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GEOMETRIC PROPERTIES OF SOME FAMILIAR DIFFUSIONS IN $\mathbb{R}^n$

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Consider a convex domain B in $\mathbb{R}^n$ and denote by $p(t, x, y)$ the transition probability density of Brownian motion in B killed at the boundary of B. The main result in this paper, in particular, shows that the function $s \ln s^n p(s^2, x, y)$, $(s, x, y) \in \mathbb{R}_+ \times B^2$, is concave.

1. Introduction. Let B be a convex domain in $\mathbb{R}^n$ and suppose V: $B \to [0, +\infty[$ is a continuous function. Moreover, let $p(t, x, y)$, $(t, x, y) \in \mathbb{R}_+ \times B^2$, denote the fundamental solution of the diffusion equation

$$\partial\psi/\partial t = \tfrac{1}{2}\Delta\psi - V(x)\psi(t, x), \qquad (t, x) \in \mathbb{R}_+ \times B, \tag{1.1}$$

with the Dirichlet boundary condition zero, that is,

$$\lim_{x \to x_0} \psi(t, x) = 0, \qquad (t, x_0) \in \mathbb{R}_+ \times \partial B. \tag{1.2}$$

The function $\psi(t, x) = p(t, x, y)$ satisfies (1.1) and (1.2) and

$$\lim_{t \to 0^+} p(t, x, y) = \delta(x - y).$$

If the potential V vanishes and the domain B equals $\mathbb{R}^n$, the corresponding fundamental solution is denoted by $e(t, x, y)$ and we have

$$e(t, x, y) = (2\pi t)^{-n/2} \exp\left(-|x - y|^2/2t\right).$$

The purpose of this paper is to study various convexity properties of appropriate functionals of p. To begin with, recall that, if the potential V is convex, then the mapping $(x, y) \to \ln p(t, x, y)$ is concave for every fixed $t > 0$ (Brascamp and Lieb [5]). Here we will impose a much stronger condition on the potential, namely that the function $V^{-1/2}$ is concave. More precisely, this means that either $V = 0$ or the function V is strictly positive with $V^{-1/2}$ concave.

The main result in this paper states that the mapping

$$s \ln s^n p(s^2, x, y), \qquad (s, x, y) \in \mathbb{R}_+ \times B^2 \tag{1.3}$$

is concave. In the special case $p = e$ the mapping in (1.3) is simply equal to a constant plus $-|x - y|^2/2s$.

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Now set

$$w(t,x,y) = \int_0^t p(s,x,y)\,ds, \qquad (t,x,y) \in \mathbb{R}_+ \times B^2.$$

If $y \in B$ is fixed, the function $v(t,x) = w(t,x,y)$ solves the equation

$$\partial v/\partial t = \tfrac{1}{2}\Delta v - V(x)v + \delta(x-y)$$

and v approaches zero at the boundary points $(\partial(\mathbb{R}_+ \times B)) \setminus \{(0,y)\}$. By exploiting the concavity of the map in (1.3), we show that all the level sets $\{(t,x,y) \in \mathbb{R}_+ \times B^2;\ w(t,x,y) > r\}$, $r \geq 0$, are convex for any $n \geq 2$. Moreover, if $n \geq 3$, the function $w^{-1/(n-2)}$ is convex. Similar properties are proved for the Green function $w(+\infty, x, y)$ in the author's paper [3]. Note that, if $p = e$ and $n \geq 3$, then $w(+\infty, x, y)$ equals a positive constant times $|x-y|^{-(n-2)}$.

Finally set

$$m(t,x,y) = \sup_{0<s\leq t} p(s,x,y), \qquad (t,x,y) \in \mathbb{R}_+ \times B^2.$$

By using the fact that the function $w^{-1/(n-2)}$ is convex for $n \geq 3$, it follows that $m^{-1/n}$ is convex for every $n \geq 1$. If $p = e$, $m(+\infty, x, y)$ equals a positive constant times $|x-y|^{-n}$.

