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THE SHARP MARKOV PROPERTY OF LÉVY SHEETS

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This paper examines the question of when a two-parameter process X of independent increments will have Lévy's sharp Markov property relative to a given domain D. This property states intuitively that the values of the process inside D and outside D are conditionally independent given the values of the process on the boundary of D. Under mild assumptions, X is the sum of a continuous Gaussian process and an independent jump process. We show that if X satisfies Lévy's sharp Markov property, so do both the Gaussian and the jump process. The Gaussian case has been studied in a previous paper by the same authors. Here, we examine the case where X is a jump process. The presence of discontinuities requires a new formulation of the sharp Markov property. The main result is that a jump process satisfies the sharp Markov property for all bounded open sets. This proves a generalization of a conjecture of Carnal and Walsh concerning the Poisson sheet.

1. Introduction. Lévy's sharp Markov property is one of the most appealing of the many analogues of the Markov property proposed for random fields: A continuous process $X = \{X(t),\ t \in \mathbb{R}^d\}$ satisfies Lévy's sharp Markov property relative to a set D in $\mathbb{R}^d$ if $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$ are conditionally independent given the boundary field $\mathscr{H}^0(\partial D)$, where $\mathscr{H}^0(D) = \sigma\{X(t),\ t \in D\}$. In spite of its attractiveness, it is satisfied less often than one might expect. The Brownian sheet, for instance—intuitively a Markov process—fails to satisfy it for most nice regions D (see [10, 11, 24]), and Constantinescu and Thalheimer [9] have shown that if $d \geq 2$ and if X is ergodic and invariant under Euclidean transformations, then X cannot satisfy the sharp Markov property for all open sets. Euclidean-invariant processes which do satisfy it, such as the free field [20], have values which are generalized functions.

The boundary field $\mathscr{H}^0(\partial D)$ is too small in these cases to be a splitting field for $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$, where $\overline{D}^c$ denotes the complement of the closure of D. (A field $\mathscr{S}$ is a *splitting field* for two fields $\mathscr{A}$ and $\mathscr{B}$ if $\mathscr{A}$ and $\mathscr{B}$ are conditionally independent given $\mathscr{S}$.) The *germ field* $\mathscr{G}(\partial D) = \bigcap \mathscr{H}^0(O)$, where the intersection is over all open sets $O \supset \partial D$, is larger, and one says that X satisfies *Lévy's Markov property* (without the "sharp")—or the *germ-field Markov property*—relative to D if $\mathscr{G}(\partial D)$ is a splitting field for $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$. This is the most-studied version. However, in some applications,

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such as in Nelson's construction of quantum fields from Markov random fields [20], it is vital to have the sharp, rather than the germ field, Markov property.

Let us make one observation before we proceed: For a general open set D, the boundaries of D and $\overline{D}^c$ need not be equal, so that the Markov property we described above is not symmetric in the two open sets. In fact, what one usually wants to look at is the common boundary of the two, but even then, we can often confine ourselves to an even smaller set. Thus we will use the phrase "Lévy's sharp Markov property" rather informally to refer to the conditional independence relative to a field generated by the restriction of X to some subset of the boundary. When we state theorems, we will be careful to specify the splitting fields explicitly.

This paper is a study of Lévy's sharp Markov property for two-parameter processes of independent increments, often called *Lévy sheets* when the process is continuous in probability. This class of processes appears naturally in the solution of Cabaña's vibrating string problem [4]. Consider a guitar left outdoors in a desert during a sandstorm. Many small grains of sand will be blown against the strings of the guitar, and they will begin to vibrate. We might ask "What tune will the guitar play?" That depends on the way the sand blows. Two reasonable assumptions about this are:

1. The number of grains of sand that hit the string in disjoint intervals of space or time are independent.
2. Each individual grain of sand only makes a very small contribution to the motion of the string.

The cumulative effect of the impacts will be Gaussian, and the motion of the string will be continuous. After a recentering and limiting argument, we can see that the displacement $u(t, x)$ of point x at time $t \geq 0$ is a solution of the stochastic wave equation

$$\frac{\partial^2}{\partial t^2} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = W(dt, dx),$$

where $W(dt, dx)$ is a space–time white noise representing the impact of the sand grains. If we assume for convenience that the string is infinite, we can solve the equation explicitly (see [23], Chapter 3, pages 392–394; several properties of the stochastic wave equation are given in [6, 7]). For $t \geq |x|$, the solution can be written in terms of a Brownian sheet:

$$u(t, x) = B^1(t - x) + B^2(t + x) + W(t - x, t + x),$$

where B^1, B^2 and W are independent, B^1 and B^2 are (one-parameter) deterministic time changes of Brownian motions and $\{W(t_1, t_2), (t_1, t_2) \in \mathbb{R}_+^2\}$ is a Brownian sheet. (We could write the solution in terms of a Brownian sheet which was not 0 on the axes and avoid B^1 and B^2; they simply allow us to reduce to a standard Brownian sheet, which vanishes on the axes.)

A variation on this problem leads to the class of processes of independent increments. Assume that in addition to the sand, small rocks arrive from time to time. The rocks cause a significant displacement of the string. This shows up in the mathematical model in the form of discontinuities of the solution. Under minor hypotheses, if the arrival of the rocks satisfies the same independence relations as the arrival of the sand grains, the motion of the string is a solution of

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = Y(dt,dx), \tag{1}$$

where Y is an infinitely divisible random measure. The solution for $t \geq |x|$ can be expressed as

$$u(t,x) = X^1(t-x) + X^2(t+x) + X(t-x, t+x),$$

where X^1, X^2 and X are processes of independent increments. As above, X^1 and X^2 are well-understood processes, and the most interesting part of the solution comes from the sheet $\{X(t_1,t_2), (t_1,t_2) \in \mathbb{R}_+^2\}$.

To see where Lévy's sharp Markov property enters, suppose one observes the solution of the equation in some interval, an interval which might change with time. This means that one observes the solution in some domain D in space–time. One could then try to estimate the behavior of the string outside the interval of observation.

It is natural to ask whether we really need to observe the solution in the entire interval, or if we could not get *just as good an estimate* by observing the process, say, at the end points of the interval, or by observing the process on the boundary of the space–time domain D. This is exactly the case if Lévy's sharp Markov property holds.

The principal question addressed in this paper is, "For a process of independent increments, when is the sharp field of ∂D a splitting field for $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$?" [Some care is necessary in the formalization of this statement. For continuous processes, $\mathscr{H}^0(\partial D)$ is the right notion for the boundary field but this is not the case in general, as we will see in Example 2.4. In Section 3.1 we will replace it by a related field which we call the *uniform sharp field* $\mathscr{H}(\partial D)$. In this context, "sharp field" refers to $\mathscr{H}(\partial D)$ rather than to $\mathscr{H}^0(\partial D)$.]

Under mild assumptions, a process of independent increments is the sum of a continuous Gaussian process and an independent jump process. We will see (Theorem 3.4) that these two parts can be examined separately. The canonical examples of Gaussian and jump processes of independent increments are the Brownian sheet and the Poisson sheet respectively.

It was thought for some time that the *only* sets for which the Brownian sheet satisfied the sharp Markov property were finite unions of rectangles. This is true for regions with piecewise-smooth boundaries [24], but Dalang and Russo [10] presented a different class of domains with respect to which the Brownian sheet did satisfy the sharp Markov property, and it was shown in [11] that in fact, *most* domains have this property. The situation is different for sheets with no Gaussian part. It was shown by Carnal and Walsh [8] that the Poisson sheet satisfies the sharp Markov property relative to bounded relatively convex domains, and it was conjectured that this should be true for all bounded domains. That conjecture *is* true, and in fact holds in much greater generality. We will show in this paper that sheets of independent increments with no Gaussian part satisfy Lévy's sharp Markov property relative to *all* bounded domains.

The key observation in [8] is that the discontinuities of the Poisson sheet propagate along lines, and that one can get a lot of information by observing them as they cross the boundary. The techniques, however, were restricted to domains with a simple boundary. We use the same ideas here. We are able to extend the results of [8] to all bounded open sets because we can use the characterization of the minimal splitting field given in [11], Theorem 3.3.

The theorem comes in two forms. If we consider only processes with positive jumps, such as the Poisson sheet, we can show the following theorem (see Section 3.1).

THEOREM 1.1. *Let $\{X(t),\ t \in \mathbb{R}_+^2\}$ be a Lévy sheet which has no Gaussian part and which satisfies Assumption B. If all jumps of X are positive, then for every open set D which is bounded or has bounded complement, $\mathscr{H}^0(\partial D)$ is a splitting field for $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$.*

Assumption B is given in Section 2.4 and says roughly that X is continuous in probability and the distribution of jump points of X is absolutely continuous. We give counterexamples in Section 2.1 to show that the sharp Markov property may not hold without these restrictions.

Merzbach and Nualart [19] have proved a special case of Theorem 1.1, in which the region D satisfies certain regularity conditions and X has a locally finite number of jumps; they do this by showing that the germ and sharp fields are equal under their assumptions. This approach cannot lead to Theorem 1.1 since in general, the germ field is strictly larger than the sharp field (see Example 2.3).

Somewhat unexpectedly, Theorem 1.1 fails for Lévy sheets with both positive and negative jumps (see Example 2.4), though it is clearly valid if X has only negative jumps. The reason for this is that the boundary field $\mathscr{H}^0(\partial D)$ turns out to be smaller than expected. This leads us to define the *uniform sharp field* $\mathscr{H}(A)$ (see Section 3.1), which is determined by the values of the process on the set A, but is in general strictly larger than $\mathscr{H}^0(A)$. Nevertheless, the equality $\mathscr{H}(A) = \mathscr{H}^0(A)$ holds if X is continuous, or if X is right continuous with only positive jumps, or if A is open (see Proposition 3.3). From now on, when we say *X satisfies Lévy's sharp Markov property relative to a set D*, we will mean that $\mathscr{H}(D)$ and $\mathscr{H}(\overline{D}^c)$ are conditionally independent given $\mathscr{H}(\partial D)$. Then we have the following result, which is contained in Theorem 3.9.

THEOREM 1.2. *Let $\{X(t),\ t \in \mathbb{R}_+^2\}$ be a right continuous Lévy sheet which has no Gaussian part and which satisfies Assumption B. Then X satisfies Lévy's sharp Markov property relative to all open sets D which are bounded or have bounded complement, in the sense that $\mathscr{H}(\partial D)$ is a splitting field for $\mathscr{H}(D)$ and $\mathscr{H}(\overline{D}^c)$.*

Since $\mathscr{H}(\partial D) = \mathscr{H}^0(\partial D)$ for right continuous processes with positive jumps, Theorems 1.1 and 1.2 say exactly the same thing for the Poisson sheet. Note that Theorem 1.1 holds for any version of the process—indeed, for any set A, the (completed) fields $\mathscr{H}^0(A)$ are the same for any version of the process—while Theorem 1.2 requires a right continuous version.

In a way, it is rather surprising that there is so little difference in the theorems for the two types of processes, since processes with only positive jumps behave much more simply than the others. Indeed, if we return to our physical analogy of the string, we see that observing the string at time 1 corresponds to observing the process of independent increments on the line $s_1 + s_2 = 1$, and conversely, observing the process on the boundary of the triangle corresponds to observing the string at time 1, that is, taking a picture of that segment at time 1. Thus the Markov property of the sheet for the triangle $\{s \in \mathbb{R}_+^2 : s_1 + s_2 < 1\}$ corresponds to a Markov property of the string at time 1, that is, the Markov property for the (one-parameter) process $t \mapsto u(t, \cdot)$.

If X is the Poisson sheet, the impacts on the string are all positive, that is, all the rocks hit the string from below, and each generates a square wave of height $+1$ (we think of the string as stretchable enough to bend itself into a square wave). Suppose that at time $\tau = 1$, the string looks like Figure 1.

With just this snapshot at time 1, we can tell the entire history of the string. Indeed, two rocks hit: one at $x = 0.1$ and time $\tau = 0.8$ (because waves travel at speed 1), and the other at $x = -0.5$ and time $\tau = 0.9$. No other rocks hit the string (assuming that outside the interval -0.7 to 0.7 the string is at rest). So clearly this snapshot at time $\tau = 1$ contains as much information as we would have gotten by observing the string from times $\tau = 0$ to $\tau = 1$, and so, for the purpose of estimating future behavior of the string, it is sufficient. Hence the sharp Markov property of the Poisson sheet (for the triangle).

Now consider the case of the signed Poisson sheet, in which the impulses can be either positive or negative. Suppose the snapshot is the same as above (Figure 1). Then we do not have as much information as if we had observed the string from times 0 to 1. Indeed, there are two possible histories:

History 1. Rocks hit from *below* at ($x = 0.1$, $\tau = 0.8$) and at ($x = -0.5$, $\tau = 0.9$).

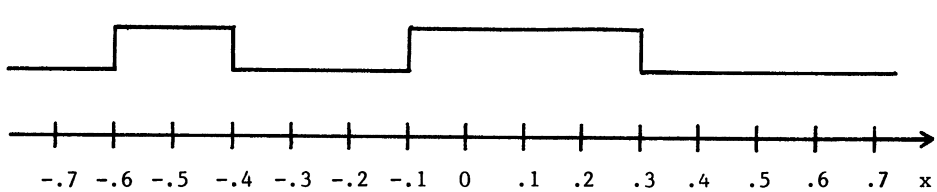

Fig. 1. *A snapshot of the vibrating string at time* 1.

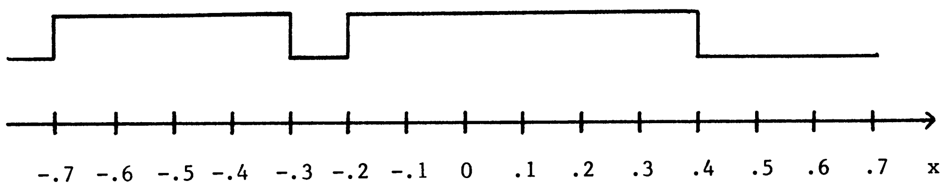

FIG. 2. *The string at time* $\tau = 1.1$, *assuming History* 1.

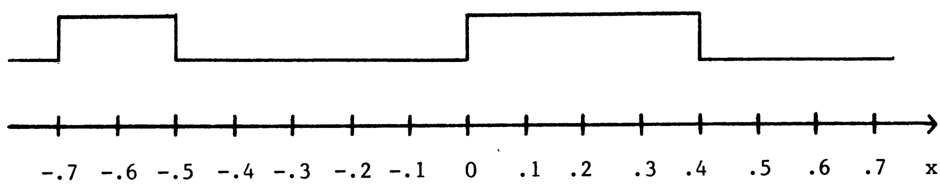

FIG. 3. *The string at time* $\tau = 1.1$, *assuming History* 2.

HISTORY 2. A rock hit from *below* at ($x = -0.15$, $\tau = 0.55$) and another hit from *above* at ($x = -0.25$, $\tau = 0.85$).

In this case the snapshot gives us less information than we would get by observing the string from times 0 to 1. We still have to ask, "Does this make a significant difference in estimating future behavior?" The answer is *yes*. Indeed, with History 1, the string at time $\tau = 1.1$ will look like Figure 2, whereas with History 2, it will look like Figure 3.

So clearly the snapshot is not sufficient to predict the future of the string, and the Markov property will not hold.

This might appear to contradict Theorem 1.2. However, there is one important bit of information present in the sheet which is absent in our string analogy: Whereas the string stretches continuously, the sheet has genuine discontinuities, and we can make use of its values at the points of discontinuity. This, as we shall see, is vital (see Example 2.4).

