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THE DISTRIBUTION OF VECTOR-VALUED RADEMACHER SERIES

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Let $X = \sum \varepsilon_n x_n$ be a Rademacher series with vector-valued coefficients. We obtain an approximate formula for the distribution of the random variable $\|X\|$ in terms of its mean and a certain quantity derived from the K-functional of interpolation theory. Several applications of the formula are given.

1. Results. In [6], Montgomery-Smith calculated the distribution of a scalar Rademacher series $\sum \varepsilon_n a_n$. The principal result of the present paper extends the results of [6] to the case of a Rademacher series $\sum \varepsilon_n x_n$ with coefficients (x_n) belonging to an arbitrary Banach space E. Its proof relies on a deviation inequality for Rademacher series obtained by Talagrand [9]. A somewhat curious feature of the proof is that it appears to exploit in a nontrivial way (see Lemma 2) the platitude that every separable Banach space is isometric to a closed subspace of ℓ_∞. The principal result is applied to yield a precise form of the Kahane–Khintchine inequalities and to compute certain Orlicz norms for Rademacher series.

First we recall some notation and terminology from interpolation theory (see, e.g., [1]). Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two Banach spaces which are continuously embedded into some larger topological vector space. For $t > 0$, the K-functional $K(x, t; E_1, E_2)$ is the norm on $E_1 + E_2$ defined by

$$K(x, t; E_1, E_2) = \inf\{\|x_1\|_1 + t\|x_2\|_2 \colon x = x_1 + x_2,\ x_i \in E_i\}.$$

For a sequence $(a_n) \in \ell_2$, we shall denote the K-functional $K((a_n), t; \ell_1, \ell_2)$ by $K_{1,2}((a_n), t)$ for short. For $1 \le p < \infty$, a sequence (x_n) in a Banach space $(E, \|\cdot\|)$ is said to be weakly ℓ_p if the scalar sequence $(x^*(x_n))$ belongs to ℓ_p for every $x^* \in E^*$. The collection of all weakly ℓ_p sequences is a Banach space, denoted $\ell_p^w(E)$, with the norm given by $\ell_p^w((x_n)) = \sup_{\|x^*\| \le 1} \|(x^*(x_n))\|_p$ [where $\|(a_n)\|_p = (\sum |a_n|^p)^{1/p}$]. If $(x_n) \in \ell_2^w(E)$, we define the following:

$$K_{1,2}^w((x_n), t) = \sup_{\|x^*\| \le 1} K_{1,2}((x^*(x_n)), t).$$

Observe that $K_{1,2}^w((x_n), t)$ is a continuous increasing function of t. In fact, it is a Lipschitz function with Lipschitz constant at most $\ell_2^w((x_n))$.

Next we set up some function space notation. Let (Ω, Σ, P) be a probability space. A Rademacher (or Bernoulli) sequence (ε_n) is a sequence of independent identically distributed random variables such that $P(\varepsilon_n = 1) =$

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$P(\varepsilon_n = -1) = \frac{1}{2}$. For a random variable Y defined on Ω, its decreasing rearrangement Y^* is the function on $[0,1]$ defined by $Y^*(t) = \inf\{s > 0: P(|Y| > s) \leq t\}$. For $0 < p < \infty$, the weak-L_p norm of Y, denoted $\|Y\|_{p,\infty}$, is given by $\|Y\|_{p,\infty} = \sup_{0<t<1} t^{1/p}Y^*(t)$. As usual, $\|Y\|_p$ denotes $(\mathbb{E}|Y|^p)^{1/p}$. Let Ψ be an Orlicz function on $[0,\infty)$. The Orlicz norm, $\|Y\|_\Psi$, is given by $\|Y\|_\Psi = \inf\{c > 0: \mathbb{E}\Psi(|Y|/c) \leq 1\}$. We shall be particularly interested in the Orlicz functions $\Psi_q(t) = e^{t^q} - 1$ for $2 < q < \infty$. The weak-ℓ_p norm of the scalar sequence (a_n) is defined by $\|(a_n)\|_{p,\infty} = \sup n^{1/p}a_n^*$, where (a_n^*) is the decreasing rearrangement of $(|a_n|)$.

Finally, we shall write $A \approx B$ to mean that there is a constant $C > 0$ such that $(1/C)A \leq B \leq CA$. We shall try to indicate in each case whether the implied constant is absolute or whether it depends on some parameter, typically $p \in [1,\infty)$, entering into the expressions for A and B.

