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THE CRITICAL VALUE FOR THE UNIFORM NEAREST PARTICLE PROCESS[1]

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We prove that the critical value for the uniform nearest particle system equals 1.

Introduction. The uniform nearest particle system with rate λ (UNPS(λ)) is the nearest particle system with rates $\beta(l, r) = \lambda/(l + r - 1)$. This is a particle system on $\{0, 1\}^Z$ where occupied sites die at rate 1 independently of the rest of the configuration and in unoccupied intervals particles are born at a rate λ uniformly spread on the interval. The UNPS processes are attractive; consequently, there exists a critical value λ_c such that a UNPS(λ) with all states initially occupied survives for $\lambda > \lambda_c$ and dies out for $\lambda < \lambda_c$. The attractiveness of the system also entails [see Theorem 2.3, page 135 of Liggett (1985)] that when the process is supercritical, there is a nontrivial invariant measure.

In this paper we prove:

Theorem 1. *A UNPS*(λ) *survives for* λ *strictly greater than* 1.

Theorem 5.5 of Liggett [(1985), page 347] states that a NPS with $\sum_{l+r-1=n}\beta(l, r) = b(n) \leq 1$ for each n must die out. This theorem implies that a UNPS(λ) dies out for $\lambda \leq 1$. Hence, Theorem 1 shows that $\lambda_c = 1$. Bramson and Gray (1981) give the upper bound $4 \log 2$ for the critical value but their method handles a much wider class of nearest particle systems, many nonattractive cases, than the method presented here.

Remarks. The proof in this paper relies heavily on the attractiveness of the processes but easily generalizes to a class of attractive nearest particle systems. Consider the nearest particle system with rates $\beta(l, r)$ given by

$$\beta(l, r) = \frac{\lambda}{l + r - 1} \int_{l-1}^{l} f\left(\frac{x}{l + r - 1}\right) dx,$$

where f: $[0, 1] \to \infty$ is a positive function, symmetric about $1/2$ and integrating to 1. If f also has the property that $xf(x)$ is increasing, then the nearest

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particle system is attractive. To see this, note that $\beta(l, r)$ can be rewritten as

$$\lambda \int_{l-1}^{l} \left(\frac{x}{l+r-1} \right) f\left(\frac{x}{l+r-1} \right) \frac{dx}{x},$$

which is decreasing in r and therefore by symmetry in l. If in addition to the above, the function f satisfies $\int_{0+} x^{-\alpha} f(x)\, dx < \infty$ for some strictly positive α, then the method of this paper can be applied to show that the critical value of λ is 1.

An attractive process which survives for λ_0 also survives for any $\lambda > \lambda_0$. Accordingly, we need only present a proof of survival that is valid for $\lambda = 1 + \varepsilon$ and ε sufficiently small.

This paper uses ideas given by Bramson (1989) who showed that for each $\lambda > 1$, there is a nearest particle system which survives and for which the interval birth rates, $\{b(n),\ n = 1, 2, \dots\}$ are bounded by λ. Bramson solved the open problems 6 and 13 from Liggett [(1985), Chapter 7]. This paper addresses the open problems 14 and 16 of the same section.

We make extensive use of coupling throughout this paper. It is assumed that the reader is familiar with pages 122–130 of Liggett (1985).

1. We introduce and examine finite NPS modified to facilitate an inductive procedure.

We now fix ε small and positive and proceed to define the finite systems. Let η_t^N: $t \geq 0$ be a UNPS$(1 + \varepsilon)$, modified so that: (i) deaths on $[-2^N, 2^N]^C$ are suppressed; (ii) births on $[-2^N, -2^N + \varepsilon 2^N]$ and $[2^N - \varepsilon 2^N, 2^N]$ are suppressed; (iii) η_0^N is equal to 1 on $[-2^N, 2^N]^C$ and equal to 0 elsewhere.

For a subinterval of $[-2^N, 2^N]$, I, we define $\eta_t^{N,I}$ to be a uniform nearest particle system that differs from η_t^N only in that births outside I are suppressed. The process $\eta_t^{N,I'}$ is a Markov process with the same jump rates as $\eta_t^{N,I}$, the same set of possible states as $\eta_t^{N,I}$ but so that $\eta_0^{N,I'}$ need not necessarily equal 0 on I.

Let $A^{\varepsilon,N} = [-2^{N-1}, -2^{N-1} + \varepsilon 2^{N-1}]$, $B^{\varepsilon,N} = [2^{N-1} - \varepsilon 2^{N-1}, 2^{N-1}]$.

For notational convenience we write $\eta_t^{N,A}$ $(\eta_t^{N,A'})$ for $\eta_t^{N,A^{\varepsilon,N}}$ $(\eta_t^{N,A^{\varepsilon,N}})$. Processes $\eta_t^{N,B}$ and $\eta_t^{N,B'}$ are defined similarly.

