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A FUNCTIONAL APPROACH TO THE STATIONARY WAITING TIME AND IDLE PERIOD DISTRIBUTIONS OF THE GI / G / 1 QUEUE

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The GI/G/1 queueing model is regarded as a functional which maps the service and interarrival time distributions onto output quantities of interest, such as the stationary waiting time distribution. For the case where the input distributions have densities, techniques from infinite-dimensional analysis are used to obtain derivatives and Taylor series expansions for the functionals. These yield approximations to the output distributions which can be viewed as nonparametric alternatives to parametric approximations such as those provided by infinitesimal perturbation analysis or the phase method.

1. Introduction. The stochastic model under consideration in this paper is that of the standard GI/G/1 queue. As the notation suggests, a precise description can be given for this queueing model which fixes the mechanism by which the queue evolves, while leaving the distributions of the times between customer arrivals and of the customer service times unspecified. Quantities of interest, for example, the distribution of customer waiting times, then depend on these unspecified distributions and the GI/G/1 queueing model can be regarded as a functional which maps the input distributions of interarrival and service times onto the relevant output quantity.

Such an approach allows for investigation of probabilistic features of the stochastic model through investigation of mathematical properties of the functional. Continuity, with respect to an appropriate topology, corresponds to a qualitative robustness result. Loosely, continuity implies that small changes in the input give rise to small changes in the output (where small refers to the topology in question). Robustness results are important in connection with approximations which may be made in applications. Often an easy explicit expression for the output is only available for certain special input, and in practice a particular input may be approximated by a new input of this special sort. Continuity of the functional is then required to justify using the new output as an approximation for the original output.

In this paper, we establish a suitable differentiability result for the two GI/G/1 functionals described in the next section. This gives a local approximation for the change in output arising from a small change in input, and thus differentiability yields a quantitative robustness result.

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A natural further step is to consider higher derivatives in the hope of obtaining better approximations, and this leads to the question of convergence and whether asymptotic expansions can be found. Our results show that this can be done, under conditions referring to the distance between the input distributions to the original and the perturbed models.

Continuity for the GI/G/1 queue has been considered by Kennedy (1972) and Borovkov (1976) among others. The idea of viewing a stochastic model as a functional and then finding an appropriate derivative has been used for renewal theory in Grübel (1989a) and for the random walk on the integers by Grübel and Pitts (1989).

Two other main approaches to stability and perturbation aspects of stochastic models, in particular, of queueing models, appear in the literature. These are different from each other and also from the approach of the present paper. The first of these alternative approaches also regards a stochastic model as a functional relating an output quantity μ_{out} to some input quantity μ_{in}. A quantitative assessment of the stability of the model could then be of the form

$$d_2(\mu_{\text{out}}, \mu'_{\text{out}}) \le d_1(\mu_{\text{in}}, \mu'_{\text{in}}), \qquad (1)$$

where d_1, d_2 are metrics or other distance measures. The theory of probability metrics, initiated by Zolotarev, can be used in this context if the μ's are probability distributions. Two recent monographs, Kalashnikov and Rachev (1990) and Rachev (1991), can serve as an introduction to this area of research; each provides a comprehensive list of references.

The second approach is known as infinitesimal perturbation analysis. One of the central ideas can be summarized as follows. Some measure of performance of the stochastic model, for example, the expectation of the stationary waiting time, depends on some system parameter, for example, the mean service time. These parameters are real- or vector-valued so that the dependence can be regarded as a function between finite-dimensional spaces. The local behaviour of the dependence is then represented by the derivative of this function. Using some ergodicity property of the model, estimators for the derivative can be found which are based on a single path of the system. Again, a recent and comprehensive survey, Suri (1989), is available.

The differentiability results obtained here for two GI/G/1 functionals go beyond continuity and distance bounds such as (1). The functionals are, in contrast to the situation considered in infinitesimal perturbation analysis, mappings from and to infinite-dimensional spaces, that is, we avoid parametrization of the perturbations. We should point out, however, that the above methods are more easily adapted to other models (see also the concluding remarks at the end of the paper).

In Section 2 we give definitions for the GI/G/1 queue and for the two quantities of interest. Our analysis of the functionals is based on the classical connection with random walks and turns on the easy access to the Wiener–Hopf factors provided by harmonic renewal measures. Section 2 contains a brief description of these two key steps; the second might be of interest in its own right. In Section 3, we specialize to the case where the input distributions have densities and obtain expressions for the functionals as the compositions of several easier maps. In Sections 4 and 5, we differentiate and expand the functionals. The cumulants of the stationary waiting time distribution are discussed in Section 6. Section 7 contains examples and discussion.

Our approach to the problem of obtaining approximations to the output distributions is a nonparametric one, in contrast to the phase method, where the input distributions are replaced by members of a parametric family for which the output can easily be calculated. This means that our approach necessitates the use of infinite-dimensional analysis, rather than analysis in $\mathbf{R}^d$. While this is not difficult in itself, it has not, in our experience, been widely applied in queueing theory. It is hoped that the inclusion of sketches of the mathematical details will encourage the adoption of these methods as useful additions to the applied probability toolbox.

2. GI / G / 1 queue and associated random walk. The GI/G/1 queueing model consists of a single server queue, where customer number zero arrives at $t = 0$ to an empty system and enters service immediately. For n in $\mathbf{N}$, customer number n arrives at time $\Sigma_{i=1}^{n} T_i$, where the interarrival times $\{T_n\}_{n\geq 1}$ are independent identically distributed positive random variables with distribution μ_T and with $E(T_1) < \infty$. The customers are served in order of arrival, the service time of customer n being S_n, where $\{S_n\}_{n\geq 0}$ are independent identically distributed positive random variables with distribution μ_S and with $E(S_1) < \infty$. The service times are assumed to be independent of the interarrival times. This description fixes the GI/G/1 setup, or mechanism, and hence the functionals.

Let W_n be the waiting time of customer n from the moment of arrival up to the beginning of service. The traffic intensity ρ is defined to be $E(S_1)/E(T_1)$. If ρ is smaller than 1, then the queue is stable and as $n \to \infty$, the W_n converge in distribution to a proper random variable W, called the stationary waiting time. [For these and the standard results below for the GI/G/1 queue and the associated random walk, see Asmussen (1987), Chapters VII and VIII.] Write μ_W for the distribution of the stationary waiting time W. This is our first quantity of interest.

The server's time is composed of busy periods and idle periods. Write I for the first period of time (> 0) for which the server is idle and write μ_I for its (possibly defective) distribution. The times when a customer arrives to an empty system are regeneration points for the queue so that the lengths of successive idle periods are independent identically distributed random variables, distributed as I. The distribution μ_I is our second quantity of interest.

From a functional point of view, we have two maps, the first taking (μ_S, μ_T) to μ_W and the second taking (μ_S, μ_T) to μ_I.

Let $\{X_i\}_{i\in\mathbf{N}}$ be independent identically distributed random variables, with the distribution μ_X of X_1 the same as that of $S_1 - T_1$. Put $Z_0 = 0$ and, for n in $\mathbf{N}$, put $Z_n = \Sigma_{i=1}^{n} X_i$. Let M denote the maximum of the random walk $\{Z_n\}_{n\geq 0}$; M is finite (for almost all sample paths) because of $E(X_1) < 0$. Then

$$W =_{\mathscr{D}} M,$$

where $=_{\mathscr{D}}$ means has the same distribution as. This relation arises from Lindley's equation [see, e.g., Asmussen (1987), III.7].

The analysis of the distribution of M is facilitated by the analysis of the distributions of the ladder variables. These will also provide access to the idle period distribution. The first (strict) ascending ladder epoch τ_+ associated with the random walk $\{Z_n\}_{n\geq 0}$ and the first (weak) descending ladder epoch τ_- are defined by

$$\tau_+ = \inf\{n > 0: Z_n > 0\}, \qquad \tau - = \inf\{n > 0: Z_n \leq 0\},$$

where $\inf \varnothing = \infty$. The first (strict) ascending ladder height Y_+ and the first (weak) descending ladder height Y_- are given by $Y_+ = Z_{\tau+}$ on $\{\tau_+ < \infty\}$ and $Y_- = Z_{\tau-}$ on $\{\tau_- < \infty\}$, and are undefined otherwise. Write μ_+ and μ_- for the (possibly defective) distributions of Y_+ and Y_-, respectively. These are called the (strong) right and (weak) left Wiener–Hopf factors of the step distribution μ_X. In the case of a stable queue, where the step distribution of the random walk has negative mean, τ_+ and Y_+ are defective, while τ_- and Y_- are proper [Feller (1971), XII.2]. Weak right and strong left Wiener–Hopf factors are defined analogously and are denoted by μ_+^w and μ_-^s, respectively.

From Asmussen [(1987), IV, Proposition 2.6] we have that, for the random walk associated with a stable queue, the distribution μ_M of the maximum M is given by

$$\mu_M = (1 - \mu_+(\mathbf{R})) \sum_{k=0}^{\infty} \mu_+^{*k}, \tag{2}$$

where μ_+^{*k} is the k-fold convolution of μ_+ and $\mu_+^{*0} = \delta_0$, the unit mass at zero. This gives a simple explicit relation between the stationary waiting time distribution and the right Wiener–Hopf factor of the associated random walk.

The idle period distribution μ_I is similarly related to the left Wiener–Hopf factor,

$$\mu_I = (\mu_-^s)^{\vee}, \tag{3}$$

where $\check{\mu}(B) = \mu(\{x: -x \in B\})$ for B in the collection $\mathscr{B}$ of all Borel sets in $\mathbf{R}$ [Feller (1971), VI.9].

Access to the Wiener–Hopf factors is achieved via their harmonic renewal measures as explained below.