Now we can state the principal result of the paper.

MAIN THEOREM. *Let $X = \Sigma\varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space E. Then, for $t > 0$, we have*

$$P\big(\|X\| > 2\mathbb{E}\|X\| + 6K_{1,2}^w((x_n),t)\big) \leq 4e^{-t^2/8}, \tag{1}$$

and, for some absolute constant c, we have

$$P\big(\|X\| > \tfrac{1}{2}\mathbb{E}\|X\| + cK_{1,2}^w((x_n),t)\big) \geq ce^{-t^2/c}. \tag{2}$$

The proof of the main theorem will be deferred until the end of the paper in order to proceed at once with the applications.

COROLLARY 1. *Let $X = \Sigma\varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $0 < t \leq \frac{1}{10}$, we have*

$$S^*(t) \approx \mathbb{E}\|X\| + K_{1,2}^w\Big((x_n), \sqrt{\log(1/t)}\Big), \tag{3}$$

where S denotes the real random variable $\|X\|$. The implied constant is absolute.

PROOF. Inequalities (1) and (2) give rise to the inequalities $S^*(4e^{-t^2/8}) \leq 2\mathbb{E}\|X\| + 6K_{1,2}^w((x_n),t)$ and $S^*(ce^{-t^2/c}) \geq \frac{1}{2}\mathbb{E}\|X\| + cK_{1,2}^w((x_n),t)$, respectively, whence (3) follows for all sufficiently small t by an appropriate change of variable. To see that the lower estimate implicit in (3) is valid in the whole range $0 < t < \frac{1}{10}$, we recall from [2] that $E\|X\|^2 \leq 9\mathbb{E}^2\|X\|$. Hence, by the Paley–Zygmund inequality (see, e.g., [4], page 8]), for $0 < \lambda < 1$, we have

$$\begin{aligned} P(\|X\| > \lambda\mathbb{E}\|X\|) &\geq (1-\lambda)^2\frac{\mathbb{E}^2\|X\|}{\mathbb{E}\|X\|^2} \\ &\geq \frac{1}{9}(1-\lambda)^2, \end{aligned}$$

whence $P(\|X\|) > (1 - 3/\sqrt{10})\mathbb{E}\|X\|) \geq \frac{1}{10}$, which easily implies (3). □

Kahane [4] proved that if $P(\|X\| > t) = \alpha$, where X is a Rademacher series in a Banach space, then $P(\|X\| > 2t) \leq 4\alpha^2$. By iteration this implies $P(\|X\| > st) \leq \frac{1}{4}(4\alpha)^s$, for $s = 2^n$. According to our next corollary, the exponent s in the latter result may be improved to be a certain multiple of s^2.

COROLLARY 2. *Let $X = \Sigma \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $t > 0$ and $s \geq 1$, we have*

$$P(\|X\| > st) \leq \left(\frac{1}{c_1} P(\|X\| > t) \right)^{c_1 s^2}$$

for some absolute constant c_1.

PROOF. By choosing $c_1 < c$, where c is the constant which appears in (2), the result becomes trivial whenever $P(\|X\| > t) \geq c$. Hence we may assume that $P(\|X\| > t) < c$. Choose $\alpha > 0$ such that $P(\|X\| > t) = ce^{-\alpha^2/c}$. Then (2) gives $t \geq \frac{1}{2}\mathbb{E}\|X\| + cK^w_{1,2}((x_n), \alpha)$. Thus,

$$\begin{aligned} st &\geq \frac{s}{2}\mathbb{E}\|X\| + scK^w_{1,2}((x_n), \alpha) \\ &\geq 2\mathbb{E}\|X\| + 6K^w_{1,2}\left((x_n), \frac{cs\alpha}{6}\right) \end{aligned}$$

provided $s \geq \max(4, 6/c)$. Now (1) gives

$$\begin{aligned} P(\|X\| > st) &\leq 4e^{-(cs\alpha)^2/288} \\ &= 4\left(\frac{1}{c}(ce^{-\alpha^2/c})\right)^{c^2s^2/288} \\ &= 4\left(\frac{1}{c}(P\|X\| > t)\right)^{c^3s^2/288}, \end{aligned}$$

which gives the result. □

Our next corollary, which is the vector-valued version of a recent result of Hitczenko [3], is a rather precise form of the Kahane–Khintchine inequalities.