Note that η_t^N, $\eta_t^{N,A}$ and $\eta_t^{N,B}$ are irreducible, continuous time, finite state Markov chains. They thus possess unique invariant probability measures.

The notation $|\eta_t^N|$ refers to the number, at time t, of occupied sites of the process for which death is not suppressed. Statements such as $\eta_t^{N,A} = (\neq)0$, mean that at time t, all nonfixed sites for the process are (are not) unoccupied.

Throughout, terms like $k(\varepsilon)$ and K_ε will denote constant which depend purely on ε and not on N (though it may only be possible to define them given N sufficiently large). These constants may tend to 0 or ∞ as ε tends to 0, but will always be strictly in $(0, \infty)$.

The following lemma enables us to relate the invariant measure of η^N to that of η^{N-1}.

LEMMA 1.1 (Fundamental coupling). *Let* $\eta_t^{N,A}$, $\eta_t^{N,B}$ *and* $\{\eta_t^{N-1,j}\}$ *be independent processes with the* $\{\eta_t^{N-1,j}\}$ *equal in distribution to* η_t^{N-1}. *Define the following stopping times*:

$$T_1 = \inf\{t\colon \eta_t^{N,A}, \eta_t^{N,B} \neq 0\},$$

$$S_i = \inf\{t > T_1\colon \eta_t^{N,A} \text{ or } \eta_t^{N,B} = 0\}.$$

For $i > 1$, $T_i = \inf\{t > S_{i-1}\colon \eta_t^{N,A}, \eta_t^{N,B} \neq 0\}$. *We can couple the above processes with* η_t^N *so that for* $T_j \le t \le S_j$, *we have*

$$\eta_{t-T_j}^{N-1,j} \subset \eta_t^N \quad \text{on } [-2^{N-1}, 2^{N-1}].$$

PROOF. We note that Theorem 1.5 of Liggett [(1985), page 127] and the attractiveness of the processes allow the coupling of $\eta_t^{N,A}$, $\eta_t^{N,B}$ and η_t^N so that for all t, $\eta_t^{N,A} \cup \eta_t^{N,B} \subset \eta_t^N$. At the stopping time T_j, we will trivially have $\eta_{t-T_j}^{N-1,j} \cap [-2^{N-1}, 2^{N-1}] \subset \eta_t^N$. We again use Theorem 1.5 of Liggett (1985) to couple η_t^N and $\eta_t^{N-1,j}$ so that for all times t in (T_j, S_j),

$$\eta_{t-T_j}^{N-1,j} \cap [-2^{N-1}, 2^N] \subset \eta_t^N. \qquad \square$$

The lemma below describes the induction step for the proof of Theorem 1.

LEMMA 1.2. *Let* $\alpha_N = E[(1/N^2)\int_0^{N^2} I_{\eta_t^N(0)=1}\, dt]$ *and*

$$P_N = P\left[\forall\, t \in [\sqrt{N}, N^2], \exists\, j\colon \eta_t^{N,A}(j) \neq 0\right]$$
$$= P\left[\forall\, t \in [\sqrt{N}, N^2], \exists\, j_1\colon \eta_t^{N,B}(j) \neq 0\right].$$

Then $\alpha_N \ge (N^2 - \sqrt{N})/N^2 \alpha_{N-1} P_N^2$.

PROOF. Without loss of generality, we may assume that η_t^N is coupled with independent processes $\eta_t^{N,A}$, $\eta_t^{N,B}$ and $\{\eta_t^{N-1,j}\}$ as in Lemma 1.1. Let D_j be the event $\{[\sqrt{N}, N^2] \subset [T_j, S_j]\}$. Then on D_j,

$$\int_0^{N^2} I_{\eta_t^N(0)=1}\, dt \ge \int_{T_j}^{N^2-\sqrt{N}+T_j} I_{\eta_t^{N-1,j}(0)=1}\, dt.$$

The stopping times T_j, S_j are independent of the process $\eta_t^{N-1,j}$, so

$$(*) \qquad \begin{aligned} E\left[I_{D_j}\int_0^{N^2} I_{\eta_t^N(0)=1}\, dt\right] &\ge E\left[I_{D_j}\int_{T_j}^{N^2+T_j-\sqrt{N}} I_{\eta_t^N(0)=1}\, dt\right] \\ &= P[D_j]\, E\left[\int_0^{N^2-\sqrt{N}} I_{\eta_t^{N-1,j}(0)=1}\, dt\right]. \end{aligned}$$