The method of proof in this paper is based on Brunn–Minkowski theory and functional integration as in [5] (cf. also Borell [1]). In the time-stationary case, the author's works [3] and [4] treat properties closely related to those discussed above using two different differential methods. Interestingly enough, even here the condition that $V^{-1/2}$ is concave enters in a natural way.

2. An inequality of the Brunn–Minkowski type. The investigations below strongly depend on an inequality of the Brunn–Minkowski type, which we describe next.

In what follows, $\lambda = (\lambda_0, \lambda_1)$ stands for an arbitrary but fixed vector with strictly positive components and such that $\lambda_0 + \lambda_1 = 1$. If $x_0, x_1 \in \mathbb{R}^n$, let $x_\lambda = \lambda_0 x_0 + \lambda_1 x_1$ and, if $A_0, A_1 \subseteq \mathbb{R}^n$, let $A_\lambda = \{x_\lambda\colon x_0 \in A_0 \text{ and } x_1 \in A_1\}$.

THEOREM 2.1. *Suppose* $\Psi\colon [0, +\infty[^2 \to [0, +\infty[$ *is a continuous, positively homogenous function of degree one, increasing in each variable separately, and such that* $\Psi(\xi, \eta) = 0$, *if* $\xi = 0$ *or* $\eta = 0$. *Moreover, let* $\Omega_0, \Omega_1 \subseteq \mathbb{R}^n$ *be open and suppose* $f_j\colon \Omega_j \to [0, +\infty[$, $j = 0, 1, \lambda$, *are continuous functions.*

The following assertions are equivalent:

$$\text{(i)} \qquad \int_{A_\lambda} f_\lambda(x)\,dx \geq \Psi\left(\int_{A_0} f_0(x)\,dx, \int_{A_1} f_1(x)\,dx\right)$$

for all open $A_i \subseteq \Omega_i$, $i = 0, 1$;

$$\text{(ii)} \qquad f_\lambda(x_\lambda)\prod_{k=1}^n a_\lambda^{(k)} \geq \Psi\left(f_0(x_0)\prod_{k=1}^n a_0^{(k)}, f_1(x_1)\prod_{k=1}^n a_1^{(k)}\right)$$

for all $x_0 \in \Omega_0$, $x_1 \in \Omega_1$ *and all vectors*

$$a_0 = \left(a_0^{(1)}, \dots, a_0^{(n)}\right), \qquad a_1 = \left(a_1^{(1)}, \dots, a_1^{(n)}\right) \in \mathbb{R}^n$$

with nonnegative components.

For a proof of Theorem 2.1, see the author's paper [2]. Here, in order to make the paper more self-contained, we will repeat a proof of Theorem 2.1 (cf. [2], Remark, page 119).

LEMMA 2.1. *Let* $f_j\colon \mathbb{R} \to [0, +\infty[$, $j = 0, 1, \lambda$, *be continuous functions with compact supports and suppose* Ψ *is as in Theorem* 2.1. *If*

$$f_\lambda(x_\lambda) a_\lambda \geq \Psi(f_0(x_0) a_0, f_1(x_1) a_1)$$

for all $x_0, x_1 \in \mathbb{R}$ *and all* $a_0, a_1 \geq 0$, *then*

$$\int_{\mathbb{R}} f_\lambda(x)\, dx \geq \Psi\left(\int_{\mathbb{R}} f_0(x)\, dx, \int_{\mathbb{R}} f_1(x)\, dx \right).$$

PROOF. Since Ψ vanishes on the boundary of $[0, +\infty[^2$, we may assume that $f_0 \neq 0$ and $f_1 \neq 0$. In what follows, suppose $i = 0$ or 1 and let $B_i \subseteq \{f_i > 0\}$ be a nonempty, finite union of nonempty open intervals. It is enough to prove that

$$\int_{\mathbb{R}} f_\lambda(x)\, dx \geq \Psi\left(\int_{B_0} f_0(x)\, dx, \int_{B_1} f_1(x)\, dx \right).$$