COROLLARY 3. *Let $X = \Sigma \varepsilon_n x_n$ be a Rademacher series in a Banach space. Then, for $1 \leq p < \infty$, we have*

$$(\mathbb{E}\|X\|^p)^{1/p} \approx \mathbb{E}\|X\| + K^w_{1,2}\left((x_n), \sqrt{p}\right).$$

The implied constant is absolute.

PROOF. We may assume that $p \geq 2$. It follows from a result of Borell [2] that $(\mathbb{E}\|X\|^{2p})^{1/2p} \leq \sqrt{3}(\mathbb{E}\|X\|^p)^{1/p}$. Since $\frac{1}{2}\|Y\|_p \leq \|Y\|_{2p,\infty} \leq \|Y\|_{2p}$ for every random variable Y (as is easily verified), it follows (letting S denote the random variable $\|X\|$) that $\frac{1}{2}\|S\|_p \leq \|S\|_{2p,\infty} \leq \sqrt{3}\|S\|_p$. So it suffices to prove that $\|S\|_{p,\infty} \approx \mathbb{E}S + K^w_{1,2}((x_n), \sqrt{p})$ to obtain the desired conclusion. By Corollary 1, we have

$$\begin{aligned}\|S\|_{p,\infty} &\approx \mathbb{E}S + \sup_{0<t<1} t^{1/p} K^w_{1,2}\big((x_n), \sqrt{\log(1/t)}\big)\\ &= \mathbb{E}S + \sup_{0<t<1}\Big\{t^{1/p} \sup_{\|x^*\|\le 1} K_{1,2}\big((x^*(x_n)), \sqrt{\log(1/t)}\big)\Big\}\\ &= \mathbb{E}S + \sup_{\|x^*\|\le 1}\Big\{\sup_{0<t<1} t^{1/p} K_{1,2}\big((x^*(x_n)), \sqrt{\log(1/t)}\big)\Big\}.\end{aligned}$$

To evaluate the expression in brackets, we make use once more (see Corollary 2) of the elementary inequality $K_{1,2}((a_n), s) \le \max(1, s/t) K_{1,2}((a_n), t)$. Thus,

$$\begin{aligned}&\sup_{0<t\le e^{-p}} t^{1/p} K_{1,2}\big((x^*(x_n)), \sqrt{\log(1/t)}\big)\\ &\quad\le \left(\sup_{0<t\le e^{-p}} t^{1/p}\sqrt{\frac{\log(1/t)}{p}}\right) K_{1,2}\big((x^*(x_n)), \sqrt{p}\big)\\ &\quad= e^{-1} K_{1,2}\big((x^*(x_n)), \sqrt{p}\big).\end{aligned}$$

Moreover,

$$\sup_{e^{-p}<t<1} t^{1/p} K_{1,2}\big((x^*(x_n)), \sqrt{\log(1/t)}\big) \le K_{1,2}\big((x^*(x_n)), \sqrt{p}\big).$$

Finally, we obtain

$$\begin{aligned}\frac{1}{e} K^w_{1,2}\big((x_n), \sqrt{p}\big) &\le \sup_{\|x^*\|\le 1}\Big\{\sup_{0<t<1} K_{1,2}\big((x^*(x_n)), \sqrt{\log(1/t)}\big)\Big\}\\ &\le K^w_{1,2}\big((x_n), \sqrt{p}\big),\end{aligned}$$

which gives the desired result. □

Our final application is to the calculation of the Orlicz norms $\|S\|_{\psi_q}$ for $2 < q < \infty$. The proof will use the scalar version of the result, which was obtained by Rodin and Semyonov [8] (see also [7]). (Recall that by a result of Kwapień [5], $\|S\|_{\psi_q} \approx \mathbb{E}\|X\|$ in the range $0 < q \le 2$.)

COROLLARY 4. *Let $X = \sum \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $2 < q < \infty$, we have*

$$\|S\|_{\psi_q} \approx \mathbb{E}\|X\| + \sup_{\|x^*\|\le 1} \big\|(x^*(x_n))\big\|_{p,\infty},$$

where $1/p + 1/q = 1$ and S denotes $\|X\|$. The implied constant depends only on q.