2. Processes of independent increments.

2.1. *Notation and examples.* Let us denote the first quadrant in the plane by $\mathbb{R}_+^2$; we will refer to the horizontal and vertical axes as the *x-axis* and the *y-axis* respectively. We will use s and t for elements of $\mathbb{R}_+^2$, and use s_i and t_i, $i = 1, 2$, for their coordinates; we will usually reserve u and v for real variables, to use in place of s_i and t_i when the subscripts are too numerous.

The usual partial order in the plane is denoted by "$\leq$": $s \leq t$ if $s_1 \leq t_1$ and $s_2 \leq t_2$. If $s \leq t$ are in $\mathbb{R}_+^2$, we will write $(s, t]$ for the rectangle $(s_1, t_1] \times (s_2, t_2]$. R_t is the rectangle $(0, t]$. (Unless we specifically say otherwise, "rectangle" means "rectangle with sides parallel to the axes.") For any $t \in \mathbb{R}^2$ and any set

$\Lambda \subset \mathbb{R}^2$, put $t + \Lambda = \{t + s: s \in \Lambda\}$. Let Q_i, $i = 1, \dots, 4$, be the four right-half-open quadrants of $\mathbb{R}^2$:

$$Q_1 = \{t: t_1 \geq 0, t_2 \geq 0\}, \qquad Q_2 = \{t: t_1 < 0, t_2 \geq 0\},$$
$$Q_3 = \{t: t_1 < 0, t_2 < 0\}, \qquad Q_4 = \{t: t_1 \geq 0, t_2 < 0\}$$

("right-half-open" refers to the fact that the indicator function of the Q_i are right continuous as defined below).

We say that a function $f(t_1, t_2)$ of two variables has *quadrantal limits* if for each point $t = (t_1, t_2)$ the four limits

$$f(t^{(j)}) \equiv \lim_{s \to t,\, s \in t + Q_j} f(s), \qquad j = 1, \dots, 4,$$

all exist. We say that f is *right continuous* if $f(t) = f(t^{(1)})$ for all t. We define the *jump of f at t* to be

$$\square f(t) = f(t) - f(t^{(2)}) - f(t^{(3)}) + f(t^{(4)}).$$

Let $\{X(t), t \in \mathbb{R}_+^2\}$ be a (two-parameter) process defined on some probability space $(\Omega, \mathscr{F}, P)$, and let $R = (s, t]$ be a rectangle in $\mathbb{R}_+^2$. The *increment* of X over R is $\Delta_R X = X(t_1, t_2) - X(t_1, s_2) - X(s_1, t_2) + X(s_1, s_2)$. X is said to have *independent increments* if for all finite sets of disjoint rectangles $R^1, \dots, R^n$, the random variables $\Delta_{R^1} X, \dots, \Delta_{R^n} X$ are independent. Notice that this says nothing whatsoever about the values of X on the axes. Indeed, if $\hat{X}(u, v) = X(u, v) - X(u, 0) - X(0, v) + X(0, 0)$, then $\hat{X} = 0$ on the axes, and $\Delta_R \hat{X} = \Delta_R X$ for all rectangles $R \subset \mathbb{R}_+^2$. Thus we can—and will—assume in what follows that *all processes of independent increments vanish on the axes*.

For $A \subset \mathbb{R}_+^2$, we define the *sharp field of A* by $\mathscr{H}^0(A) = \sigma\{X(t), t \in A\}$, completed by adjoining all subsets of $\mathscr{F}$ of probability 0.

Before proceeding to detail our hypotheses, let us give four counterexamples involving jump processes which show the need for some restrictions. Indeed, though all processes of independent increments satisfy Lévy's sharp Markov property for finite unions of rectangles (this result is due to Russo [21], Theorem 7.5; see also [11], Corollary 4.2), some restrictions on the process are necessary if the Markov property is to hold for a larger class of sets.

EXAMPLE 2.1. Let $D \subset \mathbb{R}_+^2$ be the triangular domain with vertices at $(0, 0)$, $(4, 4)$ and $(8, 0)$ and let $a = (1, 2)$, $b = (2, 1)$, $c = (3, 1)$ and $d = (3, 2)$ be points of $\mathbb{R}_+^2$. Let ξ_a, ξ_b, ξ_c and ξ_d be i.i.d. random variables taking values 0, 1 and 2, each with positive probability, and define a measure Y on $\mathbb{R}_+^2$ by $Y = \xi_a \delta_a + \xi_b \delta_b + \xi_c \delta_c + \xi_d \delta_d$, where δ_t is the unit point mass at t. Set $X(t) = Y(R_t)$. X is a right-continuous process of independent increments. Notice that the event $\Lambda = \{\xi_b = 1\} = \{Y(\{b\}) = 1\}$ is in both $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$. Indeed, for any point $e = (2, v)$, where $v > 2$, we have $Y(\{b\}) = X(b) - X(b^{(2)}) = X(e) - X(e^{(2)})$. However, Λ is not in $\mathscr{H}^0(\partial D)$. To see this, consider the events $\Lambda_1 = \{\xi_a = \xi_b = \xi_c = 1,\ \xi_d = 0\}$ and $\Lambda_2 = \{\xi_a = \xi_c = 0,\ \xi_b = 2,\ \xi_d = 1\}$. The restriction of X to ∂D is the same on Λ_1 and Λ_2, but $\Lambda_1 \subset \Lambda$ while $\Lambda_2 \cap \Lambda = \varnothing$. Thus Λ cannot be in $\mathscr{H}^0(\partial D)$. All splitting fields contain $\mathscr{H}^0(D) \cap \mathscr{H}^0(\overline{D}^c)$

([18], Section 6), so $\mathscr{H}^0(\partial D)$ is not a splitting field, and X does not satisfy Lévy's sharp Markov property on D.

This example and the ones below use the fact that if X has a jump at a point t, then it has a jump discontinuity which goes horizontally off to ∞ from t, and a second one which goes vertically. The difficulty here arises because X has jumps at fixed points, and the resulting discontinuities can cross on the boundary or even coincide. This cannot happen (with a smooth boundary) if, for instance, X is continuous in probability. We will actually assume slightly more than this (see Assumption B in Section 2.4). In addition, if X is continuous in probability, then $\mathscr{H}^0(D) = \mathscr{H}^0(\overline{D})$, which is a fact we will use below without comment.

The next example is taken from [8] and shows that we cannot expect the sharp Markov property to hold for smooth unbounded regions (though it will hold for most unbounded regions with irregular boundary: see [11], Theorems 4.1 and 5.6]). It involves the Poisson sheet, which we can define as follows. Let points be randomly distributed in $\mathbb{R}_+^2$ such that with probability 1, there are only a finite number of points in any bounded set; for any bounded Borel set $A \subset \mathbb{R}_+^2$, the number of points in A, denoted $\Pi(A)$, is a Poisson random variable with parameter equal to the area of A, and if $A_1, \dots, A_n$ are disjoint, $\Pi(A_1), \dots, \Pi(A_n)$ are independent. Then Π is a *Poisson point process* [16], and the *Poisson sheet* $X(t)$ is the process defined by $X(t) = \Pi(R_t)$. The sample paths of the Poisson sheet are constant except on countably many lines where there are jump discontinuities. In fact, there are countably many points $Z_1, Z_2, \dots$ such that $X(t) = \sum_{i \in \mathbb{N}} I_{\{Z_i \le t\}}$. In particular, $\Box X(t) \neq 0$ implies $\Box X(t) = 1$ and $t = Z_i$ for some i. Each point Z_i creates a jump discontinuity which propagates along the vertical and horizontal half-lines starting at Z_i. This property of the sample paths is typical of jump processes.

The *signed Poisson sheet* $\hat{X}$ is closely related. Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables, independent of the Z_i, with $P\{\xi_j = 1\} = P\{\xi_j = -1\} = 1/2$. Then $\hat{X}$ is defined by

$$\hat{X}(t) = \sum_{i \in \mathbb{N}} \xi_i I_{\{Z_i \le t\}}.$$

EXAMPLE 2.2. Let D be the region below the diagonal of $\mathbb{R}_+^2$ and let X be the Poisson sheet. If $Z_1, Z_2, \dots$ is a sequence which enumerates the points where $\Box X(t) = 1$, note that if $Z_i = t = (t_1, t_2)$ for some i, then for all $v \ge t_2$, $X(t_1, v) - X((t_1, v)^{(2)}) = 1$. This is because each point Z_i gives rise to a jump discontinuity of size 1 along the vertical line starting at t. Let us write R_1 instead of $R_{(1,1)}$. The random variable $N \equiv X(D \cap R_1)$ is clearly in $\mathscr{H}^0(D)$, since it equals $\#\{t \in D: \Box X(t) = 1\}$, and it is also in $\mathscr{H}^0(\overline{D}^c)$; indeed, it is equal to $\#\{t \in \partial D \cap R_1: X(t) - X(t^{(2)}) = 1\}$, which is measurable with respect to $\mathscr{H}^0(\overline{D}^c)$. By symmetry, so is $\tilde{N} \equiv X((\overline{D})^c \cap R_1)$. Let $\tilde{X}$ be the reflection of X about the diagonal: $\tilde{X}(u, v) = X(v, u)$. Note that $\tilde{X}$ and X coincide on ∂D and that $\tilde{N} = \tilde{X}(D \cap R_1)$. Thus if we only look at X on the diagonal, we cannot distinguish the event $\{N = 1,\ \tilde{N} = 0\}$ from $\{N = 0,\ \tilde{N} = 1\}$, so they are clearly not $\mathscr{H}^0(\partial D)$-measurable. Thus the sharp field fails to contain $\mathscr{H}^0(D) \cap \mathscr{H}^0(\overline{D}^c)$, and the sharp Markov property does not hold.

Notice that the germ field $\mathscr{G}(\partial D)$ is strictly larger than $\mathscr{H}^0(\partial D)$: One can distinguish the two events by looking at X in any neighborhood of the boundary. The region D is unbounded, however. Here is an example of a bounded region in which $\mathscr{G}(\partial D)$ is strictly larger than $\mathscr{H}^0(\partial D)$.

Example 2.3. Let A be a Cantor set in $[0,1]$ with positive Lebesgue measure. Let $I_1, I_2, \dots$ be the disjoint open intervals whose union is $(0,1) \setminus A$. Set $D_1 = \bigcup_{n \in \mathbb{N}} I_n \times I_n$. The boundary of D_1 is the union of the boundaries of the $I_n \times I_n$ plus the set $K = \{(x,x): x \in A\}$. There is positive probability that a discontinuity of X will intersect K. Just as in Example 2.2, one cannot tell whether the discontinuity is propagating vertically or horizontally by observing the restriction of X to ∂D, but one can tell it by observing X in any neighborhood of ∂D.

It is easy to modify the boundary in this example (by rounding off the corners) so that it is the union of two C^∞ monotone curves, and still have $\mathscr{G}(D) \neq \mathscr{H}^0(D)$.

The next example is more subtle. It shows that the boundary field $\mathscr{H}^0(\partial D)$ is not quite what we expect. Some events which one might think are in it may not be.

Example 2.4. Let D be the triangle bounded by the axes and the line $y = 1 - x$ and let $\hat{X}$ be the *signed Poisson sheet*. We claim that for the process $\hat{X}$ and the set D, $\mathscr{H}^0(D)$ and $\mathscr{H}^0(\overline{D}^c)$ are not conditionally independent given $\mathscr{H}^0(\partial D)$.

Let F be the event "$u \mapsto \hat{X}(u, 1-u)$ jumps up one, then down one in the interval $(0, \frac{1}{2})$, and then does the same in the interval $(\frac{1}{2}, 1)$." Let $U_1 < U_2 < U_3 < U_4$ be the x-coordinates of the four jumps.

Then F is clearly in $\mathscr{H}^0(\partial D)$, as are the U_j. In fact the trace of $\mathscr{H}^0(\partial D)$ on F is just the completion of $\sigma\{U_j,\ j = 1, \dots, 4\}$. Indeed, let $t = (u, 1-u) \in \partial D$. Then $\hat{X}(t) = 0$ if $u \in (0, U_1) \cup (U_2, U_3) \cup (U_4, 1)$ and $\hat{X}(t) = 1$ if $u \in (U_1, U_2) \cup (U_3, U_4)$. We can take open intervals here because the probability that one of the U_j equals u is 0.

Write $D = D_1 \cup D_2 \cup D_3$, where $D_2 = R_{(1/2,1/2)}$ and D_1 and D_3 are the triangles above and to the right, respectively, of D_2. Note that on the event F, exactly two of the Poisson points Z_i are in D, since each Z_i in D gives rise to a pair of discontinuities which intersect ∂D, and there are evidently four such. (The distribution of the Z_i is absolutely continuous, and it is easy to see that no two of these points fall on the same line, so that discontinuities from different Z_i give rise to jumps at different places on ∂D. See Corollary 2.8.) Moreover, a moment's reflection shows that there are two different ways F can happen. Either there is a point in D_1 at $(U_1, 1 - U_2)$ and a point in D_3 at

$(U_3, 1 - U_4)$, both having mass $+1$, or else there are two points in D_2, a point of mass $+1$ at $(U_1, 1 - U_4)$ and a point of mass -1 at $(U_2, 1 - U_3)$. Thus $F = F_1 \cup F_2$, where F_1 is the first of these possibilities and F_2 is the second. F_1 and F_2 are clearly in $\mathscr{H}^0(D)$, and in $\mathscr{H}^0(\overline{D}^c)$ as well, since $F_1 = F \cap \{\hat{X}((U_2, 1 - U_2)^{(1)}) = 1\}$, and $F_2 = F \cap \{\hat{X}((U_2, 1 - U_2)^{(1)}) = 0\}$.

We claim that if $B \in \mathscr{H}^0(\partial D)$, $P(B) > 0$ and $B \subset F$, then $P\{B \cap F_1\} > 0$. Let us accept this claim for the moment. It implies that $F_2 \notin \mathscr{H}^0(\partial D)$, for if it were, we could take $B = F_2$ and we would have $0 = P\{\varnothing\} = P\{B \cap F_1\} > 0$, a contradiction. Since $F_2 \in \mathscr{H}^0(D) \cap \mathscr{H}^0(\overline{D}^c)$, it follows that $\mathscr{H}^0(\partial D)$ is not a splitting field, and the sharp Markov property does not hold.

To establish our claim, note that given F_1, the vector $U = (U_1, \dots, U_4)$ has a strictly positive conditional joint density f_1—which could easily be computed—on the set $A = \{u \in \mathbb{R}^4 \colon u_1 \le u_2 \le u_3 \le u_4\}$. Similarly, given F_2, U has a strictly positive joint density f_2. Then by Bayes' theorem,

$$P\{F_1 | U\} = \frac{P\{F_1\} f_1(U)}{P\{F_1\} f_1(U) + P\{F_2\} f_2(U)} > 0.$$

If $B \in \mathscr{H}^0(\partial D)$ and $B \subset F$, then B is $\sigma(U)$-measurable, and so

$$P\{B \cap F_1\} = \int_B I_{F_1}\, dP = \int_B P\{F_1 | U\}\, dP > 0,$$

which verifies the claim.