Furthermore, by rescaling $\Psi(\xi, \eta)$ in each variable, if necessary, there is no restriction in assuming

$$\int_{B_i} f_i(x)\, dx = 1.$$

Now we introduce the distribution function

$$F_i(x) = \int_{-\infty}^{x} 1_{B_i}(y) f_i(y)\, dy, \qquad x \in \mathbb{R}.$$

Moreover, we denote by G_i the inverse of the function F_i restricted to the set B_i. Then $(f_i \circ G_i) G_i' = 1$ and if D_i stands for the domain of definition of G_i,

$$f_\lambda(\lambda_0 G_0(s) + \lambda_1 G_1(s))(\lambda_0 G_0'(s) + \lambda_1 G_1'(s)) \geq \Psi(1, 1), \qquad s \in D_0 \cap D_1.$$

By integrating this inequality over the set $D_0 \cap D_1$, it follows that

$$\int_{\mathbb{R}} f_\lambda(x)\, dx \geq \Psi(1, 1)$$

since the set $]0, 1[\setminus (D_0 \cap D_1)$ is finite. This completes the proof of Lemma 2.1. □

PROOF OF THEOREM 2.1. (i) ⇒ (ii): Suppose first that the $a_i^{(k)}$, $i = 0, 1$, are positive and put

$$A_i = x_i + \varepsilon \prod_{k=1}^{n} \left[-\tfrac{1}{2}a_i^{(k)}, \tfrac{1}{2}a_i^{(k)} \right], \qquad \varepsilon > 0.$$

We now multiply the inequality in (i) by ε^{-n} and let ε tend to zero to obtain the inequality in (ii). The latter inequality is trivial if some of the $a_i^{(k)}$ vanish.

(ii) ⇒ (i): First suppose that $\Omega_j = A_j = \mathbb{R}^n$ and that $f_j\colon \mathbb{R}^n \to [0, +\infty[$ has compact support for every $j = 0, 1, \lambda$. If the inequality in (ii) is true, we get the inequality in (i) by using Lemma 2.1 and the principle of mathematical induction on n. Clearly, the general case is an immediate consequence of this special case.

This, finally, ends our proof of Theorem 2.1. □

3. The main result. Given an open subset A of B, we define

$$u(t, x; A) = \int_A p(t, x, y)\, dy, \qquad (t, x) \in \mathbb{R}_+ \times B.$$

THEOREM 3.1. *Suppose $V^{-1/2}$ is concave. If $A_0, A_1 \subseteq B$ are open and $s_0, s_1 > 0$, $x_0, x_1 \in B$, then*

$$u^{s_\lambda}(s_\lambda^2, x_\lambda; A_\lambda) \geq u^{\lambda_0 s_0}(s_0^2, x_0; A_0) u^{\lambda_1 s_1}(s_1^2, x_1; A_1).$$

The proof of Theorem 3.1 depends on two simple lemmas.

LEMMA 3.1. *For any $a > 0$, the function x^{a+1}/t^a, $t > 0$, $x > 0$, is convex.*

LEMMA 3.2. *The function $t \ln(x/t)$, $t > 0$, $x > 0$, is concave.*

The proofs of Lemmas 3.1 and 3.2 are straightforward and they are omitted here.

PROOF OF THEOREM 3.1. Let $(X(t))_{t \geq 0}$ be Brownian motion in $\mathbb{R}^n$ and denote by $T_B = \inf\{t > 0;\ X(t) \notin B\}$ its first exit time of B. Then, by the Feynman–Kac formula,

$$u(t, x; A) = E_x\left[\exp\left(-\int_0^t V(X(s))\, ds\right); T_B > t,\ X(t) \in A\right].$$

