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THE INFINITE SECRETARY PROBLEM WITH RECALL

By Amy L. Rocha

San Jose State University

The infinite secretary problem, in which an infinite number of rankable items arrive at times which are i.i.d., uniform on $(0, 1)$, is modified to allow for a fixed period of recall of length α, $0 \le \alpha \le 1$. The goal is to find the maximum probability of best choice, $v = v(\alpha)$, as well as an optimal stopping time $\tau^* = \tau^*(\alpha)$.

A differential-delay equation is derived, the solution of which yields $v(\alpha)$ and $\tau^*(\alpha)$, the latter given in terms of a constant t^* $[= t^*(\alpha)]$. For $\alpha \ge 1/2$, the complete solution to the problem is obtained. For $0 < \alpha < 1/2$, $\nu(\alpha)$ cannot be put in closed form, so upper and lower bounds for $v(\alpha)$ and $t^*(\alpha)$ are obtained and are investigated for α near 0 and near $1/2$, where the solutions are known. We also find asymptotic expansions of $v(\alpha)$ and $t^*(\alpha)$ about $\alpha = 0$ and $\alpha = 1/2$.

Finally, the solution to the finite, n-item length-m recall problem introduced by Smith and Deely is shown to converge to the solution of the infinite problem when $m/n \to \alpha$.

0. Introduction. We shall formulate and solve a modification of the standard finite secretary problem. The standard problem is the following: A finite and known number, n, of rankable items arrive one by one in random order, all $n!$ permutations being equally likely. At any time the observer knows only the relative ranks of the items which have arrived thus far. As an item arrives, the observer may reject it and go on to examine the next arrival, or he may select it, in which case the process stops. If the last item is presented, it must be accepted. Once an item is rejected, the observer is not allowed to recall it at a later stage. The goal of the observer is to find a strategy under which the best item (the one of smallest rank) is selected with maximum probability.

Our modification is to allow recall of certain previous arrivals and to let the number n of arrivals be infinite. These modifications were introduced by Smith and Deely (1975) and by Gianini and Samuels (1976), respectively. By combining them we obtain a simpler method for finding directly the limit of Smith and Deely's result as n becomes infinite. We also evaluate this limit from their result and verify that it agrees with the result which we obtain directly.

For m fixed, Smith and Deely allowed recall (i.e., selection) of any one of the $m - 1$ previous arrivals in addition to the current arrival. Thus, at any stage k, the observer may select any one of the items which have arrived thus far if

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$1 \le k \le m$, and he may select any one of the m items which arrived at times j, $k - m + 1 \le j \le k$, for $m + 1 \le k \le n$. As in the standard problem, the goal is to maximize the probability of selecting the best item. We shall denote this maximum probability by $v = v^{(n)} = v^{(n,m)}$.

They showed that the optimal rule is: Stop at the first time from r^* onward when the best item so far is about to be lost. Here r^* is a fixed constant which depends on n and m. They derived recursion equations, the solution of which determines r^* and v. From these equations they obtained asymptotic results for v and r^* as n and m tend to infinity with $m/n \to \alpha$, $1/2 \le \alpha \le 1$. Their result for v is

$$v \to 2 - \alpha + \ln \alpha. \tag{0.1}$$

For the case $0 \le \alpha < 1/2$, they merely state that Mucci's technique [Mucci (1973a, b)] of approximating difference equations by differential equations can be used to give the asymptotic probabilities and stopping rules but they do not indicate what these are or what differential equations are indeed obtained.

Gianini and Samuels (1976) formulated an infinite problem which is the analogue of another variation of the standard finite secretary problem discussed above. In this version of the finite problem, the goal is to minimize the expectation of some prescribed positive increasing function, $q(\cdot)$, of the actual rank of the individual selected. Recall is not allowed. [See, for example, Chow, Moriguti, Robbins and Samuels (1964).] For the infinite analogue, an infinite sequence of rankable individuals (with rank 1 the best) arrive at times which are i.i.d. and uniformly distributed on $(0, 1)$. As they arrive, only their relative ranks are observable. The goal is to minimize the expectation of $q(\cdot)$ without recall.

Letting $f(t)$ be the minimal expectation among all stopping rules that are greater than t, they derived, by a direct analysis of the infinite problem, a differential equation which must be satisfied by $f(\cdot)$ as well as an optimal stopping rule for the problem. This differential equation is identical to that obtained by Mucci by an asymptotic analysis of the difference equation for the finite problem. In a separate paper, Gianini (1977) shows that the infinite problem is in fact, in a strong sense, the "limit" of a corresponding sequence of "finite secretary problems."

We will analyze a modification of the infinite problem of Gianini and Samuels, which can be considered the infinite analogue of the finite m-recall problem posed by Smith and Deely. The infinite model we shall use is that formulated by Gianini and Samuels. Our goal, however, is to find the maximum probability of selecting the candidate of absolute rank 1 in the presence of a limited form of recall. Specifically, at any time t, we can select any arrival in the interval $[[t - \alpha]^+, t]$, where $0 < \alpha < 1$ is fixed. We shall pose and solve this problem independently of the analogous finite problem.

In addition, we shall obtain the limit as n and m become infinite of what is, in essence, the solution to the difference equations derived by Smith and Deely for the finite problem. We shall show that it converges uniformly to the solution of the differential equations obtained directly for the infinite problem. That is, the solution of the infinite problem is indeed the limit of the solution of the finite problem.

1. The infinite model. The best, second best, and so on, of an infinite sequence of rankable items arrive at times $U_1, U_2, \dots,$ respectively, which are independent and identically distributed, uniform on $(0, 1)$. This completely defines the model in the sense that all of the other random variables to be considered are functions of $\mathbf{U} = (U_1, U_2, \dots)$.

At any time t, $0 < t < 1$, we are, in effect, able to observe only the relative ranks of those items which have arrived thus far. Additionally, for the best choice problem (which we will consider), we need only keep track of the arrival time of the current relative best: the so-called "candidate." Thus, for each $t \in (0, 1]$, let $Z(t)$ denote the arrival time of the best item to arrive in $(0, t]$. Formally, we define

$$K(t) = \min\{j\colon U_j \le t\}$$

and

$$Z(t) = U_{K(t)}.$$

Our innovation is to introduce a fixed *recall time* α, with $0 \le \alpha \le 1$. An item which arrives at time t can be held until time $t + \alpha$, at which time it must be either selected or discarded. Equivalently, at time t, the observer is allowed to select any item from the interval $[[t - \alpha]^+, t]$. Once an item is selected the process stops.

Our goal is to find a stopping time, adapted to $Z(\cdot)$, which maximizes the probability of best choice. That is, letting $\mathscr{C}$ denote the set of stopping times adapted to $Z(\cdot)$, we seek, for each $\alpha \in [0, 1]$,

$$v_\alpha \equiv \max_{\tau \in \mathscr{C}} P\big([\tau - \alpha]^+ \le U_1 \le \tau\big), \tag{1.1}$$

as well as an optimal stopping time $\tau^* = \tau^*(\alpha)$ (if any), such that

$$v_\alpha = P\big([\tau^* - \alpha]^+ \le U_1 \le \tau^*]\big). \tag{1.2}$$

Remark 1. Since the class $\mathscr{C}$ is infinite, the optimal stopping time need not exist. We shall directly exhibit the form of the rule $\tau^*(\alpha)$, thereby proving existence.