Definition 2.1. For any (possibly defective) probability measure μ on $(\mathbf{R}, \mathscr{B})$, define the associated harmonic renewal measure $\nu(\mu)$ by

$$\nu(\mu) = \sum_{k\geq 1} \mu^{*k}/k.$$

There is a simple connection between the step distribution and the Wiener–Hopf factors given by

$$\begin{aligned} \nu(\mu_+)(B) &= \nu(\mu_X)(B \cap (0,\infty)), \\ \nu(\mu_-)(B) &= \nu(\mu_X)(B \cap (-\infty, 0]) \end{aligned} \tag{4}$$

for all B in $\mathscr{B}$. This follows from the Spitzer–Baxter equations and is discussed in Grübel (1989b). Hence $\nu(\mu_+)$ and $\nu(\mu_-)$ are the traces of $\nu(\mu_X)$ on the positive and nonpositive half-lines, respectively. Similar relations exist for the weak right and strong left factors.

Putting these ideas together, we find that the stationary waiting time functional can be decomposed as follows:

$$(\mu_S, \mu_T) \mapsto \mu_X \mapsto \nu(\mu_X) \mapsto \nu(\mu_+) \mapsto \mu_+ \mapsto \mu_W. \tag{5}$$

Similarly, for the idle period functional,

$$(\mu_S, \mu_T) \mapsto \mu_X \mapsto \nu(\mu_X) \mapsto \nu(\mu_-^s) \mapsto \mu_I. \tag{6}$$

The details for the decompositions remain to be given. In particular, since the step distribution and the left Wiener–Hopf factor are nondefective for a stable queue, their harmonic renewal measures are infinite measures and appropriate normalizations will be required.

3. Decomposition of the functionals. When the input distributions to the queue have densities, a normalization is available which handles the infinite harmonic renewal measures. This is contained in Grübel (1986) and a precise statement is given later in Lemma 3.1. Suppose that μ_S and μ_T have densities f_S and f_T, respectively. These can be regarded as elements of

$$L^1 = \left\{ f\colon \mathbf{R} \to \mathbf{C}\colon f \text{ is Borel measurable, } \|f\|_1 = \int |f(x)|\,dx < \infty \right\}.$$

With convolution $*$ as multiplication,

$$f * g(t) = \int f(t-x) g(x)\,dx,$$

L^1 is a commutative Banach algebra without a unit; see Rudin [(1974, Chapters 10 and 11] for the general theory of Banach algebras and further details. Writing μ_f for the (complex-valued) measure with density f, we can embed L^1 in

$$A^1 = \left\{ \mu_f + \alpha\delta_0\colon f \in L^1, \alpha \in \mathbf{C} \right\}$$

which is a Banach algebra with a unit. We will repeatedly use the norm inequality

$$\|a * b\|_1 \le \|a\|_1 \|b\|_1 \quad \text{for all } a \text{ and } b \text{ in } A^1. \tag{7}$$

We will also often write $f + \alpha\delta_0$ instead of $\mu_f + \alpha\delta_0$.

For analysis of mappings between Banach algebras, the Gelfand transform is most important as it permits the representation of our algebra as a function algebra. We give a brief sketch and, again, refer the reader to Rudin's book for details.

Let $\mathscr{I}$ be the space of maximal ideals I or, equivalently, the set of nonzero complex homomorphisms ψ on the Banach algebra A. If A is semisimple, then each $a \in A$ is characterized by its Gelfand transform $\tilde{a}\colon \mathscr{I} \to \mathbf{C}$, where

$\tilde{a}(I) = \psi_I(a)$. In the case of A^1, $\mathscr{I}$ can be identified with $\mathbf{R} \cup \{\infty\}$ and

$$(f + \alpha\delta_0)^{\sim}(\theta) = \begin{cases} \psi_{I(\theta)}(f + \alpha\delta_0) = (f + \alpha\delta_0)^{\wedge}(\theta), & \text{for } \theta \in \mathbf{R}, \\ \psi_{I(\infty)}(f + \alpha\delta_0) = \alpha, & \text{for } \theta = \infty, \end{cases}$$

where $\hat{\mu}(\theta) = \int e^{i\theta x}\mu(dx)$ denotes the usual Fourier transform.

We require various subalgebras which, in probabilistic terms, are characterized by the existence of certain moments. Let

$$A^{1,n} = \{f + \alpha\delta_0 \colon f \in L^1, \alpha \in \mathbf{C}, \|f\|_{1,n} < \infty\},$$

where

$$\|f\|_{1,n} = \int (1 + |x|)^n |f(x)|\, dx.$$

These spaces are commutative Banach algebras again, and their maximal ideals arise as the respective intersections of the maximal ideals of A^1.

Multiplication together with the norm inequality (7) leads to power series in A and analytic functions $\Psi\colon D \to A$, $D \subset A$. The Gelfand transformation provides a most useful connection between such a function and its $\mathbf{C}$ to $\mathbf{C}$ counterpart $\Psi_{\mathbf{C}}$. In particular, for A^1 we have

$$(8) \qquad (\Psi(f + \alpha\delta_0))^{\sim}(\theta) = \Psi_{\mathbf{C}}\big((f + \alpha\delta_0)^{\sim}(\theta)\big),$$

provided that $\Psi_{\mathbf{C}}$ is analytic on an open subset containing $(f + \alpha\delta_0)^{\sim}(\theta)$, θ in $\mathbf{R} \cup \{\infty\}$. This means that if $\Psi_{\mathbf{C}}$ is analytic on an open subset Ω in $\mathbf{C}$, then Ψ is defined on $A_\Omega = \{a \in A^1\colon \tilde{a}(\mathscr{I}) \subseteq \Omega\}$. This will be used below for the exponential function $\exp_{\mathbf{C}}$, with $\Omega = \mathbf{C}$, the (principal branch of the) logarithm, with $\Omega = G = \{z \in \mathbf{C}\colon \operatorname{Re}(z) > 0 \text{ or } \operatorname{Im}(z) \neq 0\}$ and the inverse function $z \mapsto 1/z$ with $\Omega = \{z \in \mathbf{C}\colon z \neq 0\}$.

Several linear operators are also required. Let $R_+\colon A^1 \to A^1$, $R_-\colon A^1 \to A^1$ be given by

$$R_+(f + \alpha\delta_0) = fI_{(0,\infty)}, \qquad R_-(f + \alpha\delta_0) = fI_{(-\infty,0)},$$

where I_B is the indicator function of the set B.

For f in L^1, let $\check{f}$ be defined by

$$\check{f}(t) = f(-t)$$

for all t in $\mathbf{R}$. Define $(f + \alpha\delta_0)^{\vee}$ to be $\check{f} + \alpha\delta_0$.

For f in $L^{1,1}$, define Σf by

$$(\Sigma f)(t) = \begin{cases} \int_t^\infty f(x)\, dx, & \text{if } t \geq 0, \\ -\int_{-\infty}^t f(x)\, dx, & \text{otherwise}. \end{cases}$$

For $a = \alpha\delta_0 + f$ in $A^{1,1}$, define Σa by

$$\Sigma a = \Sigma f.$$

Using Fubini's theorem we obtain

$$\|\Sigma(f+\alpha\delta_0)\|_1 \le \int |t|\,|f(t)|\,dt \le \|f+\alpha\delta_0\|_{1,1}, \tag{9}$$

which shows that Σ is a bounded linear map from $A^{1,1}$ to A^1. Applying the first inequality in (9), we find for a in $A^{1,1}$,

$$\|\Sigma a + a\|_1 \le \|a\|_{1,1}. \tag{10}$$

In the context of a stable GI/G/1 queue, the input (f_S, f_T) is regarded as an element of $A^{1,1} \times A^{1,1}$. From (5) and (6), the first stage in the decomposition of the functionals maps (f_S, f_T) onto $f = f_S * \check{f}_T$, a density of the step distribution of the associated random walk. Put $v(f) = \sum_{k=1}^{\infty} f^{*k}/k$, so that $v(f)\colon \mathbf{R} \to [0,\infty]$ and

$$\nu(\mu_X)(B) = \int_B v(f)(x)\,dx$$

for all B in $\mathscr{B}$, where we interpret the integral of a nonnegative function g, say, which possibly takes the value infinity, as the supremum of the integrals of simple functions s with $0 \le s \le g$. Since, for a stable queue, if $X_1 = S_1 - T_1$, we have $E(|X_1|) < \infty$ and $E(X_1) < 0$, Lemma 1 of Heyde (1964) implies that, in this case, $\nu(\mu_X)$ is finite on compact sets. We can therefore select a version of $v(f)$ which is finite everywhere. However, $v(f)$ is not in L^1 because, as f is nondefective, $\nu(\mu_X)$ is infinite.

Restricting $v(f)$ to the positive (negative) half-line gives a density for $\nu(\mu_+)$ ($\nu(\mu^s_-)$). For any probability measure μ, $\mu(B) \le \nu(\mu)(B)$ for all B in $\mathscr{B}$, so that if $\nu(\mu_+)$ and $\nu(\mu^s_-)$ have densities, then so do μ_+ and μ^s_-, and it is not necessary to distinguish between densities of strong and weak Wiener–Hopf factors. Write f_+, f_-, $v(f_+)$ and $v(f_-)$ for densities of μ_+, μ^s_-, $\nu(\mu_+)$ and $\nu(\mu^s_-)$, respectively.

By (3) the idle period distribution has a density f_I given by $f_I = (f_-)^{\vee}$. From (2), the stationary waiting time distribution is of the form $\alpha_W\delta_0 + \mu_{W,f}$, where α_W is the limiting probability that customer n finds the queue empty on arrival and where $\mu_{W,f}$ is concentrated on $(0,\infty)$. The measure $\mu_{W,f}$ is $(1-\mu_+(\mathbf{R}))\sum_{k=1}^{\infty}\mu_+^{*k}$ and is finite (because μ_+ is defective). Arguing as for $\nu(\mu_X)$ above, it has a density which we denote f_W. Thus if the input has densities, then both our output quantities can be regarded as elements of A^1.