Now $\eta_t^{N-1,j}$ is an attractive process with an initial configuration of all zeros. Therefore, $P[\eta_t^{N-1,j}(0) = 1]$ and $(1/t)\int_0^t P[\eta_s^{N-1,j}(0) = 1]\, ds$ are increasing functions of t. Accordingly, for each j,

$$E\left[\int_0^{N^2-\sqrt{N}} I_{\eta_t^{N-1,j}(0)=1}\,dt\right] \geq (N^2-\sqrt{N})\alpha_{N-1}.$$

Summing $(*)$ over j and substituting the above inequality gives

$$E\left[\int_0^{N^2} I_{\eta^{N_t}(0)=1}\,dt\right] \geq (N^2-\sqrt{N})\alpha_{N-1}\sum_j P[D_j].$$

Since the processes $\eta_t^{N,A}$ and $\eta_t^{N,A}$ are independent, the sum $\Sigma P[D_j]$ is equal to $(P_N)^2$. Therefore the sum on the right-hand side is greater than or equal to $(N^2-\sqrt{N})P_N^2\alpha_{N-1}$. Dividing this inequality by N^2, we obtain the statement of the lemma. □

As we show in Section 4, Lemma 1.2 effectively reduces our problem to showing that P_N tends to 1 sufficiently quickly.

2. We now proceed to introduce a coupling of our UNPS with an integer splitting process. We also establish and use a martingale property of the integer splitting process. These results will be used in Section 3 to show that P_N tends to 1 very quickly.

We define an integer interval splitting process (IISP): The random nested collection of n intervals $\{I_{n,j}: 1 \leq j \leq n\}$, $n \geq 1$, is an integer interval splitting process when: (i) all the intervals have integer endpoints; and (ii) we obtain the collection $\{I_{n+1,j}: 1 \leq j \leq n+1\}$ from $\{I_{n,j}: 1 \leq j \leq n\}$ by choosing one of the $I_{n,j}$ at random (i.e., all the intervals have chance $1/n$ of being chosen) and splitting it into two intervals. The splitting point is chosen at random from the internal integers of the interval. The process is run until the stopping time $T = \inf\{n\colon \text{there exists } j \leq n \text{ so that } I_{n,j} \text{ has length } 1\}$.

Below, Lemmas 2.1 and 2.2 establish a martingale property of IISP's. Lemma 2.3 introduces a coupling of IISP's and UNPS.

LEMMA 2.1. *Denote the length of an interval I by $|I|$. If interval I with integer endpoints is split at a random interior integer into two subintervals I_A and I_B, then* $E[|I_B|^{-1/2}+|I_A|^{-1/2}] \leq 4|I|^{-1/2}$.

PROOF. Without loss of generality, we suppose $I=[0,n]$ where n is at least 2. Then

$$\begin{aligned}E\left[|I_B|^{-1/2}+|I_A|^{-1/2}\right] &= \frac{1}{n-1}\sum_{j=1}^{n-1} j^{-1/2}+(n-j)^{-1/2}\\ &= |I|^{-1/2}\frac{1}{n-1}\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^{-1/2}+\left(1-\frac{j}{n}\right)^{-1/2}.\end{aligned}$$

The function $x \to x^{-1/2} + (1-x)^{-1/2}$ is convex on [0, 1], so

$$\left(\frac{j}{n}\right)^{-1/2} + \left(1 - \frac{j}{n}\right)^{-1/2} \le \int_{j-1/2}^{j+1/2} \left(\frac{x}{n}\right)^{-1/2} + \left(1 - \frac{x}{n}\right)^{-1/2} dx,$$

therefore

$$E\left[|I_B|^{-1/2} + |I_A|^{-1/2}\right] \le |I|^{-1/2} \frac{1}{n-1} \sum_{j=1}^{n-1} \int_{j-1/2}^{j+1/2} \left(\frac{x}{n}\right)^{-1/2} dx$$
$$= |I|^{-1/2} \frac{1}{n-1} \int_{1/2}^{n-1/2} \left(\frac{x}{n}\right)^{-1/2} + \left(1 - \frac{x}{n}\right)^{-1/2} dx.$$

Since the function $x \to x^{-1/2} + (1-x)^{-1/2}$ is symmetric about $1/2$ and decreasing on $[0, 1/2]$,

$$|I|^{-1/2} \frac{1}{n-1} \int_{1/2}^{n-1/2} \left(\frac{x}{n}\right)^{-1/2} + \left(1 - \frac{x}{n}\right) dx$$
$$< |I|^{-1/2} \frac{1}{n} \int_0^n \left(\frac{x}{n}\right)^{-1/2} + \left(1 - \frac{x}{n}\right)^{-1/2} dx$$
$$= 4|I|^{-1/2}. \qquad \square$$