Notice that the problem in Example 2.4 comes from the fact that $\hat{X}(U_2, 1 - U_2)$ is not $\mathscr{H}^0(\partial D)$-measurable [if it were, F_1 and F_2 would also be in $\mathscr{H}^0(\partial D)$]. Now if $V_1, V_2, \dots$ enumerate the jumps of $u \mapsto \hat{X}(u, 1 - u)$, then $\mathscr{H}(\partial D) = \mathscr{H}^0(\partial D) \vee \sigma\{X(V_j, 1 - V_j)\colon j = 1, 2, \dots\}$ is indeed a splitting field. It is *a sharp boundary field*, for it is generated by the restriction of $\hat{X}$ to the boundary. This is really the field we would like to have. The problem is simply that $\mathscr{H}^0(\partial D)$ is too small. The situation is somewhat analogous to the problem of defining fields for the process $\{\chi(u),\ u \ge 0\}$, where χ is the indicator function of the set of jumps of a standard Poisson process. The field $\sigma\{\chi(u),\ u \ge 0\}$ is trivial since for each u, $P\{\chi(u) = 0\} = 1$. The solution is to take the fields generated, not by χ, but by the Poisson process itself. In the same vein, we will define a "*uniform sharp boundary field*" in Section 3.1 to circumvent this problem. In Example 2.4, the uniform sharp field is in fact $\mathscr{H}(\partial D)$.

2.2. *Background*. Let $\{X(t),\ t \in \mathbb{R}_+^2\}$ be a process of independent increments which vanishes on the coordinate axes. We will make the following assumption in order to rule out the pathology of Example 2.1.

ASSUMPTION A. *The map $t \mapsto X(t)$ is continuous in probability.*

A process of independent increments satisfying Assumption A is called a *Lévy sheet* (or *Lévy process* in the terminology of [14, 1]). This imposes a structure on X, which we will detail in this section. Since these results are essential to us but for the most part are not new, we only give an outline of the proofs for convenience of the reader. For more details, the reader is referred to [1].

Since the increments of X are independent, Assumption A is equivalent to a.s. continuity of X at each fixed $t \in \mathbb{R}_+^2$. In particular, $X(t)$ has an infinitely divisible distribution for each $t \in \mathbb{R}_+^2$. The same is true for any increment of X: Let $R \subset \mathbb{R}_+^2$ be a rectangle. Then $\Delta_R X$ is infinitely divisible. Set $\phi_R(u) = E\{e^{iu\Delta_R X}\}$ and $\psi_R(u) = \log \phi_R(u)$. By the Lévy–Khintchine representation ([14], Theorem 3.4.1), there exist constants $\gamma(R)$ and $\sigma^2(R)$, and a nonnegative measure $\mu_R(dx)$ on $\mathbb{R}$, called the *Lévy measure* of $\Delta_R X$, such that

$$\psi_R(u) = i\gamma(R)u - \frac{1}{2}\sigma^2(R)u^2 + \int_{\{|x|<1\}} (e^{iux} - 1 - iux)\mu_R(dx) \tag{2}$$
$$+ \int_{\{|x|\geq 1\}} (e^{iux} - 1)\mu_R(dx).$$

The Lévy measure μ_R satisfies

$$\mu_R\{x: |x| > 1\} < \infty, \qquad \mu_R\{0\} = 0 \quad \text{and} \quad \int_{\{|x|\leq 1\}} x^2 \mu_R(dx) < \infty$$

(recall that any measure satisfying these properties is the Lévy measure of an infinitely divisible distribution). In the case $R = R_t$, $\Delta_{R_t} X = X(t)$, and we set $\gamma_t = \gamma(R_t)$, $\sigma_t^2 = \sigma^2(R_t)$ and $\mu_t = \mu_{R_t}$.

Note that if R^j, $j = 1, \dots, m$, are disjoint rectangles, then the X_{R^j} are independent, and consequently $\sum_j X_{R^j}$ has log characteristic function $\sum_j \psi_{R^j}$. This implies that $\gamma(R)$, $\sigma^2(R)$ and μ_R are additive in R. It follows that they can be extended to become additive functions on the class of finite unions of rectangles.

By Assumption A, $t_n \to t$ implies $\phi_{t_n}(u) \to \phi_t(u)$ and $\psi_{t_n}(u) \to \psi_t(u)$, $\forall n \in \mathbb{N}$. By [1], Theorem 3.1, it follows that $t \mapsto \gamma_t$ and $t \mapsto \sigma_t^2$ are continuous, $t \mapsto \mu_t((x, y])$ is continuous at t when either $0 < x < y \leq \infty$ or $-\infty \leq x < y < 0$.

The map $t \mapsto \gamma_t$ may not have bounded variation. However, if we replace $X(t)$ by $X'(t) = X(t) - \gamma_t$, we have an a.s. continuous process of independent increments for which $\gamma_t' = 0$. In addition, replacing X by X' will not affect σ-fields defined using X. So we assume *without loss of generality* that:

- $t \mapsto \gamma_t$ is continuous and is the distribution function of a finite signed measure γ on $\mathbb{R}_+^2$.

Turning to σ^2, note that $\Delta_R \sigma_t^2 = \sigma^2(R) \geq 0$, so $t \mapsto \sigma_t^2$ has positive planar increments, which implies that it is the distribution function of a measure. Thus σ^2 can be extended to a measure $A \mapsto \sigma^2(A)$ on the Borel sets of $\mathbb{R}_+^2$.

In the case of μ, note that if $0 < x \leq y$ and R is a rectangle, if we define $\nu(R \times (x, y]) \equiv \mu_R((x, y]) \geq 0$, then ν is additive in R for fixed x and y and additive in the interval $(x, y]$ for fixed R, so it is in fact additive in the sets $R \times (x, y]$. The same is true if we take $x < y < 0$. Moreover, if $t_n \downarrow t$ and

$y_n \downarrow y > x > 0$, then

$$\begin{aligned}
&|\nu(R_{t_n} \times (x, y_n]) - \nu(R_t \times (x, y])| \\
&\quad \le \nu(R_t \times (y, y_n]) + \nu((R_{t_n} \setminus R_t) \times (x, y_1]) \\
&\quad \le \mu_t((y, y_n]) + |\mu_{t_n}((x, y_1]) - \mu_t((x, y_1])| \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$. It follows that for any $x > 0$, $(t, y) \mapsto \nu(R_t \times (x, y])$ is the distribution function of a measure on $\mathbb{R}_+^2 \times \mathbb{R}$. The same argument holds if $x < y < 0$, and we see that ν extends to a σ-finite measure on the Borel subsets of $\mathbb{R}_+^2 \times \mathbb{R}$, such that $\nu(\mathbb{R}_+^2 \times \{0\}) = 0$.

Let $(U, \mathscr{U})$ be a measurable space and let ν be a σ-finite measure on $U \times \mathbb{R}$ such that:

- $\nu(U \times \{0\}) = 0$;
- there exists $A_n \in \mathscr{U}$ such that $\bigcup_n A_n = U$, for which $\nu(A_n \times \{x: |x| > 1\}) < \infty$ and $\int_{\{|x| \le 1\}} x^2 \nu(A_n \times dx) < \infty$.

That is, for each A_n, $\nu(A_n, \cdot\,)$ is a Lévy measure. We may assume without loss of generality that the A_n are increasing. We will call such measures *generalized Lévy measures*. The above considerations show that ν defined by $\nu([0, t] \times [x, \infty)) = \mu_t([x, \infty))$ *is a generalized Lévy measure for* $U = \mathbb{R}_+^2$. (When $U = \mathbb{R}_+^2$, we shall always assume that $A_n = [0, n] \times [0, n]$.)

If R is a rectangle, we define $Y(R) = \Delta_R X$. By additivity, Y extends to an additive measure on all finite unions of rectangles. [$Y(\cdot, \omega)$ may not be a measure for each ω, but it becomes a measure if we consider it as a set function with values in L^0, the space of random variables with the topology of convergence in probability. This will be evident from the representation below.] Note that $Y(A)$ and $Y(B)$ are independent if A and B are disjoint.

Let $\mathscr{F}_t = \sigma\{X(s),\ s \le t\}$ be the filtration generated by the process X, completed by adjoining all subsets of sets of probability 0 in $\mathscr{F}$. Since X is continuous in probability, it is not difficult to show that the completed filtration is right continuous. Note that if A is a bounded finite union of rectangles on $\mathbb{R}_+^2$ which does not intersect R_t, then by the independence property of Y, $Y(A)$ is independent of $\mathscr{F}_t$.

PROPOSITION 2.1. (i) *With probability* 1, *X has quadrantal limits along the rationals.*

(ii) *There exists a version of X which is almost surely right continuous and has quadrantal limits.*

PROOF. In the one-parameter case, this is proved in [14], Theorem 8.7.2, and the idea for generalizing this martingale proof to more than one parameter is explained in [1], (1.17). □

If the Lévy measure ν has *compact support*, then $\int_{\mathbb{R}} x^2 \nu(R_t \times dx) < \infty$, so that we can write ψ_t in the form

$$(3)\quad \psi_t(u) = i\gamma(R_t)u - \frac{1}{2}\sigma^2(R_t)u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(R_t \times dx)$$

for a possibly different measure γ.

PROPOSITION 2.2. *Suppose that the Lévy measure ν has support in $\mathbb{R}_+^2 \times [-N, N]$ for some N and the log characteristic function of $X(t)$ is given by* (3) *for all t. Then*:

(i) *for each $t \in \mathbb{R}_+^2$, $X(t)$ has moments of all orders*;
(ii) *X is a two-parameter martingale if and only if $\gamma = 0$*;
(iii) *if $\gamma = 0$, then for all $p > 1$ there is a constant C_p such that for all $t \in \mathbb{R}_+^2$,*

$$(4)\qquad E\Big\{\sup_{s \le t} |X(s)|^p\Big\} \le C_p E\{|X(t)|^p\}.$$

PROOF. This short proof is included for the convenience of the reader.

(i) Let $p \ge 1$ be an integer, fix $t \in \mathbb{R}_+^2$ and let $\kappa_t(dx) = x^2 \nu(R_t \times dx)$. Then κ_t is a finite measure of compact support, and if we set $g(u, x) = (e^{iux} - 1 - iux)x^{-2}$, then g is analytic in both u and x, and hence the integral in (3) is infinitely differentiable. The other two terms are also differentiable, so it follows that ψ_t (and hence ϕ_t) is p times differentiable. If p is even, this implies that the pth moment exists and

$$E\{X(t)^p\} = (-1)^p \frac{\partial^p}{\partial u^p} \phi_t(0) < \infty.$$

(ii) Take $p = 1$ and note that $iE\{X(t)\} = (\partial/\partial u)\phi_t(0) = i\gamma(R_t)$. Since $X(0) = 0$, X cannot be a martingale unless $\gamma(R_t) = 0$ for all t, which implies that $\gamma = 0$. Conversely, if $\gamma = 0$ and $s \le t$, let $A = R_t \setminus R_s$. Then $X(t) - X(s) = Y(A)$, and, since X has independent increments, $Y(A)$ is independent of $\mathscr{F}_t$, so $E\{X(t) - X(s) | \mathscr{F}_t\} = E\{Y(A)\} = i(\partial/\partial u)\phi_A(0) = 0$.

(iii) This follows from Cairoli's two-parameter martingale maximal inequality [5]. □

Under Assumption A, the exponent function ψ_t is the sum of two terms: $i\gamma_t u - \frac{1}{2}\sigma_t^2 u^2$ and

$$(5)\qquad \int_{\{|x|<1\}} (e^{iux} - 1 - iux)\mu_t(dx) + \int_{\{|x| \ge 1\}} (e^{iux} - 1)\mu_t(dx),$$

so the characteristic function ϕ_t is the product of two corresponding factors. Each of these factors separately defines a process of independent increments, $X^g(t)$ and $X^j(t)$. The first is Gaussian, and the second is a "jump process." It is possible to construct the sample paths of these processes directly from the sample paths of $X(t)$ (see [1], Theorem 4.6), thus defining all three processes on the same probability space as $X(t)$, in such a way that $X(t) = X^g(t) + X^j(t)$. This yields the two-parameter analogue of the classical Lévy decomposition of single-parameter processes of independent increments. Since this decomposition is important for our results, we will give a simple proof of it below.

2.3. *Representations.* One key element in our study of the sharp Markov property of jump processes is the fact that the sample paths of these processes are entirely determined by their jumps, via a representation using a Poisson random measure. This representation is briefly recalled here.

Let μ be a finite measure on a subset $U \subset \mathbb{R}_+^2 \times \mathbb{R}$. A *Poisson random measure* Π_μ is a mapping from the Borel subsets of U to L^0 with the following properties:

- for each Borel set $A \subset U$, $\Pi_\mu(A)$ is a Poisson random variable with parameter $\mu(A)$;
- if $A_1, \dots, A_n$ are disjoint, $\Pi_\mu(A_1), \dots, \Pi_\mu(A_n)$ are independent.

Since Π_μ only takes on integer values, it is a sum of unit point masses. (If we concentrate on the points rather than the point masses, we would speak of a *Poisson point process* rather than a Poisson measure.)

Π_μ is a sum of unit masses, so it is countably additive for each $\omega \in \Omega$. If f is a positive Borel function on $\mathbb{R}_+^2 \times \mathbb{R}$ then $\int f(t,x)\Pi_\mu(dt \times dx)$ is a compound Poisson random variable, with characteristic function

$$(6)\quad E\left\{\exp\left[iu\int f(t,x)\Pi_\mu(dt \times dx)\right]\right\} = \exp\left[\int (e^{iuf(t,x)} - 1)\mu(dt \times dx)\right].$$

[To see this, note that if $f = I_A$ is the indicator function of a set, the equality follows from the fact that $\Pi_\mu(A)$ is Poisson with parameter $\mu(A)$, and this extends to simple functions by the independence properties of Π_μ. Then (6) follows in the limit.]

If μ is σ-finite, there exist disjoint sets U_n whose union is $\mathbb{R}_+^2 \times \mathbb{R}$ such that $\mu_n \stackrel{\text{def}}{=} \mu|_{U_n}$ is finite. Let (Π_{μ_n}) be a sequence of independent Poisson random measures corresponding to the μ_n, and let $\Pi_\mu = \sum_{n=1}^\infty \Pi_{\mu_n}$. Then Π_μ is a σ-finite measure which is a sum of unit masses, and it is again a Poisson random measure. Moreover, (6) remains true in the sense that if one side exists, so does the other and they are equal.

Let ν be a generalized Lévy measure on $\mathbb{R}_+^2 \times \mathbb{R}$, and let $\Pi = \Pi_\nu$ be a corresponding Poisson random measure. Set $\gamma = \sigma^2 = 0$ for the moment, and let $\psi_t(u)$ be the log characteristic function associated with ν as in (2):

$$(7)\quad \begin{aligned}\psi_t(u) = &\int_{\{|x|\geq 1\}} (e^{iux} - 1)\nu(R_t \times dx)\\ &+ \int_{\{|x|<1\}} (e^{iux} - 1 - iux)\nu(R_t \times dx).\end{aligned}$$

THEOREM 2.3. *The integral* $\int_{R_t \times \{|x|<1\}} x(\Pi(ds \times dx) - \nu(ds \times dx))$ *exists as the limit as* $\varepsilon \downarrow 0$ *of* $\int_{R_t \times \{\varepsilon<|x|<1\}} x(\Pi(ds \times dx) - \nu(ds \times dx))$. *The convergence is almost surely uniform for* t *in compact sets. The process* $\{\hat{X}(t),\ t \in \mathbb{R}_+^2\}$ *defined by*

$$\hat{X}(t) = \int_{R_t \times \{|x| \geq 1\}} x\Pi(ds \times dx) + \int_{R_t \times \{|x|<1\}} x(\Pi(ds \times dx) - \nu(ds \times dx))$$

is a right continuous process of independent increments with characteristic function $\phi_t(u) = e^{\psi_t(u)}$.