The normalization anticipated at the beginning of this section is given in the following lemma.

LEMMA 3.1. *Let f be a proper probability density function satisfying*

$$\int t^2 f(t)\,dt < \infty \tag{11}$$

and

$$\int t f(t)\,dt < 0.$$

Let f_1 be a density of the exponential distribution with mean 1. *Then*

$$(-\Sigma f - f + \delta_0) * (\check{f}_1 - \delta_0) = f - \delta_0.$$

Further, $v(f) - v(\check{f}_1)$ *is in* A^1 *with*

$$v(f) - v(\check{f}_1) = -\log(-\Sigma f - f + \delta_0).$$

This follows easily from the proof of Theorem 1 in Grübel (1986). We need a corresponding result for defective probability densities.

LEMMA 3.2. *Let f be a defective probability density function. Then $v(f)$ is in A^1 and*

$$v(f) = -\log(\delta_0 - f).$$

PROOF. Since $\|f\|_1 < 1$, we have $\Sigma_{k=1}^{\infty}\|f^{*k}\|_1/k \le \Sigma_{k=1}^{\infty}\|f\|_1^k/k < \infty$ using (7), so that $v(f)$ is in L^1.

For all θ in $\mathbf{R} \cup \{\infty\}$, $|\tilde{f}(\theta)| \le \|f\|_1 < 1$ implies that $\mathrm{Re}[(\delta_0 - f)\tilde{}\,(\theta)] > 0$ and so $(\delta_0 - f)\tilde{}\,(\mathscr{I}) \subseteq G$. Thus $\delta_0 - f$ is in A_G and has a logarithm in A^1. Using (8), the fact that $|\tilde{f}(\theta)| < 1$ and the multiplicativity and continuity of $\psi_{I(\theta)}$, we have

$$[-\log(\delta_0 - f)]\tilde{}\,(\theta) = \sum_{k \ge 1} (\psi_{I(\theta)}(f))^k/k = (v(f))\tilde{}\,(\theta).$$

This holds for all θ in $\mathbf{R} \cup \{\infty\}$ and the result now follows because A^1 is semisimple. □

In the case of a stable queue, f_+ is defective and, applying Lemma 3.2 and (8), we find

$$\begin{aligned}(\delta_0 - f_+)\tilde{}\,(0) &= (\exp\{-v(f_+)\})\tilde{}\,(0)\\ &= \exp_{\mathbf{C}}\{-v(f_+)\hat{}\,(0)\}.\end{aligned}$$

Comparing this with (2), if $w = \alpha_W\delta_0 + f_W$ is the element of A^1 corresponding to the stationary waiting time distribution, then

$$\alpha_W = (1 - \hat{f}_+(0)) = \exp_{\mathbf{C}}\{-v(f_+)\hat{}\,(0)\}. \tag{12}$$

Spitzer's identity [Feller (1971), XVIII, Theorem 2] applied to the random walk associated with a stable queue yields

$$\tilde{w}(\theta) = [\exp\{v(f_+) - v(f_+)\hat{}\,(0)\delta_0\}]\tilde{}\,(\theta)$$

for all θ in $\mathbf{R}$. Together with (12) and identification via transforms, this yields

$$w = \exp\{v(f_+) - v(f_+)\hat{}\,(0)\delta_0\}.$$

The stationary waiting time functional can then be decomposed as follows for the case where f_S and f_T are densities satisfying (11) for the service and interarrival times of a stable queue:

$$(13)\quad \underset{A^{1,1}\times A^{1,1}}{(f_S, f_T)} \mapsto \underset{A^{1,1}}{f} \mapsto \underset{A^1}{-\Sigma f - f + \delta_0} \mapsto_{-\log} \underset{A^1}{v(f) - v(\check{f}_1)} \mapsto_{R_+} \underset{A^1}{v(f_+)} \mapsto \underset{A^1}{v(f_+) - v(f_+)\hat{}(0)\delta_0} \mapsto_{\exp} \underset{A^1}{w}.$$

The idle period functional can be written

$$(14)\quad \underset{A^{1,1}\times A^{1,1}}{(f_S, f_T)} \mapsto \underset{A^1}{-\Sigma f - f + \delta_0} \mapsto_{\log} \underset{A^1}{v(\check{f}_1) - v(f)} \mapsto_{R_-} \underset{A^1}{v(\check{f}_1) - v(f_-)} \mapsto_{\exp} \underset{A^1}{-\Sigma f_- - f_- + \delta_0} \mapsto_{*(\check{f}_1 - \delta_0)} \underset{A^1}{f_- - \delta_0} \mapsto \underset{A^1}{f_I - \delta_0},$$

where we have applied Lemma 3.1 to f_- for the fourth and fifth component maps $\mapsto_{\exp}$ and $\mapsto_{*(\check{f}_1-\delta_0)}$.

We are only interested in the values of the functionals in the case where both arguments are probability densities. However, the functional is defined in an open subset of $A^{1,1} \times A^{1,1}$. Observe that the composite maps are nonlinear, since the exponential and logarithm maps are nonlinear, and they contain steps (R_+, R_-) which are not (1, 1).

4. Derivatives of the functionals. We consider differentiability of the functionals in the sense of Fréchet differentiability of maps between Banach spaces [see, e.g., 2.1 in Chapter 1 of Cartan (1971)]. Let B_1 and B_2 be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively and let U be a (nonempty) open subset of B_1. A map $\Phi: U \to B_2$ is Fréchet differentiable at a in U if there exists a bounded linear operator Φ'_a such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|y\|_1 < \delta$ implies

$$\|\Phi(a+y) - \Phi(a) - \Phi'_a(y)\|_2 < \varepsilon\|y\|_1.$$

We say that Φ has Fréchet derivative Φ'_a at a. The derivative of Φ is a linear operator which can be used in a local approximation to the nonlinear operator Φ, that is, for y small, $\Phi(a+y) \approx \Phi(a) + \Phi'_a(y)$. If the map Φ' sending a to Φ'_a is continuous, then we say that Φ is continuously differentiable. If $\Phi: B_1 \to B_2$ is itself a bounded linear operator, then it is obviously differentiable everywhere with derivative itself.

Since the Fréchet derivative obeys the chain rule [Cartan (1971), 1.2], we can establish differentiability of the queueing functionals by showing that each of the component maps in the decompositions (13) and (14) are differentiable.

Gelfand transforms can be used to find derivatives of certain functionals on Banach algebras [see Rudin (1974), 10.36]. Let Ω be open in $\mathbf{C}$ and $\Psi_{\mathbf{C}}: \Omega \to \mathbf{C}$ be analytic on Ω. Put $A_\Omega = \{a \in A^1: \tilde{a}(\mathscr{I}) \subseteq \Omega\}$. If $\Psi: A_\Omega \to A^1$ is such that, for all a in A_Ω, $(\Psi(a))\tilde{}(\theta) = \Psi_{\mathbf{C}}(\tilde{a}(\theta))$ for all θ in $\mathbf{R} \cup \{\infty\}$, then Ψ is differentiable at a with

$$(15)\quad (\Psi'_a(x))\tilde{}(\theta) = \Psi'_{\mathbf{C}}(\tilde{a}(\theta))\tilde{x}(\theta) \quad \text{for all } x \text{ in } A^1 \text{ and } \theta \text{ in } \mathbf{R} \cup \{\infty\}.$$

For the logarithm and exponential functions on A^1, for example, this implies

$$\exp'_a(x) = (\exp a) * x \quad \text{and} \quad \log'_a(x) = a^{*-1} * x,$$

provided that $\log a$ is defined.

We are now in a position to prove theorems concerning the differentiability of the queueing functionals. We only consider the functionals at points which are pairs of probability densities. This leads us to consider what might be called curves in the respective Banach spaces. In general, let Φ map B_1 to B_2, where B_1 and B_2 are Banach spaces, and let $x^{(\varepsilon)}$ be in B_1 for $0 \le \varepsilon \le 1$ and

$$\frac{1}{\varepsilon}(x^{(\varepsilon)} - x^{(0)}) \to y$$

in B_1 as $\varepsilon \downarrow 0$, so that $x^{(\varepsilon)}$ approaches $x^{(0)}$ smoothly. As an example of a curve, if $x^{(\varepsilon)} = (1-\varepsilon)x^{(0)} + \varepsilon x^{(1)}$ for $0 \le \varepsilon \le 1$ and $x^{(0)}$ and $x^{(1)}$ in B_1, then $y = x^{(1)} - x^{(0)}$ and so $x^{(\varepsilon)}$ approaches $x^{(0)}$ along y. The probabilistic interpretation of this type of curve is that we have, for small ε, an infinitesimal perturbation of the queueing model corresponding to the $x^{(0)}$-input in the direction of the $x^{(1)}$-input. Suppose that Φ is differentiable at $x^{(0)}$ with derivative $\Phi'_{x^{(0)}}$ there. Then

$$\frac{1}{\varepsilon}(\Phi(x^{(\varepsilon)}) - \Phi(x^{(0)})) \to \Phi'_{x^{(0)}}(y)$$

in B_2 as $\varepsilon \downarrow 0$.

In Theorem 4.1 we assume that we have one queue for each ε, $0 \le \varepsilon \le 1$, and that, as $\varepsilon \downarrow 0$, $(f_S^{(\varepsilon)}, f_T^{(\varepsilon)})$ converges to $(f_S^{(0)}, f_T^{(0)})$ in an appropriate way.