LEMMA 2.2. *Consider the IISP* $\{I_{n,j}\}$, $n \ge 1$, *with* $I_{1,1} = A^{\varepsilon, N}$. *Let* $T = \inf\{n \ge 1: j \le n \text{ with } |I_{n,j}| = 1\}$. *Then*

$$M_n = \left(\sum_{j=1}^{n \wedge T} \left(|I_{n \wedge T, j}| / (\varepsilon 2^{N-1})\right)^{-1/2}\right) \left(\prod_{j=1}^{(n \wedge T)-1} (1 + 3/j)\right)^{-1}$$

is a positive supermartingale with respect to the natural filtration of the IISP, $\{F_n\}$. *Here* $a \wedge b$ *denotes the minimum of* a *and* b. *The product over no factors is defined to be* 1.

PROOF. This proof is simply an integer version of the argument used in Lemma 2.4 of Peyriere (1979).

We may, of course, suppose that $n < T$. Define V_n to be $\sum_{j=1}^{n \wedge T} |I_{n \wedge T, j}|^{-1/2}$. Lemma 2.1 applied to the interval $I_{n,j}$ yields

$$E\left[V_{n+1} | F_n, I_{n,j} \text{ splits}\right] \le V_n + (4-1)|I_{n,j}|^{-1/2}.$$

Since each interval is chosen with probability $1/n$, it follows that

$$E\left[V_{n+1} | F_n\right] \le V_n + \frac{1}{n} \sum_j 3|I_{n,j}|^{-1/2} = V_n\left(1 + \frac{3}{n}\right).$$

The result follows. $\square$

We define a continuous time integer interval splitting process to be a continuous time Markov chain whose discrete time jump Markov chain is an

IISP as defined at the start of this section and which has a jump rate equal to $1 + \varepsilon$ times the number of intervals of the process or equivalently, each interval splits at rate $1 + \varepsilon$. We will also refer to such processes as IISP's.

LEMMA 2.3. *For a subinterval of* $[2^N, 2^N]$, I, *there is a coupling of a continuous time IISP* $\{I_t^j\}$ *with* $I_0^1 = I$, *and a process* $\eta_t^{N,I}$ *so that* $T < T_I$, *where* $T = \inf\{t\colon \exists\, j \text{ with } |I_t^j| = 1\}$ *and* $T_I = \inf\{t\colon \exists\, i, i+1 \in I \text{ s.t. } \eta_t^{N,J}(i) = \eta_t^{N,I}(i+1) = 1\}$.

PROOF. On $[0, T]$ the continuous time IISP may be identified with a spin system λ_t, taking values in $\{0,1\}^I$ by $\lambda(x) = 1$ if and only if x is an endpoint of an interval I_t^j for some j.

The spin system λ_t has flip rates

$$c_2(x, \lambda) = \begin{cases} (1+\varepsilon)\beta(l, r), & \text{if } \lambda(x) = 0, \\ 0, & \text{otherwise}. \end{cases}$$

It follows directly from Theorem 1.5 of Liggett [(1985), page 127] that the processes λ_t and $\eta_t^{N,I}$ may be coupled so that $\eta_t^{N,I} \subset \lambda_t$ on $[0, T]$. The result easily follows. □

Lemma 2.3 is useful in obtaining bounds for T_I defined above. For example, taking $I = A^{\varepsilon, N}$, we obtain:

LEMMA 2.4. *The stopping time* TA $(= T_{A^{\varepsilon,N}})$ *satisfies* $P[T_A \le N/16] \le K_\varepsilon 2^{-kN}$ *for some* k *positive and all* N *large enough.*

PROOF. Given Lemma 2.3, it will suffice to prove the corresponding statement for the stopping time T. If the time T at which an interval of length 1 is created is less than $N/16$, then either (i) the number of splits by time $N/16$ is greater than $e^{3N/32}$ or (ii) the embedded discrete time IISP creates an interval of length 1 before the $e^{3N/32}$th split.

If (ii) occurs, then upon the creation of the interval of length 1, the supermartingale of Lemma 2.2 associated with the embedded discrete time IISP must have greater than $(1/\varepsilon 2^{N-1})^{-1/2^\varepsilon} \prod_{j=1}^{3N/32} (1 + 3/j)^{-1}$. This term is greater than $\varepsilon^{1/2} c e^{N((\log 2)/2 - 9/32)}$, where c does not depend on N. By Doob's optional sampling theorem, this event has probability majorized by $1/(c\varepsilon^{1/2} e^{N((\log 2)/2 - 9/32)})$. Since $(\log 2)/2$ is greater than $9/32$, this term is of the form demanded by our theorem. The probability of (i) is less than

$$\begin{aligned} \frac{1}{e^{3N/32}} E[\#\text{ of splits in } (0, N/16)] &= \frac{1}{e^{3N/32}} \int_0^{N/16} (1+\varepsilon) e^{(1+\varepsilon)t}\, dt \\ &\le \frac{1}{e^{3N/32}} e^{(1+\varepsilon)N/16}. \end{aligned}$$

This, too, is exponentially small for ε small enough and the lemma follows. □

Similar bounds for other UNPS will be used in Section 3 without further calculation.

