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ERGODICITY OF CRITICAL SPATIAL BRANCHING PROCESSES IN LOW DIMENSIONS

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We consider two critical spatial branching processes on $\mathbb{R}^d$: critical branching Brownian motion, and the Dawson–Watanabe process. A basic feature of these processes is that their ergodic behavior is highly dimension-dependent. It is known that in low dimensions, $d \le 2$, the unique invariant measure with finite intensity is δ_0, the unit point mass on the empty state. In high dimensions, $d \ge 3$, there is a one-parameter family of nondegenerate invariant measures. We prove here that for $d \le 2$, δ_0 is the *only* invariant measure. In our proof we make use of sub- and super-solutions of the partial differential equation $\partial u/\partial t = (1/2)\Delta u - bu^2$.

1. Introduction and main results. Critical branching Brownian motion and the Dawson–Watanabe process are two closely related models of random motion and branching. The basic ergodic theory for these processes is the same as that of a wide class of processes on $\mathbb{R}^d$ and $\mathbb{Z}^d$. This class includes: the voter model ([2], [4], [16]), branching random walks and general cluster models ([9], [19], [20], [22]), interacting diffusions ([3], [29]), generalized potlatch and smoothing [17], coupled random walk models [25], the binary contact path process [15] and the "linear" processes of Chapter 9 of [24]. For these processes, the long-term behavior differs sharply in low and high dimensions. (For models with "nearest neighbor interactions" low means $d \le 2$ and high means $d \ge 3$.)

In high dimensions, each process has a one-parameter family of invariant measures, indexed by the "intensity" or some preserved quantity of the system. The situation is different in low dimensions. In this case, the only extremal invariant measures for the voter model and the interacting diffusions of [3] are "degenerate" ones, point masses on traps. This is also true, *subject to certain side conditions*, for the other models listed above. These side conditions are used to overcome the fact that for these models, either the coordinate functions or the number of particles in bounded sets can be unbounded. (In contrast, the voter model and the diffusions of [3] have state space $I^{\mathbb{Z}^d}$, I compact.) Nevertheless, it has been conjectured for most models (see, e.g., [5], [7], [10], [17], [23] and [24]) that in the low dimensional setting *there are no*

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nondegenerate invariant measures, with or without side conditions. In this paper we resolve this question for the two processes we consider here. We begin by defining them.

Critical branching Brownian motion η_t is a system of particles which undergo random motion and branching on $\mathbb{R}^d$ according to the following rules.

- Each particle lives an exponentially distributed lifetime with parameter $2b$.
- At the end of its lifetime, a particle disappears and is replaced by zero or two particles, each possibility occurring with equal probability.
- During its lifetime, a particle moves according to standard Brownian motion.
- All particle lifetimes, motions and branching are independent of one another.

The parameter b is a positive, finite constant. It is convenient to view η_t as a measure on $\mathbb{R}^d$. Thus, if the particles of η_t are located at $\mathrm{r}_1(t), \mathrm{r}_2(t), \ldots$, then

$$\int f(x)\eta_t(dx) = \sum_i f(\mathrm{r}_i(t)).$$

For $x \in \mathbb{R}^d$, let η_t^x denote the process starting from a single particle at x at time zero.

A construction of the "single ancestor" processes η_t^x can be found in [11]. Systems of infinitely many branching Brownian motions are constructed by superposition. A formal treatment can be found in [8]; see also [9]. For our purposes, the following informal description will suffice. Let ζ be a configuration of particles in $\mathbb{R}^d$, with particle locations $\{\mathrm{r}_i\}$. We assume that initial configurations ζ are *locally finite*, which means that for every compact $K \subset \mathbb{R}^d$, $\zeta(K) < \infty$. Given ζ, with particle locations $\{\mathrm{r}_i\}$, and a family of independent single ancestor processes $\{\eta_t^{\mathrm{r}_i}\}$, we define η_t^ζ, the process with initial configuration ζ, by

$$\eta_t^\zeta = \sum_i \eta_t^{\mathrm{r}_i}. \tag{1.1}$$

This also works if ζ is random: We simply require that ζ and the family $\{\eta_t^{\mathrm{r}_i}\}$ be independent. [Note that if $\zeta(K) = \infty$ for some compact K, then $P(\eta_t^\zeta(U) = \infty) = 1$ for all $t > 0$ and open sets U.]