THEOREM 4.1. *For $0 \le \varepsilon \le 1$, let $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$ be probability densities satisfying* (11). *Let $\rho^{(\varepsilon)}$ be the traffic intensity of the queue with service time and interarrival time densities $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$, respectively. Suppose that $\rho^{(\varepsilon)} < 1$ and let $w^{(\varepsilon)}$ give the associated stationary waiting time distribution.*

Suppose

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(f_S^{(\varepsilon)} - f_S^{(0)}) = g_S \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(f_T^{(\varepsilon)} - f_T^{(0)}) = g_T$$

in $A^{1,1}$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(w^{(\varepsilon)} - w^{(0)}) = w^{(0)} * \left[(R_+h)\hat{\ }(0)\delta_0 - R_+h\right] \quad \text{in } A^1,$$

where $h = (-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} * (-\Sigma g - g)$, $f^{(0)} = f_S^{(0)} * \check{f}_T^{(0)}$ and $g = g_S * \check{f}_T^{(0)} + f_S^{(0)} * \check{g}_T$.

PROOF. Write Φ for the functional mapping (f_S, f_T) onto w. Then Φ can be decomposed as in (13). Write T for the map from A^1 to A^1, where, for a in A^1,

$$T(a) = a - \hat{a}(0)\delta_0.$$

Then T is linear and bounded.

The bilinear map from $A^{1,1} \times A^{1,1}$ where $(a, b) \mapsto a * \check{b}$ is differentiable everywhere with derivative at (a, b) is given by $(x, y) \mapsto x * \check{b} + a * \check{y}$ for all (x,y) in $A^{1,1} \times A^{1,1}$.

The map

$$a \mapsto (-\Sigma - Id)(a - \delta_0),$$

where Id is the embedding $A^{1,1} \to A^1$, is differentiable everywhere with derivative at a given by

$$x \mapsto (-\Sigma - Id)x$$

for all x in $A^{1,1}$.

The derivatives of the maps exp, log and the linear bounded maps R_+ and T have been covered in the discussion preceding the theorem.

Write $f^{(0)}$, $f_+^{(0)}$ for the step density and first ascending ladder height density of the queue with input $f_S^{(0)}$ and $f_T^{(0)}$.

Since each of the component maps in the decomposition of the functional is differentiable at the relevant elements of $A^{1,1} \times A^{1,1}$, $A^{1,1}$ or A^1 (e.g., $-\log$ is differentiable at $-\Sigma f^{(0)} - f^{(0)} + \delta_0$), the chain rule implies that Φ itself is differentiable at $(f_S^{(0)}, f_T^{(0)})$ and

$$\begin{aligned}\Phi'_{(f_S^{(0)}, f_T^{(0)})}(x, y) &= \exp'_{T(\nu(f_+^{(0)}))} \circ T \circ R_+ \circ \left(-\log'_{-\Sigma f^{(0)} - f^{(0)} + \delta_0}\right) \circ (-\Sigma - Id)\left(f_S^{(0)} * \check{y} + x * \check{f}_T^{(0)}\right) \\ &= \exp\left(T\left(\nu\left(f_+^{(0)}\right)\right)\right) * T\left(R_+\left[-\left(-\Sigma f^{(0)} - f^{(0)} + \delta_0\right)^{*-1} * (-\Sigma z - z)\right]\right),\end{aligned}$$

where $z = f_S^{(0)} * \check{y} + x * \check{f}_T^{(0)}$, $x, y \in A^{1,1}$. Writing h for $(-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} * (-\Sigma g - g)$, where g is $f_S^{(0)} * \check{g}_T + g_S * \check{f}_T^{(0)}$, and evaluating the derivative at (g_S, g_T) gives

$$\Phi'_{(f_S^{(0)}, f_T^{(0)})}(g_S, g_T) = w^{(0)} * \left[(R_+h)^\wedge(0)\delta_0 - R_+h\right]. \qquad \square$$

Employing the same methods as for Theorem 4.1 we can prove the following theorem for the idle period distribution for a stable queue.

THEOREM 4.2. *For $0 \le \varepsilon \le 1$, let $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$ be probability densities satisfying* (11). *Let $\rho^{(\varepsilon)}$ be the traffic intensity of the queue with service time and interarrival time densities $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$, respectively. Suppose that $\rho^{(\varepsilon)} < 1$ and let $f_I^{(\varepsilon)}$ be the associated idle period distribution.*

Suppose

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f_S^{(\varepsilon)} - f_S^{(0)}\right) = g_S \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f_T^{(\varepsilon)} - f_T^{(0)}\right) = g_T$$

in $A^{1,1}$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f_I^{(\varepsilon)} - f_I^{(0)}\right) = \left(f_I^{(0)} - \delta_0\right) * \left((R_-h)^\vee\right) \quad \text{in } A^1,$$

where $h = (-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} * (-\Sigma g - g)$, $f^{(0)} = f_S^{(0)} * \check{f}_T^{(0)}$ and $g = g_S * \check{f}_T^{(0)} + f_S^{(0)} * \check{g}_T$.

PROOF. Use the chain rule again, noting that the maps $\check{}$ and $x \mapsto x * (\check{f}_1 - \delta_0)$ are both linear and bounded. □

Note that $\rho^{(0)} < 1$ so that there exists $\varepsilon_0 > 0$ such that $\sup_{0 \le \varepsilon \le \varepsilon_0} \rho^{(\varepsilon)} < 1$. Since Fréchet differentiability of a functional at $(f_S^{(0)}, f_T^{(0)})$ implies its continuity there, we obtain continuity of the queueing functionals (as maps from $A^{1,1} \times A^{1,1}$ to A^1) as immediate corollaries to Theorems 4.1 and 4.2.

If the base point of the derivative is an M(1)/M(β)/1 queue (where $\beta > 1$ in order to have the traffic intensity $\rho = 1/\beta < 1$), some straightforward calculations show that $(-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} = \delta_0 + r$, where $r(x) = e^{(1-\beta)x}$ for $x > 0$, $r(x) = 0$ otherwise. This is due to the fact that the step distribution of the associated random walk is a mixture of an exponential distribution and an exponential distribution with mean 1 reflected at the origin. The latter is in the kernel of the operator $\Sigma + Id$. Note that mean 1 for interarrival times can be achieved for arbitrary queues by a simple time scaling.

5. Expansions for the functionals. Let $\Phi\colon U \to B_2$ be differentiable throughout U. We can then regard $a \mapsto \Phi'_a$ as a map on U with values in the Banach space $\mathscr{L}(B_1, B_2)$ of bounded linear operators from B_1 to B_2. If this map is differentiable at a, then this derivative is called the second derivative $\Phi_a^{(2)}$ of Φ at a.

This process can be repeated to find higher derivatives of Φ. The nth derivative $\Phi_a^{(n)}$ of Φ at a, if it exists, is a symmetric, bounded n-linear map from $B_1 \times \cdots \times B_1$ to B_2; see Cartan [(1971), 1.5] for more details.

Suppose that Φ has derivatives up to order $n - 1$ throughout U. Write $\Phi_a^{(n-1)}$ for the $(n-1)$st derivative of Φ at a in U. Then the nth derivative $\Phi_a^{(n)}$ of Φ at a can be regarded as a bounded n-linear form such that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|y\|_1 < \delta$ implies

$$\sup_{\|x_i\|_1 \le 1,\, 1 \le i \le n-1} \Big\| \Phi_{a+y}^{(n-1)}(x_1, \dots, x_{n-1}) - \Phi_a^{(n-1)}(x_1, \dots, x_{n-1}) - \Phi_a^{(n)}(x_1, \dots, x_{n-1}, y) \Big\|_2 \le \varepsilon \|y\|_1.$$

Bounded linear maps have zero second and higher derivatives everywhere. The following lemma can be used to obtain higher derivatives of the exponential and logarithm maps on A^1.

LEMMA 5.1. *Let Ω be an open set in $\mathbf{C}$ and let $\Psi_{\mathbf{C}}\colon \Omega \to \mathbf{C}$ be analytic on Ω. Suppose that $\Psi\colon A_\Omega \to A^1$ is such that for a in A_Ω,*

$$(\Psi(a))\tilde{}\,(\theta) = \Psi_{\mathbf{C}}(\tilde{a}(\theta))$$

for all θ in $\mathbf{R} \cup \{\infty\}$,. Then Ψ has derivatives of all orders at a in A_Ω with, for n in $\mathbf{N}$ and θ in $\mathbf{R} \cup \{\infty\}$,

$$[\Psi_a^{(n)}(y_1, \dots, y_n)]\tilde{}\,(\theta) = \Psi_{\mathbf{C}}^{(n)}(\tilde{a}(\theta))\tilde{y}_1(\theta) \cdots \tilde{y}_n(\theta) \tag{16}$$

for all $(y_1, \dots, y_n)$ in $B_1 \times \cdots \times B_1$.

PROOF. The proof is by induction on n. Equation (15) gives the result for the case $n = 1$. Now suppose that (16) holds for $n = k$ in $\mathbf{N}$. Let θ be in $\mathbf{R} \cup \{\infty\}$. Since a is in A_Ω, $\tilde{a}(\theta)$ must be in Ω. We know that $\Psi_{\mathbf{C}}^{(k)}$ is analytic in Ω and so it has a Taylor series expansion about $\tilde{a}(\theta)$. Thus there exists R, with $0 < R < \infty$, such that, for z in $\mathbf{C}$, $|z| < R$,

$$\Psi_{\mathbf{C}}^{(k)}(\tilde{a}(\theta) + z) = \sum_{j=0}^{\infty} c_j z^j,$$

where $c_j = \Psi_{\mathbf{C}}^{(k+j)}(\tilde{a}(\theta))/j!$. Note that, if $0 < R_1 < R$, then

$$R_1^2 \sum_{j=2}^{\infty} |c_j| R_1^{j-2} \le \sum_{j=0}^{\infty} |c_j| R_1^j < \infty,$$

so that $\sum_{j=2}^{\infty} |c_j| R_1^{j-2} < \infty$.