3. This section is devoted to the proof that P_N tends to 1 very quickly. The key observation is that until the first time that two adjacent nonfixed sites are occupied, a finite UNPS$(1+\varepsilon)$ grows at least as fast as a particular birth and death chain. We then define two sets of states, state 1 and state 2, for the process $\eta_t^{N,A}$. Loosely speaking, if the process is in state 1, it will have a large number of sites in $A^{\varepsilon,N}$ occupied and far away from each other. It will also have a very high probability of reaching state 2 (in which an extremely large number of sites in $A^{\varepsilon,N}$ are occupied) before becoming extinct on $A^{\varepsilon,N}$. Conversely, the process starting from state 2 will, with very high probability, reach state 1 before becoming extinct. In this way we can show that P_N is close to 1.

We can introduce the birth and death chain:

$B(t)$ will denote the continuous time birth and death chain on the nonnegative integers (starting at 2 unless otherwise stated) with transition rates

$$p(1,2), p(0,1) = 0, \qquad p(1,0) = 1.$$

For $n > 1$, $p(n,n+1) = (n-1)(1+\varepsilon)$, $p(n,n-1) = n$. The next lemma follows simply from a comparison of jump rates and thus no proof is given.

LEMMA 3.1. *Consider $\eta_t^{N,I'}$ with $|\eta_0^{N,I'}| \geq 2$. Let T_I be the first time that two adjacent, nonfixed sites are occupied by $\eta_t^{N,I'}$. There is a coupling of $\eta_t^{N,I'}$ and $B(t)$ so that $|\eta_t^{N,I'}| \geq B(t)$ for $t \in [0, T_I]$.*

It is easy to see that $B(t)$ can escape to infinity with positive probability. As n increases, the transition rates at state n resemble more and more those of a continuous time branching process where particles split into two particles at rate $1+\varepsilon$ and expire with rate 1. These processes grow with exponential rate ε with positive probability, so the following result is not surprising.

LEMMA 3.2. *For each $0 < \delta < \varepsilon$, there exists a positive constant c_δ such that for all t sufficiently large,*

$$P\left[B(t) > e^{\delta t}\right] > c_\delta.$$

PROOF. Fix δ' in (δ, ε). Fix n, even, so large that for all $r \geq n/2$, $(1+\delta')r < (1+\varepsilon)(r-1)$. Let $B'(t)$ be the birth and death process with transition rates $p(m, m+1) = m(1+\delta')$, $p(m, m-1) = m$. If B and B' are both started at n, then we can couple the two processes so that for all $t \leq T_{n/2}$, $B'(t) \leq B(t)$, where $T_{n/2}$ is the first hitting time of $n/2$ by B'. Now it is easy to see that for each t positive, there is a constant C_δ (valid for all n greater than 4) so that $P[B'(t) > e^{\delta(t+1)}] > C_\delta$. Thus $P[B(t) > e^{\delta(t+1)}] > C_\delta - P[T_{n/2} < \infty]$. This latter quantity is greater than $C_\delta/2$ for n large enough. By the Markov property, we see that for t large enough, $P[B(t) > e^{\delta t} | B(0) = 2] > C_\delta/2P[B(1) = n | B(0) = 2]$ and the result follows. □

We now proceed to make use of Lemmas 2.4, 3.1 and 3.2. For notational convenience we suppose that $\log^2 N$ and $\varepsilon 2^{N-1}/\log^2 N$ are integer valued. We divide $A^{\varepsilon, N}$ into $\log^2 N$ subintervals of equal length, $J_1, J_2, \dots, J_{\log^2 N}$.

For $i = 1, 2, \dots, \log^2 N$, let $\chi_t^{N,i}$ be copies of η_t^{N, J_i} which are independent. The following lemma is proved by the same reasoning employed in showing Lemma 1.1 and so its proof is omitted.

LEMMA 3.3. *There is a coupling of the processes* $\chi_t^{N,i}$ *and* $\eta_t^{N,A}$ *so that for all* t, $\bigcup_i \chi_t^{N,i} \subset \eta_t^{N,A}$.