We first introduce certain useful classes of stopping times. Let

$$\hat{\mathscr{C}} = \{\tau \in \mathscr{C}\colon \tau < 1 \Rightarrow Z(\tau) = \tau - \alpha, \text{a.s.}\}.$$

For each t, $0 < t < 1$, let

$$\mathscr{C}_t = \{\tau \in \mathscr{C}\colon \tau > t \text{ a.s.}\}$$

and let

$$\hat{\mathscr{C}}_t = \hat{\mathscr{C}} \cap \mathscr{C}_t.$$

Thus, $\hat{\mathscr{C}}$ is the set of those stopping times which stop and select a candidate before time 1 only when that candidate would become unavailable should we continue observations—that is, we stop only when the candidate must be either selected or discarded. $\mathscr{C}_t$ is the set of those stopping times which do not stop before time t. $\hat{\mathscr{C}}_t$ is the subset of $\hat{\mathscr{C}}$ for which, in addition, $\tau > t$. Clearly, $\hat{\mathscr{C}} = \hat{\mathscr{C}}_\alpha$.

Now, given any $\tau \in \mathscr{C}$, there is a $\sigma \in \hat{\mathscr{C}}_\alpha$, namely,

$$\sigma = 1 \wedge \inf\{t \geq \tau : Z(t) = t - \alpha\}$$

with at least as large a success probability as τ. That is, it is clearly suboptimal to stop and select a candidate which would remain available should we continue observations. In particular, it follows that $\tau^* \in \hat{\mathscr{C}}_\alpha$ and the following lemma obtains:

LEMMA 1. $v_\alpha = \max_{\tau \in \hat{\mathscr{C}}_\alpha} P(\tau - \alpha \leq U_1 \leq \tau)$.

Now, for each fixed t, $0 \leq t \leq 1$, define the stopping time τ_t by

$$\tau_t = \begin{cases} \inf_{t+\alpha<s<1} \{s : Z(s) = s - \alpha\}, & \text{if the set is nonempty,} \\ 1, & \text{otherwise,} \end{cases} \tag{1.3}$$

where, in addition, we associate with the stopping time τ_t the restriction that selection is limited to *items arriving after time t*. Thus, for a given t, τ_t ignores all arrivals up to and including time t and selects the *first* candidate, if any, arriving after time t, which is still relatively best at the end of its recall period —or stops at time 1 and selects the best if still available—whichever comes first.

REMARK 2. For $t < 1 - \alpha$, restricting selection to $(t, 1]$ plays no role as $\tau_t - \alpha > t$ by definition of τ_t. However, for $t \geq 1 - \alpha$, $\tau_t \equiv 1$ and the restriction limits allowable selection to $(t, 1]$. Thus, for $t, s > 1 - \alpha$, $t \neq s$, τ_t and τ_s correspond to different strategies in that τ_t can select from $(t, 1]$ and τ_s from $(s, 1]$. Defining strategies in this way for $1 - \alpha < t \leq 1$ has the advantage of simplifying later arguments.

Let $\mathscr{T} = \{\tau_t : 0 \leq t \leq 1 - \alpha\}$. Observe that $\mathscr{T} \subset \hat{\mathscr{C}}_\alpha$ and $\{\tau_s : t \leq s \leq 1 - \alpha\} \subset \hat{\mathscr{C}}_{t+\alpha}$. We will now prove that the optimal time τ^* belongs to $\mathscr{T}$.

LEMMA 2. $v_\alpha = \max_{\tau \in \mathscr{T}} P(\tau - \alpha \leq U_1 \leq \tau) = \max_{0 \leq t \leq 1-\alpha} P(\tau_t - \alpha \leq U_1 \leq \tau_t)$.

PROOF. For any time t, $0 < t < 1$, we define the utility of *stopping* at time t, given $Z(t)$, by

$$u(t) \equiv u_\alpha(t, Z(t)) \equiv P\big([t-\alpha]^+ \leq U_1 \leq t | Z(t)\big) = \begin{cases} t, & \text{if } Z(t) \geq t - \alpha, \\ 0, & \text{if } Z(t) < t - \alpha. \end{cases}$$

Similarly, we define the maximum expected utility of *continuing* at time t, given $Z(t)$, by

$$v(t) \equiv v_\alpha(t, Z(t)) \equiv \max_{\tau \in \mathscr{C}_t} P\left([\tau - \alpha]^+ \le U_1 \le \tau | Z(t)\right).$$

Clearly, behaving optimally, we should stop at the first time t (if any) for which the utility of stopping exceeds the expected utility of continuing. Hence, the optimal stopping rule τ^* must be of the form

$$\tau^* = \begin{cases} \inf\{t: u(t) \ge v(t)\}, & \text{if the set is nonempty,} \\ 1, & \text{otherwise.} \end{cases}$$

Define, for $0 \le t \le 1$,

$$\varphi_\alpha(t) \equiv \max_{\tau \in \mathscr{C}_t} P(\{t < U_1\} \cap \{\tau - \alpha \le U_1 \le \tau\}). \tag{1.4}$$

$\varphi_\alpha(t)$ is the maximum probability of best choice over all stopping rules which stop after time t and which ignore all arrivals in $(0, t]$.

For $1 - \alpha \le t \le 1$, we easily see that

$$\varphi_\alpha(t) = P(U_1 > t) = 1 - t, \qquad 1 - \alpha \le t \le 1. \tag{1.5}$$

For $0 \le t < 1 - \alpha$, it follows by the argument leading to Lemma 1 that

$$\varphi_\alpha(t) = \max_{\tau \in \hat{\mathscr{C}}_{t+\alpha}} P(\tau - \alpha \le U_1 \le \tau), \qquad 0 \le t < 1 - \alpha. \tag{1.6}$$

Also, since $\mathscr{C}_t \subseteq \mathscr{C}_s$ for $t > s$, $\varphi_\alpha(t)$ is a *decreasing* (i.e., nonincreasing) function of t, and in particular, if $0 \le t \le 1 - \alpha$, then by (1.5),

$$\varphi_\alpha(t) \ge \varphi_\alpha(1 - \alpha) = \alpha, \qquad 0 \le t \le 1 - \alpha. \tag{1.7}$$

Consider now the case where $Z(t) \le t - \alpha$, for $t > \alpha$. Should we continue, we are forced to seek a new candidate among those items arriving after time t. Thus, by independence of $Z(t)$ with the arrival time (and true rank) of the best arrival in $(t, 1]$, it follows that when $Z(t) \le t - \alpha$, $v(t) \equiv \varphi_\alpha(t)$ a.s. In particular, on the set where $Z(t) = t - \alpha$, $v(t) \equiv \varphi_\alpha(t)$, a.s.