Suppose $\varepsilon > 0$ is given, and y is such that $a + y$ is in A_Ω and

$$\|y\|_1 < \min\left\{R_1, \varepsilon \Big/ \left(\sum_{j=2}^{\infty} |c_j| R_1^{j-2}\right)\right\},$$

interpreting $\varepsilon/0$ as ∞. Then, because $|\tilde{y}(\theta)| \le \|y\|_1 < R_1$,

$$\Psi_{\mathbf{C}}^{(k)}(\tilde{a}(\theta) + \tilde{y}(\theta)) - \Psi_{\mathbf{C}}^{(k)}(\tilde{a}(\theta)) - \Psi_{\mathbf{C}}^{(k+1)}(\tilde{a}(\theta))\tilde{y}(\theta) = \sum_{j=2}^{\infty} c_j (\tilde{y}(\theta))^j. \tag{17}$$

Because $\Psi_{\mathbf{C}}^{(k+1)}$ is analytic on Ω, Rudin [(1974), 10.26] implies that we can find b in A^1 such that $\tilde{b}(\theta) = \Psi_{\mathbf{C}}^{(k+1)}(\tilde{a}(\theta))$ for all θ in $\mathbf{R} \cup \{\infty\}$. We substitute this into (17) and multiply each side by $\prod_{l=1}^{k} \tilde{x}_l(\theta)$, where x_l is in A^1 with $\|x_l\|_1 \le 1$, $l = 1, \dots, k$. Using the inductive hypothesis, we see that the left-hand side of the resulting equation is the transform of

$$\Psi_{a+y}^{(k)}(x_1, \dots, x_k) - \Psi_a^{(k)}(x_1, \dots, x_k) - b * y * x_1 * \cdots * x_k. \tag{18}$$

The right-hand side of the resulting equation is the transform of

$$x_1 * \cdots * x_k * y * y * \sum_{j=2}^{\infty} c_j y^{*(j-2)}, \tag{19}$$

because $\sum_{j=2}^{\infty} c_j (\tilde{y}(\theta))^{j-2}$ is the transform of $\sum_{j=2}^{\infty} c_j y^{*(j-2)}$, an element of A^1 (since $\|y\|_1 < R_1$ and $\sum_{j=2}^{\infty} c_j R_1^{j-2} < \infty$). Since A^1 is semisimple, we can identify (18) and (19). The norm of (19) is at most

$$\prod_{l=1}^{k} \|x_l\|_1 \|y\|_1^2 \sum_{j=2}^{\infty} |c_j| R_1^{j-2} \le \varepsilon \|y\|_1.$$

Observe that the $(k+1)$-linear form

$$(x_1, \dots, x_k, y) \mapsto b * x_1 * \cdots * x_k * y$$

is bounded and hence so is $\Psi_a^{(k+1)}(x_1, \dots, x_k, y)$. Thus (16) holds for $n = k + 1$. Hence by induction it holds for all n in $\mathbf{N}$. □

LEMMA 5.2. *Let y_i be in A^1, $1 \le i \le n$.*
(i) *For a in A^1,* $\exp_a^{(n)}(y_1,\dots,y_n) = (\exp a) * y_1 * \cdots * y_n$.
(ii) *For a in A_G,* $\log_a^{(n)}(y_1,\dots,y_n) = (-1)^{n-1}(n-1)!a^{*-n} * y_1 * \cdots * y_n$.

The maps $a \mapsto \exp_a^{(n)}$ and $a \mapsto \log_a^{(n)}$ are continuous.

To calculate the nth derivatives of composite functionals we use the following result which is Formula A in Fraenkel (1978). Let $(B_i, \|\cdot\|_i)$ be Banach spaces for $i = 1, 2, 3$ and let U and V be open subsets of B_1 and B_2, respectively. Let Φ: $U \to B_2$ and Ψ: $V \to B_3$ be continuously differentiable up to order n throughout U and V, respectively. Let a be in U and $\Phi(a)$ be in V. Then $\Psi \circ \Phi$ has nth derivative at a given by

$$(20)\qquad \begin{aligned}(\Psi \circ \Phi)_a^{(n)}(x_1,\dots,x_n) &= \sum_{j=1}^{n} \sum_{\beta_1+\cdots+\beta_j=n} \sum_{\sigma} \frac{1}{j!\beta_1!\cdots\beta_j!} \\ &\quad \times \Psi_{\Phi(a)}^{(j)}\Big(\Phi_a^{(\beta_1)}(x_{\sigma(1)},\dots,x_{\sigma(\beta_1)}),\dots,\Phi_a^{(\beta_j)}(x_{\sigma(n-\beta_j+1)},\dots,x_{\sigma(n)})\Big),\end{aligned}$$

where Σ_σ denotes summation over the $n!$ permutations of $\{1,\dots,n\}$. This allows us to find higher derivatives of the queueing functionals.

Expansions are then obtained using Taylor's theorem [see, e.g., Cartan (1971), Chapter 1, Theorem 5.6.2]. Let $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ be Banach spaces and let U be an open subset of B_1. Suppose Φ: $U \to B_2$ is $(N+1)$-times differentiable throughout U. Let a be in U and let h in B_1 be such that $a + th$ is in U for all $0 \le t \le 1$. Then, for all N in $\mathbf{N}$,

$$(21)\qquad \Phi(a+h) = \Phi(a) + \sum_{n=1}^{N} \big(\Phi_a^{(n)}(h,\dots,h)\big)/n! + r_N,$$

where

$$\|r_N\|_2 \le \frac{1}{(N+1)!} \sup_{t\in[0,1]} \big\|\Phi_{a+th}^{(N+1)}\big\| \, \|h\|_1^{N+1}.$$

We now apply the Taylor series expansion to the waiting time and idle period functionals. To simplify calculations we consider these as starting with input $f = f_S * \check{f}_T$; the resulting operators in the decomposition, except for exp and log, then have zero second and higher derivatives. If we start with the pair (f_S, f_T) and vary one component only, then the first step in the decomposition is linear, too, and the results given below are easily adapted to this situation. We need two preliminary results.

LEMMA 5.3. *Suppose B_1 is a Banach space and B_2 is a Banach algebra, with multiplication in B_2 denoted $*$, and suppose a is in an open subset U of B_1. Let Ψ be a map from U to B_2 with derivatives up to and including order n in U.*

Then the map $\exp \circ \Psi$ *has nth derivative at a given by, for* $x_1, \dots, x_n$ *in* B_1,

$$\begin{aligned}(\exp \circ \Psi)_a^{(n)}(x_1, \dots, x_n) \\ = \exp(\Psi(a)) * \sum_{j=1}^{n} \sum_{\beta_1 + \cdots + \beta_j = n} \sum_{\sigma} \frac{1}{j!\beta_1! \cdots \beta_j!} \\ \times \Psi_a^{(\beta_1)}(x_{\sigma(1)}, \dots, x_{\sigma(\beta_1)}) * \cdots * \Psi_a^{(\beta_j)}(x_{\sigma(n-\beta_j+1)}, \dots, x_{\sigma(n)}),\end{aligned}$$

where Σ_σ *denotes summation over the* $n!$ *permutations of* $\{1, \dots, n\}$.

PROOF. This is obtained by substituting the expression for $\exp_a^{(n)}$ in Lemma 5.2 into (20). □

LEMMA 5.4. *With notation as in Lemma* 5.3,

$$\begin{aligned}(\exp \circ \Phi)_a^{(n)}(x, \dots, x) \\ = n! \exp(\Phi(a)) * \sum_{j=1}^{n} \sum_{\tau} \prod_{i=1}^{l} \frac{1}{(n_i!)^{r_i} r_i!} (\Phi_a^{(n_i)}(x, \dots, x))^{*r_i},\end{aligned}$$

where Σ_τ *is summation over all positive integers* $n_1, \dots, n_l$ *and* $r_1, \dots, r_l$ *with* $n_1 < n_2 < \cdots < n_l$, $\Sigma_{i=1}^{l} r_i n_i = n$ *and* $\Sigma_{i=1}^{l} r_i = j$ *and* Π *denotes multiplication in* B_2.

PROOF. Put $x_1 = \cdots = x_n = x$ in the expression in Lemma 5.3. Each term in the inner sum gives the same value so

$$\begin{aligned}(\exp * \Phi)_a^{(n)}(x, \dots, x) = \exp(\Phi(a)) * \sum_{j=1}^{n} \sum_{\beta_1 + \cdots + \beta_j = n} \frac{n!}{j!\beta_1! \cdots \beta_j!} \\ \times \Phi_a^{(\beta_1)}(x, \dots, x) * \cdots * \Phi_a^{(\beta_j)}(x, \dots, x).\end{aligned}$$

If $\{\beta_1, \dots, \beta_j\}$ consists of r_i copies of n_i, $i = 1, \dots, l$, where $\Sigma_{i=1}^{l} r_i = j$ and $\Sigma_{i=1}^{l} r_i n_i = n$, then any other $\{\beta_1, \dots, \beta_j\}$ consisting of r_i copies of n_i, $i = 1, \dots, l$, will give rise to the same term. There are $j!/r_1! \cdots r_l!$ such terms. This gives the lemma. □

In the following, write

$$h = \left(-\Sigma f^{(0)} - f^{(0)} + \delta_0\right)^{*-1} * (-\Sigma g - g),$$

where $g = f^{(1)} - f^{(0)}$. As in the proof of Theorem 4.1, write T for the map taking a to $a - \hat{a}(0)\delta_0$. For n in $\mathbf{N}$, let v_n be the element of A^1 defined by

$$v_n = \sum_{j=1}^{n} \sum_{\tau} \sum_{i=1}^{l} \frac{(TR_+(h^{*n_i}))^{*r_i}}{n_i^{r_i} r_i!},$$

where Σ_τ denotes the same summation as in Lemma 5.4 and Π denotes convolution product. The next theorem gives the Taylor series expansion for the stationary waiting time functional.