PROOF. Define a process $\hat{X}_1(t) = \int_{R_t \times \{|x| \geq 1\}} x\Pi(ds \times dx)$, and, for $0 < \varepsilon < 1$, set

$$\hat{X}_\varepsilon(t) = \int_{R_t \times \{\varepsilon<|x|<1\}} x(\Pi(ds \times dx) - \nu(ds \times dx)).$$

Since $\nu(R_t \times \{|x| \geq 1\})$ is finite, so is $\Pi(R_t \times \{|x| \geq 1\})$, which means that with probability 1, the restriction of Π to $R_t \times \{|x| \geq 1\}$ is the sum of finitely many point masses, hence $\hat{X}_1(t)$ is well defined. The independence property of Π implies that $\{\hat{X}_1(t),\ t \in \mathbb{R}_+^2\}$ is a process of independent increments. It is clearly right continuous in t. By (6), its log characteristic function is

$$(8) \qquad \psi_t^1(u) = \log E\{e^{iu\hat{X}_1(t)}\} = \int_{\{|x| \geq 1\}} (e^{iux} - 1)\nu(R_t \times dx).$$

Both ν and Π are finite on $R_t \times \{\varepsilon < |x| < 1\}$. Let $r = 1/\varepsilon$ and note that $\{\hat{X}_{1/r}(t),\ (t, r) \in \mathbb{R}_+^2 \times [1, \infty)\}$ is a right-continuous process of independent, mean zero increments. Thus it is a *three-parameter* martingale. (The reason for taking $1/\varepsilon$ instead of ε as a parameter was simply to get the partial order right.) The process $\hat{X}_{1/r}$ is independent of $\hat{X}_1$ and, as it is a compound Poisson process plus a drift, we can apply (6) again to see that its log characteristic function is

$$(9) \qquad \psi_t^\varepsilon(u) = \log E\{e^{iu\hat{X}_\varepsilon(t)}\} = \int_{\{\varepsilon<|x|<1\}} (e^{iux} - 1 - iux)\nu(R_t \times dx).$$

By (8) and (9), $\varepsilon \mapsto \hat{X}_1(\cdot) + \hat{X}_\varepsilon(\cdot)$ converges in law to a process with characteristic function $\phi_\cdot(u)$. Since this process has independent increments, it follows that for each t, $\varepsilon \mapsto \hat{X}_1(t) + \hat{X}_\varepsilon(t)$ also converges a.s. and in L^2. The uniform convergence now follows from Proposition 2.2(iii). □

Let σ^2 be a σ-additive nonnegative measure on $\mathbb{R}_+^2$ and let γ be a σ-additive signed measure on $\mathbb{R}_+^2$ such that $t \mapsto \sigma^2(R_t)$ and $t \mapsto \gamma(R_t)$ are continuous and vanish on the coordinate axes. A *Brownian sheet based on* σ^2 is a real-valued Gaussian process $W = \{W(t),\ t \in \mathbb{R}_+^2\}$ with mean 0 and covariance $E\{W(s)W(t)\} = \sigma^2(R_s \cap R_t)$. W has a right-continuous version by Proposition 2.1, but in fact the sample paths of W have additional regularity.

PROPOSITION 2.4. *W has a version with continuous sample paths.*

PROOF. This is proved in [1], Theorem 3.2, but the following proof is simpler. Assume that W is the right continuous version whose existence is guaranteed by Proposition 2.1. We will prove continuity in $R_{(N,N)}$ for an arbitrary N, which is sufficient. For each point s with rational coordinates, $v \mapsto W(s_1, v)$ is a Gaussian square-integrable martingale, so it has a version which is a deterministic time change of a Brownian motion. Since its quadratic variation is continuous, the time change is continuous. Thus $v \mapsto W(s_1, v)$ has a continuous version, which must equal the original process a.s. for all (s_1, v) since W is right continuous. It follows that for a.e. ω, $v \mapsto W(s_1, v)$ is continuous for all rationals s_1, and similarly, $u \mapsto W(u, s_2)$ is continuous for all rational s_2. With probability 1, then, W is continuous on all vertical or horizontal lines with one rational coordinate. Since the sample paths of W are right continuous with quadrantal limits by Proposition 2.1, Lemma 3.1 below implies that W has continuous sample paths. □

An immediate consequence of Theorem 2.3 is the following.

COROLLARY 2.5. *Let W be a Brownian sheet based on σ^2 and let Π_ν be a Poisson random measure which is independent of W. The process*

$$X(t) = W(t) + \gamma(R_t) + \int_{\{|x|<1\}} x(\Pi_\nu(R_t \times dx) - \nu(R_t \times dx)) + \int_{\{|x|\geq 1\}} x\Pi_\nu(R_t \times dx) \tag{10}$$

is a right continuous process of independent increments with log characteristic function given by (2).

Note that $\Pi = \Pi_\nu$ can be recovered from the sample paths of X. Indeed, Π has a mass at (t, x) if and only if $\Box X(t)$ is nonzero and $\Box X(t) = x$. It is thus possible to reconstruct Π from X. To be explicit, define a function h on the space D of right continuous functions on $\mathbb{R}_+^2$ with quadrantal limits by

$$h_B(x) = \sum_{\{(t,y)\in B\}} \Box x(t) I_{\{\Box x(t) = y\}}.$$

One easily checks that this function is Borel (for the Skorokhod topology, see Section 3.1) and that $\Pi(B, \omega) = h_B(X(\cdot, \omega))$.

Once Π is known, the integral in (10) can be calculated, added to $\gamma(R_t)$ and subtracted from $X(t)$ to obtain W. Thus all the quantities in the decomposition (10) can be obtained from the paths of X.

The above analysis assumes we start with the Poisson random measure Π. If we start with a process $\tilde{X}$ of independent increments with generalized Lévy measure ν, we do not know in advance that there is a Poisson measure defined on the same probability space as $\tilde{X}$, but we do know that if the process satisfies Assumption A, there there is another probability space on which there is a Poisson measure Π_ν, a Brownian sheet W and a process X defined via (10) which has the same law as $\tilde{X}$. We can construct a measure $\tilde{\Pi}_\nu$ from $\tilde{X}$ using the map h defined above. If we compare this with what happens with the process X of the same law, we see that $\tilde{\Pi}_\nu$ must have the same law as Π_ν and is thus a Poisson random measure [clearly, $\nu(R_t \times B)$ is the expected number of jumps of X with size in B that will occur in R_t]. Armed with this, we can construct a jump process $\tilde{X}^j$ as an integral, and the process $\tilde{X} - \tilde{X}^j$ is necessarily Gaussian and independent of $\tilde{X}^j$. We now have the following.

THEOREM 2.6. *Let $\{X(t),\ t \in \mathbb{R}_+^2\}$ be a right-continuous Lévy sheet with log characteristic function given by (2). Then there exist processes X^g and X^j and a Poisson random measure $\tilde{\Pi}$, all measurable with respect to $\sigma\{X(t),\ t \in \mathbb{R}_+^2\}$, such that*

$$X(t) = X^j(t) + X^g(t), \qquad \forall\, t \in \mathbb{R}_+^2,\ a.s.,$$

where X^g is Gaussian, independent of X^j and has log characteristic function $i\gamma_t u - \frac{1}{2}\sigma_t^2 u^2$, and $X^j(t) = Y^j(R_t)$, where Y^j is an L^0-valued measure with the Poisson representation

$$(11)\quad Y^j(A) = \int_{\{|x| \geq 1\}} x\tilde{\Pi}(A \times dx) + \int_{\{|x| < 1\}} x\big(\tilde{\Pi}(A \times dx) - \nu(A \times dx)\big).$$

Note that it follows from the representation above that the set function $Y(A)$ which we defined on finite unions of rectangles in Section 2.2 does indeed extend to an L^0-valued measure. Indeed, we can write $Y = Y^g + Y_1^j + Y_2^j$, where Y_1^j and Y_2^j are given by the first and second integrals, respectively, on the right-hand side of (11). Then the Gaussian part Y^g takes values in L^2 and is σ^2-continuous ([13], Definition 1.2.3), so it extends to an L^2-valued measure ([13], Theorem 1.5.2), and hence is an L^0-valued measure. The same is true of Y_2^j, while Y_1^j is a sum of point masses, finite on each compact set and is hence a signed measure for a.e. ω. Thus the sum of the three is an L^0-valued measure.

2.4. *A closer look at the jumps.* Let σ^2 be a Radon measure on $\mathbb{R}_+^2$ and let γ be a signed Radon measure on $\mathbb{R}_+^2$. Let ν be a generalized Lévy measure on $\mathbb{R}_+^2 \times \mathbb{R}$. For the rest of this article we will make the following hypothesis.

ASSUMPTION B. *The two measures γ and σ^2 are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+^2$, and for each set C which is relatively compact in $\mathbb{R} \setminus \{0\}$, $\nu(dt \times C)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+^2$.*

The corresponding process X of independent increments then satisfies Assumption A and can be written $X = X^g + X^j$, where $X^g(t) = W(t) + \gamma(R_t)$

is Gaussian, and X^j is the uniform limit of the $X_\varepsilon = \hat{X}_1 + \hat{X}_\varepsilon$ defined in the proof of Theorem 2.3. Because of the absolute continuity of the measures, the sample paths of X^g are continuous by Proposition 2.4. By Theorem 2.3 we can represent X_ε in terms of the Poisson random measure Π_ν as

$$X_\varepsilon(t) = \int_{\{|x| \geq 1\}} x \Pi_\nu(R_t \times dx) + \int_{\{\varepsilon < |x| < 1\}} x \Pi_\nu(R_t \times dx)$$
$$- \int_{\{\varepsilon < |x| < 1\}} x \nu(R_t \times dx).$$

Now $t \mapsto \nu(R_t \times dx)$ is continuous in the weak topology, and Π_ν is a sum of point masses, so that $X_\varepsilon(t) - \int_{\{\varepsilon < |x| < 1\}} x\nu(R_t \times dx)$ is a finite sum of terms of the form $f_{s,x}(t) \equiv x I_{[s,\infty)}(t)$ for $|x| > \varepsilon$, where $[s, \infty) = [s_1, \infty) \times [s_2, \infty)$.

Note that $f_{s,x}$ has discontinuities of two kinds. At s, $\square f_{s,x}(s) = x$. Along the horizontal and vertical half-lines emanating from s, $f_{s,x}$ also has a discontinuity of size x, but $\square f_{s,x} = 0$ there. And $f_{s,x}$ is continuous elsewhere. We use the term *jump* of a function g to refer to a discontinuity at which $\square g \neq 0$. We will say that the discontinuity *propagates* along the two half-lines.

The jumps of X_ε correspond to the masses of Π_ν: If Π_ν puts a mass at (t, x), then X_ε has a jump of size x at t. This jump gives rise to a discontinuity of size x which propagates along the horizontal and vertical half-lines which emanate from s. As we shall see below [Corollary 2.8(i)], under our assumptions, there cannot be more than one jump on any horizontal or vertical line, so that this propagating discontinuity will not change size or be cancelled by another. It follows that with probability 1 the sample paths of X_ε satisfy:

(i) $\square X_\varepsilon(t) = 0$ except for a countable set of points (t^n);
(ii) if $\square X_\varepsilon(t^n) = x$, then

$$X_\varepsilon(t_1^n, v) - X_\varepsilon\big((t_1^n, v)^{(2)}\big) = x, \qquad v \geq t_2^n,$$
$$X_\varepsilon(u, t_2^n) - X_\varepsilon\big((u, t_2^n)^{(4)}\big) = x, \qquad u \geq t_1^n;$$

(iii) other than those discontinuities, X_ε is continuous;
(iv) $X_\varepsilon = 0$ on $\partial\mathbb{R}_+^2$.

Since X_ε converges uniformly to X^j and X^g is continuous, *the sample paths of X have the same four properties* [note, however, that the set of t where $\square X_\varepsilon(t) = 0$ is discrete whereas this need not be true for X].

Let (T^n) be a sequence of random variables with values in $\mathbb{R}_+^2 \cup \{\infty\}$ which exhausts the jumps of X, that is, X jumps at each T^n and all the jumps of X are contained in the set $\{T^n,\ n = 1, 2, \dots\}$. By setting $T^n = \infty$ on the set $\{\exists\ j < n\colon T^n = T^j\}$, we can have the T^n represent distinct jumps. Then let $\Phi = \sum_j \phi_{T^j}$ and let $\Phi_\varepsilon = \sum_j I_{\{|\square X(T^j)| > \varepsilon\}} \delta_{T^j}$; Φ is a random measure which marks the jumps of X, and Φ_ε marks the jumps of modulus greater than ε. As remarked in Section 2.2, they are both Poisson random measures. Set $\Psi = \sum_{j \neq k} \delta_{T^j} \times \delta_{T^k}$ and let $\Psi_\varepsilon = \sum_{j \neq k} I_{\{|\square X(T^j)| > \varepsilon,\, |\square X(T^k)| > \varepsilon\}} \delta_{T^j} \times \delta_{T^k}$. Notice that

$\Psi = \Phi \times \Phi - \sum_j \delta_{T^j} \times \delta_{T^j}$, and Ψ_ε is the same for the jumps of modulus greater than ε. Here is one consequence of Assumption B.

LEMMA 2.7. *Let $A \subset \mathbb{R}_+^2 \times \mathbb{R}_+^2$ be a set of Lebesgue measure* 0. *Then*

$$P\{\Psi(A) = 0\} = 1.$$

PROOF. We first remark that because of the Poisson distribution of the jumps, given that $\Phi_\varepsilon(R_t) = k$, the distribution of the points where the jumps occur is the same as that of k i.i.d. random variables with values in R_t, each having a distribution ρ given by $\rho(B) = \nu((B \cap R_t) \times \{|x| > \varepsilon\})/\nu(R_t \times \{|x| > \varepsilon\})$. The only thing we have to know about ρ is that it is absolutely continuous with respect to Lebesgue measure on R_t, which follows from Assumption B. Since $\rho \times \rho$ is absolutely continuous on $R_t \times R_t$, any pair (S_i, S_j) with $i \neq j$ and the joint distribution $\rho \times \rho$ satisfies $P\{(S_i, S_j) \in A\} = 0$, and it follows that $P\{\Psi_\varepsilon(A \cap R_t) = 0 | \Psi_\varepsilon(R_t) = k\} = 1$. This is true for all k and all $t \in \mathbb{R}_+^2$, so $P\{\Psi_\varepsilon(A) = 0\} = 1$. Now let $\varepsilon \to 0$ and the conclusion follows. □

There are several consequences of this lemma which we will use repeatedly below. Each discontinuity of X propagates both horizontally and vertically from a jump of X, along two half-lines to ∞. We can extend these half-lines to full lines, which we will call *extended discontinuities*.

COROLLARY 2.8. (i) *P{two jumps fall on the same horizontal or vertical line}* = 0.

(ii) *Let $B \subset \mathbb{R}_+^2$ be a set of Lebesgue measure* 0. *Then*

$$P\{\textit{two extended discontinuities cross in } B\} = 0.$$

(iii) *Let $\phi(u)$ and $\psi(u)$ be Borel functions on $\mathbb{R}_+$ with values in $\mathbb{R}_+^2$: $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$. Suppose that the four curves $\{(\psi_i(u), \psi_j(u))\colon u \geq 0\}$, $i, j = 1, 2$, have Lebesgue measure* 0 *in $\mathbb{R}_+^2$. Then*

$$P\{\exists\, u \in \mathbb{R}^+\colon \phi(u) \textit{ and } \psi(u) \textit{ are on different extended discontinuities}\} = 0.$$

PROOF. (i) Take $A = \{(s,t)\colon s_1 = t_1 \text{ or } s_2 = t_2\}$ in Lemma 2.7.

(ii) Take $A = \{(s,t)\colon (s_2, t_1) \in B\}$. To see that A is a null set, note that if B is a rectangle in R_t, the (four-dimensional) Lebesgue measure of $A \cap (R_t \times R_t)$ is bounded by $t_1 t_2$ times the (two-dimensional) Lebesgue measure of B. This also holds for unions of rectangles, and it follows that for any $t \in \mathbb{R}_+^2$ and any Borel set B, $|A \cap (R_t \times R_t)| \leq t_1 t_2 |B \cap R_t|$, which vanishes if $|B| = 0$.