By Lemma 1, $\tau^* \in \hat{\mathscr{C}}_\alpha$. Therefore, for $0 \le t \le \alpha$, $v(t) > u(t)$ since $\tau^* > \alpha$; and for $t > \alpha$, $v(t) > u(t)$ if $Z(t) \ne t - \alpha$. The utility of continuing is larger than the utility of stopping in each case. Hence, the only interesting case is when $t > \alpha$ and $Z(t) = t - \alpha$; but in this case, $v(t) = \varphi_\alpha(t)$ and $u(t) = t$. Thus, τ^* is given by

$$\tau^* = \begin{cases} \inf\{t > \alpha: Z(t) = t - \alpha \text{ and } t \ge \varphi_\alpha(t)\}, & \text{if the set is nonempty,} \\ 1, & \text{otherwise.} \end{cases} \tag{1.8}$$

Now, since $\varphi_\alpha(t)$ is a decreasing function of t with $\varphi_\alpha(1) = 0$, there exists (a unique) $\hat{t}$ [$= \hat{t}(\alpha)$] such that $\varphi_\alpha(t) > t$ for $t < \hat{t}$, and $\varphi_\alpha(t) < t$ for $t > \hat{t}$. Let $t_\alpha^* = \hat{t} \vee \alpha$, that is,

$$t_\alpha^* = \inf\{t \ge \alpha: \varphi_\alpha(t) \le t\}. \tag{1.9}$$

It follows that

$$(1.10)\qquad \tau^* = \begin{cases} \inf\{t > t_\alpha^*: Z(t) = t - \alpha\}, & \text{if the set is nonempty,} \\ 1, & \text{otherwise.} \end{cases}$$

Observe that, in the case where $1/2 \le \alpha < 1$, $t_\alpha^* = \alpha$ since, by (1.5), $\varphi_\alpha(t) = 1 - t \le t$ for $t \ge \alpha$. Letting

$$(1.11)\qquad t_\alpha = t_\alpha^* - \alpha,$$

we see that $0 \le t_\alpha < 1 - \alpha$ and $\tau^* = \tau_{t_\alpha} \in \mathscr{T}$. The result follows. □

For $0 \le t \le 1$, define the probability that the rule τ_t selects the overall best item by

$$(1.12)\qquad \psi_\alpha(t) \equiv P[\tau_t \text{ "selects the best"}].$$

Observe that

$$(1.13)\qquad \psi_\alpha(t) = P[\tau_t = \min(U_1 + \alpha, 1)], \quad \text{for } 0 \le t \le 1 - \alpha$$

and that

$$(1.14)\qquad \psi_\alpha(t) = P[U_1 > t] = 1 - t, \quad \text{for } 1 - \alpha \le t \le 1.$$

We will now prove the main result of this section:

THEOREM 1. *For each* $\alpha \in [0, 1]$,

$$(1.15)\qquad v_\alpha = \max_{0 \le t \le 1} \psi_\alpha(t) = \psi_\alpha(t_\alpha),$$

where t_α *is the unique time at which* $\psi_\alpha(t)$ *achieves its maximum given by* $t_\alpha = t_\alpha^* - \alpha$, *where*

$$(1.16)\qquad t_\alpha^* = \begin{cases} \alpha, & \text{if } \alpha \ge \frac{1}{2}, \\ \psi_\alpha(t_\alpha^*), & \text{if } \alpha \le \frac{1}{2}. \end{cases}$$

Moreover, the solution t_α^* *to* (1.16) *is unique. The optimal stopping time is then uniquely given by* $\tau^* = \tau_{t_\alpha}$, *where* τ_{t_α} *is of the form* (1.3).

For $\alpha = 0$, $\psi_0(t) = -t \ln t$, $t_0 = t_0^* = e^{-1}$ *and* $v_0 = e^{-1}$.

For $\alpha > 0$, $\psi_\alpha(t)$ *satisfies the differential equation*

$$(1.17)\qquad \psi_\alpha'(t) = \frac{\psi_\alpha(t + \alpha)}{t + \alpha} - 1, \qquad 0 < t < 1 - \alpha$$

with boundary condition

$$(1.18)\qquad \psi_\alpha(t) = 1 - t, \qquad 1 - \alpha \le t \le 1.$$

PROOF. That $v_\alpha = \max_{0 \le t \le 1} \psi_\alpha(t)$ follows from Lemma 2, the definition of $\psi_\alpha(t)$ and the fact that $\psi_\alpha(1 - \alpha) = \alpha > \psi_\alpha(t)$, $1 - \alpha < t \le 1$. (That is, the maximum is not affected by taking the maximum over the larger interval.) If we can show there exists a point $t_\alpha \in [0, 1]$ at which $\psi_\alpha(t)$ achieves its maximum, then $v_\alpha = \psi_\alpha(t_\alpha)$ and the optimal rule becomes $\tau^* = \tau_{t_\alpha}$, again by definition of $\psi_\alpha(t)$. For $\alpha > 0$, we will prove the existence of t_α by proving that $\psi_\alpha(t)$ is a continuous (bounded) function of t on [0, 1].

In the classical no-recall case corresponding to $\alpha = 0$,

$$\psi_0(t) \equiv P[U_1 > t; Z(U_1) < t] = \int_t^1 \frac{t}{z}\,dz = -t \ln t, \qquad 0 < t < 1. \tag{1.19}$$

Since $\psi_0(t)$ achieves its maximum value of e^{-1} at $t = e^{-1}$, it follows that $t_0 = e^{-1}$ and $v_0 = e^{-1}$, which are the well known limits in the finite no-recall best choice problem. [See, for example, Lindley (1961) or Gilbert and Mosteller (1966).]

For $\alpha > 0$ and $t \geq 1 - \alpha$, the boundary condition (1.18) is immediate from (1.14). The case with $t < 1 - \alpha$ is more complicated. For $0 < \delta \leq \alpha$ and $t + \delta \leq 1 - \alpha$,

$$\{\tau_t \text{``selects the best'' but } \tau_{t+\delta} \text{``doesn't''}\} = \{t < U_1 \leq t + \delta\} \tag{1.20}$$

while

$$\begin{aligned} &\{t < Z(t+\alpha) < t + \delta\} \cap \{\tau_{t+\alpha} \text{``selects the best''}\} \\ &\quad \supset \{\tau_{t+\delta} \text{``selects the best'' but } \tau_t \text{``doesn't''}\} \\ &\quad \supset \{t < Z(t+\alpha+\delta) < t+\delta\} \cap \{\tau_{t+\alpha+\delta} \text{``selects the best''}\}. \end{aligned} \tag{1.21}$$

From the definition of $\psi_\alpha(\cdot)$ and using (1.20), we obtain

$$\psi_\alpha(t) = \psi_\alpha(t+\delta) + \delta - P\{\tau_{t+\delta} \text{``selects the best'' but } \tau_t \text{``doesn't''}\}, \tag{1.22}$$

from which we obtain, using (1.21), the extremely useful inequalities

$$\delta\left[1 - \frac{\psi_\alpha(t+\alpha)}{t+\alpha}\right] < \psi_\alpha(t) - \psi_\alpha(t+\delta) < \delta\left[1 - \frac{\psi_\alpha(t+\alpha+\delta)}{t+\alpha+\delta}\right]. \tag{1.23}$$

From the inequalities (1.23) and the boundedness of $\psi_\alpha(t)$, we see that $\psi_\alpha(t)$ is continuous on $0 \leq t < 1 - \alpha$, and is left-continuous at $t = 1 - \alpha$, so is continuous on all of $(0, 1]$. Moreover, by dividing (1.23) through by δ and letting $\delta \to 0$, it follows that $\psi_\alpha(t)$ satisfies the differential equation (1.17).