THEOREM 5.5. *For* $i = 0, 1$, *let* $f_S^{(i)}$, $f_T^{(i)}$, $w^{(i)}$ *and* $\rho^{(i)}$ *be service time and interarrival time densities, stationary waiting time distribution and traffic intensity, respectively, for a* GI/G/1 *queue, where* $f_S^{(i)}$ *and* $f_T^{(i)}$ *satisfy* (11) *and* $\rho^{(i)} < 1$. *Let* $f^{(i)} = f_S^{(i)} * \check{f}_T^{(i)}$. *Put* $f^{(t)} = (1-t)f^{(0)} + tf^{(1)}$ *and suppose that*

$$c = 3 \sup_{0 \le t \le 1} \left\| \left(-\Sigma f^{(t)} - f^{(t)} + \delta_0 \right)^{*-1} \right\|_1 \| f^{(1)} - f^{(0)} \|_{1,1} < 1.$$

Then

$$w^{(1)} = w^{(0)} + w^{(0)} * \sum_{n=1}^{N} (-1)^n v_n + r_N \quad \textit{with } \|r_N\|_1 = o(c^N) \textit{ as } N \to \infty.$$

PROOF. Write Φ for the functional mapping f onto w. Then, from (13), Φ can be decomposed as

$$f \mapsto -\Sigma f - f + \delta_0 \mapsto_{-\log} v(f) - v(\check{f}_1) \mapsto_{R_+} v(f_+)$$
$$\mapsto_T v(f_+) - (v(f_+))^{\wedge}(0)\delta_0 \mapsto_{\exp} w,$$

where f_1 is an exponential density with parameter 1. Let

(22) $\Phi_1 \colon a \mapsto -\Sigma a - a + \delta_0, \qquad \Phi_2 = -\log \circ \Phi_1, \qquad \Phi_3 = T \circ R_+ \circ \Phi_2$

so that $\Phi = \exp \circ \Phi_3$. In order to apply (21) to Φ, we need to find its nth derivative. Assume that a is in the open subset $\{a \in A^{1,1} \colon -\Sigma a - a + \delta_0 \in A_G\}$. By Lemma 5.4, for x in $A^{1,1}$,

$$\Phi_a^{(n)}(x, \dots, x) = n! \exp(\Phi_3(a)) * \sum_{j=1}^{n} \sum_{\tau} \prod_{i=1}^{l} \frac{1}{(n_i!)^{r_i} r_i!} \times \left(\Phi_{3,a}^{(n_i)}(x, \dots, x) \right)^{*r_i}. \tag{23}$$

Since $T \circ R_+$ has first derivative itself and zero second and higher derivatives, when we apply (20) to Φ_3, only the $j = 1$ term, when $\beta_1 = n$, is nonzero, and each term in the inner sum is the same. We find

$$\Phi_{3,a}^{(n)}(x, \dots, x) = T \circ R_+ \left(\Phi_{2,a}^{(n)}(x, \dots, x) \right). \tag{24}$$

Observe that $\Phi'_{1,a}(x) = -\Sigma x - x$ for all a and that Φ_1 has zero second and higher derivatives. Applying (20) to $\Phi_2 = -\log \circ \Phi_1$ (this time only the $j = n$ term is nonzero, when $\beta_1 = \cdots = \beta_n = 1$) and using Lemma 5.2(ii), we have

$$\Phi_{2,a}^{(n)}(x, \dots, x) = (-1)^n (n-1)! (-\Sigma a - a + \delta_0)^{*-n} * (-\Sigma x - x)^{*n}. \tag{25}$$

Substituting (24) and (25) into (23) gives

$$\Phi_a^{(n)}(x, \dots, x) = (-1)^n n! \Phi(a) * \sum_{j=1}^{n} \sum_{\tau} \prod_{i=1}^{l} \frac{1}{n_i^{r_i} r_i!} \times \left(TR_+ \left[\left\{ (-\Sigma a - a + \delta_0)^{*-1} * (-\Sigma x - x) \right\}^{*n_i} \right] \right)^{*r_i}.$$

Putting $a = f^{(0)}$ and with g and v_n as defined before the statement of the theorem, we find

$$\Phi^{(n)}_{f^{(0)}}(g, \dots, g) = (-1)^n n! w^{(0)} * v_n$$

for n in $\mathbf{N}$. Substituting this into (21), we obtain the form of the expansion as stated in the theorem.

We now consider the remainder term. We have

$$\begin{aligned}\Phi^{(n)}_{3,a}(x_1, \dots, x_n) &= (-1)^n (n-1)! TR_+\big[(-\Sigma a - a + \delta_0)^{*-n} * (-\Sigma x_1 - x_1) \\ &\qquad * \cdots * (-\Sigma x_n - x_n)\big]\end{aligned}$$

for $x_1, \dots, x_n$ in $A^{1,1}$.

Using Lemma 5.3,

$$\begin{aligned}\Phi^{(n)}_{a}(x_1, \dots, x_n) &= (-1)^n \Phi(a) * \sum_{j=1}^{n} \sum_{\beta_1 + \cdots + \beta_j = n} \sum_{\sigma} \frac{1}{j! \beta_1 \cdots \beta_j} \\ &\quad \times \Big\{TR_+\big[(-\Sigma a - a + \delta_0)^{*-\beta_1} * (-\Sigma - Id)(x_{\sigma(1)}) \\ &\qquad * \cdots * (-\Sigma - Id)(x_{\sigma(\beta_1)})\big]\Big\} \\ &\quad * \cdots * \Big\{TR_+\big[(-\Sigma a - a + \delta_0)^{*\beta_j} * (-\Sigma - Id)(x_{\sigma(n-\beta_j+1)}) \\ &\qquad * \cdots * (-\Sigma - Id)(x_{\sigma(n)})\big]\Big\},\end{aligned}$$

where Id is the embedding $A^{1,1} \to A^1$.

We have

$$\|TR_+\| \le \|T\| \, \|R_+\| = 2.$$

From (7),

$$\big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-k}\big\|_1 \le \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_1^k,$$

and, by (10) for x in $A^{1,1}$,

$$\| - \Sigma x - x\|_1 \le \|x\|_{1,1}.$$

We note that there are $\binom{n-1}{j-1}$ ways to choose distinct $\beta_1, \dots, \beta_j$ with $\beta_i > 0$ for all i and $\beta_1 + \cdots + \beta_j = n$. Putting these observations together gives

$$\begin{aligned}&\big\|\Phi^{(n)}_{f^{(t)}}(x_1, \dots, x_n)\big\|_1 \\ &\quad \le \|w^{(t)}\|_1 \sum_{j=1}^{n} \frac{2^j n!}{j!} \binom{n-1}{j-1} \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_1^n \|x_1\|_{1,1} \cdots \|x_n\|_{1,1}.\end{aligned}$$

For $\|x_i\|_{1,1} \le 1$, $i = 1, \dots, n$,

$$\big\|\Phi^{(n)}_{f^{(t)}}(x_1, \dots, x_n)\big\|_1 \le (n-1)! 3^n \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_1^n.$$

Substituting this into the bound on the norm of r_N given in the Taylor series expansion (21), we find $\|r_N\|_1 = o(c^N)$ as $N \to \infty$. □

The range of the curve $t \mapsto (-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}$ is a compact subset of A^1, which implies that the supremum in the definition of c is finite. In particular, the condition of the theorem is satisfied if $f^{(0)}$ and $f^{(1)}$ are close enough to each other. If both $\Sigma f^{(0)} + f^{(0)}$ and $\Sigma f^{(1)} + f^{(1)}$ have 1-norm smaller than 1, then the supremum can be estimated by $(1 - \max\{\|\Sigma f^{(0)} + f^{(0)}\|_1, \|\Sigma f^{(1)} + f^{(1)}\|_1\})^{-1}$. Note that if f is a step density for an M(1)/G/1 queue with mean service time m_S, $m_S < 1$, then $\|\Sigma f + f\|_1 = m_S$. Thus if $f^{(0)}$ and $f^{(1)}$ are both step densities for M(1)/G/1 queues with mean service time m_S, we have

$$\sup_{0 \le t \le 1} \left\|\left(-\Sigma f^{(t)} - f^{(t)} + \delta_0\right)^{*-1}\right\|_1 \le 1/(1 - m_S),$$

and applying the mean value theorem [see Cartan (1971)], we obtain

$$\|w^{(1)} - w^{(0)}\|_1 \le 2\|f^{(1)} - f^{(0)}\|_{1,1}/(1 - m_S).$$

The approximation arising from the expansion with $N = 2$ is

$$w^{(0)} + w^{(0)} * \left[(R_+h)^\wedge(0)\delta_0 - R_+h\right] + \tfrac{1}{2}w^{(0)} * \left[R_+(h^{*2}) - \left(R_+(h^{*2})\right)^\wedge(0)\delta_0 + \left(R_+h - (R_+h)^\wedge(0)\delta_0\right)^{*2}\right].$$

Thus we can obtain successive approximations to the stationary waiting time distribution of a queue by expanding about, for example, an M/M/1 queue.

For the expansion of the idle period functional, define, for n in $\mathbf{N}$,

$$v_n^- = \sum_{j=1}^{n} (-1)^j \sum_{\tau} \prod_{i=1}^{l} \frac{(R_-(h^{*n_i}))^{*r_i}}{n_i^{r_i} r_i!},$$

where Σ_τ denotes summation as in Lemma 5.4 and Π denotes convolution product. This is the same as v_n with TR_+ replaced with R_- and with an extra factor $(-1)^j$, which arises because we have log rather than $-\log$ in the decomposition of the idle period functional.