The Dawson–Watanabe process $\hat{\eta}_t$ is the "diffusion limit" of critical branching Brownian motion obtained by speeding up the branching rate, decreasing the mass of particles and packing more particles into less space. If we let ${}^n\eta_t$ denote branching Brownian motion with lifetime parameter $2bn$, and suppose that $(1/n)({}^n\eta_0)$ converges to a measure $\hat{\eta}_0$ as $n \to \infty$, then $(1/n)({}^n\eta_t)$ converges to a measure-valued diffusion $\hat{\eta}_t$, the Dawson–Watanabe process. We will not give the details of this construction, but refer the reader to [26], [5], [6] and [18] for extensive treatments of the construction of $\hat{\eta}_t$. In Section 2, we will give the Laplace functional of $\hat{\eta}_t$, which is enough to justify the calculations we need to make.

Before proceeding further, we review the basic ergodic theory of η_t and $\hat{\eta}_t$. Let $|\cdot|$ denote Lebesgue measure, $\mathscr{L}$ denote law, and $\Rightarrow$ denote weak convergence. A measure μ has *finite intensity* if there exists $\theta < \infty$ such that

$\int \zeta(E)\,d\mu(\zeta) \le \theta|E|$ for all measurable $E \subset \mathbb{R}^d$. Let ζ_t be either η_t or $\hat{\eta}_t$. A measure μ is an *invariant measure* for ζ_t if it is concentrated on locally finite configurations (measures) and has the property that $\mathscr{L}(\zeta_0) = \mu$ implies $\mathscr{L}(\zeta_t) = \mu$ for all $t \ge 0$. Let δ_0 denote the unit point mass on the empty configuration or measure.

THEOREM 0. (a) *Assume $d \le 2$. If $\mathscr{L}(\zeta_0)$ has finite intensity, then*

$$\mathscr{L}(\zeta_t) \quad \Rightarrow \quad \delta_0 \quad \text{as } t \to \infty. \tag{1.2}$$

(b) *Assume $d \ge 3$. There is a one-parameter family of invariant measures $\{\nu_\theta,\ 0 \le \theta < \infty\}$. The measures ν_θ are translation invariant, $\int \zeta(K)\,d\nu_\theta(\zeta) = \theta|K|$, and for compact K_1, K_2, $\int \zeta(K_1)\zeta(x + K_2)\,d\nu_\theta(\zeta) \to \theta^2|K_1|\,|K_2|$ as $x \to \infty$. Moreover, if $\mathscr{L}(\zeta_0)$ is translation invariant, ergodic and $E\zeta_0(K) = \theta|K|$, then*

$$\mathscr{L}(\zeta_t) \Rightarrow \nu_\theta \quad \text{as } t \to \infty. \tag{1.3}$$

Proofs of the various statements in Theorem 0 can be found in [8], [9] and [12] for η_t, and in [5] and [6] for $\hat{\eta}_t$.