Continuity of $\psi_\alpha(t)$ and (1.23) imply that the function $\psi_\alpha(t+\alpha)/(t+\alpha)$ is continuous and strictly decreasing on $[0, 1-\alpha]$. This, together with the differential equation (1.17), implies that $\psi_\alpha(\cdot)$ is unimodal and concave with a unique maximum at

$$t_\alpha \equiv t_\alpha^* - \alpha, \tag{1.24}$$

where t_α^* satisfies (1.16). An easy calculation shows that $\psi_\alpha(0) \geq \alpha$ which, with concavity, implies uniqueness of the solution to (1.16). □

Remark 3. Uniqueness of t_α and t_α^* imply that t_α, as given by (1.11) and (1.9), is in fact the same as that given by (1.24) and (1.16). Indeed, it can be shown that $\varphi_\alpha(t)$ as given in (1.4) satisfies (1.17) for $t \in (t_\alpha, 1 - \alpha)$ with boundary condition (1.18) so that $\varphi_\alpha(t) \equiv \psi_\alpha(t)$ for $t_\alpha \leq t \leq 1$. [See Rocha (1988).]

The differential equation (1.17) is easily solved on $[[1 - 2\alpha]^+, 1 - \alpha]$ by substituting the known value $\psi_\alpha(t+\alpha) = 1 - t - \alpha$, from (1.18), into the right-hand side of (1.17). The solution is

$$\psi_\alpha(t) = 2 - 2t - \alpha + \ln(t+\alpha), \qquad [1-2\alpha]^+ \le t \le 1-\alpha. \tag{1.25}$$

This yields the complete solution for $\alpha \ge 1/2$, namely,

$$t_\alpha = 0, \qquad v_\alpha = 2 - \alpha + \ln\alpha, \quad \text{if } \alpha \ge 1/2. \tag{1.26}$$

For $\alpha < 1/2$, we can, in principle, use the differential equation (1.17), together with the boundary condition (1.18), to successively solve for $\psi_\alpha(t)$ on intervals of length α (i.e., first on $[1-2\alpha, 1-\alpha]$, then on $[1-3\alpha, 1-2\alpha]$ and so on) until we find the interval containing t_α^*. We then use (1.17) once more to determine $v_\alpha = \psi_\alpha(t_\alpha^* - \alpha)$.

Unfortunately, for $\alpha \le 1/2$, the solution for v_α cannot be put in closed form: For $t < 1 - 2\alpha$, we encounter integrals of the form

$$\int \frac{\ln(t+2\alpha)}{t+\alpha}\,dt. \tag{1.27}$$

We can still find t_α^*, by using (1.25), as long as $\psi_\alpha(1-2\alpha) \ge (1-2\alpha)$—which holds for $\alpha \ge 0.260303$—but even then we are faced with an integral like (1.27) to find v_α.

In the next two sections, we will estimate v_α and t_α^* for α near 0 and α near 1/2, where the solutions are known. In Section 2 we will improve the bounds on $\psi(\cdot)$ given in (1.23); we will obtain these bounds probabilistically. We will then use these improved bounds to get upper and lower bounds for v_α and t_α^*. In Section 3, we will find analytically, asymptotic expansions for v_α and t_α^*, directly from the differential equation (1.17) and boundary condition (1.18).

2. Upper and lower bounds on v_α and t_α^*. An improvement on (1.21), valid for $0 < \delta \le \alpha$ and $t + \delta \le 1 - \alpha$, is

$$\begin{aligned}
&\bigcup_{k=1}^{r}\Bigg[\left\{t + \frac{k-1}{r}\delta < Z\left(t+\alpha+\frac{k-1}{r}\delta\right) \le t + \frac{k}{r}\delta\right\}\\
&\qquad \cap \Bigg[\left\{t+\alpha+\frac{k-1}{r}\delta < U_1 \le t+\alpha+\delta\right\}\\
&\qquad\qquad \cup \{\tau_{t+\alpha+\delta} \text{ “selects the best”}\}\Bigg]\Bigg]\\
&\quad \supset \{\tau_{t+\delta}\text{“selects the best” but } \tau_t\text{“doesn’t”}\}\\
&\quad \supset \bigcup_{k=1}^{r}\Bigg[\left\{t+\frac{k-1}{r}\delta < Z\left(t+\alpha+\frac{k}{r}\delta\right) \le t + \frac{k}{r}\delta\right\}\\
&\qquad\qquad \cap \left\{t+\alpha+\frac{k}{r}\delta < U_1 \le t+\alpha+\delta\right\}\Bigg]\\
&\quad \cup \big[\{t < Z(t+\alpha+\delta) \le t+\delta\} \cap \{\tau_{t+\alpha+\delta}\text{“selects the best”}\}\big].
\end{aligned} \tag{2.1}$$

This enables us to improve (1.23) to

$$\begin{aligned}
&\delta\left(1 - \frac{1}{r}\sum_{k=1}^{r}\left[\frac{\delta(1-(k-1)/r) + \psi_\alpha(t+\alpha+\delta)}{t+\alpha+((k-1)/r)\delta}\right]\right)\\
(2.2)\qquad &< \psi_\alpha(t) - \psi_\alpha(t+\delta)\\
&< \delta\left(1 - \frac{1}{r}\sum_{k=1}^{r}\left[\frac{\delta(1-k/r)}{t+\alpha+(k/r)\delta}\right] - \frac{\psi_\alpha(t+\alpha+\delta)}{t+\alpha+\delta}\right).
\end{aligned}$$

Letting $r \to \infty$ yields

$$\begin{aligned}
&2\delta - \left[(t+\alpha+\delta) + \psi_\alpha(t+\alpha+\delta)\right]\ln\left(1 + \frac{\delta}{t+\alpha}\right)\\
(2.3)\qquad &\le \psi_\alpha(t) - \psi_\alpha(t+\delta)\\
&\le 2\delta - (t+\alpha+\delta)\ln\left(1+\frac{\delta}{t+\alpha}\right) - \delta\frac{\psi_\alpha(t+\alpha+\delta)}{t+\alpha+\delta}.
\end{aligned}$$

Now, for $0.260303 \le \alpha \le 0.5$, the solution, t_α^*, to the equation $\psi_\alpha(t) = t$ belongs to the interval $[1-2\alpha, 1-\alpha]$, where $\psi_\alpha(t) = 2 - 2t - \alpha + \ln(t+\alpha)$, and thus t_α^* can be explicitly calculated from

$$(2.4)\qquad \psi_\alpha(t_\alpha^*) = 2 - 2t_\alpha^* - \alpha + \ln(t_\alpha^* + \alpha) = t_\alpha^*.$$