In what follows, N stands for an integer greater than or equal to two and we define

$$u_N(t, x; A) = E_x\left[\exp\left(-\frac{t}{N}\sum_{k=1}^{N} V\left(X\left(\frac{k}{N}t\right)\right)\right); X\left(\frac{k}{N}t\right) \in B,\right.$$

$$\left. k = 1, \ldots, N-1,\ X(t) \in A\right].$$

Noting that the boundary of B is regular,

$$u(t,x;A) = \lim_{N \to +\infty} u_N(t,x;A).$$

Accordingly, from this, it is enough to show that

$$u_N^{s_\lambda}(s_\lambda^2, x_\lambda; A_\lambda) \geq u_N^{\lambda_0 s_0}(s_0^2, x_0; A_0) u_N^{\lambda_1 s_1}(s_1^2, x_1; A_1) \tag{3.1}$$

for every fixed N. To this end, we introduce

$$q_N(t,x,y) = \left(\frac{2\pi t}{N}\right)^{-n/2} \exp\left(-\frac{N}{2t}|y-x|^2 - \frac{t}{N}V(y)\right), \quad (t,x,y) \in \mathbb{R}_+ \times B^2,$$

so that, by the Markov property of Brownian motion,

$$u_N(t,x;A) = \int_{B^{N-1}\times A} q_N(t,x,\xi_1) q_N(t,\xi_1,\xi_2) \cdot \cdots \cdot q_N(t,\xi_{N-1},y)\, d\xi_1 \cdot \cdots \cdot d\xi_{N-1}\, dy.$$

In order to simplify notation, let

$$Q_N(t,x,\hat{\xi}) = q_N(t,x,\xi_1) q_N(t,\xi_1,\xi_2) \cdot \cdots \cdot q_N(t,\xi_{N-1},\xi_N)$$

for every $t > 0$, $x \in B$, and $\hat{\xi} = (\xi_1|\xi_2| \cdots |\xi_N) \in B^N$. Then

$$u_N(t,x;A) = \int_{B^{N-1}\times A} Q_N(t,x,\hat{\xi})\, d\hat{\xi}.$$

Now consider the function

$$r_N(t,x,y) = t^{n/2} q_N(t,x,y), \qquad (t,x,y) \in \mathbb{R}_+ \times B^2.$$

From the definitions we have

$$r_N^s(s^2,x,y) = \left(\frac{2\pi}{N}\right)^{-sn/2} \exp\left(-\frac{N}{2s}|y-x|^2 - \frac{s^3}{N}V(y)\right), \qquad s > 0.$$

Here, by Lemma 3.1, the function $|y-x|^2/2s$, $(s,x,y) \in \mathbb{R}_+ \times B$ is convex. Further, since $V^{-1/2}$ is concave, Lemma 3.1 implies that the function $s^3 V(y)$, $(s,y) \in \mathbb{R}_+ \times B$, is convex. Thus, given $y_0, y_1 \in B$, it follows that

$$r_N^{s_\lambda}(s_\lambda^2, x_\lambda, y_\lambda) \geq r_N^{\lambda_0 s_0}(s_0^2, x_0, y_0) r_N^{\lambda_1 s_1}(s_1^2, x_1, y_1).$$

Moreover, if the vectors $a_0 = (a_0^{(1)}, \dots, a_0^{(n)})$, $a_1 = (a_1^{(1)}, \dots, a_1^{(n)}) \in \mathbb{R}^n$ have nonnegative components, Lemma 3.2 gives

$$\left(\frac{a_\lambda^{(k)}}{s_\lambda}\right)^{s_\lambda} \geq \left(\frac{a_0^{(k)}}{s_0}\right)^{\lambda_0 s_0} \left(\frac{a_1^{(k)}}{s_1}\right)^{\lambda_1 s_1}, \qquad k = 1, \dots, n.$$

Consequently,

$$\left(q_N(s_\lambda^2, x_\lambda, y_\lambda)\prod_{k=1}^{n} a_\lambda^{(k)}\right)^{s_\lambda}$$
$$\geq \left(q_N(s_0^2, x_0, y_0)\prod_{k=1}^{n} a_0^{(k)}\right)^{\lambda_0 s_0}\left(q_N(s_1^2, x_1, y_1)\prod_{k=1}^{n} a_1^{(k)}\right)^{\lambda_1 s_1}.$$