It follows easily from (1.2) that for $d \le 2$, if μ is an invariant measure with finite intensity, then necessarily $\mu = \delta_0$. However, this does not rule out the possibility of an invariant measure with *infinite intensity*. For instance, it is conceivable that the tendency toward local extinction might somehow be balanced by a densely populated initial configuration, resulting in a nondegenerate invariant measure. (See [28], where this actually occurs for a *subcritical* model in which branching and spatial motion are highly dependent.) This question was explored (but not settled) for $\hat{\eta}_t$ in [7]. Theorem 3.1 in [7] says the following for $d = 1$ and $\hat{\eta}_0$ a stable random measure with exponent γ, $0 < \gamma < 1$. If $\gamma > 1/2$, then $\hat{\eta}_t(E) \to_p 0$ as $t \to \infty$ for every bounded Borel set E, whereas if $\gamma < 1/2$, $\hat{\eta}_t(E) \to_p \infty$ for every open set E. ($\to_p$ denotes convergence in probability.) If $\gamma = 1/2$, then $\hat{\eta}_t(E) \to_p \hat{\eta}_\infty(E)$ as $t \to \infty$, where $P(\eta_\infty(E) = 0) = P(\eta_\infty(E) = \infty) = 1/2$ for bounded open sets E. This shows that no (hypothetical) nondegenerate invariant measure for $\hat{\eta}_t$ can have a stable random measure in its domain of attraction.

To state our results we use the following terminology. A *finite ball* $B \subset \mathbb{R}^d$ is a ball in the Euclidean norm of finite, positive radius; we reserve B for such a ball. The family $\{\zeta_t(B)\}$ is *stochastically bounded* as $t \to \infty$ if $\limsup_{t \to \infty} P(\zeta_t(B) \ge M) \to 0$ as $M \to \infty$. Moreover, we say that:

- ζ_t *becomes extinct* if for every B, $\zeta_t(B) \to_p 0$ as $t \to \infty$;
- ζ_t is *unstable* if for every B, $\{\zeta_t(B)\}$ is not stochastically bounded as $t \to \infty$;
- ζ_t *explodes* if for every B, $\zeta_t(B) \to_p \infty$ as $t \to \infty$.

[If $\zeta_s(B) = \infty$ at some $s < \infty$, then the same is true at all $t > s$, so this possibility is included in the above scheme. There is an interesting remark on page 794 of [26] concerning this phenomenon.] Also, define

$$\phi(x,r) = \begin{cases} \dfrac{1}{r}\exp\left(\dfrac{-x^2}{r}\right), & d = 1, \\ \dfrac{1}{r\log r}\exp\left(\dfrac{-|x|^2}{r}\right), & d = 2. \end{cases} \tag{1.4}$$

We will show that no matter what the initial state, extinction and instability are the only possibilities. Hence, there can be only one invariant measure, the degenerate one concentrated on the empty state. We also give a necessary and sufficient condition for ζ_t to explode.

THEOREM 1. *Assume $d \le 2$, and ζ_t is either critical branching Brownian motion or the Dawson–Watanabe process. The following dichotomy holds: If*

$$\int \phi(x,r)\zeta_0(dx) \to_p 0 \quad \textit{as } r \to \infty, \tag{1.5}$$

then ζ_t becomes extinct; otherwise ζ_t is unstable. Furthermore, ζ_t explodes if and only if

$$\int \phi(x,r)\zeta_0(dx) \to_p \infty \quad \textit{as } r \to \infty. \tag{1.6}$$

COROLLARY . *For $d \le 2$, δ_0 is the only invariant measure.*

The proof of Theorem 1 for both processes centers around analyzing the heat equation given in (2.1) below. The solutions of (2.1) correspond in a simple way to the Laplace functionals of η_t and $\hat{\eta}_t$. In Section 2, Proposition 1 gives sub- and super-solutions for (2.1). In Section 3 we derive the proof of Theorem 1 using the estimates of Proposition 1. We note that although only binary critical branching Brownian motion is considered here, Theorem 1 holds just as well when the branching mechanism has a finite second moment (and is critical). The argument remains the same, although the estimates in Section 2 become somewhat less explicit. Also note that a simple computation using Theorem 1 exhibits the three types of behavior produced by the initial stable random measures treated in [7].