Substituting $\delta = \alpha$ and $t = t_\alpha = t_\alpha^* - \alpha$ into (2.3) and using the fact that $v_\alpha = \psi_\alpha(t_\alpha) = \psi_\alpha(t_\alpha^* - \alpha)$, $\psi_\alpha(t_\alpha + \alpha) = \psi_\alpha(t_\alpha^*) = t_\alpha^*$ and $\psi_\alpha(t_\alpha + 2\alpha) = \psi_\alpha(t_\alpha^* + \alpha) = 1 - (t_\alpha^* + \alpha)$, for such α, we obtain the following bounds on v_α:

$$\begin{aligned}
&t_\alpha^* + 2\alpha - \ln\left(1 + \frac{\alpha}{t_\alpha^*}\right)\\
(2.5)\qquad &\le v_\alpha \le t_\alpha^* + 3\alpha - (t_\alpha^* + \alpha)\ln\left(1 + \frac{\alpha}{t_\alpha^*}\right) - \frac{\alpha}{t_\alpha^* + \alpha},\\
&\qquad\qquad 0.260303 \le \alpha \le 0.5.
\end{aligned}$$

Solving (2.4) for t_α^* and then substituting the result into (2.5), we obtain the values of α near $\alpha = 0.5$ given in Table 1. By comparison, when $\alpha = 0.5$, $t_\alpha^* = 0.5$ and $v_\alpha = 3/2 - \ln 2 \cong 0.8068528194$.

TABLE 1
t_α^* is explicitly calculated from (2.4) using Newton's method. The upper and lower bounds are obtained by substitution of this value into (2.5)

α	t_α^*	Lower bound for v_α	Upper bound for v_α
0.400	0.4971594375	0.7068366205	0.7216941183
0.490	0.4999747042	0.7968528182	0.7987391753
0.499	0.4999997497	0.8058528194	0.8060455150

Now, we can furthermore use (2.4) to obtain a power series expansion of t_α^* about $\alpha = 1/2$:

$$t_\alpha^* = \tfrac{1}{2} - \tfrac{1}{4}\left(\tfrac{1}{2} - \alpha\right)^2 - \tfrac{7}{24}\left(\tfrac{1}{2} - \alpha\right)^3 - \tfrac{79}{192}\left(\tfrac{1}{2} - \alpha\right)^4 + O\left(\left(\tfrac{1}{2} - \alpha\right)^5\right). \tag{2.6}$$

This, in turn, can be substituted into the left- and right-hand sides of (2.5) to yield the following series expansions for the lower and upper bounds, respectively, for v_α:

$$\begin{aligned} t_\alpha^* + 2\alpha - \ln\left(1 + \frac{\alpha}{t_\alpha^*}\right) &= \left(\frac{3}{2} - \ln 2\right) - \left(\frac{1}{2} - \alpha\right) \\ &\quad - \frac{1}{8}\left(\frac{1}{2} - \alpha\right)^4 + O\left(\left(\frac{1}{2} - \alpha\right)^5\right) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &t_\alpha^* + 3\alpha - (t_\alpha^* + \alpha)\ln\left(1 + \frac{\alpha}{t_\alpha^*}\right) - \frac{\alpha}{t_\alpha^* + \alpha} \\ &\quad = \left(\frac{3}{2} - \ln 2\right) + \left(-\frac{3}{2} + \ln 2\right)\left(\frac{1}{2} - \alpha\right) \\ &\quad\quad + \left(-\frac{5}{8} + \frac{1}{4}\ln 2\right)\left(\frac{1}{2} - \alpha\right)^2 \\ &\quad\quad + \left(-\frac{7}{48} + \frac{7}{24}\ln 2\right)\left(\frac{1}{2} - \alpha\right)^3 + \left(-\frac{49}{128} + \frac{79}{192}\ln 2\right)\left(\frac{1}{2} - \alpha\right)^4 \\ &\quad\quad + O\left(\left(\frac{1}{2} - \alpha\right)^5\right). \end{aligned} \tag{2.8}$$

Considering now α near 0, we note that the inequalities (2.5) are not valid and the inequalities (2.3) are not helpful. To obtain sharper bounds on v_α and t_α^* for small α, we need some additional relations. Observe that

$$\begin{aligned} &\{t < U_1 < t + \alpha\} \cup \left[\{U_1 > t + \alpha\} \cap \{Z(U_1 - \alpha) < t\}\right] \\ &\quad\subset \{\tau_t \text{“selects the best”}\} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\{U_1 < t\} \cup \left[\{U_1 > t + \alpha\} \cap \{t < Z(U_1) < U_1 - \alpha\}\right] \\ &\quad\subset \{\tau_t \text{“does } \textit{not} \text{ select the best”}\} \end{aligned} \tag{2.10}$$

both obtain. By taking probabilities in (2.9) and (2.10) and direct computation, we obtain the inequalities

$$\begin{aligned} \alpha + t\ln(1 - \alpha) - t\ln t < \psi_\alpha(t) < \alpha - (t + \alpha)\ln(t + \alpha), \\ 0 < t < 1 - \alpha. \end{aligned} \tag{2.11}$$

From (2.11), we easily obtain

$$e^{-1} < t_\alpha^* < e^{-1} + \alpha. \tag{2.12}$$

(Recall that from Theorem 1, $t_0 = t_0^* = e^{-1}$ is optimal for $\alpha = 0$.)

Now, we can increase the sets on the left sides of (2.9) and (2.10), thereby improving the bounds in (2.11). Specifically, the left sides of (2.9) and (2.10) can be enlarged by taking the unions with the sets

$$\begin{aligned}\bigcup_{k=1}^{r-1}\Bigg[&\{U_1 > t+\alpha\} \cap \left\{U_1 - \alpha - \frac{k}{r}\alpha < Z(U_1 - \alpha) < U_1 - \alpha - \frac{k-1}{r}\alpha\right\}\\ &\cap\left\{Z\left(U_1 - \alpha - \frac{k}{r}\alpha\right) < t\right\} \cap \left\{U_1 - \alpha < Z\left(U_1 - \frac{k}{r}\alpha\right)\right\}\Bigg]\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\bigcup_{k=1}^{r}\Bigg[&\{U_1 > t+\alpha\} \cap \left\{U_1 - \frac{k}{r}\alpha < Z(U_1) < U_1 - \frac{k-1}{r}\alpha\right\}\\ &\cap\left\{t < Z\left(U_1 - \frac{k}{r}\alpha\right) < U_1 - \alpha - \frac{k}{r}\alpha\right\}\Bigg],\end{aligned} \tag{2.14}$$

respectively. Note that the choice of sets in (2.13) and (2.14) is in no way unique. The particular events used here have the advantage of having easily computed probabilities and the resulting improved bounds are

$$\begin{aligned}&\alpha + t\ln(1-\alpha) - t\ln t + \sum_{k=1}^{L} C_k\\ &\quad< \psi_\alpha(t) < \alpha - (t+\alpha)\ln(t+\alpha) + \sum_{k=1}^{U} D_k, \qquad 0 < t < 1-\alpha,\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}C_k = \frac{t}{k}\Bigg[&-\ln(1-\alpha) + \ln\left(t + \frac{k}{r}\alpha\right) + \frac{k}{r}\ln\left(1 - \frac{k}{r}\alpha\right) - \frac{k}{r}\ln(t+\alpha)\\ &+\left(1 - \frac{k}{r}\right)\ln\left(1 - \alpha - \frac{k}{r}\alpha\right) - \left(1 - \frac{k}{r}\right)\ln t\Bigg],\end{aligned} \tag{2.16}$$

$$\begin{aligned}D_k = {}&\frac{\alpha}{r}\ln\left(t + \alpha + \frac{k}{r}\alpha\right)\\ &+\frac{t+\alpha}{k}\left[\ln\left(1 - \frac{k}{r}\alpha\right) - \ln(t+\alpha) + \ln\left(t + \alpha + \frac{k}{r}\alpha\right)\right],\end{aligned} \tag{2.17}$$

$$L = \min\left(r-1, \left[\frac{r}{\alpha}(1-\alpha-t)\right]\right) \tag{2.18}$$