Therefore, for any $b_0 = (b_0^{(1)},\dots,b_0^{(nN)})$, $b_1 = (b_1^{(1)},\dots,b_1^{(nN)}) \in [0, +\infty[^n$, we have

$$Q_N(s_\lambda^2, x_\lambda, \hat{\xi}_\lambda)\prod_{k=1}^{nN} b_\lambda^{(k)}$$
$$\geq \left(Q_N(s_0^2, x_0, \hat{\xi}_0)\prod_{k=1}^{nN} b_0^{(k)}\right)^{\lambda_0 s_0/s_\lambda}\left(Q_N(s_1^2, x_1, \hat{\xi}_1)\prod_{k=1}^{nN} b_1^{(k)}\right)^{\lambda_1 s_1/s_\lambda}.$$

The inequality (3.1) now follows at once from Theorem 2.1, which completes the proof of Theorem 3.1. □

COROLLARY 3.1. *Suppose $V^{-1/2}$ is concave. Then the mapping $s \ln s^n p(s^2, x, y)$, $(s, x, y) \in \mathbb{R}_+ \times B^2$, is concave. Equivalently, if $s_0, s_1 > 0$, $x_0, y_0, x_1, y_1 \in B$ and $a_0^{(k)}, a_1^{(k)} \geq 0$, $k = 1,\dots,n$, then*

$$\left(p(s_\lambda^2, x_\lambda, y_\lambda)\prod_{k=1}^{n} a_\lambda^{(k)}\right)^{s_\lambda}$$
$$\geq \left(p(s_0^2, x_0, y_0)\prod_{k=1}^{n} a_0^{(k)}\right)^{\lambda_0 s_0}\left(p(s_1^2, x_1, y_1)\prod_{k=1}^{n} a_1^{(k)}\right)^{\lambda_1 s_1}. \tag{3.2}$$

PROOF. By Theorem 2.1 the inequality (3.2) is equivalent to the inequality in Theorem 3.1. Furthermore, in view of Lemma 3.2, the inequality (3.2) just means that the mapping $s \ln s^n p(s^2, x, y) \in \mathbb{R}_+ \times B^2$ is concave. This completes the proof of Corollary 3.1. □

4. An application.

THEOREM 4.1. *Suppose $V^{-1/2}$ is concave and set*

$$w(t, x, y) = \int_0^t p(s, x, y)\, ds, \qquad (t, x, y) \in \mathbb{R}_+ \times B^2.$$

If $n \geq 2$, the function w is quasiconcave, that is, the level sets $\{w > r\}$, $r \geq 0$, are convex. Moreover if $n \geq 3$ the function $w^{-1/(n-2)}$ is convex.

We do not know if the function w in Theorem 4.1 is quasiconcave for $n = 1$.

Proof of Theorem 4.1. Setting $a_j^{(n)} = s_j$, $j = 0, 1, \lambda$, in (3.2), it follows easily that

$$s_\lambda p(s_\lambda^2, x_\lambda, y_\lambda) \prod_{k=1}^{n-1} a_\lambda^{(k)} \geq \min\left(s_0 p(s_0^2, x_0, y_0) \prod_{k=1}^{n-1} a_0^{(k)}, s_1 p(s_1^2, x_1, y_1) \prod_{k=1}^{n-1} a_1^{(k)} \right).$$

Therefore, if $n \geq 2$, Theorem 2.1 implies that, for any $t_0, t_1 > 0$,

$$w\left(\left(\lambda_0 t_0^{1/2} + \lambda_1 t_1^{1/2}\right)^2, x_\lambda, y_\lambda\right) \prod_{k=1}^{n-2} a_\lambda^{(k)}$$
$$\geq \min\left(w(t_0, x_0, y_0) \prod_{k=1}^{n-2} a_0^{(k)}, w(t_1, x_1, y_1) \prod_{k=1}^{n-2} a_1^{(k)} \right).$$