2. Probability and pde estimates. In this section we consider the finite processes η_t^x and $\hat{\eta}_t^x$, and concentrate on obtaining useful estimates of their Laplace functionals. Fundamental for each process is the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - bu^2. \tag{2.1}$$

For critical branching Brownian motion, the function $u(x,t)$ determined by

$$E\exp(-\lambda\eta_t^x(B)) = 1 - u(x,t), \qquad \lambda \in [0,\infty), \tag{2.2a}$$

is the solution of (2.1) satisfying

$$u(x,0) = (1 - e^{-\lambda})1_B(x), \qquad x \in \mathbb{R}^d. \tag{2.2b}$$

(As before, B denotes a finite ball.) For the Dawson–Watanabe process, the function $u(x,t)$ determined by

$$E\exp(-\lambda\hat{\eta}_t^x(B)) = \exp(-u(x,t)), \qquad \lambda \in [0,\infty), \tag{2.3a}$$

is the solution of (2.1) satisfying

$$u(x,0) = \lambda 1_B(x), \qquad x \in \mathbb{R}^d. \tag{2.3b}$$

These facts are well known (see [5], [6], [11] and [18]).

We obtain estimates of the functions $u(x,t)$ by utilizing the standard method of sub- and super-solutions. This technique has been used to study the asymptotic behavior of $u(x,t)$ in both probabilistic ([21]) and analytic ([1], [13] and [14]) contexts. Sub- and super-solutions to (2.1) are given in Lemmas 1 and 2, and are combined in Proposition 1 to provide the fundamental estimates we will use to prove Theorem 1. We start with super-solutions.

LEMMA 1. *Let A be a constant, $A \geq 2e^4/b$, and set*

$$\bar{u}(x,t) = \begin{cases} \dfrac{A}{t}\exp\left\{\dfrac{-x^2}{4t}\right\}, & d = 1, \\ \dfrac{A}{t\log t}\exp\left\{-\left(1 - \dfrac{1}{\log t}\right)\dfrac{|x|^2}{2t}\right\}, & d = 2. \end{cases} \tag{2.4}$$

Then

$$\frac{\partial \bar{u}}{\partial t} - \frac{1}{2}\Delta\bar{u} + b\bar{u}^2 \geq 0 \tag{2.5}$$

for all $x \in \mathbb{R}^d$ and $t \geq e^4$.

PROOF. Suppose that $d = 1$. Direct computation yields

$$\frac{\partial \bar{u}}{\partial t} - \frac{1}{2}\Delta\bar{u} + b\bar{u}^2 = \bar{u}\left(-\frac{3}{4t} + \frac{x^2}{8t^2} + \frac{bA}{t}e^{-x^2/4t}\right).$$

By considering separately the two cases $x^2 \geq 8t$ and $x^2 \leq 8t$, and using the fact that $A \geq e^2/b$, it is easy to see that the right-hand side above is always nonnegative. Now suppose $d = 2$. Direction computation yields

$$\begin{aligned} &\frac{\partial \bar{u}}{\partial t} - \frac{1}{2}\Delta\bar{u} + b\bar{u}^2 \\ &\quad = \bar{u}\left(-\frac{2}{t\log t} + \frac{|x|^2(1 - 2/\log t)}{2t^2\log t} + \frac{bA}{t\log t}\exp\left\{-\left(1 - \frac{1}{\log t}\right)\frac{|x|^2}{2t}\right\}\right) \\ &\quad \geq \bar{u}\left(-\frac{2}{t\log t} + \frac{|x|^2}{4t^2\log t} + \frac{bA}{t\log t}e^{-|x|^2/2t}\right), \end{aligned}$$

where we have used the assumption $t \geq e^4$. Again, considering separately the cases $|x|^2 \geq 8t$ and $|x|^2 \leq 8t$, using the assumption $A \geq 2e^4/b$, we find that the right-hand side above is always nonnegative. □

The argument for sub-solutions is even easier, so we omit the proof of the following:

LEMMA 2. *Let a be a constant,* $0 < a < (2b)^{-1}$, *and let*

$$\underline{u}(x,t) = \begin{cases} \dfrac{a}{t}\exp\left\{\dfrac{-x^2}{2t}\right\}, & d = 1, \\ \dfrac{a}{t\log t}\exp\left\{\dfrac{-|x|^2}{2t}\right\}, & d = 2. \end{cases} \tag{2.6}$$

Then

$$\frac{\partial \underline{u}}{\partial t} - \frac{1}{2}\Delta\underline{u} + b\underline{u}^2 \leq 0 \tag{2.7}$$

for all $x \in \mathbb{R}^d$ *and* $t > 1$.

