*The Annals of Probability*
1992, Vol. 20, No. 4, 2117–2140

# INTERNAL DIFFUSION LIMITED AGGREGATION[1]

BY GREGORY F. LAWLER, MAURY BRAMSON AND DAVID GRIFFEATH

*Duke University, University of Wisconsin and University of Wisconsin*

We study the asymptotic shape of the occupied region for an interacting lattice system proposed recently by Diaconis and Fulton. In this model particles are repeatedly dropped at the origin of the $d$-dimensional integers. Each successive particle then performs an independent simple random walk until it "sticks" at the first site not previously occupied. Our main theorem asserts that as the cluster of stuck particles grows, its shape approaches a Euclidean ball. The proof of this result involves Green's function asymptotics, duality and large deviation bounds. We also quantify the time scale of the model, establish close connections with a continuous-time variant and pose some challenging problems concerning more detailed aspects of the dynamics.

**1. Introduction.** We study the following simple multidimensional lattice dynamic that starts with only the origin of the $d$-dimensional integers $\mathbb{Z}^d$ occupied and grows a random cluster over time.

THE RULE. One by one, particles perform independent simple symmetric $d$-dimensional discrete-time random walks $X^i(t)$, $i = 1, 2, \dots$ . Each particle starts from the origin $\mathbf{0}$ and moves until it reaches a site that has not been visited previously, at which point it stops. Let $A(n)$ denote the cluster of occupied sites after the $n$th particle stops. By convention, $A(0) = \{\mathbf{0}\}$. Thus the process $A(n)$ is a Markov chain with transition probabilities

$$A \to A \cup \{x\} \quad \text{with probability } h(A^c, x),$$

where $h_z(B, C)$ denotes the probability that a random walk starting from $z$ first hits the set of sites $B$ at some site in set $C$, and we abbreviate $h(B, x) = h_{\mathbf{0}}(B, \{x\})$.

We first learned about this rule from Persi Diaconis (private communication). In reference [3], Diaconis and Fulton have formulated a way to "multiply" subsets of a commutative ring $R$ that (in a special case) makes reference to an underlying discrete-time Markov chain $X(t)$ with state space $R$. If $R$ is the $d$-dimensional integers and $X(t)$ is simple random walk, repeated "multiplication" of $\{\mathbf{0}\}$ by itself gives rise to the process $A(n)$.

Their model is most succinctly described as an internal variant of the celebrated Witten–Sander [12] rule for *diffusion limited aggregation* (DLA). The Witten–Sander model starts with only $\mathbf{0}$ occupied and repeatedly sends

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Received June 1991; revised December 1991.

[1]Partially supported by research grants to each of the authors from the NSF.

*AMS* 1980 *subject classification*. Primary 60K35.

*Key words and phrases.* Growth model, random walk, interacting particle system, diffusion limited aggregation.

random walks "in from $\infty$": Each walk stops as soon as it neighbors the previously occupied cluster. DLA exhibits intriguing dendritic growth that is widely believed to enjoy fractal characteristics. See reference [12], or Plate 12 in Toffoli and Margolus [11] for pictures. Very roughly, tendrils result from the propensity of exterior wandering particles to first hit the neighborhood of extreme sites in the occupied cluster. By way of contrast, in the Diaconis–Fulton rule particles diffusing through the interior of the occupied cluster are most likely to stop at unoccupied sites that are closest to $\mathbf{0}$. In other words, whereas DLA tends to exacerbate irregularities, $A(n)$ tends to eliminate them and so might well be expected to grow like an expanding ball. We will call this model *internal DLA*.

For a preliminary indication that $A(n)$ is well behaved, consider the one-dimensional situation. If $d = 1$, the occupied cluster of the model is always an (integer) interval. The dynamics of this interval $[l, r]$ are governed by Friedman's "safety campaign" urn model [7]. Namely, simple gambler's ruin considerations show that the next particle adjoins the left and right ends with probabilities $(r + 1)/(l + r + 2)$ and $(l + 1)/(l + r + 2)$, respectively ($l = r = 0$ at time $n = 0$). Whereas the celebrated Polya urn scheme reinforces inequities, Friedman's urn has a central tendency. In our setting this means that with overwhelming probability both the left and right edges of $A(n)$ are close to $n/2$. In fact, the fluctuations of the edges about $n/2$ are Gaussian, of order $C\sqrt{n}$ for a computable constant $C$ (cf., Freedman [6]). Evidently internal DLA on the integers is quite simple and amenable to explicit analysis. So, except for a small remark toward the end of the paper, *we assume henceforth that the dimension of the lattice is at least 2*.

How does $A(n)$ grow for $d \geq 2$? As already indicated, one expects the occupied cluster to exhibit a fairly regular and predictable shape once it is reasonably large. With a little work, isotropy of Brownian motion and the invariance principle imply that the hitting distribution of simple random walk for large lattice spheres is asymptotically uniform. So if $A(n)$ has a limiting shape after suitable normalization, then it is not too hard to show that this shape must be a Euclidean ball. Roughly speaking, if the boundary of the asymptotic shape were not isotropic, then a disproportionately high density of particles would stick along regions of the boundary that were closest to the origin. Computer simulations also indicate that internal DLA produces a spherical asymptotic shape. For instance, if we simulate the two-dimensional chain $A(n)$ until time $n = \pi \cdot 100^2$, the maximal ($l^\infty$) distance from any occupied site to the lattice ball $\{\|x\| \leq 100\}$ rarely exceeds 3.

There is by now a sizable literature on "shape theory" for lattice interactions; a very nice introduction is provided by Durrett [4]. Lattice systems for which asymptotic shape results can be proved include: branching random walks, first-passage percolation, the Richardson and Williams–Bjerknes growth models, contact processes and various epidemic and forest-fire dynamics. The limiting shape for branching random walks, described by a Legendre transform (Biggins [1]), is not isotropic. Shapes of the other systems just mentioned are characterized implicitly by means of the subadditive ergodic theorem, so rigorous quantitative descriptions are lacking. However, current empirical consensus suggests that none of these shapes is a Euclidean ball: Cluster growth retains the effect of lattice anisotropy even in the limit. In this connection one should mention Eden's famous process [5], arguably the oldest and simplest of all growth models. Identifying each site $x \in \mathbb{Z}^d$ with a $d$-dimensional unit cube in $\mathbb{R}^d$ centered at $x$, Eden's rule recursively pastes a new cube to a randomly chosen boundary face of the occupied cluster. Early Monte Carlo simulations suggested that Eden's model in $\mathbb{Z}^2$ might be asymptotically circular. Since then, Kesten (unpublished) has proved that the limiting shape is not a ball if $d$ is sufficiently large, and recent supercomputer experiments indicate persuasively that even for $d = 2$ the limit differs slightly from a ball. Evidently the issue of isotropic shape is a delicate one.

Our primary objective in this paper is to show that the asymptotic shape of an internal DLA cluster is, in fact, a Euclidean ball. We are motivated to some extent by a hope that the Markov chain $A(n)$ will provide a relatively tractable prototype for the spread of certain complex spatial structures. In addition, as indicated by the previous paragraph, our result provides a rare instance where the geometry of the limiting shape can be identified explicitly. To the best of our knowledge this is the first lattice rule for which isotropic growth has been rigorously established.

Let $\omega_d$ be the volume of the $d$-dimensional Euclidean ball of radius 1, and let $\mathfrak{B}_r = \{x \in \mathbb{Z}^d\colon \|x\| < r\}$ be the $d$-dimensional "lattice ball" of radius $r$. Extend the process $A(\cdot)$ to real $t \geq 0$ by setting $A(t) = A(\lfloor t \rfloor)$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to $t$. Our main result is:

THEOREM 1. *At time $\omega_d n^d$, internal DLA occupies a set of sites close to a $d$-dimensional ball of radius $n$. More precisely, for any $\varepsilon > 0$,*

$$\mathfrak{B}_{n(1-\varepsilon)} \subset A(\omega_d n^d) \subset \mathfrak{B}_{n(1+\varepsilon)} \quad \text{for all sufficiently large } n \tag{1.1}$$

*with probability* 1.

Our argument for Theorem 1 is divided into two parts. Part A, the main step, shows that for any $\varepsilon > 0$, with probability 1,

$$\mathfrak{B}_{n(1-\varepsilon)} \subset A(\omega_d n^d) \quad \text{for all sufficiently large } n. \tag{1.2}$$

Making use of (1.2), Part B shows that with probability 1,

$$A(\omega_d n^d) \subset \mathfrak{B}_{n(1+\varepsilon)} \quad \text{for all sufficiently large } n. \tag{1.3}$$

The proofs of (1.2) and (1.3) rely on Green's function asymptotics for simple random walk, as well as large deviation bounds. We have collected these preliminary results in Section 2, making frequent reference to Lawler [10] for needed random walk estimates. Parts A and B are completed in Sections 3 and 4, respectively.