TABLE 2
The bounds obtained are for $r = 100$, using the method outlined above

α	Lower bound for t_α^*	Upper bound for t_α^*	Lower bound for v_α	Upper bound for v_α
0.001	0.3685107832	0.3685113314	0.3685121424	0.3685126953
0.010	0.3741219108	0.3741711301	0.3742578092	0.3743114118
0.100	0.4229465858	0.4246721619	0.4360199240	0.4401380709
0.400	0.4969611948	0.4971682574	0.6148702554	0.7314882162

and

$$U = \min\left(r, \left[\frac{r}{\alpha}(1 - \alpha - t)\right]\right). \tag{2.19}$$

We can use (2.15) to get upper and lower bounds for v_α—call them v_U and v_L, respectively—as follows:

1. Choose r.
2. Denoting the left and right sides of (2.15) by $\psi_L(t)$ and $\psi_U(t)$, respectively, get upper and lower bounds for t_α^* by finding the roots t_L and t_U of $\psi_L(t) = t$ and $\psi_U(t) = t$, respectively.
3. Get a lower bound for v_α by evaluating $v_L \equiv \max[\psi_L(t_L - \alpha), \psi_L(t_U - \alpha)]$.
4. Get an upper bound for v_α by evaluating $v_U \equiv \psi_U(t_U - \alpha) + (t_U - t_L) \times (1 - \psi_L(t_U)/t_U)$. That v_U is indeed an upper bound for v_α follows by substituting $t = t_\alpha^* - \alpha$ and $\delta = t_U - t_\alpha^*$ into the rightmost inequality in (1.23), and using the fact that $\psi_L(t) < \psi_\alpha(t) < \psi_U(t)$.

For $r = 100$, we obtain the bounds given in Table 2. Remarkably, for $\alpha = 0.001$, the bounds obtained determine v_α to six decimal places. On the other hand, when $\alpha = 0.1$, the accuracy is only to the first decimal place. For $\alpha = 0.4$, the bounds break down, as expected.

3. Asymptotic analysis. An alternative approach to the probabilistic bounds obtained in the last section is to do a classical asymptotic analysis of the perturbed differential equation and boundary condition, (1.17) and (1.18). [See, for example, Nayfeh (1973); this is the approach used in Rocha (1988).] For ease of notation we will henceforth write $\psi(t, \alpha)$ in place of $\psi_\alpha(t)$.

To investigate $\psi(t, \alpha)$ and t_α^* for α near 0, we will seek solutions of the form

$$\psi(t, \alpha) = \psi_0(t) + \psi_1(t)\alpha + \psi_2(t)\alpha^2 + \cdots \tag{3.1}$$

and

$$t_\alpha^* = t_0 + t_1\alpha + t_2\alpha^2 + \cdots, \tag{3.2}$$

where $t_0, t_1, t_2, \ldots$ are constants and the functions $\psi_0(t), \psi_1(t), \psi_2(t), \ldots$ depend on t alone. Substituting (3.1) into the differential equation (1.17), using

Taylor's theorem to expand the functions $\psi_i(t+\alpha)$ for $i = 0, 1, 2$, as

$$\psi_i(t+\alpha) = \psi_i(t) + \psi_i'(t)\alpha + \psi_i''(t)\alpha^2 + O(\alpha^3) \tag{3.3}$$

[which we can expect to be valid for $0 < t < 1 - \alpha$, even though $\psi''(t, \alpha)$ is discontinuous at $t = 1 - \alpha$—see Hardy (1958), page 289], and then equating like powers of α, yields a set of first-order linear ordinary differential equations. These can be solved explicitly using the boundary condition at $t = 1 - \alpha$, yielding

$$\psi(t,\alpha) = -t\ln t + (1-t)\alpha + \left(\frac{1}{4t} - \frac{3}{4}t\right)\alpha^2 + O(\alpha^3). \tag{3.4}$$

We next substitute (3.2) in place of t in (3.4). Expanding the result in powers of α and using the equation

$$\psi(t_\alpha^*, \alpha) = t_\alpha^* \tag{3.5}$$

to equate like powers of α enables us to solve for the coefficients in (3.2). In this way we obtain

$$t_\alpha^* = e^{-1} + (1 - e^{-1})\alpha - \left(\frac{e + e^{-1}}{4}\right)\alpha^2 + O(\alpha^3). \tag{3.6}$$

Finally, since $v_\alpha = \psi_\alpha(t_\alpha, \alpha) = \psi_\alpha(t_\alpha^* - \alpha, \alpha)$, we can use (3.6) and (3.4) to obtain

$$v_\alpha = e^{-1} + (1 - e^{-1})\alpha + \left(\frac{e - e^{-1}}{4}\right)\alpha^2 + O(\alpha^3). \tag{3.7}$$

In principle, asymptotically as $\alpha \to 0$, (3.7) provides a more accurate estimate of v_α than the bounds obtained in Section 2—which from Table 2 can be seen to differ by about $O(\alpha^2)$. However, lacking explicit bounds on the error term in (3.7), we are unable to gauge the accuracy of the approximation for specific values of α. Hence, for particular values of α, the bounds obtained in Section 2 must be considered more reliable than the polynomial approximations given here. Table 3 is obtained from the second-degree approximating polynomials in (3.6) and (3.7), and should be compared with Table 2 of the last section. As noted, it is rather fortuitous that these estimates fall within the bounds provided there.

TABLE 3

The estimates given here were obtained from the second-order approximating polynomials in α obtained from the asymptotic expansions about $\alpha = 0$, given in (3.6) *and* (3.7)

α	Estimate of t_α^*	Estimate of v_α
0.001	0.3685107902	0.3685121493
0.010	0.3741234927	0.3742594068
0.100	0.4233760939	0.4369675030

TABLE 4

The estimates given here were obtained from the fourth-order approximating polynomials in $(\frac{1}{2} - \alpha)$ obtained from the asymptotic expansions about $\alpha = \frac{1}{2}$, given in (3.8) and (3.9). The actual value of t_α^*, shown for comparison, is from Table 1 of Section 2

α	$(\frac{1}{2} - \alpha)$	t_α^*	Estimate of t_α^*	Estimate of v_α
0.400	0.100	0.4971594375	0.4971671875	0.7071986528
0.490	0.010	0.4999747042	0.4999747042	0.7968531540
0.499	0.001	0.4999997497	0.4999997497	0.8058528198

In a similar manner, to investigate v_α and t_α^* for α near $\frac{1}{2}$, $\alpha < \frac{1}{2}$, we assume they have expansions in powers of $(\frac{1}{2} - \alpha)$ and carry through the analysis as above. We obtain

$$(3.8)\quad t_\alpha^* = \tfrac{1}{2} - \tfrac{1}{4}\left(\tfrac{1}{2} - \alpha\right)^2 - \tfrac{7}{24}\left(\tfrac{1}{2} - \alpha\right)^3 - \tfrac{79}{192}\left(\tfrac{1}{2} - \alpha\right)^4 + O\left(\left(\tfrac{1}{2} - \alpha\right)^5\right)$$

and

$$(3.9)\quad v_\alpha = \left(\tfrac{3}{2} - \ln 2\right) - \left(\tfrac{1}{2} - \alpha\right) + \tfrac{1}{3}\left(\tfrac{1}{2} - \alpha\right)^3 + \tfrac{1}{8}\left(\tfrac{1}{2} - \alpha\right)^4 + O\left(\left(\tfrac{1}{2} - \alpha\right)^5\right).$$

Note that (3.8) was obtained previously in (2.6).