PROOF. It is easily verified that $\|f\|_{\psi_q} \approx \sup_{0<t<1}(\log(1/t))^{-1/q} f^*(t)$. Hence, by Corollary 1, we have

$$\begin{aligned}
\|S\|_{\psi_q} &\approx \mathbb{E}\|X\| + \sup_{0<t<1} (\log(1/t))^{-1/q} K^w_{1,2}((x_n), t) \\
&\approx \mathbb{E}\|X\| + \sup_{0<t<1} \left\{ (\log(1/t))^{-1/q} \sup_{\|x^*\|\le 1} K_{1,2}((x^*(x_n)), t) \right\} \\
&\approx \mathbb{E}\|X\| + \sup_{\|x^*\|\le 1} \left\{ \sup_{0<t<1} (\log(1/t))^{-1/q} K_{1,2}((x^*(x_n)), t) \right\} \\
&\approx \mathbb{E}\|X\| + \sup_{\|x^*\|\le 1} \left\| \sum \varepsilon_n x^*(x_n) \right\|_{\psi_q} \\
&\approx \mathbb{E}\|X\| + \sup_{\|x^*\|\le 1} \|(x^*(x_n))\|_{p,\infty},
\end{aligned}$$

where the last line follows from the result of Rodin and Semyonov [8]. □

2. Proof of the main result. The principal ingredient in the proof of the main theorem is the following deviation inequality of Talagrand [9].

THEOREM A. *Let $X = \sum_{n=1}^N \varepsilon_n x_n$ be a finite Rademacher series in a Banach space and let M be a median of $\|X\|$. Then, for $t > 0$, we have*

$$P\left(\left| \left\| \sum_{n=1}^N \varepsilon_n x_n \right\| - M \right| > t \right) \le 4e^{-t^2/8\sigma^2},$$

where $\sigma = \ell_2^w((x_n)_{n=1}^N)$.

LEMMA 1. *Let $X = \sum_{n=1}^N \varepsilon_n x_n$ be a finite Rademacher series in a Banach space E. Then, for $t > 0$, we have*

$$P\Big(\|X\| > 2\mathbb{E}\|X\| + 3K\big((x_n)_{n=1}^N, t; \ell_1^w(E), \ell_2^w(E)\big)\Big) \le 4e^{-t^2/8}.$$

PROOF. It follows from Theorem A that, for all $y_1, \dots, y_N$ in E, we have

$$P\left(\left\| \sum \varepsilon_n y_n \right\| > 2\mathbb{E}\left\| \sum \varepsilon_n y_n \right\| + t\ell_2^w((y_n)) \right) \le 4e^{-t^2/8}. \tag{4}$$

On the other hand, since $\max\|\sum \varepsilon_n y_n\| = \ell_1^w((y_n))$, we have the trivial estimate

$$P\left(\left\| \sum \varepsilon_n y_n \right\| > \ell_1^w((y_n)) \right) = 0. \tag{5}$$

Let $x_n = x_n^{(1)} + x_n^{(2)}$, for $1 \le n \le N$; let $X^{(1)} = \sum \varepsilon_n x_n^{(1)}$; and let $X^{(2)} = \sum \varepsilon_n x_n^{(2)}$. Then

$$\begin{aligned}
&\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}\|X^{(2)}\| \\
&\quad \le \ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}\|X^{(1)}\| + 2\mathbb{E}\|X\| \\
&\quad \le 3\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}\|X\| \\
&\quad \le 2\mathbb{E}\|X\| + 3\big(\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)}))\big).
\end{aligned}$$

Let Q denote $2\mathbb{E}\|X\| + 3(\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})))$. Then, by (4) and (5) and by the above inequality, we have

$$\begin{aligned} P(\|X\| > Q) &\le P\Big(\|X^{(1)}\| + \|X^{(2)}\| > \ell_1^w\big((x_n^{(1)})\big) + t\ell_2^w\big((x_n^{(2)})\big) + 2\mathbb{E}\|X^{(2)}\|\Big) \\ &\le P\Big(\|X^{(1)}\| > \ell_1^w\big((x_n^{(1)})\big)\Big) + P\Big(\|X^{(2)}\| > 2\mathbb{E}\|X^{(2)}\| + t\ell_2^w\big((x_n^{(2)})\big)\Big) \\ &\le 0 + 4e^{-t^2/8}. \end{aligned}$$

The desired conclusion now follows from the definition of the K-functional. □

LEMMA 2. *Let $x_1, \dots, x_N$ be elements of the Banach space ℓ_∞. Then, for $t > 0$, we have*

$$K\Big((x_n)_{n=1}^N, t; \ell_1^w(\ell_\infty), \ell_2^w(\ell_\infty)\Big) \le 2K_{1,2}^w\Big((x_n)_{n=1}^N, t\Big).$$