Sections 5 and 6 discuss two variants of internal DLA. The first variant is simply the same process running on the time scale of individual random walk steps. In essence, Theorem 2 asserts that $(d\omega_d/(d+2))n^{d+2}$ steps are needed to grow a lattice ball of radius $n$. The second variant is a continuous time system in which particles are dropped at the origin according to a rate 1 Poisson stream and execute independent rate 1 continuous time random walks thereafter until they stick at a site not previously occupied by any other particle. For this model, let $B_t$ denote the set of sites occupied by one or more particles at time $t$. By means of a coupling argument, we prove in Theorem 3 that $B_t$ satisfies the same shape result (1.1) as $A(n)$ (with $n$ replaced by $t$) provided $d \geq 3$. Our proof breaks down for the two-dimensional continuous time system. Indeed, Monte Carlo simulation and heuristic reasoning suggest an interesting "hydrodynamic" effect in that case. We conclude Section 6 with a computer graphic and a formal calculation to illustrate the intriguing behavior when $d = 2$. Finally, in Section 7, we mention models with several sources of particles and also offer a few tentative remarks concerning the asymptotic order of fluctuations at the edge of the growth cluster.

**2. Preliminaries.** We first review a few facts from probabilistic potential theory. Let $\mathbf{P}_y$ and $\mathbf{E}_y$ be the probability law and expectation operator, respectively, of a random walk $X(t)$ starting from $y \in \mathbb{Z}^d$. For $d \geq 3$, the *Green's function* $G$ is defined by

$$G(y,z) = \mathbf{E}_y\left[\sum_{t=0}^{\infty} 1_{\{X(t)=z\}}\right], \qquad z \in \mathbb{Z}^d.$$

As $\|z\| \to \infty$, $G(0,z)$ is asymptotic to constant $\cdot \|z\|^{2-d}$, which is not surprising since the latter is the Green's function of Brownian motion. We will require estimates for $G_n$, the Green's function of random walk stopped upon leaving $\mathfrak{B}_n$. Introduce $\tau_n = \min\{t\colon X(t) \notin \mathfrak{B}_n\}$ and let

$$G_n(y,z) = \mathbf{E}_y\left[\sum_{t=0}^{\tau_n - 1} 1_{\{X(t)=z\}}\right], \qquad z \in \mathfrak{B}_n.$$

Since the total number of visits to the origin starting at a point $z \in \mathfrak{B}_n$ can be split into those that occur before $\tau_n$ and those after $\tau_n$,

$$G(\mathbf{0},z) = G(z,\mathbf{0}) = G_n(z,\mathbf{0}) + \mathbf{E}_z\big[G(X(\tau_n),\mathbf{0})\big],$$

the asymptotics for $G$ suggest that $G_n(\mathbf{0},z)$, which equals $G_n(z,\mathbf{0})$ by symmetry, should be approximately constant $\cdot (\|z\|^{2-d} - n^{2-d})$.

The Green's function $G_n$ is defined equally well for $d = 2$, although $G$ is not. Nevertheless the same type of analysis can be carried out using the potential kernel

$$a(y,z) = \lim_{t\to\infty} \mathbf{E}_y\left[\sum_{s=0}^{t} \big(1_{\{X(s)=y\}} - 1_{\{X(s)=z\}}\big)\right].$$

In this case $a(\mathbf{0},z)$ is asymptotic to constant $\cdot \ln\|z\|$, one can show that

$$-a(\mathbf{0},z) = -a(z,\mathbf{0}) = G_n(z,\mathbf{0}) - \mathbf{E}_z\big[a(X(\tau_n),\mathbf{0})\big]$$

and $G_n(\mathbf{0}, z)$ behaves like constant $\cdot$ $(\ln n - \ln\|z\|)$. A precise development of these ideas, including an analysis of the error in the asymptotics, can be found in reference [10].

In Lemma 1 we collect the facts about Green's function asymptotics that we will need. Two estimates for $G_n(\mathbf{0}, z)$ will be used when $d \geq 3$: one when $z$ is small compared to $n$; the other when $z$ is large. Error terms denoted by $O$ or $o$ may depend on both $n$ and $z$, but are uniform in the unspecified variable. To avoid exceptional cases when $z = \mathbf{0}$, let us denote $[\![z]\!] = \max\{\|z\|, 1\}$.

LEMMA 1. *Let $z \in \mathfrak{B}_n$. Then*

$$(2.1)\quad G_n(\mathbf{0}, z) = \frac{2}{\pi} \ln \frac{n}{[\![z]\!]} + o\left(\frac{1}{[\![z]\!]}\right) + O\left(\frac{1}{n}\right), \qquad d = 2,$$

$$(2.2\text{a})\quad = G(\mathbf{0}, z) + O(n^{2-d}), \qquad d \geq 3,$$

$$(2.2\text{b})\quad = \frac{2}{d-2} \frac{1}{\omega_d}([\![z]\!]^{2-d} - n^{2-d}) + O([\![z]\!]^{1-d}), \qquad d \geq 3.$$

*Moreover, if $z \in \mathfrak{B}_{n(1-\varepsilon)}$, then the following elementary inequalities hold*:

$$(2.3)\quad G_{\varepsilon n}(\mathbf{0}, \mathbf{0}) \leq G_n(z, z) \leq G_{2n}(\mathbf{0}, \mathbf{0}).$$

PROOF. See Theorem 1.6.6, Proposition 1.6.7 and Proposition 1.5.9 of [10]. $\square$

Our next preliminary result is a familiar stopped martingale inequality.

LEMMA 2. *If $z \in \mathfrak{B}_n$, then*

$$n^2 - \|z\|^2 \leq \mathbf{E}_z[\tau_n] \leq (n+1)^2 - \|z\|^2.$$

PROOF. Consider the martingale $M(t) = \|X(t)\|^2 - t$. By the optional sampling theorem [see (1.21) of [10] for details], $\mathbf{E}_z[M(\tau_n)] = \mathbf{E}_z[M(0)]$, that is,

$$\mathbf{E}_z\left[\|X(\tau_n)\|^2\right] - \mathbf{E}_z[\tau_n] = \|z\|^2. \qquad \square$$

Lemmas 1 and 2 pave the way for a key inequality in our derivation of the lower bound (1.2). Namely, Lemma 3 says that for each $z \in \mathfrak{B}_{n(1-\varepsilon)}$, the average value of $G_n(y, z)$ over all $y \in \mathfrak{B}_n$ is bounded above by $G_n(\mathbf{0}, z)$. The continuous analog of this inequality holds even with $\varepsilon = 0$ and is easy to verify. Suppose $\mathfrak{B}$ is the Euclidean ball of radius 1 centered at $\mathbf{0}$ and let $g(y, z)$ be the usual Green's function for Brownian motion killed upon leaving the ball. Fix $z \in \mathfrak{B}$. Since $g(y, z)$ is harmonic for $\|y\| < \|z\|$, the average value of $g(y, z)$ on any sphere about $\mathbf{0}$ of radius smaller than $\|z\|$ equals $g(\mathbf{0}, z)$. Similarly, it is easy to see that if $\|z\| < r < 1$, then the average value of $g(y, z)$ over the sphere of radius $r$ is less than $g(\mathbf{0}, z)$. [In fact, the average value of $g(y, z)$ on the sphere of radius $r$ equals $g(\mathbf{0}, z) - g_r(\mathbf{0}, z)$, where $g_r(\mathbf{0}, z)$ is the Green's function on the ball of radius $r$.] Since the average value of $g(y, z)$ on any sphere is less than or equal to $g(\mathbf{0}, z)$, the average value of $g(y, z)$ on $\mathfrak{B}$ is less than or equal to $g(\mathbf{0}, z)$.

LEMMA 3. *Fix $\varepsilon > 0$. For $n$ sufficiently large and $z \in \mathfrak{B}_{n(1-\varepsilon)}$,*

$$\sum_{y \in \mathfrak{B}_n} G_n(y, z) \le \omega_d n^d G_n(\mathbf{0}, z). \tag{2.4}$$

PROOF. Since $G_n$ is symmetric,

$$\mathbf{E}_z[\tau_n] = \sum_{y \in \mathfrak{B}_n} G_n(z, y) = \sum_{y \in \mathfrak{B}_n} G_n(y, z). \tag{2.5}$$

By Lemma 2, this quantity is at most $(n+1)^2 - \|z\|^2$. On the other hand, by (2.1),

$$\omega_d n^d G_n(\mathbf{0}, z) = 2n^2 \ln \frac{n}{[\![z]\!]} + o\left(\frac{1}{[\![z]\!]}\right) n^2 + O(n)$$

if $d = 2$. So to verify (2.4) in two dimensions, it suffices to show that for all large $n$,

$$(n+1)^2 - [\![z]\!]^2 \le 2n^2 \ln \frac{n}{[\![z]\!]} + o\left(\frac{1}{[\![z]\!]}\right) n^2 + O(n).$$

A little arithmetic shows that this is equivalent to

$$\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{n}{[\![z]\!]}\right)^2 \le 1 + 2\left(\frac{n}{[\![z]\!]}\right)^2 \ln \frac{n}{[\![z]\!]} + o\left(\frac{1}{[\![z]\!]}\right)\left(\frac{n}{[\![z]\!]}\right)^2. \tag{2.6}$$

By elementary calculus, the inequality

$$(1+\delta)\zeta^2 < 1 + 2(1-\delta)\zeta^2 \ln \zeta$$

holds for $\zeta \in [(1-\varepsilon)^{-1}, \infty)$ if $\delta = \delta(\varepsilon)$ is close enough to 0. It is therefore easy to check that (2.6) holds for $n$ large and $z \in \mathfrak{B}_{n(1-\varepsilon)}$. Hence (2.4) holds for $d = 2$.