(iii) Apply Lemma 2.7 with

$$A = \bigcup_u \big((\{\phi_1(u)\} \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times \{\phi_2(u)\})\big)$$
$$\times \big((\{\psi_1(u)\} \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times \{\psi_2(u)\})\big).$$

This is a union of four sets. $\bigcup_u ((\{\phi_1(u)\} \times \mathbb{R}^+) \times (\mathbb{R}^+ \times \{\psi_2(u)\}))$ has measure

0. Indeed, let $B = \{(\psi_2(u), \phi_1(u)): u \geq 0\}$. This has two-dimensional measure 0 by hypothesis, so apply (ii). The second set is a reflection of the first: $\bigcup_u((\{\phi_1(u)\} \times \mathbb{R}^+) \times (\{\psi_1(u)\} \times \mathbb{R}^+))$ maps into a set of the first type—which therefore has measure 0—under the measure-preserving isomorphism $(s_1, s_2, t_1, t_2) \mapsto (s_1, s_2, t_2, t_1)$. The remaining two sets are handled in similar fashion, so we conclude that A has measure 0. □

3. The sharp field is a splitting field.

3.1. *The uniform sharp field.* We saw in Example 2.4 that the definition of the sharp field is delicate when the processes involved have discontinuities. We are now going to define what we call the *uniform sharp field*.

Let $\mathscr{D}(\mathbb{R}^2_+)$ (or just $\mathscr{D}$ for short) be the space of functions on $\mathbb{R}^2_+$ which vanish on the axes, are right continuous and have quadrantal limits at every point. Equip $\mathscr{D}$ with a *Skorokhod metric* ρ, defined as follows. On the index set $T = [0, 1]^2$, set

$$\rho(f, g) = \inf_{\lambda \in \Lambda} \max(\|f - g \circ \lambda\|, \|\lambda\|),$$

where $\|f - g \circ \lambda\| = \sup_{t \in T}|f(t) - g(\lambda(t))|$ and $\|\lambda\| = \sup_{t \in T}|\lambda(t) - t|$. The space Λ is the group of all transformations $\lambda: T \to T$ of the form $\lambda(t_1, t_2) = (\lambda_1(t_1), \lambda_2(t_2))$, where $\lambda_i: [0, 1] \to [0, 1]$ is continuous, strictly increasing and leaves 0 and 1 fixed. We can extend the metric to all of $\mathbb{R}^2_+$ in the usual manner. Then $\mathscr{D}$ becomes a complete metric space ([2], and [22] page 205).

The discontinuity set of a function $f \in \mathscr{D}$ is far from arbitrary. A discontinuity occurs at $t \in \mathbb{R}^2_+$ if and only if one of the following four inequalities holds: $f(t) \neq f(t^{(2)})$, $f(t^{(4)}) \neq f(t^{(3)})$, $f(t) \neq f(t^{(4)})$ or $f(t^{(2)}) \neq f(t^{(3)})$. If either of the first two inequalities hold, t is termed a *vertical discontinuity*, and in the last two cases a *horizontal discontinuity*. It can be shown that all vertical discontinuities fall on a countable collection of vertical lines and that all horizontal discontinuities fall on a countable collection of horizontal lines (see [17], Théorème 2-1, and [22], page 205). Discontinuities are not isolated: A vertical discontinuity will propagate for a positive distance along a vertical line, and a horizontal discontinuity will propagate along a horizontal line.

LEMMA 3.1. *Suppose that $f \in \mathscr{D}$ has the following property: There is a countable set of horizontal lines with dense y-coordinates and a countable set of vertical lines with dense x-coordinates on which the restriction of f is continuous. Then f is continuous.*

PROOF. If t is a point of discontinuity of f, at least one of the four above types of discontinuities occurs at t. Suppose, for example, that $f(t) \neq f(t^{(2)})$. There exists a sequence of horizontal lines of continuity of f whose y-coordinates, which we denote v^n, decrease to t_2. Since $f(\cdot, v^n)$ is continuous at (t_1, v^n), there are $u^n < t_1$ such that $u^n \to t_1$ and $f(u^n, v^n) - f(t_1, v^n) \to 0$. But $f(u^n, v^n) \to f(t^{(2)})$ and $f(t_1, v^n) \to f(t)$, which is a contradiction. Similar arguments eliminate the possibility of the other types of discontinuity, so f is continuous at t. □

Let $\mathscr{B}(\mathscr{D})$ denote the Borel σ-field on $\mathscr{D}$. This σ-field coincides with the smallest σ-field for which all the coordinate mappings $f \mapsto f(t)$ are measurable ([22], page 205).

The process X can be defined canonically on the space $\mathscr{D}$. Let $\tilde{X}$ be the canonical process on $\mathscr{D}$: $\tilde{X}(t, f) = f(t)$, $f \in \mathscr{D}$. Then there is a probability measure $\tilde{P}$ on $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$ such that under $\tilde{P}$, $\tilde{X}$ has the same distribution as X. Note that $\tilde{P}$ lives on the subspace $\mathscr{D}_0$ of all $f \in \mathscr{D}$ which satisfy (i)–(iv) of Section 2.4. Equivalently, $\mathscr{D}_0$ is the closure in $\mathscr{D}$ of functions of the form

$$f(t) = g(t) + \sum_{i=1}^{n} f_{s^j, x^j}(t),$$

where g is continuous and vanishes on the boundary, the x^j's are real numbers and the s^j's are elements of $\mathbb{R}_+^2$ whose first coordinates are distinct and whose second coordinates are also distinct, and $f_{s^j, x^j}(t) = x^j I_{[s^j, \infty)}(t)$. [Though one might find it surprising that all elements in the closure of this set of functions satisfy (i)–(iv) of Section 2.4, the proof of this fact is straightforward from the definition of the Skorokhod metric and is left to the reader.] Note that $\mathscr{D}_0$ is a measurable subset of $\mathscr{D}$, since it is closed, so $(\mathscr{D}_0, \mathscr{B}(\mathscr{D}_0))$ is again a Blackwell space. (This is almost immediate from Blackwell's original definition of what he called Lusin spaces [3].) Discontinuities of elements of $\mathscr{D}_0$ have a very specific form. Indeed, $f \in \mathscr{D}_0$ has a discontinuity at t if and only if $f(t) \neq f(t^{(2)})$ or $f(t) \neq f(t^{(4)})$.

The theorem that follows is the key to our definition of the uniform sharp field. Let K be a σ-compact subset of $\mathbb{R}_+^2$, and let "$\overset{K}{\sim}$" be the equivalence relation on $\mathscr{D}$ defined by

$$f \overset{K}{\sim} g \Leftrightarrow f(t) = g(t), \qquad \forall\, t \in K.$$

THEOREM 3.2. *There is a unique separable sub-σ-field $\tilde{\mathscr{H}}(K)$ of $\mathscr{B}(\mathscr{D})$ whose atoms are the equivalence classes of* "$\overset{K}{\sim}$".

PROOF. Since $\mathscr{D}$ is a complete separable metric space, $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$ is a Blackwell space. By Blackwell's theorem ([3] and [12], Theorem 3.26), a separable sub-σ-field of $\mathscr{B}(\mathscr{D})$ is determined by its atoms. Thus there can be at most one separable sub-σ-field of $\mathscr{B}(\mathscr{D})$ whose atoms are the equivalence classes of "$\overset{K}{\sim}$". This means that we need only prove existence. We will do this by exhibiting a countable family of random variables which generate a σ-field $\mathscr{G}$ with the appropriate atoms.

Let us first assume K is compact. For convenience, we shall assume that the index set is $[0, 1]^2$ instead of $\mathbb{R}_+^2$. Let (t^n) be a countable dense subset of K.

Even in simple cases (see Example 2.4), the values of the $f(t^n)$ do not determine the restriction of f to K. We need to include the values of f at points of K which are also discontinuities of f. We begin by finding the countable set of vertical and horizontal lines which contain the discontinuities of a given $f \in \mathscr{D}$ (this set depends of course on f). Set

$$Z_{s_1}(f) = \sup_{0 \le s_2 \le 1} \big(f(s) - f(s^{(2)})\big) = \sup_{s_2 \in [0,1] \cap \mathbb{Q}} \big(f(s) - f(s^{(2)})\big).$$

It is easy to see that the map $(f, s_1) \mapsto Z_{s_1}(f)$ is jointly measurable from $\mathscr{D} \times \mathbb{R}$ to $\mathbb{R}$, and since $f \in \mathscr{D}$, there are only countably many s_1 for which $Z_{s_1}(f) \neq 0$; these are the x-coordinates of the lines containing the vertical discontinuities of f. We can thus enumerate them by a sequence of $\mathscr{B}(\mathscr{D})$-measurable functions $T_1^1, T_1^2, \dots$ on $\mathscr{D}$ (these need not always be distinct). The T_1^i enumerate the x-coordinates of the countable set of vertical lines on which the vertical discontinuities of f fall. Similarly, we can find $T_2^1, T_2^2, \dots$ which enumerate the y-coordinates of the horizontal discontinuities of f.

For any $s = (s_1, s_2) \in \mathbb{R}_+^2$ and $a \ge 0$, let $V(s; a)$ be the point in $\mathbb{R}_+^2$ whose first coordinate is s_1 and whose second coordinate is $\sup\{v\colon s_2 \le v \le a, (s_1, v) \in K\}$. Similarly, let $U(s; a)$ be the point in $\mathbb{R}_+^2$ whose second coordinate is s_2 and whose first coordinate is $\sup\{u\colon s_1 \le u \le a, (u, s_2) \in K\}$. We make the convention that the sup of the empty set is 0, so that $U(s; a)$ and $V(s; a)$ are either in K or in the boundary of $\mathbb{R}_+^2$. Since K is compact, both U and V are Borel functions of the pair (s, a).

Let (r^m) be an enumeration of the rationals in $[0, 1]$. Let $\mathscr{G}$ be the σ-field generated by the coordinate mappings $f \mapsto f(t^n)$, $n = 1, 2, \dots$ [where the (t^n) are the points dense in K] together with the mappings $f \mapsto f(U((r^m, T_2^n); r^p))$ and the mappings $f \mapsto f(V((T_1^n, r^m); r^p))$, $m, n, p = 1, 2, \dots$.

These are all $\mathscr{B}(\mathscr{D})$-measurable. Thus $\mathscr{G} \subset \mathscr{B}(\mathscr{D})$. It remains to show that the atoms of $\mathscr{G}$ are the equivalence classes of "$\overset{K}{\sim}$". To see this, it is enough to show that $f|_K \equiv 0$ if and only if all the random variables generating $\mathscr{G}$ vanish. This is clear in one direction: If $f|_K \equiv 0$, then, as $U(s; a)$ and $V(s; a)$ are all in $K \cup \partial\mathbb{R}_+^2$, all the variables generating $\mathscr{G}$ vanish. For the converse, suppose that $\sup_K f > 0$. Then there is a point $s \in K$ such that $f(s) > 0$. If s is an isolated point of K, it is one of the t^n, so $\sup_n f(t^n) > 0$. If not, there are two possibilities: If s is a point of continuity of f, it is a limit of t^n and $f(s) = \lim f(t^n) > 0$, so one of the $f(t^n)$ is again strictly positive. If s is a point of discontinuity, then s must be a horizontal or vertical discontinuity. Suppose it is vertical. Then $s_1 = T_1^n(f)$ for some $n \in \mathbb{N}$, so if $r > s_2$, the second coordinate of $V((T_1^n(f), s_2); r)$ is greater than or equal to s_2, and it decreases to s_2 as r decreases to s_2. By right continuity, $f(V((T_1^n(f), s_2); r)) \to f(s) > 0$, hence $f(V((T_1^n(f), r^m); r^p)) > 0$ for some m, n and p. Note that this is true even if $V((T_1^n(f), s_2); r) = s$.

If, now, K is only σ-compact, say $K = \bigcup_n K_n$, put $\mathscr{H}(K) = \bigvee_n \tilde{\mathscr{H}}(K_n)$. This is a separable sub-σ-field of $\mathscr{B}(\mathscr{D})$, and its atoms are clearly the equivalence classes of "$\overset{K}{\sim}$". □

DEFINITION. Let X be a process with sample paths in $\mathscr{D}$ a.s. and let $K \subset \mathbb{R}_+^2$ be a σ-compact set. Consider X as a random variable with values in $\mathscr{D}$. Then the *uniform sharp field* $\mathscr{H}(K)$ is the σ-field $X^{-1}(\tilde{\mathscr{H}}(K))$, completed by adjoining all null sets of $\mathscr{F}$.

REMARK 3.1. Let $(B, \mathscr{B})$ be a Blackwell space, let $\Lambda \in \mathscr{B}$ and let $\mathscr{A}$ be a separable subfield of $\mathscr{B}$. If Λ is a union of atoms of $\mathscr{A}$, then $\Lambda \in \mathscr{A}$.

To see this, let $\mathscr{A}'$ be the σ-field obtained by adjoining Λ to $\mathscr{A}$. $\mathscr{A}'$ is a separable sub-σ-field of $\mathscr{B}$, and it has the same atoms as $\mathscr{A}$, hence it equals $\mathscr{A}$ by Blackwell's theorem.

This remark makes it easy to verify measurability. For instance, let us compare $\tilde{\mathscr{H}}(K)$ with the uncompleted sharp field $\tilde{\mathscr{H}}^1(K) \equiv \sigma\{f(t): t \in K\}$ (note that there is no measure involved in this definition), and the uniform sharp field $\mathscr{H}(K)$ with the sharp field $\mathscr{H}^0(K)$.

PROPOSITION 3.3. (i) *If $J \subset K$, then $\tilde{\mathscr{H}}(J) \subset \tilde{\mathscr{H}}(K)$ and $\mathscr{H}(J) \subset \mathscr{H}(K)$.*

(ii) *$\tilde{\mathscr{H}}^1(K) \subset \tilde{\mathscr{H}}(K)$ and $\mathscr{H}^0(K) \subset \mathscr{H}(K)$ for all K.*

(iii) *If D is open, $\tilde{\mathscr{H}}(D) = \tilde{\mathscr{H}}^1(D)$ and $\mathscr{H}(D) = \mathscr{H}^0(D)$.*

(iv) *Let K be closed and consider the subset $\mathscr{D}^+ \subset \mathscr{D}_0$ of functions f such that $\Box f(t) \geq 0$ for all t. Assume $P(\mathscr{D}^+) = 1$. Then there is a Borel subset $\mathscr{D}_K^+$ of $\mathscr{D}^+$ with $P\{\mathscr{D}_K^+\} = 1$ such that the traces of $\tilde{\mathscr{H}}(K)$ and $\tilde{\mathscr{H}}^1(K)$ coincide on $\mathscr{D}_K^+$ (in particular, the two fields are equal on the subset of continuous functions in $\mathscr{D}$). Consequently, if X has only positive jumps, $\mathscr{H}(K) = \mathscr{H}^0(K)$.*

PROOF. It is only necessary to prove the statements for the σ-fields on $\mathscr{D}$, since the others are just their (completed) images. To see (i), note that $\tilde{\mathscr{H}}(J)$ and $\tilde{\mathscr{H}}(K)$ are separable subfields on $\mathscr{B}(\mathscr{D})$. Two functions which agree on K must agree on J, so every set in $\tilde{\mathscr{H}}(J)$ is a union of atoms of $\tilde{\mathscr{H}}(K)$. Now apply Remark 3.1.

