proofs of Propositions 1 and 2. We return to the specific case where X is Brownian motion on $S^3 \times H^3$, started at the center of the geodesic ball $B_m(\varepsilon)$, (R^1, R^2) and (Θ^1, Θ^2) are given as before and A is the generator of R. Let $\hat{R} = (\hat{R}^1, \hat{R}^2)$ be a pair of Bessel processes of index 3 with generator $\hat{A}$, started at the center of $B_0(\varepsilon)$. Then for

$$h(x^1, x^2) = \log\left(\frac{\sin x^1}{x^1}\right) + \log\left(\frac{\sinh x^2}{x^2}\right),$$

we have $A = \hat{A} + \sum_{i=1}^{2}(\partial_i h)\partial_i$ and $\hat{A}h + \frac{1}{2}\|\nabla h\|^2 = 0$ inside $B_0(\pi)$. Proposition 3 now applies if the processes are stopped before exiting $B_0(\pi)$. Proposition 1 follows from Corollary 1.

By symmetry, Θ_T^1 and Θ_T^2 are each uniformly distributed on S^2. Furthermore, T, R_T^1 and R_T^2 are invariant under rotations in the angular coordinates for S^3 or H^3. Therefore, (T, R_T^1, R_T^2) is independent of (Θ_T^1, Θ_T^2). By Corollary 2, T and R_T are independent. Therefore it follows that T and X_T are independent.

5. Remarks. The exit time property also holds more generally for products of S^3, H^3 and $\mathbf{R}^n$ such that the product has scalar curvature zero. For example, the exit times for $S^3(k^2) \times S^3(-k^2)$ and $S^3(1) \times S^3(1) \times H^3(-2)$ have the same distribution as the exit times for $\mathbf{R}^6$ and $\mathbf{R}^9$, respectively. Also $S^3 \times H^3 \times \mathbf{R}^n$ has the same exit time distributions as $\mathbf{R}^{6+n}$. Therefore examples of manifolds with this property can be found for any dimension greater than or equal to 6. The independence property holds for any product of S^3, H^3 and $\mathbf{R}^n$ with arbitrary (constant) curvatures.

Corollary 2 is similar to a property of Brownian motion with drift on $\mathbf{R}^n$: The first exit time and place from $B_x(\varepsilon)$ for Brownian motion with drift ∇h, started at x, are independent for every $x \in \mathbf{R}^n$ and $\varepsilon > 0$ if and only if $\Delta h + \|\nabla h\|^2 =$ constant; see Hughes and Liao (1989).

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