For $d \ge 3$, the reasoning is similar. From (2.2b), (2.5) and Lemma 2 one obtains the following counterpart of (2.6):

$$\frac{d}{d-2}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{n}{[\![z]\!]}\right)^2 \le 1 + \left\{\frac{2}{d-2} + O\left(\frac{1}{[\![z]\!]}\right)\right\}\left(\frac{n}{[\![z]\!]}\right)^d. \tag{2.7}$$

One can check that

$$\frac{d(1+\delta)}{d-2}\zeta^2 < 1 + \frac{2(1-\delta)}{d-2}\zeta^d$$

holds for $\zeta \in [(1-\varepsilon)^{-1}, \infty)$, if $\delta = \delta(\varepsilon)$ is small. It follows that (2.7) and hence (2.4) hold when $n$ is large and $[\![z]\!] \ge r$ for a suitable $r$. If $[\![z]\!] < r$, we apply

(2.2a) instead to obtain the sufficient condition

$$(n+1)^2 - [\![z]\!]^2 \le \omega_d n^d\big(G(\mathbf{0},z) + O(n^{2-d})\big).$$

If $n$ is large enough, this last inequality holds simultaneously for all $z$ with $[\![z]\!] < r$, so (2.4) holds in this case as well. □

Next, we present a rather general large deviations estimate for sums of independent indicators (i.e., $\{0,1\}$-valued random variables).

LEMMA 4. *Let $S$ be a finite sum of independent indicator random variables with mean $\mu$. For any $0 < \gamma < 1/2$, and for all sufficiently large $\mu$,*

$$\mathbf{P}\big(|S-\mu| \ge \mu^{1/2+\gamma}\big) \le 2\exp\{-\tfrac{1}{4}\mu^{2\gamma}\}.$$

PROOF. This straightforward estimate is, for example, a special case of Lemma 4.3 in Bramson and Lebowitz [2]. We include a brief derivation for the sake of completeness. Write

$$S = \sum_1^n 1_{A_k} \quad \text{with } \mathbf{P}(A_k) = p_k.$$

Then

$$\mathbf{E}[e^{\lambda(S-\mu)}] = \prod_1^n \big[(1-p_k)e^{-\lambda p_k} + p_k e^{\lambda(1-p_k)}\big].$$

By Chebyshev's inequality, for $\lambda > 0$,

$$\text{(2.8)} \quad \mathbf{P}\big(S-\mu \ge \mu^{1/2+\gamma}\big) \le \exp\{-\lambda\mu^{1/2+\gamma}\}\prod_1^n \big[(1-p_k)e^{-\lambda p_k} + p_k e^{\lambda(1-p_k)}\big].$$

For $\lambda$ small, elementary estimation shows that the bracketed quantities on the right are at most $1 + \lambda^2 p_k$. So the sum of their logarithms is at most

$$\lambda^2 \sum_1^n p_k = \lambda^2\mu.$$

Hence the left side of (2.8) is at most

$$\exp\{-\lambda\mu^{1/2+\gamma}(1-\lambda\mu^{1/2-\gamma})\}.$$

For $\mu$ large and $\lambda = \frac{1}{2}\mu^{\gamma-1/2}$, we therefore obtain

$$\mathbf{P}\big(S-\mu \ge \mu^{1/2+\gamma}\big) \le \exp\{-\tfrac{1}{4}\mu^{2\gamma}\}.$$

Analogous estimation of $\mathbf{E}[e^{-\lambda(S-\mu)}]$ yields the same upper bound for $\mathbf{P}(S-\mu \le -\mu^{1/2+\gamma})$. □

In Part B of our main argument for the upper bound (1.3), we will want to introduce *shells* $\mathscr{S}_k$, $k = 0, 1, \dots,$ defined by

$$\mathscr{S}_k = \{x : k \le \|x\| < k+1\}.$$

Repeated use will be made of the fact that a random walk $X(t)$ cannot exit the ball $\mathfrak{B}_k$, $k \geq 1$, without hitting the shell $\mathscr{S}_k$. The final lemma of this section presents two inequalities concerning hitting probabilities for shells. Inequality Lemma 5(a) is similar to the well-known "gambler's ruin" formula for one-dimensional random walks. Our estimate is hardly surprising: In order for a random walk starting in $\mathscr{S}_k$ to hit $\mathscr{S}_j$ before returning to $\mathscr{S}_k$, its radial component, essentially a one-dimensional random walk, must travel up to a distance of $\Delta = k - j$ before returning to the origin. Inequality (b) of Lemma 5 asserts that the hitting distribution of $\mathscr{S}_k$ for a random walk starting from $\mathscr{S}_j$ is fairly spread out over $\mathscr{S}_k$. It is easy to guess the amount of spread heuristically: Since the radial component of $X(t)$ must move distance $\Delta$ to hit $\mathscr{S}_k$, we would expect the remaining components to move a distance of about $\Delta$ as well. The number of points in $\mathscr{S}_k$ within distance constant $\cdot\, \Delta$ of a point in $\mathscr{S}_j$ is of order $\Delta^{d-1}$. Hence the probability that any particular point in $\mathscr{S}_k$ is hit first should be bounded above by constant $\cdot\, \Delta^{1-d}$. Below, we let $T_k = \min\{t \geq 1 \colon X(t) \in \mathscr{S}_k\}$ be the (positive) hitting time of $\mathscr{S}_k$.

LEMMA 5. *There exists a $J = J_d < \infty$ such that for $j < k$, $\Delta = k - j$:*

(a) *If $z \in \mathscr{S}_k$, then*

$$\mathbf{P}_z(T_j < T_k) \leq J\Delta^{-1}.$$

(b) *If $y \in \mathscr{S}_j$ and $B \subset \mathscr{S}_k$, then*

$$h_y(\mathscr{S}_k, B) \leq J|B|\Delta^{1-d}.$$

PROOF. Part (a) follows from Proposition 1.5.10 and Exercise 1.6.8 of reference [10]. [For $d \geq 3$, the proof applies the optional sampling theorem to the martingale $M(t) = G(X(t))$ stopped at time $\min\{T_j, T_k\}$, where $G$ is the standard Green's function on all of $\mathbb{Z}^d$. For $d = 2$ the proof is similar, using the potential kernel instead.]

It clearly suffices to prove (b) for $\Delta$ sufficiently large and singletons $B = \{z\}$. Let

$$\rho = \rho_{y,k} = \inf\{t \geq 1 \colon X(t) \in \mathscr{S}_k \cup \{y\}\}.$$

Then by a last exit decomposition,

$$h_y(\mathscr{S}_k, z) = \mathbf{P}_y(X(T_k) = z) = G_k(y,y)\mathbf{P}_y(X(\rho) = z).$$

By the reversibility of the random walk, $\mathbf{P}_y(X(\rho) = z) = \mathbf{P}_z(X(\rho) = y)$. Thus,

$$h_y(\mathscr{S}_k, z) = G_k(y,y)\mathbf{P}_z(X(\rho) = y).$$

Let $m = m_{k,\Delta} = \lfloor k - (\Delta/2) \rfloor$. If $\Delta$ is large enough, then $y \notin \mathscr{S}_m$ and

$$\begin{aligned}\mathbf{P}_z(X(\rho) = y) &= \mathbf{P}_z(T_m < T_k)\mathbf{P}_z(X(\rho) = y | T_m < T_k) \\ &\leq \mathbf{P}_z(T_m < T_k) \sup_{x \in \mathscr{S}_m} \mathbf{P}_x(X(\rho) = y).\end{aligned}$$

So, using (a),

$$h_y(\mathscr{S}_k, z) \le J\Delta^{-1} G_k(y,y) \sup_{x \in \mathscr{S}_m} \mathbf{P}_x(X(\rho) = y)$$
$$= J\Delta^{-1} \sup_{x \in \mathscr{S}_m} G_k(x,y).$$

(Here and throughout the remainder of this proof we use $J$ for an absolute constant that may change from line to line.) It therefore suffices to show that

$$\sup_{x \in \mathscr{S}_m} G_k(x,y) \le J\Delta^{2-d}. \tag{2.9}$$

To this end, let

$$V = V_{y,\Delta} = \left\{ w \in \mathbb{Z}^d : \|w - y\| < \frac{\Delta}{4} \right\}.$$