Proposition 1 is the main estimate. Recall that ϕ is defined in (1.4).

PROPOSITION 1. *Assume* $u(x,t)$ *solves* (2.1) *in* $d = 1,2$, *with* $u(x,0) = r1_B(x)$, *and* $r > 0$. *Then there are positive finite constants* a, A *depending on* r *such that*

$$a\phi(x,t) \leq u(x,t) \leq A\phi(x,8t) \tag{2.8}$$

for all $x \in \mathbb{R}^d$ *and* $t \geq e^4$.

PROOF. *The upper bound*. Let $\bar{u}(x,t)$ be as in Lemma 1, with A large enough so that

$$u(x,0) \leq \bar{u}(x,e^4).$$

By Lemma 1 and a standard maximum principle (see [27]), it follows that

$$u(x,t) \leq \bar{u}(x,t+e^4), \qquad x \in \mathbb{R}^d, t \geq 0.$$

For $t \geq e^4$, this implies (2.8).

The lower bound. Consider η_t^x, and let $\lambda = -\log(1-(r \wedge 1/2))$. Up until the death of the original particle, η_t^x consists of a single Brownian motion W_t^x starting at x. Furthermore,

$$\begin{aligned} u(x,t) &= 1 - E\exp(-\lambda\eta_t^x(B)) \geq \left(r \wedge \tfrac{1}{2}\right)P(\eta_t^x(B) = 1) \\ &\geq \left(r \wedge \tfrac{1}{2}\right)e^{-2bt}P(W_t^x \in B). \end{aligned}$$

So for $\underline{u}(x,t)$ defined as in Lemma 2 with $a > 0$ small enough,

$$u(x,4) \geq \underline{u}(x,2), \qquad x \in \mathbb{R}^d. \tag{2.9}$$

Together with Lemma 2 and the maximum principle, this implies that

$$u(x,t) \ge \underline{u}(x,t-2), \qquad x \in \mathbb{R}^d,\ t \ge 4.$$

For $t \ge 4$, this implies (2.8). □

There is a trivial upper bound on the functions $u(x,t)$ which is useful. Suppose $u(x,t)$ solves (2.1) with $u(x,0) = r1_B(x)$, and $r \in [0,1]$, $\lambda = -\log(1-r)$. Then $u(x,t) = 1 - E\exp(-\lambda\eta_t^x(B))$, and so

$$u(x,t) \le P\big(\eta_t^x(B) > 0\big) \le P\big(\eta_t^x(\mathbb{R}^d) > 0\big) = \frac{1}{1+bt}. \tag{2.10}$$

We also recall the formulas for the first two moments of $\eta_t(B)$ and $\hat{\eta}_t(B)$ (see [6] and [12]). Let $p_t(x,dy)$ denote the transition kernel of standard Brownian motion, let

$$m_t(x,B) = \int_{\mathbb{R}^d}\int_0^t p_{t-s}(x,dy)\,p_s^2(y,B)\,ds,$$

and let ζ_t be either η_t or $\hat{\eta}_t$. Then

$$E\zeta_t^x(B) = p_t(x,B) \tag{2.11}$$

and

$$E\zeta_t^x(B)^2 = p_t^2(x,B) + bm_t(x,B). \tag{2.12}$$

We will need the following modest extensions of (2.11) and (2.12). If Δ is an event depending only on ζ_0, then

$$E\zeta_t(B)1_\Delta = E1_\Delta\int_{\mathbb{R}^d} p_t(x,B)\zeta_0(dx) \tag{2.13}$$

and

$$E\zeta_t(B)^2 1_\Delta = E\left(1_\Delta\left(\int_{\mathbb{R}^d} p_t(x,B)\zeta_0(dx)\right)^2 + b1_\Delta\int_{\mathbb{R}^d} m_t(x,B)\zeta_0(dx)\right). \tag{2.14}$$