PROOF. For $1 \le n \le N$, let $x_n = (x_{n,j})_{j=1}^\infty \in \ell_\infty$. A simple convexity argument gives

$$\|(x_n)\|_{\ell_p^w(\ell_\infty)} = \sup_{1 \le j \le \infty} \left(\sum_{n=1}^N |x_{n,j}|^p \right)^{(1/p)}.$$

It follows that the mapping ϕ which associates an element $(y_n)_{n=1}^\infty \in \ell_p^w(\ell_\infty)$ with the element in $\ell_\infty(\ell_p)$ whose jth coordinate equals $(y_{n,j})_{n=1}^\infty$ is an isometry. Hence $K((x_n), t; \ell_1^w, \ell_2^w) = K(\phi((x_n)), t; \ell_\infty(\ell_1), \ell_\infty(\ell_2))$. Let $(y_n)_{n=1}^\infty \in \ell_\infty(\ell_2)$ and let $\varepsilon > 0$. For each n there exists a splitting $y_n = z_n^{(1)} + z_n^{(2)}$ such that

$$\Big\|\big(z_{n,j}^{(1)}\big)_{j=1}^\infty\Big\|_1 + t\Big\|\big(z_{n,j}^{(2)}\big)_{j=1}^\infty\Big\|_2 \le K_{1,2}\Big((y_{n,j})_{j=1}^\infty, t\Big) + \varepsilon.$$

It follows that

$$\begin{aligned} \big\|\big(z_n^{(1)}\big)\big\|_{\ell_\infty(\ell_1)} + t\big\|\big(z_n^{(2)}\big)\big\|_{\ell_\infty(\ell_2)} &= \sup_{1 \le n < \infty} \Big\|\big(z_{n,j}^{(1)}\big)_{j=1}^\infty\Big\|_1 + t \sup_{1 \le n < \infty} \Big\|\big(z_{n,j}^{(2)}\big)_{j=1}^\infty\Big\|_2 \\ &\le 2 \sup_{1 \le n < \infty} K_{1,2}\Big((y_{n,j})_{j=1}^\infty, t\Big) + 2\varepsilon \\ &\le 2K_{1,2}^w\big((y_n), t\big) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the result now follows from the definition of the K-functional. □

PROOF OF THE MAIN THEOREM. First we prove (1) for a finite Rademacher series $\sum_{n=1}^N \varepsilon_n x_n$. Since every separable Banach space embeds isometrically into ℓ_∞, we may assume that E is a closed subspace of ℓ_∞. Recall that $K_{1,2}^w((x_n), t)$ was defined as $\sup_{\|x^*\| \le 1} K_{1,2}((x^*(x_n)), t)$. By the Hahn–Banach theorem, the supremum is the same whether it is taken over elements of E^* or over elements of ℓ_∞^*. Hence (1) follows by combining Lemmas 1 and 2. The result for an infinite series follows from the result for $\sum_{n=1}^N \varepsilon_n x_n$ by taking the limit as $N \to \infty$. To prove (2), we use the result from [6] that there exists an absolute constant d such that

$$P\left(\sum \varepsilon_n a_n > dK_{1,2}((a_n), t)\right) \geq de^{-t^2/d},$$

for every sequence $(a_n) \in \ell_2$. Hence

$$\begin{aligned} P\left(\left\|\sum \varepsilon_n x_n\right\| > \frac{d}{2} K^w_{1,2}((x_n), t)\right) &\geq \inf_{\|x^*\| \leq 1} P\left(\left\|\sum \varepsilon_n x_n\right\| > dK_{1,2}((x^*(x_n)), t)\right) \\ &\geq \inf_{\|x^*\| \leq 1} P\left(\sum \varepsilon_n x^*(x_n) > dK_{1,2}((x^*(x_n)), t)\right) \\ &\geq de^{-t^2/d}. \end{aligned}$$

The Paley–Zygmund inequality now gives

$$\begin{aligned} &P\left(\|X\| > \frac{1}{2}\mathbb{E}\|X\| + \frac{d}{6} K^w_{1,2}((x_n), t)\right) \\ &\quad \geq \min\left(P\left(\|X\| > \frac{3}{4}\mathbb{E}\|X\|\right), P\left(\|X\| > \frac{d}{2} K^w_{1,2}((x_n), t)\right)\right) \\ &\quad \geq \min\left(\frac{1}{144}, de^{-t^2/d}\right). \end{aligned}$$

□

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