For (ii), note that if $t \in K$, then for any x, $\{f \in \mathscr{D}: f(t) < x\}$ is a union of atoms of $\tilde{\mathscr{H}}(K)$, so $f \mapsto f(t)$ is $\tilde{\mathscr{H}}(K)$-measurable.

Statement (iii) follows upon noting that the first pair of σ-fields are separable and are generated by $f \mapsto f(t^k)$ for any dense subset $(t^k) \subset D$, while the second pair are completions of the first two.

The last statement requires more effort. For $s \in \mathbb{R}_+^2$, set

$$(K, s] = \{t \in \mathbb{R}_+^2 : t \leq s \text{ and } [t, s] \cap K = \varnothing\}$$
$$= R_s \setminus \bigcup_{t \in K \cap R_s} R_t,$$

and let $\Gamma_{K,s}$ denote the monotone decreasing curve which is the lower left boundary of $(K, s]$ (this is the last exit line of $K \cap R_s$). Let

$$K^1 = \bigcup_{r \in \mathbb{Q}_+^2 \setminus K} \Gamma_{K,r}.$$

Then K^1 is contained in a countable union of monotone curves and has Lebesgue measure 0. Let

$$\mathscr{D}_K^+ = \{f \in \mathscr{D}^+ \colon \text{no extended discontinuities of } f \text{ cross in } K^1\}.$$

$\mathscr{D}_K^+$ is easily seen to be a Borel subset of $\mathscr{D}$, and by Corollary 2.8 we have $P\{\mathscr{D}_K^+\} = 1$. We are going to show that the traces $\mathscr{A}$ and $\mathscr{B}$ of $\tilde{\mathscr{H}}^1(K)$ and $\tilde{\mathscr{H}}(K)$ on $\mathscr{D}_K^+$, respectively, are equal. This will complete the proof.

Since $\mathscr{D}_K^+$ is a Borel subset of $\mathscr{D}$, it is again a Blackwell space, so it is sufficient to show that $\mathscr{A}$ is separable, and that $\mathscr{A}$ and $\mathscr{B}$ have the same atoms. A countable generating family of random variables is obtained as follows. Since $K \cap \Gamma_{K,s}$ is compact, we let Q'_s be a countable dense subset of this set. Q_s is obtained from Q'_s by adding in all extremities of vertical or horizontal segments of $\Gamma_{K,s}$ which belong to K. There are only countably many such extremities, so Q_s is countable. Let Q_K be a countable dense subset of K, and set $Q = Q_K \cup (\bigcup Q_s)$, where the second union is over $s \notin K$ with rational coordinates. This set is countable and dense in K, and its restriction to each curve $\Gamma_{K,s}$ is dense in $K \cap \Gamma_{K,s}$.

Now observe that for $f \in \mathscr{D}_K^+$, we have

$$f(t) = \limsup_{s \to t,\, s \in Q} f(s), \qquad \forall\, t \in K. \tag{12}$$

Indeed, this is clear if t is an isolated point of K or an accumulation point of $K \cap [t, \infty[$. Otherwise, there is $s \in \mathbb{Q}_+^2 \setminus K$ such that $t \in \Gamma_{K,s}$. There are two cases. First, if t does not lie on a vertical or horizontal segment, there is a sequence of points t^n converging to t such that no t^n is comparable to t in the partial order $\leq$. By the definition of D_K^+, we have $f(t) = \limsup f(t^n)$. Second, if t is on a vertical or horizontal segment of $\Gamma_{K,s}$, say a vertical segment, then, if t is an extremity, t is already in Q, and there is no problem. Otherwise, there are again two cases. If $K \cap \Gamma_{K,s} \cap [t, \infty)$ has t as an accumulation point, then (12) holds by right continuity. If t is not an accumulation point of this set, then there is a rational point s' such that t is an extremity of a vertical segment of $K \cap \Gamma_{K,s'}$ and we are reduced to a previous case. It follows that $\mathscr{A}$ is generated by the countable family of coordinate maps $f \mapsto f(s)$, $s \in Q$, and these maps determine $f|_K$. Thus the σ-fields $\mathscr{A}$ and $\mathscr{B}$ are equal, and the proof is complete. □

REMARK 3.2. The sharp field and the uniform sharp field coincide for the open sets D and $\bar{D}^c$ by (iii) above. The sharp boundary field and uniform sharp boundary field coincide if the process is continuous, such as the Brownian sheet, or has only positive jumps, such as the Poisson sheet. The only case when the two do not coincide is that in which the process has both positive and negative jumps. The germ field is generally strictly larger than the uniform sharp field. See Examples 2.2 and 2.3.

PROOF OF THEOREM 1.1. From Proposition 3.3(iv), it follows that Theorem 1.1 is a consequence of Theorem 1.2, which we will prove in the next section (see Theorem 3.9). Notice that Theorem 1.2 is stated only for the right continuous version of the process, whereas Theorem 1.1 is stated for an arbitrary version. However, it is easy to see that the right continuity is not necessary in Theorem 1.1. Indeed, suppose that X is a right continuous version and that X_* is any other version. Let $\mathscr{H}^0(A) \equiv \sigma\{X(t),\ t \in A\}$ and $\mathscr{H}^0_*(A) \equiv \sigma\{X_*(t),\ t \in A\}$, where both σ-fields are completed by adjoining all null sets of $\mathscr{F}$. Then $\mathscr{H}^0(A) = \mathscr{H}^0_*(A)$. Indeed, if $t \in A$, $X(t) = X_*(t)$ a.s., so a set of the form $\{X_*(t) \in C\}$ differs from $\{X(t) \in C\}$ by a null set, and is hence in $\mathscr{H}^0(A)$. As these sets generate their respective σ-fields, we have $\mathscr{H}^0_*(A) \subset \mathscr{H}^0(A)$ and $\mathscr{H}^0(A) \subset \mathscr{H}^0_*(A)$. Since the statement of Theorem 1.1 depends only on the σ-fields $\mathscr{H}^0$, it must hold for any version of X. □

3.2. *The sharp Markov property for jump processes.* Let $X = \{X(t),\ t \in \mathbb{R}^2_+\}$ be a right continuous Lévy process satisfying Assumption B. As we have seen in Section 2.1, X may or may not satisfy the sharp Markov property with respect to a given open set D. In the case that X is a Brownian sheet based on Lebesgue measure, necessary and sufficient conditions on the boundary of a Jordan domain were given in [11]. Here we consider the case where D is bounded or has bounded complement. We are first going to show that X can have the sharp Markov property only if both its Gaussian and jump parts do too. We then study in detail the sharp Markov property of the jump part of X, or equivalently, we will assume that X has no Gaussian part.

Let D_1 be an arbitrary *bounded* open set, and let D_2 be the complement of its closure. Let $\dot{D}_i$ be the interior of the closure of D_i, $i = 1, 2$, and set $\Gamma = \partial \dot{D}_1 = \partial \dot{D}_2$ ($\dot{D}_1$ contains D_1 and may be different from D_1, but $\dot{D}_2 = D_2$; and Γ may be smaller than ∂D_1).

THEOREM 3.4. *Suppose X is a Lévy sheet which satisfies Assumption B and has the sharp Markov property with respect to D_1 in the sense that for some dense subset Λ of Γ, $\mathscr{H}(\Lambda)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$. Then both its Gaussian part X^g and its jump part X^j (given in Theorem 2.6) satisfy the same sharp Markov property.*

Before proving this, we need some of the notation introduced in [11] and three lemmas. For $F \subset \mathbb{R}^2_+$, let S^V and S^H be respectively the *vertical* and *horizontal shadows of F*:

$$S^V(F) = \{(t_1, t_2) \in \mathbb{R}^2_+ : \exists\, (s_1, s_2) \in F \text{ with } s_1 = t_1,\ s_2 \geq t_2\}, \tag{13}$$

$$S^H(F) = \{(t_1, t_2) \in \mathbb{R}^2_+ : \exists\, (s_1, s_2) \in F \text{ with } s_2 = t_2,\ s_1 \geq t_1\}. \tag{14}$$

Fix attention on the vertical shadows and define the vertical shadows of the $\dot{D}_i$ on each other by

$$S_1 = \dot{D}_1 \cap S^V(\dot{D}_2), \qquad S_2 = \dot{D}_2 \cap S^V(\dot{D}_1).$$

If $G \subset \mathbb{R}^2_+$, define a "hitting time" T_G by

$$T_G(t) = \begin{cases} \inf\{v \geq t_2: (t_1, v) \in G\}, & \text{if } \{\ \} \neq \varnothing, \\ +\infty, & \text{otherwise,} \end{cases}$$

and define maps p and τ with domain $S_1 \cup S_2$ by

$$p(t) = \begin{cases} T_{\dot{D}_2}(t), & \text{if } t \in S_1, \\ T_{\dot{D}_1}(t), & \text{if } t \in S_2. \end{cases}$$

Finally, set $\tau(t) = (t_1, p(t))$. Note that p never takes the value $+\infty$ and that τ projects $S_1 \cup S_2$ onto Γ. Let Γ^V be this projection. Had we fixed attention on the horizontal rather than the vertical shadows, this procedure would have led to the subset Γ^H of Γ. A subset of Γ which will be important in the proof of our results is

$$\Gamma^0 = \Gamma^V \cup \Gamma^H.$$

A point $t \in \mathbb{R}^2_+$ has four half-neighborhoods:

$$B_L(t,\varepsilon) = \{s: s_1 < t_1, |s - t| < \varepsilon\}, \qquad B_R(t,\varepsilon) = \{s: s_1 > t_1, |s - t| < \varepsilon\},$$

$$B_A(t,\varepsilon) = \{s: s_2 > t_2, |s - t| < \varepsilon\}, \qquad B_B(t,\varepsilon) = \{s: s_2 < t_2, |s - t| < \varepsilon\}.$$

Let $\Lambda \subset \mathbb{R}^2_+$. We say that a point $t \in \Lambda$ is *left-isolated* in Λ if there exists $\varepsilon > 0$ such that $\Lambda \cap B_L(t,\varepsilon) = \varnothing$, and, similarly, t is *right-isolated* (respectively *isolated from above*, *isolated from below*) in Λ if there exists $\varepsilon > 0$ such that $\Lambda \cap B_R(t,\varepsilon) = \varnothing$ [respectively, $\Lambda \cap B_A(t,\varepsilon) = \varnothing$, $\Lambda \cap B_B(t,\varepsilon) = \varnothing$].

LEMMA 3.5. (i) Γ^0 *is dense in* Γ.

(ii) *The vertical projection* $\hat{F}_1$ *on the x-axis of the set* F_1 *of points of* Γ *which are left-isolated or right-isolated in* Γ *is countable.*

(iii) *The horizontal projection* $\hat{F}_2$ *on the y-axis of the set* F_2 *of points of* Γ *which are isolated from below or from above in* Γ *is also countable.*

[(ii) *and* (iii) *are true for any subset of* $\mathbb{R}^2_+$.]

PROOF. (i) Let $t \in \Gamma$. Then there exist sequences $(t^n) \subset \dot{D}_1$ and $(s^n) \subset \dot{D}_2$ which each converge to t. Suppose that t is not in the closure of Γ^0. Then there is $\varepsilon > 0$ such that

$$A_\varepsilon = \{(u,v): |u - t_1| \leq 2\varepsilon, |v - t_2| \leq 2\varepsilon\} \cap \Gamma^0 = \varnothing.$$

We claim that for all large enough n,

$$\{t_1^n\} \times [t_2^n - \varepsilon, t_2^n + \varepsilon] \subset \dot{D}_1. \tag{15}$$

Indeed, $\dot{D}_1$ is open, so for large n there is a $\delta_n > 0$ such that

$$[t_1^n - \delta_n, t_1^n + \delta_n] \times \{t_2^n\} \subset \dot{D}_1 \cap ([t_1 - \varepsilon, t_1 + \varepsilon] \times [t_2 - \varepsilon, t_2 + \varepsilon]).$$

But then

$$C(n,\varepsilon) = ([t_1^n - \delta_n, t_1^n + \delta_n] \times [t_2^n - \varepsilon, t_2^n + \varepsilon]) \cap \dot{D}_2 = \varnothing. \tag{16}$$

For otherwise, if $(u, v) \in C(n, \varepsilon)$ there must be a point $q^n \in \Gamma^V$ which lies on the segment connecting (u, v) and (u, t_2^n): Either $v < t_2^n$, in which case we can take $q^n = \tau(u, v)$, or $v > t_2^n$, in which case we take $q^n = \tau(u, t_2^n)$. Since $q^n \in A_\varepsilon$, this is a contradiction. So (16) holds, and since $\partial \dot{D}_2 = \Gamma$, (16) implies $(\{t_1^n\} \times [t_2^n - \varepsilon, t_2^n + \varepsilon]) \cap \Gamma = \varnothing$. Property (15) follows.

Now consider the sequence $(s^n) \subset \dot{D}_2$. If we apply the same argument as above with the role of first and second coordinates interchanged—in particular, if we interchange horizontal and vertical projections and use Γ^H in place of Γ^V—we see that there exists $\varepsilon' > 0$ such that for all large enough n,

$$(17) \qquad (s_1^n - \varepsilon', s_1^n + \varepsilon') \times \{s_2^n\} \subset \dot{D}_2.$$

But for large n the segments in (17) intersect those in (15), which is impossible. Thus (i) follows by contradiction.

(ii) and (iii) Let Λ be any subset of $\mathbb{R}_+^2$. It is sufficient to prove (ii) and (iii) for $\Lambda \cap R_t$, for all $t \in \mathbb{R}_+^2$, so we can assume that Λ is bounded. By symmetry, it is enough to prove that the vertical projection of the set of left-isolated points is countable. For these, it is enough to prove that for any $\varepsilon > 0$, the set $\tilde{\Lambda}_\varepsilon$ of $t \in \Lambda$ for which $B_L(t, \varepsilon) \cap \Lambda = \varnothing$ has a finite vertical projection on the x-axis. Suppose not. Since Λ is bounded, there would be a point $t = (t_1, t_2)$ and a sequence $(t^n) \in \tilde{\Lambda}_\varepsilon$ of left-isolated points converging to t whose first coordinates t_1^n are distinct. For each pair $m < n$, either $t_1^n < t_1^m$ or $t_1^n > t_1^m$. If the former, then $t^n \notin B_L(t^m, \varepsilon)$, which implies that $|t^m - t^n| > \varepsilon$. The same would be true if we had the latter inequality. This is a contradiction since the sequence (t^n) converges. □

Let Q_i, $i = 1, \dots, 4$, be the four right-half-open quadrants of $\mathbb{R}^2$ defined in Section 2.1.

LEMMA 3.6. (i) $\mathscr{H}^0(\Gamma) = \mathscr{H}^0(\Gamma^0) \subset \mathscr{H}(\Gamma^0) \subset \mathscr{H}(\Gamma) \subset \mathscr{H}(D_2)$, *and the three inclusions may be strict.*

(ii) $\mathscr{H}(\Gamma^0) \subset \mathscr{H}(\dot{D}_1) \cap \mathscr{H}(\dot{D}_2)$.

(iii) *If* $\mathscr{H}(\Gamma^0)$ *is a splitting field for* $\mathscr{H}(D_1)$ *and* $\mathscr{H}(D_2)$, *so is* $\mathscr{H}(\Gamma)$; *and if* $\mathscr{H}(\Gamma)$ *is a splitting field, so is* $\mathscr{H}(\partial D_1)$.