The above lemma enables us to argue that if the probability of an increasing event is bounded away from zero for all the $\chi_t^{N,i}$, then the probability of the event for $\eta_t^{N,A}$ is very large indeed. The following lemma helps ensure that at a fixed time t, the probability that $\eta_t^{N,A} \neq 0$ is small.

LEMMA 3.4. *Let* t_0 *be a fixed time in the interval* $[\sqrt{N}, N/16]$. *For each* k, $P[\chi_{t_0}^{N,k} \neq 0] \geq f(\varepsilon) > 0$ *for some* $f(\varepsilon)$ *strictly greater than zero and depending on* N *or* t_0.

PROOF. Consider the continuous time integer valued process

$$Y(t) = \left| \chi_t^{N,k} \right|.$$

Y is not quite a Markov process though it jumps with constant rate depending on the corresponding state of η: (a) When in state 0, Y jumps to 1 with rate $\varepsilon/2\log^2 N$. (b) When in state 1, Y jumps to 0 with rate 1 and to 2 with rate which depends on $\chi_t^{N,k}$ but is always at least $\varepsilon/2\log^2 N$.

Define the stopping time $T_k = \inf\{t\colon Y(t) = 2\}$. If $T_j > \sqrt{N}$, then one of the following must have occurred: (i) the time spent by Y at 1 before a jump to 2 is greater than $\sqrt{N}/2$; (ii) the number of jumps from 0 to 1 is less than $\sqrt{N}/\log^4 N$ for the first $\sqrt{N}/2$ units of time spent at 0; (iii) the first $\sqrt{N}/\log^4 N - 1$ times that Y jumps from 0 to 1 are followed by returns to 0 before any visits to 2.

It is easy to see that $P[T_k < \sqrt{N}]$ must tend to 1 as N tends to infinity. Let T^k be the first time that $\chi_k^{N,k}$ has two adjacent nonfixed sites which are occupied. Using the proof of Lemma 2.4, we can see that $P[T^k - T_k < N/16]$ tends to 1 as N becomes large.

From T_k until T^k, Y can be coupled with $B(t)$ so that $Y(t + T_k) \geq B(t)$. The birth and death process B does not depend on N and escapes to ∞ with a positive probability depending on ε, $p(\varepsilon)$. For $t_0 < N/16$, it is easily seen, therefore, that

$$P\left[\eta_{t_0}^{N,j} \neq 0\right] \geq P\left[T_j \leq \sqrt{N}\right]\left[p(\varepsilon) - P\left[T^A \leq N/16\right]\right] \geq f(\varepsilon) > 0. \qquad \square$$

We say a state η is in state 1, if there are $f(\varepsilon)\log^2 N/8$ occupied points in $A^{\varepsilon,N}$ which are each more than $(\varepsilon 2^{N-1}/\log^2 N)$ away from the rest.

LEMMA 3.5. *For* $[\sqrt{N} \le t_0 \le N/16]$, $P[\eta_{t_0}^{N,A}$ *is not in state* $1] \le P[B(\log^2 N, f(\varepsilon)) \le (\log^2 N/4) f(\varepsilon)] \le K_{1,\varepsilon} e^{-k_1(\varepsilon)\log^2 N}$ *for N sufficiently large. Here* $B(\log^2 N, f(\varepsilon))$ *denotes a binomial random variable with parameters* $\log^2 N$ *and* $f(\varepsilon)$.

PROOF. Lemma 3.3 states that the process $\eta_t^{N,A}$ may be coupled so that $\bigcup_{i=1}^{\log^2 N} \chi_t^{N,i} \subset \eta_t^{N,A}$. Thus by Lemma 3.4, the number of j such that $\chi_{t_0}^{N,j} \ne 0$ is stochastically greater than a binomial random variable with parameters $\log^2 N$ and $f(\varepsilon)$. By Chernoff's inequality, outside of a set of probability $K_{1,\varepsilon} e^{-k_1(\varepsilon)\log^2 N}$ there will be more than $\log^2(N) f(\varepsilon)/4j$'s with $\chi_{t_0}^{N,j} \ne 0$. If this occurs, then it is easy to see that we can find at least $\log^2(N) f(\varepsilon)/8$ occupied points each at least $\varepsilon 2^{N-1}/\log^2 N$ apart. □

LEMMA 3.6. *Let* $\eta_t^{x,y}$ *be a UNPS*$(1+\varepsilon)$ *process on the whole of* Z^1 *with initial configuration*

$$\eta_0^{x,y}(x) = \eta_0^{x,y}(y) = 1,$$

$$\eta_0^{x,y}(z) = 0 \quad \text{for } z \in Z^1/\{x, y\}.$$

Let $T_{x,y}$ *be the stopping time* $\inf\{t\colon \eta_t^{x,y}$ *has adjacent occupied sites*$\}$. *If* $y - x > \varepsilon 2^{N-1}/\log^2 N$, *then*:

(i) $P[T_{x,y} < N/32] \le K_e 2^{-N/128}$.