As with the expansion about $\alpha = 0$, without specific bounds on the error terms in (3.8) and (3.9), we must view the bounds from Table 1 as more reliable for given values of α—even though these bounds differ by $(-\frac{1}{2} + \ln 2)(\frac{1}{2} - \alpha) + O((\frac{1}{2} - \alpha)^2)$, as can be seen from (2.7) and (2.8). Table 4, obtained from the fourth-order approximating polynomials from (3.8) and (3.9), should be compared with Table 1 of the last section.

4. The limit of the finite problem. We will now prove that the solution to the finite secretary problem with recall of length m converges to the solution of the corresponding infinite problem when $m/n \to \alpha$. We first derive a variant of the recursion equations of Smith and Deely (1975) in order to relate the solution of the finite problem more closely to our solution of the infinite problem.

Thus, consider the finite secretary problem with recall of length m. Smith and Deely proved that the optimal stopping rule has the following form: Stop at the first time after some fixed $r^* \geq m$ at which the current candidate is about to be lost, and otherwise stop at time n. In particular, they showed that it suffices to consider the following class of stopping rules: For fixed k, $0 \leq k \leq n$, define the rule t_k by

$$(4.1)\quad t_k = \begin{cases} \inf\{j\colon (k+m) \wedge n \leq j \leq n \text{ and the current candidate} \\ \qquad \text{is at } j - m + 1\}, & \text{if the set is nonempty,} \\ n, & \text{otherwise,} \end{cases}$$

where, corresponding to the rule t_k, we additionally restrict selection to arrivals in $\{k+1,\dots,n\}$. (Note that the latter condition only affects the rules for which $k \ge n-m+1$ where $t_k \equiv n$; moreover, the rules for $k \ge n-m+1$ are clearly suboptimal.) Furthermore, define

$$P(k) = P(t_k \text{"selects the best"}). \tag{4.2}$$

Clearly $P(k)$ and t_k are the finite analogues of $\psi_\alpha(t)$ and τ_t of Section 1. Not surprisingly, we have the following theorem analogous to Theorem 1.

THEOREM 2. *$P(k)$ is the unique solution to the recursion equation*

$$P(k) = \begin{cases} \dfrac{n-k}{n}, & n-m \le k \le n, \\ P(k+1) + \dfrac{1}{n} - \dfrac{P(k+m)}{k+m}, & 0 \le k \le n-m-1. \end{cases} \tag{4.3}$$

Moreover, let

$$r^* = \min_{m \le k \le n} \{k \colon P(k) \le k/n\}. \tag{4.4}$$

Then

$$v^{(n,m)} = P(r^*-m) = \max_{0 \le k \le n} P(k) \tag{4.5}$$

and the optimal time is given by t_{r^*-m}, defined by (4.1) *with $k = r^* - m$.*

PROOF. Since Smith and Deely proved that the optimal rule has the form (4.1), it follows by definition of $P(k)$ that $v^{(n,m)} = \max_{0 \le k \le n} P(k)$.

For $n-m \le k \le n$, $P(k) = P(t_k$ "selects the best") $= P$(best item arrives in $\{k+1,\dots,n\}) = (n-k)/n$.

The rest of the proof of (4.3) is just like the derivation of the left-hand side of (1.23) but with equality holding. For $0 \le k \le n-m-1$, we have

$$\begin{aligned} P(k) &= P(k+1) + P(t_k \text{"selects the best"}; t_{k+1} \text{"doesn't select the best"}) \\ &\quad - P(t_{k+1} \text{"selects the best"}; t_k \text{"doesn't select the best"}) \\ &= P(k+1) + \frac{1}{n} - P(\{\text{best of } \{1,2,\dots,k+m\} \text{ is at } k\} \\ &\qquad\qquad \cap \{t_{k+m} \text{"selects the best"}\}) \\ &= P(k+1) + \frac{1}{n} - \frac{P(k+m)}{k+m}, \quad \text{by independence.} \end{aligned}$$

This proves (4.3).

Now taking probabilities in the set inequality

$$\begin{aligned}&\{\text{best of } \{1,2,\dots,k+m\} \text{ is at } k+1\}\\&\quad\cap\{t_{k+m}\text{“selects the best”}\}\\&\quad\supset\{\text{best of } \{1,2,\dots,k+m+1\} \text{ is at } k+1\}\\&\qquad\cap\{t_{k+m+1}\text{“selects the best”}\},\end{aligned}$$

we obtain the inequality

$$\frac{P(k+m)}{k+m} \geq \frac{P(k+m+1)}{k+m+1},$$

valid for all k, $0 \leq k \leq n-m-1$. It follows that $P(k+m)/(k+m)$ is a decreasing function of k. In particular, together with (4.3) this shows that $P(\cdot)$ is unimodal with a unique maximum at $r^* - m$, where r^* is given by (4.4). The optimal rule is thus t_{r^*-m}. □

By comparing the recursion equations (4.3) with those of Smith and Deely [(1975), page 360, equations (3.5)–(3.8)], we see that the function $F(k)$, which they considered, is just our $nP(k)$ for $r^* - m \leq k \leq n$. In essence, they proved that if $k/n \to t$ and $m/n \to \alpha \geq 1/2$, then $P(k) \to \psi_\alpha(t)$, $r^*/n \to t^*_\alpha$ and $v^{(n,m)} \to v_\alpha$, where $\psi_\alpha(t)$, t^*_α and v_α are the analogous quantities for the infinite α-recall problem. In the remainder of this section we will prove that this result holds for all α, $0 < \alpha < 1$. We first need a definition and a lemma.

For each pair, (n, m), define the piece-wise linear function $f^{(n,m)}(t)$ on [0, 1] by $f^{(n,m)}(k/n) = P^{(n,m)}(k) = P(k)$, $k = 0, 1, 2, \dots, n$, and define $f^{(n,m)}(t)$ elsewhere by linear interpolation. Then the following lemma holds:

LEMMA 3. *Fix $\beta > 0$. The family of functions $\{f^{(n,m)}(t)\colon n, m = 1, 2, 3, \dots; m/n \geq \beta\}$ is a bounded, equicontinuous family on* [0, 1].