PROOF. (i) Lemma 3.5(i) and the continuity in probability of X imply the inclusion $\mathscr{H}^0(\Gamma) \subset \mathscr{H}^0(\Gamma^0)$. The reverse inclusion is obvious and the next two inclusions follow from Proposition 3.3(ii) and (i), respectively. The first inclusion is strict in Example 2.4. To see that the second inclusion can be strict, consider the signed Poisson sheet $\hat{X}$ of Example 2.4 on the following domain. Let $A \subset [0, 1]$ be a Cantor set of positive Lebesgue measure, and let $I_1, I_2, \dots$ be the disjoint open intervals whose union is $(0, 1) \setminus A$. Set $G_n = \{(u, v): u \in I_n, 1 - v \in I_n\}$ and let $D_1 = \bigcup_n G_n$. Then Γ^0 is the union of the boundaries of the G_n minus their vertices, while Γ is Γ^0 plus the set $K = \{(x, 1 - x): x \in A\}$. There is positive probability that a discontinuity of $\hat{X}$ will intersect K. Just as in Example 2.4, one cannot tell whether the discontinuity is propagating vertically or horizontally from $\mathscr{H}(\Gamma^0)$. Indeed, let F be the event "$u \mapsto$

$\hat{X}(u, 1-u)$ jumps up one, then down one in the interval $(0, \frac{1}{2})$, and then does the same in the interval $(\frac{1}{2}, 1)$, all four jumps occur for $u \in A$, and the first two are due to vertically propagating discontinuities." One can determine whether or not $\omega \in F$ from the values of $\hat{X}(t, \omega)$, $t \in \Gamma$ [indeed, ω belongs to F if and only if $\hat{X}((v, 1-v), \omega) = \hat{X}((v, 1-v)^{(4)}, \omega)$ at each of the first two discontinuities of $u \mapsto \hat{X}((u, 1-u), \omega)$], but by arguments similar to those in Example 2.4, one cannot determine this by looking only at the values of $X_t(\omega)$, $t \in \Gamma^0$.

Though the last inclusion can clearly be strict, the fact that it occurs at all is delicate [$\mathscr{H}(\Gamma)$ is not always included in $\mathscr{H}(D_1)$ for instance, as in the example above]. To prove it, we assume, as we can, that X is canonically defined on $\mathscr{D}_0$. We are going to use Blackwell's theorem and Remark 3.1, so we need only check that there is a Borel subset $\tilde{\mathscr{D}}$ of $\mathscr{D}_0$ with $P(\tilde{\mathscr{D}}) = 1$, such that for $t \in \Gamma$ and $f \in \tilde{\mathscr{D}}$, $f(t)$ can be determined from the restriction of f to D_2.

Let F_1, $\hat{F}_1$, F_2 and $\hat{F}_2$ be as in Lemma 3.5. For each point $r \in \mathbb{Q}_+^2 \setminus \Gamma$, set

$$(\Gamma, r] = \{t \in \mathbb{R}_+^2 : t \le r \text{ and } (t, r] \cap \Gamma = \varnothing\},$$

let C_r be the monotone decreasing curve which is the lower left boundary of $(\Gamma, r]$ and set $C = \bigcup_r C_r$. This set is a countable union of monotone decreasing curves and has Lebesgue measure 0. Now let $\tilde{\mathscr{D}}$ be the subset of $f \in \mathscr{D}_0$ such that:

(a) no extended discontinuities of f cross in C;
(b) if $t_1 \in \hat{F}_1$, then $f(t) - f(t^{(2)}) = 0$;
(c) if $t_2 \in \hat{F}_2$, then $f(t) - f(t^{(4)}) = 0$.

By Corollary 2.8(ii) and Lemma 3.5, $P(\tilde{\mathscr{D}}) = 1$, and $\tilde{\mathscr{D}}$ is easily seen to be a Borel subset of $\mathscr{D}_0$ [if $(T^n(f)$ enumerates the jumps of f, then (a)–(c) are easily expressed by requiring that these jumps lie in some Borel set].

Fix $t \in \Gamma$ and $f \in \tilde{\mathscr{D}}$. Assume first that $t \notin C$. Then there is a sequence (s^n) converging to t with $s^n \in \Gamma$ and $s_i^n > t_i$, $i = 1, 2$, for all n. But each s^n is a limit point of D_2, so there is also a sequence (r^n) of elements of D_2 converging to t with $r_i^n > t_i$, $i = 1, 2$, for all n. By right continuity, $f(t) = \lim_{n \to \infty} f(r^n)$ and so the value of $f(t)$ can be determined from the restriction of f to D_2.

Now assume $t \in C_r \cap \Gamma$, for some $r \in \mathbb{Q}_+^2 \setminus \Gamma$. If $(t, r] \subset D_2$, then again by right continuity, $f(t)$ can be determined from $f|_{D_2}$. So we assume that $(t, r] \subset \dot{D}_1$. Since $\Gamma = \partial D_2$, there is a sequence $s^n \to t$ with $s^n \in D_2$, for all n.

If t is isolated from above in Γ, then we can choose these (s^n) such that $s_2^n \uparrow t_2$. Now *either* t is also isolated from the right in Γ, in which case t is a point of continuity of f by (b) and (c), so $f(t) = \lim_{n \to \infty} f(s^n)$, *or* t is not isolated from the right in Γ, in which case we can choose these (s^n) so that in addition, $s_1^n \downarrow t_1$, $s_1^n > t_1$. By (c), we get $f(t) = \lim_{n \to \infty} f(s^n)$.

A similar argument applies if t is isolated from the right. So we now assume that t is neither isolated from above nor from the right in Γ. In this case, there are two sequences (s^n) and (w^n) of elements of D_2, both converging to t, with the properties $s^n \in t + Q_2$, $w^n \in t + Q_4$, for all n. Thus $f(t^{(2)}) = \lim_{n\to\infty} f(s^n)$ and $f(t^{(4)}) = \lim_{n\to\infty} f(w^n)$. If these two limits are equal, then by (a), the common limit is $f(t)$. If they are distinct, $f(t)$ is one of these two limits, again by (a). To determine which, we use the fact that D_1 is bounded and D_2 is unbounded. For sufficiently large $u > t_1$, (u, t_2) is an element of D_2. If $f(u, t_2) = f((u, t_2)^{(4)})$, then the discontinuity passing through t is propagating vertically, so $f(t) = \lim_{n\to\infty} f(w^n)$, and otherwise, $f(t) = \lim_{n\to\infty} f(s^n)$. In all cases, we have determined $f(t)$ from $f|_{D_2}$, concluding the proof of (i).

(ii) By (i), we only need to show that $\mathscr{H}(\Gamma^0) \subset \mathscr{H}(\dot{D}_1)$. This proof follows the last part of the proof of (i). Let C, C_r, F_1, $\hat{F}_1$, F_2 and $\hat{F}_2$ be defined with Γ^0 replacing Γ: $C = \bigcup_r C_r$, where C_r is the lower left boundary of

$$(\Gamma^0, r] = \{t \in \mathbb{R}^2_+ : t \le r \text{ and } (t, r] \cap \Gamma^0 = \varnothing\},$$

and then define $\tilde{\mathscr{D}}$ as before. After exchanging the roles of D_1 and D_2 above, the proof remains the same up to the case when $t \in C_r \cap \Gamma_0$ is neither isolated from the right nor from above in Γ^0, $(t, r] \subset D_2$, and

$$f(t^{(2)}) = \lim_{n\to\infty} f(s^n) \neq \lim_{n\to\infty} f(w^n) = f(t^{(4)}).$$

Since D_1 is bounded, we cannot argue as above and need to use the fact that $t \in \Gamma^0 = \Gamma^V \cup \Gamma^H$. Assume that $t \in \Gamma^V$, since the other case is similar. In this case, there is $s \in S_1 \cup S_2$ such that $s_1 = t_1$, $s_2 < t_2$ and $t = \tau(s)$. Since $t \in C_r$ and $(t, r] \subset D_2$, it follows from the definition of τ that for some $\varepsilon > 0$, $\{t_1\} \times (t_2, t_2 + \varepsilon] \subset D_2$, and that $s \in \dot{D}_1$. There are now two possibilities.

Case 1. There is a sequence $u_n \uparrow t_2$ with $u_n > s_2$ and $(t_1, u_n) \in \dot{D}_1$. If for all n, $f(t_1, u_n) = f((t_1, u_n)^{(2)})$, then the discontinuity through t is propagating horizontally and $f(t) = \lim_{n\to\infty} f(s^n)$. Otherwise, it is propagating vertically and $f(t) = \lim_{n\to\infty} f(w^n)$.

Case 2. For some $\eta > 0$, $G = \{t_1\} \times [t_2 - \eta, t_2] \subset \Gamma$. If $G \cap F_1 \neq \varnothing$, then by (b), no vertical discontinuity passes through this set, and thus the discontinuity at t must be propagating horizontally, so $f(t) = \lim_{n\to\infty} f(s^n)$. Now if $G \cap F_1 = \varnothing$, then in particular no point of G is left-isolated in Γ^0, so there is a sequence of elements of $\Gamma^0 \cap (t + Q_3)$ which converges to t. This implies that there is a sequence (z^n) of elements of $\dot{D}_1$ converging to t with the properties $z^n \in t + Q_3$, for all n. We now compare $\lim_{n\to\infty} f(z^n)$ and $\lim_{n\to\infty} f(w^n)$. If the two limits are equal, then the discontinuity through t is propagating horizontally and $f(t) = \lim_{n\to\infty} f(s^n)$. Otherwise, it is propagating vertically and $f(t) = \lim_{n\to\infty} f(w^n)$. In all four cases, we have determined the values of $f(t)$ from $f|_{D_1}$, completing the proof of (ii).

(iii) Recall the following fact ([15], and [18], Section 6): If $\mathscr{S}$ is a splitting field for $\mathscr{A}$ and $\mathscr{B}$ and if $\mathscr{S}'$ is another σ-field such that $\mathscr{S}' \subset \mathscr{A}$ or $\mathscr{S}' \subset \mathscr{B}$, then $\mathscr{S} \vee \mathscr{S}'$ is also a splitting field for $\mathscr{A}$ and $\mathscr{B}$.

If $\mathscr{H}(\Gamma^0)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$, it then follows from the last inclusion in (i) that $\mathscr{H}(\Gamma)$ is also a splitting field. If $\mathscr{H}(\Gamma)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$, then since $\mathscr{H}(\partial D_1) = \mathscr{H}(\Gamma) \vee \mathscr{H}(\partial D_1 \setminus \Gamma)$ and $\partial D_1 \setminus \Gamma \subset \dot{D}_1$, $\mathscr{H}(\partial D_1)$ is also a splitting field. □

For any subset B of $\mathbb{R}_+$ and $d \geq 0$, we set $B^{(d)} = B \times \{d\}$ and, if $B^{(d)} \subset S_i$, $i = 1, 2$, we let

$$V(B^{(d)}) = \{(t_1, t_2) \in \mathbb{R}_+^2 : t_1 \in B, 0 \leq t_2 \leq p(t_1, d)\}.$$

We will usually use this when B is an interval, in which case $V(B^{(d)})$ is essentially a vertical strip bounded above by a piece of the boundary Γ. Let

$$\mathscr{M}(D_1) = \sigma\{Y(V(B^{(d)})) : B^{(d)} \subset S_i,\ B = [a, b],\ a < b,\ d > 0,\ i = 1, 2\} \vee \mathscr{H}^0(\Gamma),$$

where Y is the L^0-valued measure associated with X defined in Section 2.2. The following result is a slight modification of [11], Theorem 3.3.

THEOREM 3.7. *Let X be a Lévy sheet satisfying Assumption B. Then*

$$\mathscr{M}(D_1) = \mathscr{H}(D_1) \cap \mathscr{H}(D_2)$$

and this is the minimal splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$.

PROOF. This theorem was proved in [11] under slightly stronger assumptions. More precisely, X was assumed square integrable with mean 0, and the measure $F \mapsto E(X(F)^2)$ was assumed absolutely continuous with respect to Lebesgue measure. It is however straightforward to check that in the proof of [11], Theorem 3.3, L^2-convergence can be replaced by convergence in probability, and so the result remains valid under Assumption B. □

It follows from Theorem 3.7 that in order to prove that a sub-σ-field $\mathscr{A}$ of $\mathscr{H}(D_2)$ which contains $\mathscr{H}^0(\Gamma)$ is a splitting field, it is sufficient to show that the random variables $Y(V(B^{(d)}))$ are $\mathscr{A}$-measurable. This can be reduced further: By the following lemma, it is enough to show this for the sum of the jumps of X which are in $V(B^{(d)})$ and have magnitude greater than $\varepsilon > 0$. Write $Y = Y^g + Y^j$, where $Y^g(R) = \Delta_R X^g$ and $Y^j(R) = \Delta_R X^j$ for any rectangle $R \in \mathbb{R}_+^2$.

LEMMA 3.8. *Let A be a bounded Borel subset of $\mathbb{R}_+^2$. Then $Y^j(A)$ is measurable with respect to the completion of the σ-field generated by $\{\Pi(A \times dx), x \in \mathbb{R}\}$.*

PROOF. Define

$$Y_\varepsilon^j(A) = \int_{\{|x| > \varepsilon\}} x \Pi(A \times dx) - \int_{\{\varepsilon < |x| < 1\}} x \nu(A \times dx),$$

so that by (11),

$$Y^j(A) - Y^j_\varepsilon(A) = \int_{\{|x| \le \varepsilon\}} x(\Pi(A \times dx) - \nu(A \times dx)).$$

The measure ν is deterministic, so that Y^j_ε is measurable with respect to the given σ-field. If we replace R_t by A in Theorem 2.3—as we can—we see that $Y^j_\varepsilon(A) \to Y^j(A)$ as $\varepsilon \to 0$. □

PROOF OF THEOREM 3.4. By hypothesis, $\mathscr{H}(\Lambda)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$, so $\mathscr{M}(D_1) \subset \mathscr{H}(\Lambda)$. In particular, the variables $Y(V(B^{(d)}))$ are all $\mathscr{H}(\Lambda)$-measurable, where $B^{(d)} \subset S_i$, $B = [a, b]$, $a < b$, $d > 0$, $i = 1, 2$.

In order to show that X^g and X^j have the sharp Markov property with respect to D_1, we let Y^g and Y^j denote the L^0-valued measures associated with X^g and X^j and we use $\mathscr{H}^g(\Lambda)$ and $\mathscr{H}^j(\Lambda)$ to denote the uniform sharp fields associated with X^g and X^j, respectively. Since $\mathscr{H}^j(\Lambda) \subset \mathscr{H}^j(\Gamma) \subset \mathscr{H}^j(D_2)$ by Lemma 3.6(i), $\mathscr{H}^j(\Lambda)$ will be a splitting field for $\mathscr{H}^j(D_1)$ and $\mathscr{H}^j(D_2)$ if we show that $\mathscr{M}^j(D_1) \subset H^j(\Lambda)$. Since Λ is dense in Γ, we have $(\mathscr{H}^j)^0(\Gamma) \subset \mathscr{H}^j(\Lambda)$. So it will be sufficient to show that $Y^j(V(B^{(d)})) \in \mathscr{H}^j(\Lambda)$, for all $B^{(d)} \subset S_i$, $B = [a, b]$, $a < b$, $d > 0$, $i = 1, 2$.

Set $Z_u = Y(V([a, u]^{(d)}))$. Then $Z = \{Z_u,\ a \le u \le b\}$ is a one-parameter process of independent increments which is continuous in probability (since X satisfies Assumption B) and which is $\mathscr{H}(\Lambda)$-measurable by hypothesis. We can thus decompose Z into its independent Gaussian and jump parts Z^g and Z^j, so that $Z = Z^g + Z^j$. Both of these processes are $\mathscr{H}(\Lambda)$-measurable by Theorem 2.6.