THEOREM 5.6. *For $i = 0, 1$, let $f_S^{(i)}$, $f_T^{(i)}$, $f_I^{(i)}$ and $\rho^{(i)}$ be service time, interarrival time and idle period densities and traffic intensity, respectively, for a* GI/G/1 *queue, where $f_S^{(i)}$ and $f_T^{(i)}$ satisfy* (11), *and $\rho^{(i)} < 1$. Let $f^{(i)} = f_S^{(i)} * \check{f}_T^{(i)}$. Put $f^{(t)} = (1 - t)f^{(0)} + tf^{(1)}$ and suppose that*

$$c = 2 \sup_{0 \le t \le 1} \left\|\left(-\Sigma f^{(t)} - f^{(t)} + \delta_0\right)^{*-1}\right\|_1 \|f^{(1)} - f^{(0)}\|_{1,1} < 1.$$

Then

$$f_I^{(1)} = f_I^{(0)} + \left(f_I^{(0)} - \delta_0\right) * \sum_{n=1}^{N} (-1)^n\left((v_n^-)^\vee\right) + r_N$$

with

$$\|r_N\|_1 = o(c^N) \text{ as } N \to \infty.$$

The first few terms are

$$f_I^{(0)} + \left(f_I^{(0)} - \delta_0\right) * \left(\left(R_- h\right)^{\vee}\right) + \tfrac{1}{2}\left(f_I^{(0)} - \delta_0\right) * \left(\left[-R_-(h^{*2}) + (R_- h)^{*2}\right]^{\vee}\right).$$

6. Cumulants of the stationary waiting time distribution. If the stationary waiting time has $E(|W|^r) < \infty$, then the rth cumulant κ_r exists [κ_r is the coefficient of $(it)^r/r!$ in the expansion of $\log E(e^{itW})$]. From Stoyan [(1983), (5.0.5)], κ_r, if it exists, is given by

$$\kappa_r = \sum_{n=1}^{\infty} \frac{1}{n} \int_{0_+}^{\infty} t^r \mu^{*n}(dt).$$

Define $\mathscr{E}_r$: $A^{1,r} \to \mathbf{C}$ by

$$\mathscr{E}_r(f + \alpha\delta_0) = \int t^r f(t)\, dt.$$

Then we have from (4),

$$\kappa_r = \mathscr{E}_r[\nu(\mu_+)].$$

Let Φ denote the functional mapping (f_S, f_T) onto κ_r. In decomposing Φ, we need to consider the logarithm map $A^{1,r} \to A^{1,r}$. Since the maximal ideals of $A^{1,r}$ are precisely the intersections of $A^{1,r}$ with the maximal ideals of A^1, if a in $A^{1,r} \subset A^1$ has a logarithm in A^1, then it has a logarithm in $A^{1,r}$ and they are the same.

We need a preliminary lemma extending (10).

Lemma 6.1. *For all n in $\mathbf{N}$ and a in $A^{1,n+1}$,*

$$\|\Sigma a + a\|_{1,n} \le \|a\|_{1,n+1}.$$

Proof. Since $A^{1,n+1}$ is a subset of $A^{1,1}$ for all n in $\mathbf{N}$, Σa is certainly in A^1 for all a in $A^{1,n+1}$. For any k in $\mathbf{N}$ and f in $L^{1,k+1}$,

$$\int |t|^k |(\Sigma f)(t)|\, dt \le \frac{1}{k+1} \int |x|^{k+1} |f(x)|\, dx$$

by Fubini's theorem, hence $f \in L^{1,k+1}$ implies $\Sigma f \in L^{1,k}$. We have for f in $L^{1,n+1}$,

$$\begin{aligned}
\|\Sigma f + f\|_{1,n} &\le \|\Sigma f\|_{1,n} + \|f\|_{1,n} \\
&= \int \sum_{k=0}^{n} \binom{n}{k} |t|^k \left(|f(t)| + |(\Sigma f)(t)|\right) dt \\
&\le \int \sum_{k=0}^{n} \binom{n}{k} \left(|t|^k + |t|^{k+1}/(k+1)\right) |f(t)|\, dt \\
&\le \int |f(t)|\, dt + \int \sum_{k=1}^{n} \binom{n+1}{k} |t|^k |f(t)|\, dt + \int |f(t)|\, |t|^{n+1}\, dt \\
&= \int (1 + |t|)^{n+1} |f(t)|\, dt \\
&= \|f\|_{1,n+1}.
\end{aligned}$$

Now suppose a is in $A^{1,n+1}$ and $a = f + \alpha\delta_0$. Then

$$\|\Sigma a + a\|_{1,n} \le \|\Sigma f + f\|_{1,n} + |\alpha| \le \|f\|_{1,n+1} + |\alpha| = \|a\|_{1,n+1}. \qquad \square$$

In the course of the proof, we showed that, if a is in $A^{1,n+1}$, then Σa is defined and is in $A^{1,n}$ for n in $\mathbf{N}$.

Then Φ can be decomposed:

$$\underset{A^{1,r+1}\times A^{1,r+1}}{(f_S, f_T)} \mapsto \underset{A^{1,r}}{-\Sigma f - f + \delta_0} \mapsto_{-\log} \underset{A^{1,r}}{v(f) - v(\check{f}_1)} \mapsto_{R_+} \underset{A^{1,r}}{v(f_+)} \mapsto_{\mathscr{E}_r} \underset{\mathbf{C}}{\kappa_r}.$$

Note that $\mathscr{E}_r$ is linear and bounded. Using the same methods of proof as before, we obtain:

THEOREM 6.2. *Let r be in $\mathbf{N}$. For $0 \le \varepsilon \le 1$, let $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$ be probability densities with finite $(r+1)$st moments. Let $\rho^{(\varepsilon)}$ be the traffic intensity of the queue with service time and interarrival time densities $f_S^{(\varepsilon)}$ and $f_T^{(\varepsilon)}$, respectively, and suppose that $\rho^{(\varepsilon)} < 1$, $0 \le \varepsilon \le 1$. Let $\kappa_r^{(\varepsilon)}$ be the rth cumulant of the associated stationary waiting time distribution.*

Suppose

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f_S^{(\varepsilon)} - f_S^{(0)}\right) = g_S \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f_T^{(\varepsilon)} - f_T^{(0)}\right) = g_T$$

in $A^{1,r+1}$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(\kappa_r^{(\varepsilon)} - \kappa_r^{(0)}\right) = -\mathscr{E}_r[R_+ h] \quad \text{in } \mathbf{C},$$

where $h = (-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} * (-\Sigma g - g)$, $f^{(0)} = f_S^{(0)} * \check{f}_T^{(0)}$ and $g = g_S * \check{f}_T^{(0)} + f_S^{(0)} * \check{g}_T$.

Similarly, we have an expansion for the cumulants.

THEOREM 6.3. *Let r be in $\mathbf{N}$. For $i = 0, 1$, let $f_S^{(i)}$, $f_T^{(i)}$, $\kappa_r^{(i)}$ and $\rho^{(i)}$ be service time and interarrival time densities, rth cumulant of the stationary waiting time distribution and traffic intensity, respectively, for a GI/G/1 queue, where $f_S^{(i)}$ and $f_T^{(i)}$ have finite $(r+1)$st moments and $\rho^{(i)} < 1$. Let $f^{(i)} = f_S^{(i)} * \check{f}_T^{(i)}$. Put $f^{(t)} = (1-t)f^{(0)} + tf^{(1)}$ and suppose that*

$$c = \sup_{0 \le t \le 1} \left\|\left(-\Sigma f^{(t)} - f^{(t)} + \delta_0\right)^{*-1}\right\|_{1,r} \|f^{(1)} - f^{(0)}\|_{1,r+1} < 1.$$

Then

$$\kappa_r^{(1)} = \kappa_r^{(0)} + \sum_{n=1}^{N} \frac{(-1)^n}{n} \mathscr{E}_r[R_+(h^{*n})] + r_N \quad \text{with } |r_N| = o(c^N) \text{ as } N \to \infty,$$

where $h = (-\Sigma f^{(0)} - f^{(0)} + \delta_0)^{*-1} * (-\Sigma g - g)$ and $g = f^{(1)} - f^{(0)}$.

PROOF. Write Φ for the map taking f onto κ_r. Let Φ_2 be as in (22) and let $\Phi_3 = R_+ \circ \Phi_2$ so that

$$\Phi = \varepsilon_r \circ \Phi_3.$$

Second and higher derivatives of $\mathscr{E}_r$ vanish, hence,

$$\begin{aligned}\Phi_a^{(n)}(x_1, \dots, x_n) &= \mathscr{E}_r\big[\Phi_{3,a}^{(n)}(x_1, \dots, x_n)\big] \\ &= (-1)^n (n-1)! \mathscr{E}_r R_+\big[(-\Sigma a - a + \delta_0)^{*-n} * (-\Sigma x_1 - x_1) \\ &\qquad * \cdots * (-\Sigma x_n - x_n)\big],\end{aligned}$$

using (25). When a is $f^{(0)}$, inserting this into (21) gives the form of the expansion as stated in the theorem.

For the remainder term, observe that

$$|\mathscr{E}_r(f + \alpha\delta_0)| \le \int |t|^r |f(t)|\, dt \le \|f + \alpha\delta_0\|_{1,r}.$$

Hence, by Lemma 6.1,

$$\begin{aligned}&\big|\Phi_{f^{(t)}}^{(n)}(x_1, \dots, x_n)\big| \\ &\quad \le (n-1)! \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_{1,r}^n \|\Sigma x_1 + x_1\|_{1,r} \cdots \|\Sigma x_n + x_n\|_{1,r} \\ &\quad \le (n-1)! \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_{1,r}^n \|x_1\|_{1,r+1} \cdots \|x_n\|_{1,r+1}\end{aligned}$$

so

$$\|\Phi_{f^{(t)}}^{(n)}\| \le (n-1)! \big\|(-\Sigma f^{(t)} - f^{(t)} + \delta_0)^{*-1}\big\|_{1,r}^n.$$

On using the bound on $|r_N|$ given in (21), this gives the required result. □

Taking $r = 1$ gives theorems for the derivative and expansion for the expected stationary waiting time; taking $r = 2$ gives the corresponding results for the variance.