If $x \in \mathscr{S}_m$ and $\Delta$ is large enough, then the function $h(w) = G_k(x,w)$ is harmonic (with respect to the discrete Laplacian) for $\|w - y\| < 3\Delta/8$. The discrete Harnack inequality ([10], Theorem 1.7.2) then implies that for some $J < \infty$,

$$G_k(x,y) \le J G_k(x,w) \quad \text{for all } w \in V,\ x \in \mathscr{S}_m.$$

Therefore,

$$\sup_{x \in \mathscr{S}_m} G_k(x,y) \le J\Delta^{-d} \sup_{x \in \mathscr{S}_m} \sum_{w \in V} G_k(x,w)$$
$$= J\Delta^{-d} \sup_{x \in \mathscr{S}_m} \mathbf{E}_x[Y] \le J\Delta^{-d} \sup_{w \in V} \mathbf{E}_w[Y],$$

where $Y$ is the number of visits to $V$ before leaving $\mathfrak{B}_k$,

$$Y = \sum_{j=0}^{\tau_k - 1} 1_{\{X(j) \in V\}}.$$

So it in turn suffices to show that

$$\sup_{w \in V} \mathbf{E}_w[Y] \le J\Delta^2. \tag{2.10}$$

By the central limit theorem applied to the random walk at time $\Delta^2$, there exists a $\theta > 0$, not depending on $k$, such that for all $w \in V$ and all sufficiently large $\Delta$,

$$\mathbf{P}_w\left(\tau_k \le \Delta^2\right) \ge \theta.$$

A little thought therefore reveals our strategy in substituting (2.10) for (2.9). Over a time interval of length $\Delta^2$, at most $\Delta^2$ sites in $V$ can be visited. Each time we restart the process at some $w \in V$, the above bound on the exit probability holds. So $Y/\Delta^2$ has a geometric tail, that is,

$$\sup_{w \in V} \mathbf{P}_w(Y > l\Delta^2) \le (1-\theta)^l, \qquad l \in \mathbb{Z}^+.$$

This implies (2.10) and completes the proof of (b). □

**3. The lower bound.** Equipped with the necessary preliminary results, we turn to Part A of the proof of Theorem 1. To deduce (1.2), it suffices to prove that for every $\varepsilon > 0$,

$$\mathfrak{B}_{n(1-\varepsilon)} \subset A\big(\omega_d n^d(1+\varepsilon)\big) \quad \text{for all sufficiently large } n. \tag{1.2'}$$

Let each (independent) constituent random walk $X^i(t)$ of the internal DLA cluster evolve forever, even after it has left the occupied cluster, and introduce the random times

$$\begin{aligned} \sigma^i &= \min\{t \colon X^i(t) \notin A(i-1)\} \\ &= \text{the time it takes the } i\text{th particle to leave the occupied cluster}, \\ \tau_z^i &= \min\{t \colon X^i(t) = z\} = \text{the time it takes the } i\text{th walk to hit site } z, \\ \tau_n^i &= \min\{t \colon X^i(t) \notin \mathfrak{B}_n\} = \text{the time it takes the } i\text{th walk to leave } \mathfrak{B}_n. \end{aligned}$$

Denote by $E_z(n)$ the event that site $z$ does not belong to the cluster $A(n)$: $E_z(n) = \{\sigma^i < \tau_z^i \ \forall\, i \le n\}$. Then if $\Lambda$ is any set, $\{\Lambda \not\subset A(n)\} = \bigcup_{z \in \Lambda} E_z(n)$, so by Borel–Cantelli a sufficient condition for (1.2′) is

$$\sum_n \sum_{z \in \mathfrak{B}_{n(1-\varepsilon)}} \mathbf{P}\big(E_z(\omega_d n^d(1+\varepsilon))\big) < \infty. \tag{3.1}$$

Now fix $n$ and $z \in \mathfrak{B}_{n(1-\varepsilon)}$. Consider the random variables

$$\begin{aligned} N &= \sum_i 1_{\{\tau_z^i < \sigma^i\}} \\ &= \text{number of particles that visit } z \text{ before stopping (i.e., leaving the cluster)}, \\ M &= \sum_i 1_{\{\tau_z^i < \tau_n^i\}} = \text{number of walks that visit } z \text{ before leaving } \mathfrak{B}_n, \\ L &= \sum_i 1_{\{\sigma^i \le \tau_z^i < \tau_n^i\}} \\ &= \text{number of walks that visit } z \text{ before leaving } \mathfrak{B}_n \text{ but after the particle stops}, \end{aligned}$$

where the sums are over $i \le \omega_d n^d(1+\varepsilon)$. Clearly $N \ge M - L$. So for any given $a$,

$$\begin{aligned} \mathbf{P}\big(E_z(\omega_d n^d(1+\varepsilon))\big) &= \mathbf{P}(N = 0) \le \mathbf{P}(M \le a \text{ or } L \ge a) \\ &\le \mathbf{P}(M \le a) + \mathbf{P}(L \ge a). \end{aligned} \tag{3.2}$$

The notation has been chosen to reflect our basic strategy for proving (3.1): We will show that $M$ includes "more" walks on average while $L$ includes "fewer," so that the last two probabilities in (3.2) can be made quite small for a suitable $a$. Our choice for $a$ will be specified shortly.

The summands of $M$ are i.i.d., with

$$\mathbf{E}[M] = \lfloor \omega_d n^d(1+\varepsilon) \rfloor \mathbf{P}(\tau_z < \tau_n). \tag{3.3}$$

The summands of $L$ are certainly not identically distributed, nor even independent. Nevertheless, only indices $i$ such that $X^i(\sigma^i) \in \mathfrak{B}_n$ contribute to the sum and for each $y \in \mathfrak{B}_n$ there is at most one index $i$ with $X^i(\sigma^i) = y$. Note that the corresponding post-$\tau_y$ random walks are independent. So to avoid dependence in $L$, we enlarge the index set to all of $\mathfrak{B}_n$ and let

$$\tilde{L} = \sum_{y \in \mathfrak{B}_n} 1^y_{\{\tau_z < \tau_n\}},$$

where the indicators $1^y$ correspond to independent post-$\tau_y$ random walks. Taking expectations gives

$$\mathbf{E}[\tilde{L}] = \sum_{y \in \mathfrak{B}_n} \mathbf{P}_y(\tau_z < \tau_n). \tag{3.4}$$

Since $L \le \tilde{L}$, (3.2) can be replaced by

$$\mathbf{P}\big(E_z(\omega_d n^d(1+\varepsilon))\big) \le \mathbf{P}(\tilde{L} \ge a) + \mathbf{P}(M \le a). \tag{3.5}$$

In Lemma 6, we compare $M$ and $\tilde{L}$ with the aid of Lemmas 3 and 4. The main point is that for two sums of large numbers of independent random variables, the sum with the larger expectation will typically be greater.

LEMMA 6. *For fixed $\varepsilon > 0$ and $n$ sufficiently large,*

$$\begin{aligned} &\mathbf{P}\left(\tilde{L} \ge \left(1+\frac{\varepsilon}{4}\right)\mathbf{E}[\tilde{L}]\right) \le \exp\{-c_d n\}, \\ &\mathbf{P}\left(M \le \left(1+\frac{\varepsilon}{4}\right)\mathbf{E}[\tilde{L}]\right) \le \exp\{-c_d n\} \end{aligned} \tag{3.6}$$

*for all $z \in \mathfrak{B}_{n(1-\varepsilon)}$ and appropriate constants $c_d > 0$, depending on $\varepsilon$.*

The lower bound (1.2′) follows easily from (3.5) and Lemma 6. Set $a = (1+\varepsilon/4)\mathbf{E}[\tilde{L}]$ in (3.5) to get

$$\mathbf{P}\big(E_z(\omega_d n^d(1+\varepsilon))\big) \le 2\exp\{-c_d n\}$$

for $n \ge n_0$, $n_0$ appropriate. Hence

$$\sum_{n \ge n_0} \sum_{z \in \mathfrak{B}_{n(1-\varepsilon)}} \mathbf{P}\big(E_z(\omega_d n^d(1+\varepsilon))\big) \le \sum_{n \ge n_0} 2\omega_d n^d \exp\{-c_d n\} < \infty.$$

This gives us (3.1) and implies (1.2′).

PROOF OF LEMMA 6. Recall that by standard Markov chain theory,

$$\mathbf{P}_y(\tau_z < \tau_n) = \frac{G_n(y,z)}{G_n(z,z)} \quad \text{and} \quad \mathbf{P}(\tau_z < \tau_n) = \frac{G_n(0,z)}{G_n(z,z)}. \tag{3.7}$$

Therefore, by (3.3), (3.4), (3.7) and Lemma 3, for $n$ sufficiently large,

$$\begin{aligned} \mathbf{E}[M] &= \lfloor \omega_d n^d(1+\varepsilon)\rfloor \frac{G_n(0,z)}{G_n(z,z)} \\ &\geq \left(1+\frac{\varepsilon}{2}\right)\sum_{y\in\mathfrak{B}_n}\frac{G_n(y,z)}{G_n(z,z)} = \left(1+\frac{\varepsilon}{2}\right)\mathbf{E}[\tilde{L}]. \end{aligned} \tag{3.8}$$

Also, by (3.4), (3.7) and (2.5),

$$\mathbf{E}[\tilde{L}] = \frac{\mathbf{E}_z[\tau_n]}{G_n(z,z)},$$

which by (2.3) and Lemma 2 is

$$\geq \frac{n^2 - \|z\|^2}{G_{2n}(\mathbf{0},\mathbf{0})}.$$

Applying (2.1) and (2.2), one gets

$$\begin{aligned} \mathbf{E}[\tilde{L}] &\geq \beta_2\frac{n^2}{\ln n}, &\quad d=2, \\ &\geq \beta_d n^2, &\quad d\geq 3, \end{aligned}$$

for every $z\in\mathfrak{B}_{n(1-\varepsilon)}$, eventually in $n$, for suitable constants $\beta_d > 0$ depending on $\varepsilon$. On account of (3.8) one has the same lower bounds for $\mathbf{E}[M]$.