(ii) *Until* $T_{x,y}$, $|\eta_t^{x,y}|$ *behaves like the birth and death chain* $B(t)$.

PROOF. The first statement follows in the same way as Lemma 2.4, while the second statement can be seen by simply comparing rates. □

Recall that $\eta_t^{N,A}$ will always denote a uniform nearest particle system for which births outside of $A^{\varepsilon,N}$ are suppressed and whose initial state has $[-2^N, 2^N]$ completely vacant. We change our definitions so that henceforth $\eta_t^{N,A'}$ will denote a process with identical transition rates to $\eta_t^{N,A}$ but whose initial position is in state 1.

LEMMA 3.7. *Consider any process* $\eta_t^{N,A'}$. *There exist* $K_{2,\varepsilon}$ *and* $k_2(\varepsilon)$ *so that for all* $\eta_0^{N,A'}$ *in state* 1 *outside of a set of probability* $K_{2,\varepsilon} e^{-k_2(\varepsilon)\log^2 N}$:

(i) *there will be more than* $e^{\varepsilon N/64}$ *occupied sites in* $A^{\varepsilon,N}$ *at time* $N/32$; *and*

(ii) *at no time t in* $[0, N/32]$ *will* $\eta_t^{N,A'} = 0$.

PROOF. Since the process $\eta_t^{N,A'}$ is attractive, nothing is lost in assuming that our initial state has precisely $(f(\varepsilon)\log^2 N)/8$ occupied sites each more than $(\varepsilon 2^{N-1}/\log^2 N)$ apart from the other occupied sites.

Let $x_1 < y_1 < x_2 < y_2 < \cdots < y_{(f(\varepsilon)\log^2 N)/16}$ denote the occupied sites of $A^{\varepsilon, N}$. Then we may couple the process $\eta_t^{N, A'}$ with the independent processes $\eta_t^{x_i, y_i}$ as described in Lemma 3.6, so that for all time $\bigcup_i \eta_t^{x_i, y_i} \subset \eta_t^{N, A'}$.

It follows from Lemmas 3.2 and 3.6 that for each i, the probability that $|\eta_{N/32}^{x_i, y_i} > e^{\varepsilon N/64}$ is greater than $c_{\varepsilon/2} - K2^{-N/128}$. The result now follows from the independence of the $\eta_t^{x_i, y_i}$ processes. □

We say a state η is in state 2 if it is a possible state for $\eta_t^{N, A}$ at positive time and it contains greater than or equal to $e^{-\varepsilon N/64}$ occupied sites in $A^{\varepsilon N}$. So Lemma 3.7 states that outside a set of small probability a process $\eta_t^{N, A'}$ will be in state 2 at time $N/32$.

We denote any process with transition rates equal to those of $\eta_t^{N, A}$ (and therefore to $\eta_t^{N, A'}$) but starting from η in state 2, by $\psi_t^{N, A}$.

LEMMA 3.8. *Consider any process $\psi_t^{N, A}$ as described in the remarks preceding this lemma. Outside of a set of probability less than $K_{3, \varepsilon} e^{-k_3(\varepsilon)\log^2 N}$ (for all N large enough) we have:* (i) *for all* $0 \le t \le N\varepsilon/192$, $\psi_t^{N, A} \neq 0$, *and* (ii) $\psi_{N\varepsilon/192}^{N, A}$ *is in state* 1.

PROOF. Let $\eta_t^{A, 1}$ be a process such that no births occur and deaths occur independently at rate $2 + \varepsilon$ and $\eta_0^{A, 1} = \psi_0^{A, N}$. From the attractiveness of the respective processes, we can write $\eta_t^{A, 1} \cup \eta_t^{N, A} \subset \psi_t^{N, A}$, where $\eta_0^{A, 1} = \psi_0^{A, N}$ and no births occur for $\eta_t^{A, 1}$ and deaths occur independently at rate $2 + \varepsilon$.

When written in this way, we can see that

$$P\left[\exists\, 0 \le t \le N\varepsilon/192\colon \psi_t^{N, A} = 0\right] < P\left[\exists\, 0 \le t \le N\varepsilon/192\colon \psi_t^{A, 1} = 0\right]$$

$$\le \left[1 - e^{-N\varepsilon(2+\varepsilon)/192}\right]^{e^{N\varepsilon/64}}.$$

Similarly, $P[\psi_{N\varepsilon/192}^{N, A}$ is not in state 1$] \le P[\eta_{N\varepsilon/192}^{N, A}$ is not in state 1$]$.