Moreover, since $w_t' \geq 0$ and

$$\left(\lambda_0 t_0^{1/2} + \lambda_1 t_1^{1/2}\right)^2 \leq t_\lambda,$$

we have

$$w(t_\lambda, x_\lambda, y_\lambda) \prod_{k=1}^{n-2} a_\lambda^{(k)} \geq \min\left(w(t_0, x_0, y_0) \prod_{k=1}^{n-2} a_0^{(k)}, w(t_1, x_1, y_1) \prod_{k=1}^{n-2} a_1^{(k)} \right).$$

Now, if $n = 2$, clearly, the function w is quasiconcave. Further, if $n \geq 3$, we put

$$a_i^{(k)} = \varepsilon + \left(w(t_i, x_i, y_i)\right)^{-1/(n-2)}, \qquad i = 0, 1 \ (\varepsilon > 0 \text{ small})$$

and it follows at once that the function $w^{-1/(n-2)}$ is convex. This completes the proof of Theorem 4.1. □

Example 4.1. Suppose $n = 3$ and denote by μ the linear measure of a line segment S contained in B. The function

$$v(t, x) = \int_S w(t, x, y)\, d\mu(y)$$

solves the equation

$$\partial v/\partial t = \tfrac{1}{2}\Delta v - V(x)v + \mu$$

and approaches zero at the boundary points $(\partial(\mathbb{R}_+ \times B)) \setminus (\mathbb{R}_+ \times \overline{S})$. By Theorem 2.1, the function v is quasiconcave.

Example 4.2. Let $k \in \mathbb{N}_+$. The fundamental solution of the equation

$$\partial\psi/\partial t = \tfrac{1}{2}(\Delta_{x_1} + \cdots + \Delta_{x_k})\psi - (V(x_1) + \cdots + V(x_k))\psi$$

in $\mathbb{R}_+ \times B^k$ with the Dirichlet boundary condition zero equals $p(t, x_1, y_1) \cdot \cdots \cdot p(t, x_k, y_k)$. Since the function $(V(x_1) + \cdots + V(x_k))^{-1/2}$ is concave, Theorem 4.1 applies. In particular, if $nk \geq 3$, the function

$$\left(\int_0^t p^k(s, x, y)\, ds\right)^{-1/(nk-2)}, \qquad (t, x, y) \in \mathbb{R}_+ \times B^2$$

is convex. Letting $k \to +\infty$, we conclude that the function $m^{-1/n}$ is convex, where

$$m(t,x,y) = \sup_{0<s\leq t} p(s,x,y), \qquad (t,x,y) \in \mathbb{R}_+ \times B^2.$$

Alternatively, these properties may be derived more directly from Corollary 3.1.

5. Discussion. There are several Brunn–Minkowski inequalities for Brownian motion, which, so far, only have been possible to prove for convex sets. Some of them are rather close to the inequalities considered above. For example, if $n \geq 3$ and if $\tau_A = T_{\mathbb{R}^n \setminus A}$ denotes the first hitting time of $A \subseteq \mathbb{R}^n$,

$$P_{x_\lambda}[\tau_{A_\lambda} < +\infty] \geq \min\left(P_{x_0}[\tau_{A_0} < +\infty], P_{x_1}[\tau_{A_1} < +\infty]\right)$$

for all open convex sets A_0, $A_1 \subseteq \mathbb{R}^n$[3]. It is unknown whether this inequality remains true for arbitrary open sets. The same remark applies to the capacitary inequality

$$c_n^{1/(n-2)}(A_0 + A_1) \geq c_n^{1/(n-2)}(A_0) + c_n^{1/(n-2)}(A_1), \qquad n \geq 3,$$

where c_n is Newtonian capacity in $\mathbb{R}^n$[3].

We believe that a better understanding of this context would be of great interest, also in relation to the inequalities in Sections 1–4.

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