We now observe that $M$ and $\tilde{L}$ are both independent sums to which Lemma 4 applies. With $\gamma = 1/3$, it follows that

$$\mathbf{P}\left(\tilde{L}\geq \mathbf{E}[\tilde{L}] + \mathbf{E}[\tilde{L}]^{5/6}\right) \leq 2\exp\left\{-\tfrac{1}{4}\mathbf{E}[\tilde{L}]^{2/3}\right\} \leq \exp\{-c_d n\},$$

$$\mathbf{P}\left(M\leq \mathbf{E}[M] - \mathbf{E}[M]^{5/6}\right) \leq 2\exp\left\{-\tfrac{1}{4}\mathbf{E}[M]^{2/3}\right\} \leq \exp\{-c_d n\},$$

for all $z\in\mathfrak{B}_{n(1-\varepsilon)}$ once $n$ is sufficiently large, for appropriate constants $c_d > 0$. Applying (3.8), the bounds in (3.6) follow. □

**4. The upper bound.** We now turn to Part B of the proof of Theorem 1. To show (1.3), it suffices to verify that for each $\varepsilon > 0$,

$$A(\omega_d n^d) \subset \mathfrak{B}_{n(1+K\varepsilon^{1/d})} \quad \text{provided } n \text{ is sufficiently large} \tag{1.3$'$}$$

for a suitable $K = K_d < \infty$. The basic idea is to use (1.2) in conjunction with Lemma 5 to show that the number of occupied sites in each shell $\mathscr{S}_k$ outside of $\mathfrak{B}_n$ can only increase at a controlled rate. To ensure that the internal DLA cluster cannot grow too quickly, we argue that on the interval $I$ of time steps from 0 to $\omega_d n^d$, the growth of $A(i)$ outside $\mathfrak{B}_{n(1+\varepsilon^{1/d})}$ is dominated by a multitype branching process that spreads out slowly on the $n^d$ scale. Here

$0 < \varepsilon < 1$ is arbitrary, and $n_0$ should be chosen large enough so that, by Part A,

$$\mathbf{P}\left(\mathfrak{B}_{n(1-\varepsilon)} \subset A(\omega_d n^d) \text{ for } n \geq n_0\right) \geq 1 - \varepsilon.$$

Since most of the first $\lfloor \omega_d n^d \rfloor$ particles stop to fill out $\mathfrak{B}_{n(1-\varepsilon)} \subset \mathfrak{B}_n$ with only a small portion left over,

$$\mathbf{P}\left(|A(\omega_d n^d) \cap \mathfrak{B}_n^c| < K_0 \varepsilon n^d \text{ for } n \geq n_0\right) \geq 1 - \varepsilon \tag{4.1}$$

for an appropriate constant $K_0 = K_{0,d} < \infty$.

Let us relabel the particles $X^{i_j}$ that exit $\mathfrak{B}_n$ during $I$ as $Y^j$ and consider the embedded growth process $\tilde{A}(j) = A(i_j)$. That is, time is now rescaled in terms of particles that escape from $\mathfrak{B}_n$. Choose $k_0 = \lfloor n(1 + \varepsilon^{1/d}) \rfloor + 1$. Introduce $Z_k(j) = |\tilde{A}(j) \cap \mathscr{S}_{k_0+k}|$ and $\mu_k(j) = \mathbf{E}Z_k(j)$, $k \geq 1$. The quantity $\mu_k(j)$ is the average number of occupied sites in $\mathscr{S}_{k_0+k}$ after particles $Y^1, \dots, Y^j$ have stopped. We will regulate the propagation of $\tilde{A}(j)$ by estimating $\mu_k(j)$. Clearly, $\mu_1(j) \leq j$ and $\mu_k(0) \equiv 0$. For the general case we will make use of the following lemma.

LEMMA 7. *For each $j$ and $k$,*

$$\mu_k(j) \leq n^{d-1}\left[J_1 \frac{j}{k} \varepsilon^{(1-d)/d} n^{1-d}\right]^k, \tag{4.2}$$

*where $J_1 = J_{1,d} < \infty$.*

Once we establish (4.2), the demonstration of (1.3′) is straightforward. Let $F$ denote the event in (4.1). Then

$$\begin{aligned}\mathbf{P}\left(A(\omega_d n^d) \not\subset \mathfrak{B}_{n(1+K\varepsilon^{1/d})}, F\right) &\leq \mathbf{P}\left(Z_{n'}(\lfloor K_0 \varepsilon \cdot n^d \rfloor) \geq 1\right)\\ &\leq \mu_{n'}(\lfloor K_0 \varepsilon \cdot n^d \rfloor)\end{aligned}$$

for $n \geq n_0$, where $n' = \lceil (K-1)\varepsilon^{1/d} n - 1 \rceil$. ($\lceil t \rceil$ denotes the least integer greater than or equal to $t$.) Plug $j = \lfloor K_0 \varepsilon \cdot n^d \rfloor$ and $k = n'$ into (4.2) to conclude that

$$\mathbf{P}\left(A(\omega_d n^d) \not\subset \mathfrak{B}_{n(1+K\varepsilon^{1/d})}, F\right) \leq n^{d-1}\left(\frac{J_2}{K}\right)^{n'}$$

for large $n$ and $K$, and a suitable $J_2 = J_{2,d} < \infty$. If $K > J_2$, the right side goes to 0 like $\exp\{-\alpha n\}$ for some $\alpha > 0$. So if $n_1$ is large enough,

$$\sum_{n \geq n_1} \mathbf{P}\left(A(\omega_d n^d) \not\subset \mathfrak{B}_{n(1+K\varepsilon^{1/d})}, F\right) \leq \sum_{n \geq n_1} \exp\{-\alpha n\} < \infty.$$

By Borel–Cantelli,

$$\mathbf{P}\left(\limsup_n \left\{A(\omega_d n^d) \not\subset \mathfrak{B}_{n(1+K\varepsilon^{1/d})}\right\} \cap F\right) = 0. \tag{4.3}$$

Since $\varepsilon$ can be chosen as close to 0 as desired, (1.3′) follows from (4.1) and (4.3).

PROOF OF LEMMA 7. The crux of our argument goes as follows. The rate at which a shell $\mathscr{S}_{k_0+k}$ fills with particles is restricted by the rate at which particles escape from $\mathfrak{B}_n$ and penetrate the preceding shell $\mathscr{S}_{k_0+k-1}$. According to Lemma 5(b), this penetration is bounded by the proportion of $\mathscr{S}_{k_0+k-1}$ that is occupied. Arguing in this manner, we will obtain a recursion relation (4.4) for $\mu_k$ in terms of $\mu_{k-1}$. Iteration then yields (4.2).

We first condition on the time $\tau = \tau_n^{i_{l+1}}$ at which $Y^{l+1}$ exits $\mathfrak{B}_n$; this gives

$$\mu_k(l+1) - \mu_k(l) = \mathbf{E}\Big[h_{Y^{l+1}(\tau)}\big(\tilde{A}(l)^c, \mathscr{S}_{k_0+k}\big)\Big].$$

Any walk stopping in $\mathscr{S}_{k_0+k}$ must remain within the growth cluster while it hits the immediately preceding shell $\mathscr{S}_{k_0+k-1}$. Therefore, this is at most

$$\mathbf{E}\Big[h_{Y^{l+1}(\tau)}\big(\mathscr{S}_{k_0+k-1}, \tilde{A}(l)\big)\Big] \le \max_{y \in \mathscr{S}_n} \mathbf{E}\Big[h_y\big(\mathscr{S}_{k_0+k-1}, \tilde{A}(l)\big)\Big].$$

Applying the uniform upper bound of Lemma 5(b) (with $j \to n$, $k \to k_0 + k - 1$ and $\Delta \to k_0 + k - n - 1 \ge n\varepsilon^{1/d}$), we obtain

$$\mu_k(l+1) - \mu_k(l) \le J \cdot \mu_{k-1}(l) \cdot \left(\frac{1}{n\varepsilon^{1/d}}\right)^{d-1}. \tag{4.4}$$

Note that (4.4) is, in effect, a comparison with a multitype branching process with types $1, 2, \dots$ and where the growth rate for type $k$ is proportional to the number of individuals of type $k-1$.