For N large enough, we can invoke Lemma 3.6 and bound the last quantity by $K_{1, \varepsilon} e^{-k_1(\varepsilon)\log^2 N}$ and the lemma follows. □

Recall: $P_N = P[\forall\, t \in [\sqrt{N}, N^2], \eta_t^{N, A} \neq 0]$.

LEMMA 3.9. *The quantity P_N is greater than $1 - 32NK(\varepsilon)e^{-k(\varepsilon)\log^2 N}$ for some $K(\varepsilon), k(\varepsilon) > 0$.*

PROOF. We say that the process $\eta_t^{N, A}$ makes a crossing from r to s if $\eta_r^{N, A}$ and $\eta_s^{N, A}$ are both in state 1 and for each t in (r, s), $\eta_t^{N, A}$ is not identically zero. It follows from Lemma 3.5 that outside of a set of probability $K_{1, \varepsilon} e^{-k_1(\varepsilon)\log^2 N}$, $\eta_{\sqrt{N}}^{N, A}$ is in state 1. Given this event, Lemmas 3.7 and 3.8 and the Markov property ensure that $\eta_t^{N, A}$ makes crossings from $\sqrt{N} + (i - 1)(1/32 + 1/192)N$ to $\sqrt{N} + i(1/32 + 1/192)N$ for $i = 1, 2, \ldots, 192N/7 + 1$, outside of a set of probability $(1 + N192/7)[K_{2, \varepsilon} e^{-k_2(\varepsilon)\log^2 N} + K_{3, \varepsilon} e^{-k_3(\varepsilon)\log^2 N}]$. The proof of the lemma follows easily. □

4. In this section we complete the proof of Theorem 1.

Let $\mu_{N,1}$ be the unique invariant measure corresponding to the process η_t^N. Since η_t^N is attractive and starts with all its nonfixed sites vacant, it follows that

$$\mu_{N,1}(\{\eta\colon \eta(0)=1\}) \geq \frac{1}{N^2}\int_0^{N^2} P\big[\eta_t^N(0)=1\big]\,dt = \alpha_N.$$

The quantities P_N are all strictly positive, so Lemma 3.9 ensures that

$$\prod_{N=n_0}^{\infty} P_N < \infty,$$

when n_0 is large enough to ensure that P_N is defined. Lemma 1.2 and the above inequality ensure that

$$\liminf_{N\to\infty} \alpha_N \geq \prod_{N=n_0}^{\infty} \frac{N^2-\sqrt{N}}{N^2} \prod_{N=n_0}^{\infty} P_N \alpha_{n_0} > 0.$$

That is, there exists c such that $\alpha_N > c$ for all N. Therefore $\liminf_{N\to\infty} \mu_{N,1}(\{\eta\colon \eta(0)=1\}) \geq c$.

Let $\eta_t^{N'}$ be the UNPS$(1+\varepsilon)$ with 1s fixed at $[-2^N, 2^N]^c$ (in other words, with transition rates equal to those of η_t^N except that births on $[-2^N, -2^N + \varepsilon 2^N]$ and $[2^N - \varepsilon 2^N, 2^N]$ are permitted). By attractiveness, if μ_N is the unique invariant probability measure corresponding to $\eta_t^{N'}$, then $\liminf_{N\to\infty}\mu_N(\{\eta\colon \eta(0)=1\}) \geq \liminf_{N\to\infty}\mu_{N,1}(\{\eta\colon \eta(0)=1\}) \geq c$.

The state space $\{0,1\}^Z$ is compact under its cylinder topology, so there exists μ which is the weak limit of a subsequence of the μ_N's. Necessarily $\mu(\{\eta\colon \eta(0)=1\}) \geq c$ and so is nontrivial. The UNPS is a Feller process and for a fixed continuous function g, by Theorem 2.2 of Liggett (1983), $S_{-2^N,2^N}(t)g$ tends uniformly in N to $S(t)g$, where $S(t)$ is the semigroup corresponding to the unrestricted UNPS and $S_{-2^N,2^N}(t)$ is that for the processes $\eta_t^{N'}$. It now follows that for each continuous g and time t, that

$$\begin{aligned}\int S(t)g(\eta)\,d\mu(\eta) &= \lim_{N_i\to\infty}\int S(t)g(\eta)\,d\mu_{N_i}(\eta)\\ &= \lim_{N_i\to\infty}\int S_{-2^{N_i},2^{N_i}}(t)g(\eta)\,d\mu_{N_i}(\eta)\\ &= \lim_{N_i\to\infty}\int g(\eta)\,d\mu_{N_i}(\eta) = \int g(\eta)\,d\mu(\eta),\end{aligned}$$

which is to say μ is a nontrivial invariant measure. □

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