3. Proof of Theorem 1. We begin by observing that independence allows us to compute Laplace functionals of the infinite process in terms of the Laplace functionals of the finite processes. Consider branching Brownian motion η_t, with initial random particle locations $\{\mathrm{r}_i\}$. Using (1.1),

$$\begin{aligned} E\exp(-\lambda\eta_t(B)) &= E\exp\Big(-\lambda\sum_i \eta_t^{\mathrm{r}_i}(B)\Big) = E\prod_i E\big[\exp(-\lambda\eta_t^{\mathrm{r}_i}(B))\big|\eta_0\big] \\ &= E\prod_i(1-u(\mathrm{r}_i,t)) = E\exp\sum_i\log(1-u(\mathrm{r}_i,t)), \end{aligned}$$

where $u(x,t)$ satisfies (2.1) and (2.2b). Thus,

$$E\exp(-\lambda\eta_t(B)) = E\exp\int\log(1-u(x,t))\eta_0(dx). \tag{3.1}$$

A similar formula holds for the Dawson–Watanabe process (see [5] and [6]):

$$E\exp(-\lambda\hat{\eta}_t(B)) = E\exp\left(-\int u(x,t)\hat{\eta}_0(dx)\right), \tag{3.2}$$

where $u(x,t)$ satisfies (2.1) and (2.3b).

In order to treat η_t and $\hat{\eta}_t$ simultaneously, we establish the following notation for the remainder of this section. B is a finite ball, $\lambda \in (0,\infty)$, ζ_t denotes either of η_t or $\hat{\eta}_t$, $u(x,t)$ is the solution of (2.1) satisfying either (2.2b) (for $\zeta_t = \eta_t$) or (2.3b) (for $\zeta_t = \hat{\eta}_t$), and $v(x,t)$ is given by

$$v(x,t) = \begin{cases} -\log(1-u(x,t)), & \zeta_t = \eta_t, \\ u(x,t), & \zeta_t = \hat{\eta}_t. \end{cases}$$

With these definitions we may combine (3.1) and (3.2) into

$$E\exp(-\lambda\zeta_t(B)) = E\exp\left(-\int v(x,t)\zeta_0(dx)\right). \tag{3.3}$$

Note that (2.10) and the inequality $s \le -\log(1-s) \le 2s$ for small positive s imply that for large t,

$$u(x,t) \le v(x,t) \le 2u(x,t), \qquad x \in \mathbb{R}^d. \tag{3.4}$$

We break the proof of Theorem 1 into three parts, those of extinction, explosion and instability.

Extinction of ζ_t. If $\int\phi(x,t)\zeta_0(dx) \to_p 0$ as $t \to \infty$, then ζ_t becomes extinct.

PROOF. By Proposition 1 there exists $A < \infty$ such that

$$u(x,t) \le A\phi(x,8t), \qquad x \in \mathbb{R}^d,$$

for all large t. From this and (3.4) it follows that

$$\int v(x,t)\zeta_0(dx) \le 2A\int\phi(x,8t)\zeta_0(dx) \to_p 0 \quad \text{as } t \to \infty.$$

By (3.3), $E\exp(-\zeta_t(B)) \to 1$. This shows ζ_t becomes extinct. □

Explosion of ζ_t. ζ_t explodes if and only if $\int\phi(x,t)\zeta_0(dx) \to_p \infty$ as $t \to \infty$.

PROOF. ζ_t explodes if and only if $E\exp(-\zeta_t(B)) \to 0$ as $t \to \infty$. By (3.3), this is equivalent to $\int v(x,t)\zeta_0(dx) \to_p \infty$ as $t \to \infty$. But Proposition 1 and (3.4) imply that for some $0 < a$, $A < \infty$, for all large t,

$$a\int\phi(x,t)\zeta_0(dx) \le \int v(x,t)\zeta_0(dx) \le 2A\int\phi(x,8t)\zeta_0(dx).$$

This implies the above equivalence. □

Instability of ζ_t. If $\int\phi(x,t)\zeta_0(dx) \not\to_p 0$ as $t\to\infty$, then ζ_t is unstable.