7. Examples. The results of Sections 4 and 5 allow us to obtain approximations to the output, for example, the stationary waiting time distribution, for queues near those with analytically tractable input. We give the name 0-queue to the queue at which derivatives are taken. The expansion theorems yield successive approximations obtained by expanding the relevant functional about the 0-queue. If we take $f_T^{(0)}$ and $f_S^{(0)}$ to be exponential densities with mean $1/\alpha$ and $1/\beta$, respectively, then the 0-queue is an M(α)/M(β)/1 queue for which the output of our functionals is known explicitly. If the queue is stable, that is, if $\alpha < \beta$, then, from Feller [(1971), VI.9],

$$w^{(0)} = (1 - (\alpha/\beta))\delta_0 + (\alpha/\beta) f_{\beta - \alpha},$$

where f_γ is a density of an exponential distribution with mean $1/\gamma$. The idle period distribution for this queue (or more generally for a stable M(α)/G/1 queue) is known to be exponential with mean $1/\alpha$; see Asmussen [(1987), IX, Theorem 2.2].

We can take advantage of these simple explicit expressions to approximate the stationary waiting time and idle period distributions of a GI/G/1 queue by taking the derivatives of the functionals at, or expanding them about, an appropriate M/M/1 queue. A scaling argument shows that it is enough to consider queues with mean interarrival time equal to 1.

In the first example, we expand the stationary waiting time functional about an $M(1)/M(1/\rho)/1$ queue to obtain successive approximations to the stationary waiting time distribution for a $GI/M(1/\rho)/1$ queue with interarrival times distributed as a gamma distribution with index 2 and parameter 2, so that $f_T^{(1)}(t) = 4te^{-2t}$ for $t > 0$ (zero otherwise).

From Theorem 5.5, we obtain approximations

$$\text{zeroth order: } w^{(0)},$$

$$\text{first order: zeroth order } - w^{(0)} * TR_+h,$$

$$\text{second order: first order } + (1/2)w^{(0)} * \left[(TR_+h)^{*2} + TR_+(h^{*2})\right],$$

using the notation from the proof of that theorem. Write

$$\alpha_W^{(0)}\delta_0 + f_W^{(0)}, \qquad \alpha_W^{\mathrm{app}\,1}\delta_0 + f_W^{\mathrm{app}\,1} \quad \text{and} \quad \alpha_W^{\mathrm{app}\,2}\delta_0 + f_W^{\mathrm{app}\,2}$$

for the zeroth, first and second order approximations, respectively. In this simple example $w^{(1)} = \alpha_W^{(1)}\delta_0 + f_W^{(1)}$ can be found explicitly so that we can compare our approximations to the true distribution. From Asmussen [(1987), IX, Theorem 1.2], if η is the smallest positive solution of

$$\eta = E(\exp\{(\eta - 1)T/\rho\}) = \frac{4}{(2 - (\eta - 1)/\rho)^2},$$

then

$$w^{(1)} = \alpha_W^{(1)}\delta_0 + f_W^{(1)} = (1 - \eta)\delta_0 + \eta f_{(1-\eta)/\rho}.$$

For low to moderate ρ-values the second order approximation was found to be almost indistinguishable from the true output. For $\rho = 0.9$, $f_W^{(0)}$, $f_W^{\mathrm{app}\,1}$, $f_W^{\mathrm{app}\,2}$ and $f_W^{(1)}$ are shown in Figure 1 by dotted, broken, chained and solid lines, respectively. The atoms at zero and the approximations in the cases $\rho = 0.1$, 0.5 and 0.9 are given in Table 1; Table 2 lists the successive approximations to the expectation of the stationary waiting time obtained with the results of Section 6.

For our second example, let $f_S^{(0)}$ and $f_T^{(0)}$ be the service and interarrival time densities for an $M(1)/M(1/\rho)/1$ queue and let $f_S^{(1)}$ and $f_T^{(1)}$ be uniform densities on $(0, 2\rho)$ and $(0, 2)$, respectively. We aim to find an approximation to the idle period density for a GI/G/1 queue with interarrival and service densities

$$f_T^{(1/2)} = (1/2)\left(f_T^{(0)} + f_T^{(1)}\right), \qquad f_S^{(1/2)} = (1/2)\left(f_S^{(0)} + f_S^{(1)}\right).$$

Here each input distribution is an equal mixture of an exponential and a

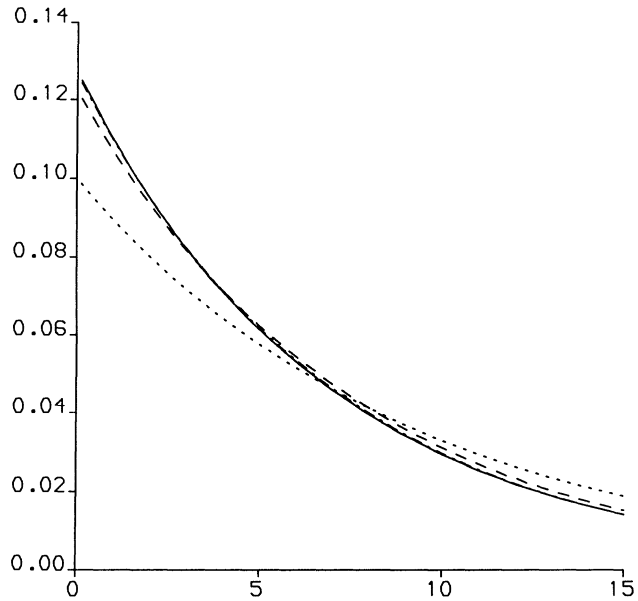

FIG. 1.

TABLE 1
Approximations to the atom at zero for the GI/M/1 *queue*

ρ	$\alpha_W^{(0)}$	$\alpha_W^{\mathrm{app}\,1}$	$\alpha_W^{\mathrm{app}\,2}$	$\alpha_W^{(1)}$
0.1	0.9	0.974380	0.970468	0.970820
0.5	0.5	0.611111	0.619341	0.618034
0.9	0.1	0.124931	0.130492	0.131783

TABLE 2
Approximations to expected stationary waiting time for the GI/M/1 *queue*

ρ	**Zeroth approx.**	**First approx.**	**Second approx.**	$E(W^{(1)})$
0.1	0.011111	0.001928	0.003057	0.003006
0.5	0.5	0.277778	0.299726	0.309017
0.9	8.1	5.856233	5.887224	5.929455

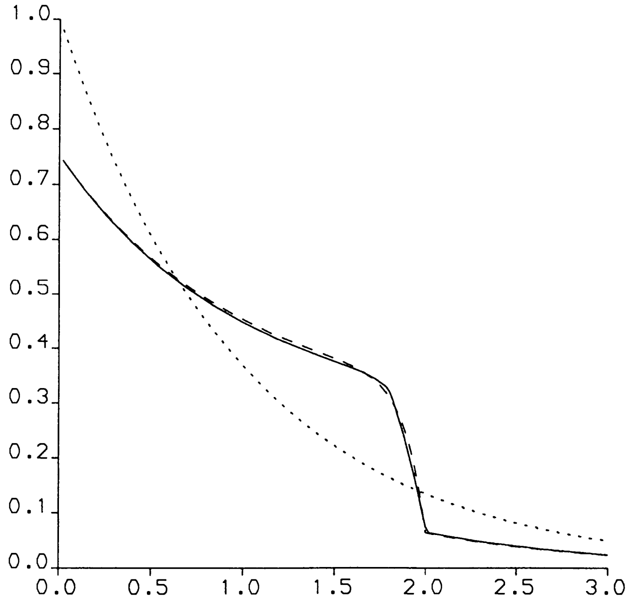

FIG. 2.

uniform distribution. We call this the 1/2-queue and write $f_I^{(1/2)}$ for the idle period density for this queue. Let f_I^{app} be the approximation resulting from Theorem 4.2,

$$f_I^{\text{app}} = f_I^{(0)} + (1/2) \quad \{\text{limit in theorem}\}.$$

For comparison, we need $f_I^{(1/2)}$. However, there is no easy explicit expression for the idle period for such a GI/G/1 queue and $f_I^{(1/2)}$ is found numerically using the methods of Grübel (1991).

Figure 2 shows $f_I^{(0)}$, f_I^{app} and $f_I^{(1/2)}$ by dotted, broken and solid lines, respectively, for $\rho = 0.1$. Even though the (1/2)-queue output is quite different from the 0-queue output in this situation, our infinite-dimensional approach yields a first order approximation which reflects the shape of the true output. Approximation via a parametric phase type family would not achieve this. Notice that the approximations can take negative values. This is consistent with Theorem 4.2 which claims that the f_I^{app} are functions that are close to $f_I^{(1/2)}$ in an L^1-sense, not that the approximations themselves are densities.

In conclusion, we have exhibited derivative and expansion theorems for the stationary waiting time and idle period distributions of the GI/G/1 queue. Similar theorems for other quantities related to the Wiener–Hopf factors, for example, the moments of the idle period distribution, the sojourn time, the expected number of customers served in a busy period, can be derived from the existing derivatives and expansions. The idle period distribution is also defined for a queue with traffic intensity greater than or equal to 1, and alternative normalizations give rise to similar results in this case.

The idea of regarding a stochastic model as a functional or operator which maps known input quantities onto output quantities of interest, and of using modern analysis techniques to obtain approximations for the latter, is certainly not tied to the particular model of this paper. It should be pointed out, however, that the success of the method depends on the tractability of the operators; changing the queueing discipline, for example, seems to imply that we have to start afresh. On the other hand, some of the building blocks of the functionals considered here appear in connection with random walks, storage processes, insurance models and many other stochastic models; in these cases the results and techniques presented in this paper do carry over in a straightforward manner.

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