Sum the last inequalities over $l = 0, \dots, j-1$ to get

$$\mu_k(j) \le J \cdot \left(\frac{1}{n\varepsilon^{1/d}}\right)^{d-1} \cdot \sum_{l=1}^{j-1} \mu_{k-1}(l).$$

We claim that iteration in $k$, with $j$ fixed, yields

$$\mu_k(j) \le \left[J \cdot \left(\frac{1}{n\varepsilon^{1/d}}\right)^{d-1}\right]^{k-1} \cdot \frac{j^k}{k!}. \tag{4.5}$$

To see this, recall that the $k = 1$ case of (4.5) is immediate. The general case follows by induction, using the elementary inequality

$$\sum_{l=1}^{j-1} l^{k-1} \le \frac{j^k}{k}.$$

Applying the simple estimate $k! \ge k^k e^{-k}$ to (4.5), we obtain (4.2). □

**5. The time scale of random walk steps.** Formulation of internal DLA dynamics in terms of the Markov chain $A(n)$ is most expedient if one wants to prove shape results such as Theorem 1, since we know that the growth model contains precisely $n$ sites at the $n$th step. However, a more natural time scale for the algorithm involves the number of individual random walk steps required to grow a ball of a given radius. This scale governs the duration of a computer simulation, for example. Recall from Section 3 that

$$\sigma^i = \min\{t\colon X^i(t) \notin A(i-1)\} = \text{the time it takes the } i\text{th particle to stop}.$$

Introduce

$$\begin{aligned}\chi(t) &= \max\{k\colon \sigma^1 + \cdots + \sigma^k \le t\}\\ &= \text{the number of stuck particles after } t \text{ random walk steps}\end{aligned}$$

and the process with time change

$$\hat{A}(t) = A(\chi(t)), \qquad t = 0, 1, \dots .$$

Of course $\hat{A}(t)$ grows like a ball because $A(n)$ does. A simple "back of the envelope" calculation identifies the growth rate on the new time scale. Namely, we know from Theorem 1 that the $k$th particle encounters a shape that is close to a ball of radius $r = r(k) = (k/\omega_d)^{1/d}$. As indicated by Lemma 2, that particle takes about $r^2$ steps to reach the edge of the ball. Thus the total number of random walk steps taken by the $\omega_d n^d$ particles needed to grow a ball of radius $n$ should be

$$(5.1) \qquad \approx \sum_{k=1}^{\lfloor \omega_d n^d \rfloor} r^2(k) \approx \left(\frac{1}{\omega_d}\right)^{2/d} \cdot \frac{d}{d+2}\left(\omega_d n^d\right)^{(d+2)/d} = \left(\frac{d\omega_d}{d+2}\right) n^{d+2}.$$

Our next result, a modification of Theorem 1, establishes asymptotic circular growth for the $\hat{A}(t)$ process at the growth rate prescribed by (5.1).

THEOREM 2. *Let $\hat{A}(t)$ be internal DLA on the time scale of individual random walk steps. For fixed $\varepsilon > 0$,*

$$\mathfrak{B}_{n(1-\varepsilon)} \subset \hat{A}(t) \subset \mathfrak{B}_{n(1+\varepsilon)} \quad \textit{for all sufficiently large } n$$

*with probability* 1, *where*

$$t = \left(\frac{d\omega_d}{d+2}\right) n^{d+2}.$$

PROOF. Recall from Section 3 that

$$\tau_n^i = \min\{t\colon X^i(t) \notin \mathfrak{B}_n\} = \text{the time it takes the } i\text{th walk to leave } \mathfrak{B}_n.$$

Set

$$n_i = (1-\varepsilon)\left(\frac{i}{\omega_d}\right)^{1/d}, \qquad N_i = (1+\varepsilon)\left(\frac{i}{\omega_d}\right)^{1/d}.$$

According to Theorem 1,

$$\tau_{n_i}^i \le \sigma^i \le \tau_{N_i}^i \quad \text{for all sufficiently large } i \text{ (almost surely)}.$$

Hence, for any $k_0$ exceeding a suitably chosen random index $\kappa$ and any $k > k_0$,

$$\sum_{i=k_0}^{k} \tau_{n_i}^i \le \sum_{i=k_0}^{k} \sigma^i \le \sum_{i=k_0}^{k} \tau_{N_i}^i. \tag{5.2}$$

We will estimate the leftmost and rightmost sums. For this purpose, we need a uniform bound on the tails of $\tau_n/n^2$. The upper bound in Lemma 2 and Chebyshev's inequality imply that

$$\sup_{n \ge 1,\, z \in \mathfrak{B}_n} \mathbf{P}_z(\tau_n > 4n^2) \le \tfrac{1}{2}.$$

Repeated application of the Markov property at times that are multiples of $4n^2$ yields the needed estimate:

$$\mathbf{P}_{\mathbf{0}}(\tau_n > n^2 s) \le C_1 e^{-C_2 s} \tag{5.3}$$

for appropriate constants $C_1, C_2 > 0$. By Lemma 2, $\mathbf{E}[\tau_n/n^2] \sim 1$. This, together with (5.3) and a little estimation, implies that for any sufficiently small $\lambda > 0$ and $i$ sufficiently large,

$$\mathbf{E}\left[\exp\left\{\lambda \frac{\tau_{n_i}^i - n_i^2}{n_k^2}\right\}\right] \le e^{C_3 \lambda^2} \quad \text{for } k \ge i,$$

where $C_3$ does not depend on $\lambda$. Hence, for large enough $k_0$,

$$\mathbf{E}\left[\exp\left\{\frac{\lambda}{n_k^2} \sum_{i=k_0}^{k} \left(\tau_{n_i}^i - n_i^2\right)\right\}\right] \le e^{C_3 \lambda^2 (k - k_0 + 1)}.$$

By Chebyshev's inequality,

$$\mathbf{P}\left(\sum_{i=k_0}^{k} \left(\tau_{n_i}^i - n_i^2\right) \ge \varepsilon (k - k_0) n_k^2\right) \le \exp\{-\lambda(\varepsilon - C_3\lambda)(k - k_0)\};$$

choosing $\lambda = \varepsilon/2C_3$, the right side is

$$\le \exp\left\{-\frac{\varepsilon^2 (k - k_0)}{4C_3}\right\}.$$

Analogous estimates apply to the lower tails of the $\tau_{n_i}^i$, so in fact,

$$\mathbf{P}\left(\left|\sum_{i=k_0}^{k} \left(\tau_{n_i}^i - n_i^2\right)\right| \ge \varepsilon (k - k_0) n_k^2\right) \le \exp\{-C_4 \varepsilon^2 (k - k_0)\}, \qquad C_4 > 0. \tag{5.4}$$

The same reasoning also gives a bound for the rightmost sum in (5.2):

$$(5.5)\quad \mathbf{P}\left(\left|\sum_{i=k_0}^{k}\left(\tau_{N_i}^i - N_i^2\right)\right| \geq \varepsilon(k-k_0)N_k^2\right) \leq \exp\{-C_4\varepsilon^2(k-k_0)\}.$$

After straightforward approximation, it follows from (5.2), (5.4) and (5.5) that if $\varepsilon$ is small, $k_0$ is large and $k \geq 2k_0$,

$$\mathbf{P}\left(\left|\sum_{i=k_0}^{k}\left(\sigma^i - \left(\frac{i}{\omega_d}\right)^{2/d}\right)\right| \geq \varepsilon k^{1+2/d},\ \kappa \leq k_0\right) \leq \exp\{-C_5\varepsilon^2 k\}, \qquad C_5 > 0.$$

Hence, applying Borel–Cantelli and then including indices $i$ with $1 \leq i < k_0$, we conclude that

$$k^{-(1+2/d)}\sum_{i=1}^{k}\left(\sigma^i - \left(\frac{i}{\omega_d}\right)^{2/d}\right) \to 0 \quad \text{a.s. as } k \to \infty.$$

Equivalently,

$$(5.6)\qquad k^{-(1+2/d)}\sum_{i=1}^{k}\sigma^i \to \frac{d}{d+2}(\omega_d)^{-2/d} \quad \text{a.s. as } k \to \infty.$$

Choosing $k = \lfloor \omega_d n^d \rfloor$ and reformulating Theorem 1 in terms of the process $\hat{A}$, a simple comparison shows that

$$(5.7)\qquad \mathfrak{B}_{n(1-\varepsilon)} \subset \hat{A}\left(\sum_{i=1}^{k}\sigma^i\right) \subset \mathfrak{B}_{n(1+\varepsilon)} \quad \text{for all sufficiently large } n.$$

Substitute (5.6) into (5.7) to obtain the desired result. □

**6. A continuous-time variant.** In contrast to the "one particle at a time" process studied so far, consider the following "simultaneous" variant more akin to the continuous-time growth models discussed in reference [4]. Particles are dropped at the origin $\mathbf{0}$ of $\mathbb{Z}^d$ in a rate 1 Poisson stream. Initially, only $\mathbf{0}$ is occupied. Each successive particle then executes an independent rate 1 continuous time simple symmetric random walk until it lands at a site that has not been visited previously by any other particle, at which site it stops. Let $B_t$ denote the set of sites that are occupied at time $t$ in this process, noting that several "active" particles may now occupy a site simultaneously. In this section we will first prove a counterpart of Theorem 1 for $B_t$ provided that the dimension is at least three and then discuss the intriguing behavior of $B_t$ in two dimensions at the level of heuristics.