STEP 1. If $\int\phi(x,t)\zeta_0(dx)$ is not stochastically bounded as $t\to\infty$, then ζ_t is unstable.

PROOF. Let $\Gamma_t(M)=\{\int\phi(x,t)\zeta_0(dx)\ge M)\}$, $M>0$. By assumption there exists $\delta>0$ such that for every finite M,

$$\limsup_{t\to\infty} P(\Gamma_t(M))\ge\delta.$$

By Proposition 1 and (3.4) there exists $a>0$ (depending on λ), such that

$$v(x,t)\ge a\phi(x,t),\qquad x\in\mathbb{R}^d$$

for all sufficiently large t. For such t,

$$\begin{aligned} E\exp\left(-\int v(x,t)\zeta_0(dx)\right) &\le E\exp\left(-a\int\phi(x,t)\zeta_0(dx)\right)\\ &\le 1-P(\Gamma_t(M))+e^{-aM}P(\Gamma_t(M)). \end{aligned}$$

From this and (3.3) we obtain

$$\liminf_{t\to\infty} E\exp(-\lambda\zeta_t(B))\le 1-\delta+e^{-aM}.$$

Let $M\to\infty$ and then $\lambda\to 0$ to conclude that ζ_t is unstable. □

Now let $I(t)=\{x: |x|^2\le t\}$. We note that Step 1 implies that if there exists $\varepsilon>0$ such that for all $N<\infty$,

$$\limsup_{t\to\infty} P\left(\int_{I^c(Nt)}\phi(x,t)\zeta_0(dx)>\varepsilon\right)\ge\varepsilon,$$

then ζ_t is unstable. This follows from the observation that $\phi(x,2t)\ge(1/4)e^{N/2}\phi(x,t)$ for $x\in I^c(Nt)$ and $t\ge 2$, and hence

$$\int\phi(x,2t)\zeta_0(dx)\ge\tfrac{1}{4}e^{N/2}\int_{I^c(Nt)}\phi(x,t)\zeta_0(dx).$$

So we can assume that for every $\varepsilon>0$ there exists an $N<\infty$ such that

$$\limsup_{t\to\infty} P\left(\int_{I^c(Nt)}\phi(x,t)\zeta_0(dx)>\varepsilon\right)<\varepsilon.$$

Thus, the proof of the instability of ζ_t will be complete with the following:

STEP 2. If for some finite N, $\int_{I(Nt)}\phi(x,t)\zeta_0(dx)\not\to_p 0$ as $t\to\infty$, then ζ_t is unstable.

PROOF. Let $G(t) = t$ for $d = 1$, $G(t) = t \log t$ for $d = 2$. Note that

$$e^{-N} \le \phi(x,t)G(t) \le 1, \qquad x \in I(Nt). \tag{3.5}$$

Also, set

$$\Delta_t = \{\varepsilon G(t) \le \zeta_0(I(Nt)) \le \varepsilon^{-1}G(t)\}.$$

On account of Step 1, we may assume $\int \phi(x,t)\zeta_0(dx)$ is stochastically bounded as $t \to \infty$. Plugging the upper and lower bounds from (3.5) into $\int_{I(Nt)}\phi(x,t)\zeta_0(dx)$, one therefore has, for appropriate $\varepsilon > 0$ and $t(n) \to \infty$, $\inf_n P(\Delta_{t(n)}) > 0$. Now define the family of processes $\{\tilde{\zeta}_{s,t},\ t \ge 0\}$, $s > 0$, branching Brownian motion or the Dawson–Watanabe process, by setting the initial states equal to $\tilde{\zeta}_{s,0}(\cdot) = \zeta_0(\cdot \cap I(Ns))$. [This simply ignores the initial particles or mass of ζ_0 not in $I(Ns)$.] Set $Z_t = \tilde{\zeta}_{t,t}(B)1_{\Delta_t}$, and observe that for any K,

$$P(\zeta_t(B) \ge K) \ge P\big(\tilde{\zeta}_{t,t}(B) \ge K\big) \ge P(Z_t \ge K). \tag{3.6}$$

The point of this construction is that we can now show ζ_t is unstable by showing Z_t is not stochastically bounded as $t \to \infty$, which we do with moment calculations.