The key observation is now that $Y^g(V([a, u]^{(d)})) = Z^g_u$ a.s. and $Y^j(V([a, u]^{(d)})) = Z^j_u$ since $Z_u = Y^g(V([a, u]^{(d)})) + Y^j(V([a, u]^{(d)}))$ is also a decomposition of Z. It follows that $Y^j(V(B^{(d)}))$ and $Y^g(V(B^{(d)}))$ are $\mathscr{H}(\Lambda)$-measurable.

Note that $\mathscr{H}(\Lambda) \subset \mathscr{H}^g(\Lambda) \vee \mathscr{H}^j(\Lambda)$, and that the last two σ-fields are independent. Since $\sigma\{Y^j(V(B^{(d)}))\} \vee \mathscr{H}^j(\Lambda)$ is independent of $\mathscr{H}^g(\Lambda)$, we have

$$\begin{aligned} Y^j(V(B^{(d)})) &= E(Y^j(V(B^{(d)}))|\mathscr{H}(\Lambda)) \\ &= E(Y^j(V(B^{(d)}))|\mathscr{H}^g(\Lambda) \vee \mathscr{H}^j(\Lambda)) \\ &= E(Y^j(V(B^{(d)}))|\mathscr{H}^j(\Lambda)), \end{aligned}$$

so $Y^j(V(B^{(d)}))$ is $\mathscr{H}^j(\Lambda)$-measurable. The proof that $\mathscr{H}^g(\Lambda)$ is a splitting field for $\mathscr{H}^g(D_1)$ and $\mathscr{H}^g(D_2)$ is similar and so is omitted. The proof is complete. □

Having shown in Theorem 3.4 that the Markov property of the Gaussian and jump parts of X can be examined separately, we are going to look more closely at the case when X is a jump process, since the basic Gaussian case was studied in [11]. We have seen in Example 2.2 that such a process X may not satisfy Lévy's sharp Markov property for unbounded sets. However, it does satisfy it for bounded open sets. The following theorem is the main result of the paper.

THEOREM 3.9. *Let $X = \{X(t),\ t \in \mathbb{R}_+^2\}$ be a right-continuous Lévy sheet satisfying Assumption* B *and having no Gaussian part. Then X has Lévy's sharp Markov property with respect to all open sets D which are bounded or have bounded complement. More precisely, if D_1 is a bounded open set, then $\mathscr{H}(\Gamma^0)$ is the minimal splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(\overline{D}_1^c)$, and both $\mathscr{H}(\Gamma)$ and $\mathscr{H}(\partial D_1)$ are splitting fields.*

For $\varepsilon > 0$ and $t \cup \mathbb{R}_+^2$, set $\tilde{Q}_i(t, \varepsilon) = (t + Q_i) \cap \{s\colon |s - t| < \varepsilon\}$. These are quarter-discs of radius ε centered at t. Let us denote the subsets of Γ^0 which are limit points of Γ^0 in one of the four quadrants by $\Gamma_i^0 = \{t \in \Gamma^0\colon \Gamma^0 \cap \tilde{Q}_i(t, \varepsilon) \neq \varnothing, \forall\, \varepsilon > 0\}$, $i = 1, \dots, 4$.

PROOF OF THEOREM 3.9. We will assume, as we can, that X is defined canonically on the (Blackwell) subspace $\mathscr{D}_0$ of the space $\mathscr{D}$, provided with its topological σ-field $\mathscr{B}(\mathscr{D})$. We are again going to use Blackwell's theorem and Remark 3.1, so we are going to restrict ourselves to a Borel subset $\mathscr{D}_1$ of $\mathscr{D}_0$ with full measure, which we now construct.

Let F_1, $\hat{F}_1$, F_2 and $\hat{F}_2$ be as in Lemma 3.5. Choose rational $0 \le a < b$ and $d \ge 0$ such that $[a, b] \times \{d\} \subset S_i$, $i = 1$ or 2, and consider $V = V([a, b] \times \{d\})$. Let ∂V^+ be the graph of the function $\phi = \phi_{a,b,d}$ defined by $\phi(u) \equiv p(u, d)$, $a \le u \le b$. It has measure 0, so by Corollary 2.8, for a.e. ω, we have for all rational a, b and d, and for all $t \in \partial V^+$ that:

(a) No extended discontinuities of $X(\cdot, \omega)$ cross at t, and, moreover, $X(\cdot, \omega)$ has at most one jump on the pair of horizontal and vertical lines passing through t.

It follows that either $X(\cdot, \omega)$ is continuous at t, or else t lies on a single propagating discontinuity. The discontinuity can propagate either vertically or horizontally, and we have from property (ii) of Section 2.4 that:

(b) $X(t, \omega) - X(t^{(2)}, \omega) = X(t^{(4)}, \omega) - X(t^{(3)}, \omega) = Y(\{t_1\} \times (0, t_2], \omega)$. If this is nonzero, then $X(t^{(4)}, \omega) = X(t, \omega)$.

(c) $X(t, \omega) - X(t^{(4)}, \omega) = X(t^{(2)}, \omega) - X(t^{(3)}, \omega) = Y((0, t_1] \times \{t_2\}, \omega)$. If this is nonzero, then $X(t, \omega) = X(t^{(2)}, \omega)$.

(d) If $t_1 \in \hat{F}_1$, $X(t, \omega) - X(t^{(2)}, \omega) = 0$.

(e) If $t_2 \in \hat{F}_2$, $X(t, \omega) - X(t^{(4)}, \omega) = 0$.

We will define functions $\psi_{a,b,d}$ and $\hat{\psi}_{a,b,d}$ below. It will be clear that they satisfy the conditions of Corollary 2.8(iii), so that we can also require that:

(f) For each rational a, b and d, for each pair $\phi = \phi_{a,b,d}$, $\psi = \psi_{a,b,d}$ and for each pair $\phi = \phi_{a,b,d}$, $\psi = \hat{\psi}_{a,b,d}$, there is no $u \geq 0$ such that $\phi(u)$ and $\psi(u)$ are on different extended discontinuities.

Now let $\mathscr{D}_1$ be the space of functions $f \in \mathscr{D}_0$ such that for all rational a, b and d, and for all $t \in \partial V^+([a,b] \times \{d\})$:

(i) no extended discontinuities of f cross at t;
(ii) if $t_1 \in \hat{F}_1$, $f(t) - f(t^{(2)}) = 0$;
(iii) if $t_2 \in \hat{F}_2$, $f(t) - f(t^{(4)}) = 0$;
(iv) for each pair $\phi = \phi_{a,b,d}$, $\psi = \psi_{a,b,d}$, and for each pair $\phi = \phi_{a,b,d}$, $\psi = \hat{\psi}_{a,b,d}$ there is no u such that $\phi(u)$ and $\psi(u)$ are on different extended discontinuities.

It is not difficult to show that $\mathscr{D}_1 \in \mathscr{B}(\mathscr{D})$ [if $(T^n(f))$ enumerates the jumps of $f \in \mathscr{D}$, then (i)–(iv) are easily expressed by requiring that these jumps lie in some Borel set], and hence $(\mathscr{D}_1, \mathscr{B}(\mathscr{D}_1))$ is a Blackwell space. By (a)–(f) above, X can be canonically defined on $\mathscr{D}_1$, and we can assume that (a)–(f) hold for *every* $\omega \in \mathscr{D}_1$.

If $\mathscr{H}(\Gamma^0)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(\overline{D}_1^c)$, then Lemma 3.6 implies that both $\mathscr{H}(\Gamma)$ and $\mathscr{H}(\partial D_1)$ are also. So by Theorem 3.7, we only need to show that $\mathscr{H}(\Gamma^0) = \mathscr{M}(D_1)$.

Theorem 3.7 and Lemma 3.6(ii) give us the inclusion $\mathscr{H}(\Gamma^0) \subset \mathscr{M}(D_1)$, so we need only show the converse. By Theorem 3.7 and Lemma 3.6(i), we only need to show that $Y(V(B^{(d)}))$ is $\mathscr{H}(\Gamma^0)$-measurable, for rational $B = [a,b]$ and $d > 0$ with $B^{(d)} \subset S_i$, $i = 1, 2$.

Since the measure Y is just the sum of the jumps of X plus a continuous part, $Y(\{t_1\} \times (0, t_2))$ is the value of the jump (there can be at most one—see Corollary 2.8) of X on the line segment $\{t_1\} \times (0, t_2)$. It follows that for any $\varepsilon > 0$, with probability 1,

$$\int_{\{|x| > \varepsilon\}} x \Pi(V([a,b] \times \{d\}) \times dx)$$

$$= \sum_{t \in \partial V^+,\, t \notin F_1} \left(X(t) - X(t^{(2)})\right) I_{\{|X(t) - X(t^{(2)})| > \varepsilon\}}.$$

By Lemma 3.8, the proof will be finished once we show that on $\mathscr{D}_1$, the right-hand side is measurable with respect to the trace of $\tilde{\mathscr{H}}(\Gamma^0)$ (recall that this is the uncompleted uniform sharp field). By Blackwell's theorem, it is enough to show that the right-hand side is Borel measurable and that on $\mathscr{D}_1$, it is a function of $X|_{\Gamma^0}$. The measurability is clear, since the right-hand side just adds the vertically propagating discontinuities of the sheet of size greater than ε which intersect the graph of the restriction of the function ϕ to the Borel set $[a,b] \setminus \hat{F}_1$. It is the fact that it is a function of $X|_{\Gamma^0}$ which is delicate.

In order to prove this, we only need to show that for each $t \in \partial V^+ \setminus F_1$, we can determine the value of $X(t) - X(t^{(2)})$ from $X|_{\Gamma^0}$.

Let us first remark that if $r \in \Gamma_i^0$, then $X(t^{(i)})$ is a function of $X|_{\Gamma^0}$, for one can compute the limit in the ith quadrant since t is an accumulation point $\Gamma^0 \cap \tilde{Q}_i(t, \varepsilon)$ for any ε.

Let $t \in \partial V^+ \setminus F_1 \subset \Gamma^0$. Since t is not left-isolated in Γ^0, then $t \in \Gamma_2^0 \cup \Gamma_3^0$. We claim that:

1. the magnitude of any vertical discontinuity propagating through t is a function of $X|_{\Gamma^0}$; indeed, it equals

$$|X(t) - X(t^{(2)})|, \quad \text{if } t \in \Gamma_2^0,$$

$$|X(t) - X(t^{(3)})|, \quad \text{if } t \in \Gamma_3^0 \setminus \Gamma_2^0;$$

2. if $t \in \Gamma_2^0 \cup \Gamma_3^0 \cup \Gamma_4^0$, then $X(t) - X(t^{(2)})$ is a function of $X|_{\Gamma^0}$.

Indeed, if $t \in \Gamma_2^0$, then $X(t) - X(t^{(2)})$ is a function of $X|_{\Gamma^0}$. If $t \notin \Gamma_2^0$, then $t \in \Gamma_3^0$; if in fact $t \in \Gamma_3^0 \cap \Gamma_4^0$, then $X(t^{(3)})$ and $X(t^{(4)})$ are functions of $X|_{\Gamma^0}$, hence so is $X(t) - X(t^{(2)}) = X(t^{(4)}) - X(t^{(3)})$ [by (b)]. If $t \in \Gamma_3^0 \setminus (\Gamma_2^0 \cup \Gamma_4^0)$, then $X(t) - X(t^{(3)})$ is a function of $X|_{\Gamma^0}$. If it is 0, X is continuous at t. If not, X has a jump on either the horizontal or the vertical line passing through t of magnitude $|X(t) - X(t^{(3)})|$. This puts us in the case where $t \in \Gamma_3^0 \setminus (\Gamma_2^0 \cup \Gamma_4^0)$ and $X(t) - X(t^{(3)}) \neq 0$. An inspection of (b) and (c) shows that in this case there is either a jump of X of size $X(t) - X(t^{(3)})$ below t or to the left of t. We need the direction of propagation of the discontinuity to determine $X(t) - X(t^{(2)})$, which will be 0 if the direction is horizontal, and will be equal to $X(t) - X(t^{(3)})$ otherwise.

There are two cases. The segment $[a, b] \times \{d\}$ is either in $\dot{D}_1$ or in $\dot{D}_2$. Suppose first that it is in $\dot{D}_2$. Then $t \in \tau(S_2)$, so that there are points of $\dot{D}_1$ above t, and consequently there are points of $\Gamma^V \subset \Gamma^0$ above t (because $\dot{D}_1$ is bounded). Define a function $\psi = \psi_{a,b,d}$ on $[a, b]$ as follows. Let (r_n) be an ordering of the positive rationals, and, for each $u \in [a, b]$, let $r(u)$ be the rational of smallest index for which $r_n > \phi(u)$ and $(u, r_n) \in \dot{D}_1$. Then let $\psi(u) = p((u, r(u)))$ (this is well defined since D_1 is bounded). The point $\zeta \equiv (t_1, \psi(t_1))$ lies above $t = (t_1, \phi(t_1))$. Now $t_1 \notin \hat{F}_1$, so as above, $\zeta \in \Gamma_2^0 \cup \Gamma_3^0$. Thus by claim 1, the magnitude of the discontinuity (if any) propagating through ζ is a function of $X|_{\Gamma^0}$. If it is nonzero, there is a discontinuity passing through $\zeta = \psi(t_1)$ as well as a discontinuity passing through t; by (f), they must be the same, and hence the discontinuity passing through t must be vertical.

Now suppose $[a, b] \times \{d\} \subset \dot{D}_1$. If $t_2 \in \hat{F}_2$, there is no discontinuity propagating horizontally through t by (e), so the discontinuity must be vertical, and we are done. Thus suppose $t_2 \notin \hat{F}_2$. Define $\hat{\psi} = \hat{\psi}_{a,b,d}$ by setting $\hat{\psi}(t_1) = \inf\{u > t_1\colon (u, t_2) \in \dot{D}_2\}$. Note that $\hat{\psi}$ is bounded since the domain D_1 is bounded. Moreover, $\hat{\psi}(t_1) > t_1$. Indeed, if $\hat{\psi}(t_1) = t_1$, there would be a sequence of u^j decreasing to t_1 such that $(u^j, t_2) \in \dot{D}_2$. As $\dot{D}_2$ is open, $\{u^j\} \times [t_2 - \varepsilon_j,$

$t_2 + \varepsilon_j] \subset \dot{D}_2$ for some $0 < \varepsilon_j < 1/j$. Now if for some j the open rectangle

$$R_j = (t_1, u^j) \times (t_2 - \varepsilon_j, t_2)$$

were included in $\dot{D}_2$, then by the definition of τ, $\{t_1\} \times (t_2 - \varepsilon_j, t_2] \subset \Gamma$ and the points on this vertical segment would be right-isolated. This is not possible since $t_1 \notin \hat{F}_1$. So, there is $s^j \in \overline{D}_1 \cap R^j$, hence there is a point $r^j = (r_1^j, r_2^j) \in \dot{D}_1 \cap R^j$. We can conclude that there exist points of Γ^0 between (r_1^j, r_2^j) and (u^j, r_2^j). For large enough j, such points are in $\tilde{Q}_4(t, \varepsilon)$. This implies that $t \in \Gamma_4^0$, a contradiction.

Consider the point $\sigma = (\hat{\psi}(t_1), t_2)$. It is not isolated from below, for $t_2 \notin \hat{F}_2$, so, modifying claim 1, the magnitude of a horizontal discontinuity at σ is a function of $X|_{\Gamma^0}$; if it is 0, the discontinuity at t is vertical (if it were horizontal, it would have to pass through σ); if it is nonzero, there is a discontinuity passing through σ. According to (f), it must be the same discontinuity as the one passing through t, which must therefore be horizontal. Thus in all cases we can see if the discontinuity through t is propagating horizontally or vertically by looking at $X|_{\Gamma^0}$. This completes the proof. □

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