It turns out that for $d \geq 3$, the shape result is unchanged if $A(n)$ is replaced by $B_t$:

THEOREM 3. *Let $B_t$ denote the continuous-time simultaneous variant of internal DLA, as described previously. Assume $d \geq 3$. For any $\varepsilon > 0$,*

$$\mathfrak{B}_{t(1-\varepsilon)} \subset B_{\omega_d t^d} \subset \mathfrak{B}_{t(1+\varepsilon)} \quad \textit{for all sufficiently large } t$$

*with probability* 1.

To analyze $B_t$, it is more convenient to consider a slightly different particle system with the same distribution for the set of occupied sites. The general construction goes as follows. Drop particles numbered $1, 2, \dots$ at successive random times $T_1, T_2, \dots,$ at respective sites $x_1, x_2, \dots$ . The particles perform independent rate 1 random walks except that if the $i$th particle is at site $x$ and there is no other particle at $x$ with number less than $i$, then that particle must remain at $x$. If, later on, a particle numbered less than $i$ reaches $x$, then the $i$th particle is "freed" to move again. However, if no particle numbered less than $i$ ever arrives at $x$, then the $i$th particle stays at $x$ forever.

Suppose we place a particle numbered 0 at the origin, let $T_1 < T_2 < \cdots$ be the arrival times of a Poisson stream and set $x_1 = x_2 = \cdots = 0$. Then it is easy to see that at every time $t$ the distribution of occupied sites in the numbered particle system agrees with $B_t$; in fact, the occupation densities of the two systems agree. For the remainder of this section we will assume that $B_t$ is represented in terms of the numbered model. One important feature of this version is that the motion of the first $n$ particles is not affected at all by the motion of the particles numbered $n + 1, n + 2, \dots$ . Hence, if we let $B_t(n)$ be the cluster of sites occupied by the first $n$ particles at time $t$, then

$$B_t(n) \subset B_t \quad \text{for each } t \geq 0, n \geq 1. \tag{6.1}$$

Also note that the sets $B_t$ and $B_t(n)$ increase as $t$ increases.

Let us now define a discrete-time growth model as follows. Let $A(0) = \{\mathbf{0}\}$. For each $n > 0$, let $y_n$ be the first site not in $A(n - 1)$ that the $n$th particle visits. Set $A(n) = A(n - 1) \cup \{y_n\}$. By induction, one can verify that in this model the $n$th particle always reaches such a new site $y_n$ and stops there forever. In other words,

$$\lim_{t \to \infty} B_t(n) = A(n). \tag{6.2}$$

The choice of notation here is not coincidental: A moment's thought will convince the reader that $A_n$ is just a labeled version of the basic internal DLA model $A(n)$ defined in Section 1. Note also that this equality does not depend on the times $T_1, T_2, \dots$ or the fact that the waiting times for the random walk are exponential. The only important feature is that each particle, "when it moves," behaves like a simple random walk independently of all that has occurred beforehand. (We could, in fact, consider a discrete-time version of $B_t$ and the analogous correspondence would hold.)

This connection between $B_t$ and $A(n)$ leads to easy proofs of some facts about the discrete-time model. For instance, suppose that $\Lambda$ is a finite subset of $\mathbb{Z}^d$ containing $n$ points, and that $x_1, \dots, x_k$ is a sequence of points in the "initial cluster" $\Lambda$. Start random walks, one at a time, from these points. Let each move until a new site is visited, at which time that site is added to the cluster and the next walk starts. After the $k$ particles have stopped, a cluster of size $n + k$ is formed. Rather surprisingly, the distribution of the final cluster is independent of the ordering of points $x_1, \dots, x_k$. To see this, start continuous-time random walks at $x_1, \dots, x_k$, and let them evolve simultaneously according to the original construction of $B_t$ (each particle moving until it finds a new site). In this manner we obtain a final cluster $B_\infty$. Now number the particles in any order and consider the cluster $B_t$ obtained from the second construction (given after Theorem 3). Essentially the same argument as given above shows that the distribution of $B_\infty$ for that particular ordering agrees with the cluster formed from the discrete-time model that drops particles one at a time in the same order. Since the distribution of $B_\infty$ in the first construction does not depend on the ordering of particles, neither can the construction by means of numbered particles. Diaconis and Fulton [3] showed this independence of ordering in a more general Markov chain context using a different argument. Our construction can easily be adapted to their more general case.

The proof of Theorem 3 uses the fact that in $d$ dimensions it takes time of order $t^d$ for the simultaneous system to accumulate the particles that are required to fill a sphere of radius $t$. However, the time it takes each random walk to reach the edge of $\mathfrak{B}_t$ is of order $t^2 = o(t^d)$ if $d \geq 3$. Thus particles reach the boundary of the occupied cluster immediately after being dropped at $\mathbf{0}$ on the relevant time scale. Because of this, in three or more dimensions $B_\infty(\lfloor t \rfloor)$ is a good approximation to $B_t$. Consequently, as we now show, Theorem 3 is a fairly easy consequence of Theorem 1 and (6.2). In the proof to follow, we use the second construction of $B_t$ with Poisson arrival times $T_1 < T_2 < \cdots$ of the particles. Also, we abbreviate $u = u(t, d) = \omega_d t^d$.

PROOF OF THEOREM 3. *The upper bound.* By asking whether at least $n = \lfloor (1 + \varepsilon/2) u \rfloor$ particles enter the system by time $u$ and letting the first $n$ particles evolve until they all stop if the answer is "no", we obtain the estimate

$$\begin{aligned}\{B_u \not\subset \mathfrak{B}_{t(1+\varepsilon)}\} &\subset \{T_n \leq u\} \cup \{B_u(n) \not\subset \mathfrak{B}_{t(1+\varepsilon)}\} \\ &\subset \{T_n \leq u\} \cup \{A(n) \not\subset \mathfrak{B}_{t(1+\varepsilon)}\},\end{aligned}$$

this last inclusion by (6.2). On account of (1.3), the last event on the right eventually fails with probability 1 as $t \to \infty$. The same is true for the first event on the right, by the strong law of large numbers. Hence the right hand inclusion of Theorem 3 holds eventually in $t$ with probability 1. Note that this proof of the upper bound applies in any dimension.

*The lower bound.* Now choose $n = \lfloor (1 - \varepsilon/2)u \rfloor$. Since $B_u(n) \subset B_u$ by (6.1),

$$\{\mathfrak{B}_{t(1-\varepsilon)} \not\subset B_u\} \subset \{\mathfrak{B}_{t(1-\varepsilon)} \not\subset B_u(n)\}.$$

Comparing $B_u(n)$ with $A(n)$, we see that the right side is contained in

$$\{\mathfrak{B}_{t(1-\varepsilon)} \not\subset A(n)\} \cup \left\{T_n > \left(1 - \frac{\varepsilon}{4}\right)u\right\} \cup \left\{T_n \le \left(1 - \frac{\varepsilon}{4}\right)u,\ B_u(n) \neq A(n)\right\}$$
$$=_{\text{def}} E_1 \cup E_2 \cup E_3.$$

By (1.2), event $E_1$ fails eventually in $t$ with probability 1. Another application of the strong law shows that the same is true for event $E_2$. Event $E_3$ is contained in

$$\left\{\text{one of } n \text{ particles takes at least time } \frac{\varepsilon}{4}u \text{ to exit } \mathfrak{B}_{t(1+\varepsilon)}\right\} \cup \{B_u \not\subset \mathfrak{B}_{t(1+\varepsilon)}\}. \tag{6.3}$$

The second event in (6.3) fails eventually on account of the already established upper bound. Our assumption that $d \ge 3$, together with (5.3), shows that the first event has probability at most

$$n\mathbf{P}_{\mathbf{0}}\left(\tau_{t(1+\varepsilon)} \ge \frac{\varepsilon}{4}u\right) \le n\mathbf{P}_{\mathbf{0}}\left(\tau_{t(1+\varepsilon)} \ge t^{5/2}\right) \le \exp\{-C\sqrt{t}\}$$

for some absolute constant $C > 0$. So another application of Borel–Cantelli implies that the first event in (6.3) fails eventually in $t$ as well. Consequently, the left inclusion of Theorem 3 holds eventually in $t$ with probability 1. □

Our proof of Theorem 3 breaks down in two dimensions because the final estimate [of $\mathbf{P}(E_3)$] fails. The reason is simple. If $d = 2$, then the time a particle needs to reach the boundary of $\mathfrak{B}_t$ is of the same order $t$ as the time required to produce enough particles to fill $\mathfrak{B}_t$. Consequently the occupied ball is covered by a "cloud" of active particles as it grows, and the growth rate is governed by a complex interaction of inflow and boundary absorption.

We have simulated the dynamics of this two-dimensional process. A sample configuration is shown in Figure 1. The occupied cluster is shaded in gray and then sites with one or more active particle are superimposed in black.