Indeed, we will show there are constants $0 < c$, $C < \infty$, depending only on B, such that for $t \ge 1$,

$$EZ_t \ge \begin{cases} c\varepsilon t^{1/2}P(\Delta_t), & d = 1, \\ c\varepsilon(\log t)P(\Delta_t), & d = 2, \end{cases} \tag{3.7}$$

and

$$EZ_t^2 \le \begin{cases} C\varepsilon^{-2}tP(\Delta_t), & d = 1 \\ C\varepsilon^{-2}(\log t)^2P(\Delta_t), & d = 2. \end{cases} \tag{3.8}$$

To make use of these estimates we apply the inequality, easily derived with Hölder's inequality,

$$P(X \ge rEX) \ge (1-r)^2\frac{(EX)^2}{EX^2}, \qquad 0 \le r \le 1, \tag{3.9}$$

where X is nonnegative. Substituting (3.7) and (3.8) into (3.9) yields

$$P\left(Z_t \ge \frac{1}{2}EZ_t\right) \ge \frac{c^2\varepsilon^4}{4C}P(\Delta_t).$$

Since $\inf P(\Delta_{t(n)}) > 0$, (3.6) implies that

$$\liminf_{n\to\infty} P\big(\zeta_{t(n)}(B) \ge (1/2)EZ_{t(n)}\big) > 0.$$

But (3.7) shows $EZ_{t(n)} \to \infty$ as $n \to \infty$, so ζ_t is unstable.

The proofs of the moment estimates are as follows. From (2.13) and the definition of Δ_t,

$$\begin{aligned} EZ_t &= E\tilde{\zeta}_t(B)1_{\Delta_t} \\ &= E1_{\Delta_t}\int_{I(Nt)} p_t(x,B)\zeta_0(dx) \\ &\geq E\big[1_{\Delta_t}\zeta_0(I(Nt))\big] \inf_{x\in I(Nt)} p_t(x,B) \\ &\geq \varepsilon G(t)P(\Delta_t) \inf_{x\in I(Nt)} p_t(x,B). \end{aligned}$$

From this it readily follows that (3.7) must hold for some positive c. Similarly, using (2.14),

$$\begin{aligned} EZ_t^2 &= E\tilde{\zeta}_t(B)^2 1_{\Delta_t} \\ &= E\left(1_{\Delta_t}\left(\int_{I(Nt)} p_t(x,B)\zeta_0(dx)\right)^2 + b1_{\Delta_t}\int_{I(Nt)} m_t(x,B)\zeta_0(dx)\right) \\ &\leq \left(\varepsilon^{-1}G(t)\sup_x p_t(x,B)\right)^2 P(\Delta_t) + \varepsilon^{-1}bG(t)P(\Delta_t)\sup_x m_t(x,B). \end{aligned}$$

Furthermore,

$$\begin{aligned} m_t(x,B) &\leq \int_{\mathbb{R}^d}\int_0^t p_{t-s}(x,dy)p_s(y,B)\sup_z p_s(z,B)\,ds \\ &= p_t(x,B)\int_0^t \sup_z p_s(z,B)\,ds, \end{aligned}$$

from which it follows that for some finite C and $t \geq e$,

$$m_t(x,B) \leq \begin{cases} C, & d=1, \\ Ct^{-1}\log t, & d=2. \end{cases}$$

Using this bound in the above estimate for EZ_t^2 yields (3.8). □

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