PROOF. Since $0 \leq P(k) \leq 1$ for all $0 \leq k \leq n$, it follows that $|f^{(n,m)}(t)| \leq 1$ for all n, m.

To show equicontinuity, it suffices to show that the derivatives, where they exist, are uniformly bounded. For given n and m, if $1 - m/n \leq t \leq 1$, then $f^{(n,m)}(t) = 1 - t$, and so $f^{(n,m)\prime}(t) = -1$. For $0 \leq k \leq n - m - 1$, we have, using (4.3) and the assumption $m/n \geq \beta$,

$$\left|\frac{f^{(n,m)}((k+1)/n) - f^{(n,m)}(k/n)}{1/n}\right| = \left|-1 + \frac{P(k+m)}{(k+m)/n}\right| \leq 1 + \frac{1}{(k+m)/n} \leq 1 + \frac{1}{\beta}. \tag{4.6}$$

This inequality proves the lemma. □

THEOREM 3. *Fix* $0 < \alpha < 1$. *Let* m $[= m(n)]$ *and* n *tend to infinity in such a way that* $m/n \to \alpha$, $0 < \alpha < 1$. *Then* $f^{(n,m)}(t)$ *converges uniformly to* $\psi_\alpha(t)$ *on* $[0,1]$, *where* $\psi_\alpha(t)$ *is the* (*unique*) *solution to the differential equation* (1.17) *with boundary condition* (1.18). *Moreover*, $r^*/n \to t_\alpha^*$ *where* t_α^* *is defined by* (1.16). *In particular*, $v^{(n,m)} \to v_\alpha$.

PROOF. Since $m/n \to \alpha$, $\beta = \sup\{m/n\} < \infty$. It follows from Lemma 3 that the (sub-)family $\{f^{(n,m)}(t)\colon m/n \to \alpha\}$ is a bounded, equicontinuous family. By the Ascoli–Arzelà theorem [see, for example, Royden (1968)], there exists a subsequence $\{f^{(n_j, m_j)}(t)\}_{j=1}^{\infty}$, which converges uniformly to a function $f(t)$, continuous on $[0,1]$. We will show that f is unique and thus that the full sequence converges to f on $[0,1]$. Consequently, we henceforth drop the subscript j and write $f^{(n,m)}(t)$ in place of $f^{(n_j,m_j)}(t)$ [keeping in mind that $m = m(n)$ here].

Now $f^{(n,m)}(t) = 1 - t$ for $1 - m/n \le t \le 1$. Hence, since $m/n \to \alpha$, it follows that $f(t) = 1 - t$ for $1 - \alpha \le t \le 1$. That is, $f(t)$ satisfies the boundary condition (1.18).

Now suppose $\varepsilon > 0$ is given. For fixed k, with $0 \le k \le n - m - 1$, repeated application of (4.3) yields

$$f^{(n,m)}\left(\frac{k}{n}\right) = 1 - \frac{k}{n} - \frac{1}{n}\sum_{j=m+k}^{n-1} \frac{f^{(n,m)}(j/n)}{j/n}. \tag{4.7}$$

Take $k/n \to t_0 \in [0, 1-\alpha]$ (maintaining $0 \le k \le n - m - 1$) and let $\varepsilon' = \varepsilon(t_0 + \alpha)/2$. Since $f^{(n,m)}(t) \Rightarrow f(t)$, $t \in [0,1]$, it follows that there exists $N_1 = N_1(\varepsilon')$, independent of t, such that for all $n \ge N_1$, we have

$$\left| f^{(n,m)}(t) - f(t) \right| < \varepsilon'. \tag{4.8}$$

Similarly, since $k/n + m/n \to t_0 + \alpha$, there exists N_2 such that for all $n \ge N_2$, $k/n + m/n \ge (t_0 + \alpha)/2$. In particular, taking $n \ge \max(N_1, N_2)$, we have

$$\left| \frac{1}{n}\sum_{j=m+k}^{n-1} \frac{f(j/n)}{j/n} - \frac{1}{n}\sum_{j=m+k}^{n-1} \frac{f^{(n,m)}(j/n)}{j/n} \right| < \varepsilon. \tag{4.9}$$

Since $t_0 + \alpha > 0$, $f(u)/u$ is bounded and continuous on $[t_0 + \alpha, 1]$. It follows that it is integrable there as well, so there exists $N_3 = N_3(t_0, \varepsilon)$, such that for all $n \ge N_3$,

$$\left| \int_{t_0+\alpha}^{1} \frac{f(u)}{u}\,du - \frac{1}{n}\sum_{j=m+k}^{n-1} \frac{f(j/n)}{j/n} \right| < \varepsilon. \tag{4.10}$$

Thus, for all $n \ge \max(N_1, N_2, N_3)$,

$$\left| \int_{t_0+\alpha}^{1} \frac{f(u)}{u}\,du - \frac{1}{n}\sum_{j=m+k}^{n-1} \frac{f^{(n,m)}(j/n)}{j/n} \right| < 2\varepsilon. \tag{4.11}$$

Letting $n \to \infty$ in (4.7), since $t_0 \in [0, 1-\alpha]$ was arbitrary, it follows that $f(t)$ satisfies the integral equation

$$f(t) = 1 - t - \int_{t+\alpha}^{1} \frac{f(u)}{u}\, du, \qquad 0 \le t \le 1 - \alpha. \tag{4.12}$$

Since $f(u)/u$ is continuous on $[t+\alpha, 1]$, it follows from (4.12) that f is differentiable on $(0, 1-\alpha)$. Differentiating (4.12) we obtain

$$f'(t) = -1 + \frac{f(t+\alpha)}{t+\alpha}, \qquad 0 < t < 1 - \alpha, \tag{4.13}$$

which is (1.17).

Finally, since there is a unique solution to the integral equation (4.13) satisfying the boundary condition (1.18), it follows that the limiting function f is unique. We have proved $f(t) = \psi_\alpha(t)$ and thus

$$f^{(n,m)}(t) \Rightarrow \psi_\alpha(t), \qquad 0 \le t \le 1. \tag{4.14}$$

Now, since $f^{(n,m)}(t) \Rightarrow \psi_\alpha(t)$ on $[0,1]$, it follows by continuity and the definition of r^* that $r^*/n \to t_\alpha^*$, where t_α^* satisfies $t_\alpha^* = \inf_{\alpha \le t \le 1}\{t: \psi_\alpha(t) \le t\}$. As already noted in Remark 3, this definition of t_α^* is equivalent to (1.16).

From (4.5), $v^{(n,m)} = P(r^* - m)$. Thus (with $m/n \to \alpha$, as above), we have

$$\begin{aligned} \lim_{n\to\infty} v^{(n,m)} &= \lim_{n\to\infty} P^{(n,m)}(r^* - m) \\ &= \lim_{n\to\infty} f^{(n,m)}\left(\frac{r^* - m}{n}\right) = \psi_\alpha(t_\alpha^* - \alpha) = v_\alpha. \end{aligned} \tag{4.15}$$

Theorem 3 is now proved. □

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DEPARTMENT OF MATHEMATICS AND
COMPUTER SCIENCE
SAN JOSE STATE UNIVERSITY
SAN JOSE, CALIFORNIA 95192-0103