Our computer experiments certainly suggest circular growth. Unfortunately we are currently unable to provide a proof of this behavior for $B_t$ in two dimensions. Figure 1 is also suggestive of a limiting "density profile" of active particles within the occupied cluster. In other words, there may well be radial limiting probabilities $\pi_k(r)$, $k \ge 0$, $0 \le r \le 1$, that a site $x$, located a proportion $r$ of the distance from $\mathbf{0}$ to the edge of the occupied cluster, has $k$ active particles. Presumably the densities $\pi \cdot (r)$ should decrease stochastically as $r$ increases.

Let us assume asymptotic circularity for $d = 2$, and present a heuristic evaluation of the asymptotic growth rate of $B_t$ in terms of certain boundary

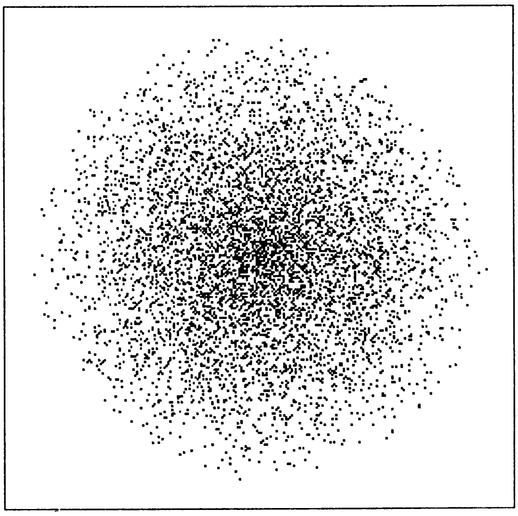

FIG. 1.

crossing probabilities for Brownian motion. That is to say, let us suppose that

$$B_t \approx \mathfrak{B}_{C\sqrt{t}} \tag{6.4}$$

and propose a plausible identification of the constant $C$. We do so by estimating $\mathbf{E}|B_T|$, the expected number of particles that stop by some large time $T$. According to (6.4), on the average, a particle dropped at time $t \le T$ will stop if its random walk $X_u$ hits the edge of $\mathfrak{B}_{C\sqrt{t+u}}$ for some $u \le T - t$. Thus, since the Poisson stream drops particles uniformly over $[0, T]$,

$$\mathbf{E}|B_T| \approx \int_0^T dt\, \mathbf{P}_0\big(\|X_u\| = C\sqrt{t+u} \text{ for some } u \le T - t\big).$$

Invoking Brownian scaling and the invariance principle, we get

$$T^{-1}\mathbf{E}|B_T| \approx \int_0^1 ds\, \mathbf{P}_0\big(\|W_v\| = C\sqrt{s+v} \text{ for some } v \le 1 - s\big), \tag{6.5}$$

where $W_v$ is a standard Wiener process. But from (6.4) we also know that

$$T^{-1}\mathbf{E}|B_T| \approx \pi C^2. \tag{6.6}$$

Clearly the right side of (6.5) decreases from 1 to 0 as $C$ increases, whereas the right side of (6.6) increases from 0 to $\infty$ as $C$ increases. We conjecture that the growth of $B_t$ in two dimensions is characterized by the ball with the unique $C$ for which these two expressions are equal.

Finally, let us note that the asymptotic growth of $B_t$ is even interesting in one dimension; it cannot be deduced immediately from the equivalence of $A(n)$ to the Friedman urn model when $d = 1$. In fact, the size of the occupied cluster is of order $\sqrt{t \log t}$ at time $t$ in this case and although we have not done so, we expect that a one-dimensional version of the heuristic just given can be fashioned into a rigorous evaluation of the exact growth rate. □

**7. Closing remarks.** We conclude our paper with some brief remarks about a couple of possible future directions for the study of internal DLA.

One natural extension of our analysis is to consider systems with several sources of particles. Suppose particles drop at locations chosen randomly from a finite collection of sources in the "one-at-a-time" version or drop according to independent Poisson streams in the continuous-time "simultaneous" variant. If the sources are well separated, then each cluster grows independently, like an expanding ball, until it collides with another. After such collisions, however, there is more and more interaction due to particles that travel from their original cluster into an adjacent one before they stop. The growth dynamics during this intermediate phase are presumably quite complicated. Once the growth cluster attains a size many times larger than the maximal intersource distance, however, the distinct locations from which particles originate become increasingly insignificant. In particular, Theorems 1 and 3 generalize in a straightforward manner to corresponding internal DLA dynamics with any finite number of sources. Interesting problems also arise for infinite particle systems with a countable collection of sources, for example, a Bernoulli random field of sources with small density $p$.

Perhaps the most basic open problem raised by our results concerns the statistical deviations of $A(n)$ from a Euclidean ball. Are the fluctuations of order $\sqrt{n}$, of order $n^{\delta}$ for some $\delta \in (0, \frac{1}{2})$, or even smaller? Clues for the answer to this challenging question are provided by two recent developments.

Etheridge and Lawler (unpublished) have studied the internal DLA growth model on a regular degree-3 tree $\mathscr{T}^3$. That more tractable graph admits an analogous shape theorem and sharp estimates can be given for the variance. On $\mathscr{T}^3$ a "ball" $\mathfrak{B}(n)$ of radius $n$ contains $3 \cdot 2^{n-1} - 2$ points. Letting $A(n)$ denote the cluster at time $n$, they show that for suitable constants $0 < c_1 < c_2 < \infty$, with probability 1, for all sufficiently large $n$.

$$\mathfrak{B}(n - c_2 \log n) \subset A(3 \cdot 2^{n-1} - 2) \subset \mathfrak{B}\big(n + c_2\sqrt{n}\big)$$

and

$$\mathfrak{B}(n - c_1 \log n) \not\subset A(3 \cdot 2^{n-1} - 2), \qquad A(3 \cdot 2^{n-1} - 2) \not\subset \mathfrak{B}\big(n + c_1\sqrt{n}\big).$$

In other words, the "inner error" is of order $\log n$ while the "outer error" is of order $\sqrt{n}$.

Analogies between the behavior of $A(n)$ on $\mathscr{T}^3$ and $\mathbb{Z}^d$ are rather dubious, however, since the size of the boundary shell $\mathscr{S}(n)$ is of the same order of magnitude as $|\mathfrak{B}(n)|$ on the tree, but is $o(|\mathfrak{B}(n)|)$ on the $d$-dimensional integers. More relevant is a comparison between the internal DLA rule and Eden's growth model [5]. As mentioned in the introduction, the latter process adds each additional particle at a site chosen uniformly from the available boundary locations. Eden's model has been simulated extensively, especially in two dimensions, and is widely believed to have subdiffusive fluctuations. More precisely, its deviations from the asymptotic deterministic shape are believed to be of order $n^\delta$ for an exponent $\delta \in (0, \frac{1}{2})$. There is now some experimental and heuristic evidence that $\delta = 1/3$ when $d = 2$; see Krug and Spohn [9] for a detailed discussion of related interface dynamics. At the beginning of the paper we described the tendency of $A(n)$ to eliminate irregularities. Intuitively, successive particles should be more likely to stop at points along the boundary that are closer to $\mathbf{0}$ and less likely to stop further away. This bias should give rise to substantially less variation than in Eden's model. Unfortunately the comparison technique that leads to the lower bound of Theorem 1 seems insufficient to establish subdiffusive fluctuations for internal DLA. But essentially the same nonrigorous interface analysis described in reference [9] also applies to an exterior smoothing dynamic known as *diffusion limited erosion* (DLE), in which particles wandering in from $\infty$ successively erase boundary sites from an occupied set. In the limit of a flat interface, DLE dynamics and internal DLA dynamics coincide. A recent report of Krug and Meakin [8] on DLE fluctuations therefore suggests that $A(n)$ should exhibit logarithmic fluctuations when $d = 2$ and a tight boundary for $d \geq 3$.

To conclude, let us reflect a bit more on the core idea of the paper. Lemma 3 presents the fundamental inequality that is used in the proof of the lower bound (1.2). Recall its statement: For every $z \in \mathfrak{B}_n$, the average value of $G_n(y,z)$ over $y \in \mathfrak{B}_n$ is bounded above by $G_n(\mathbf{0}, z)$. The proof of the analogous fact for Brownian motions, sketched in Section 2, uses very strongly the fact that the set is a ball centered at $\mathbf{0}$. In fact, this seems to give a way to characterize a Euclidean ball and its center. This is the only domain $\mathfrak{D}$ and specified point $x \in \mathfrak{D}$ with the property that for all other $z \in \mathfrak{D}$, the average value of $g(y, z)$ over $y \in \mathfrak{D}$ is bounded above by $g(x, z)$. We will not give a direct proof of this fact. Rather, we ask the reader to note that if some other set $\mathfrak{D}$ had this property, then we could prove that $\mathfrak{D}$ was also the limiting shape for internal DLA.

**Acknowledgments.** We thank Persi Diaconis for showing us his manuscript with Fulton [3] in an early handwritten draft, and for inviting us to study the shape of their model in several dimensions. Thanks also to Herbert Spohn for bringing references [8] and [9] to our attention. Finally, we are grateful to the referee for a careful reading that uncovered a number of rough spots in the original manuscript.

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GREGORY F. LAWLER
DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27706

MAURY BRAMSON
DAVID GRIFFEATH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706