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STABLE LIMITS FOR ASSOCIATED RANDOM VARIABLES

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We consider a stationary sequence of associated real random variables and state conditions which guarantee that partial sums of this sequence, when properly normalized, converge in distribution to a stable, non-Gaussian limit.

Limit theorems for jointly stable and associated random variables are investigated in detail. In the general case we assume that finite-dimensional distributions belong to the domain of attraction of multidimensional strictly stable laws and that there is a bound on the positive dependence given by finiteness of an analog to the lag covariance series.

1. Introduction. Random variables $X_1, X_2, \dots, X_n$ are *associated* if

$$\operatorname{cov}(f(X_1, X_2, \dots, X_n), g(X_1, X_2, \dots, X_n)) \geq 0 \tag{1}$$

for each pair of functions, $f, g: \mathbb{R}^n \to \mathbb{R}^1$, which are nondecreasing in each coordinate and for which the above covariance exists. An infinite collection of random variables is associated if every finite subset of that collection consists of associated random variables. These definitions are due to Esary, Proschan and Walkup (1967) (which is also our main reference for basic properties of associated random variables) and seem to be the description of positive dependence phenomena most appropriate to reliability theory [Barlow and Proschan (1981)], statistical physics [Newman (1980), (1983)] and percolation theory [Cox and Grimmett (1984)].

Let $X_1, X_2, \dots$ be associated and strictly stationary. In what follows S_n will always stand for $X_1 + X_2 + \cdots + X_n$. Suppose that $EX_1 = 0$, $EX_1^2 < +\infty$. The remarkable central limit theorem of Newman (1980) states that if

$$\sigma^2 = EX_1^2 + 2 \sum_{j=2}^{\infty} EX_1 X_j < +\infty, \tag{2}$$

then

$$\frac{S_n}{\sqrt{n}} \xrightarrow[\mathscr{D}]{} \mathscr{N}(0, \sigma^2) \quad \text{as } n \to +\infty. \tag{3}$$

This result inspired a series of limit theorems for associated random variables: the functional central limit theorem [Newman and Wright (1981)], the func-

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tional law of the iterated logarithm [Dabrowski (1985) and Dabrowski and Dehling (1988)], Berry–Esseen-type estimates [Wood (1983) and Dabrowski and Dehling (1988)], local limit theorems [Wood (1985)], the Glivenko–Cantelli lemma and invariance principles for empirical processes [Yu (1993a, b)], extensions to the nonstationary case [Cox and Grimmett (1984) and Birkel (1987)] and extensions to *weakly* associated sequences [Burton, Dabrowski and Dehling (1986)].

A natural question in this context is whether the central limit theorem holds for a stationary sequence of associated random variables whose common marginal distribution does not belong to the domain of attraction of a normal law, but rather to the domain of attraction of a p-stable limit distribution, $0 < p < 2$. The goal of this paper is to provide such a p-stable central limit theorem for stationary associated sequences. The principal difficulty is to find a suitable analog to (2) which preserves the simplicity of that condition.

While the general idea for such limit theorems remains unchanged—we need a bound on positive dependence similar to (2)—it is not clear which two-dimensional characteristics can replace covariances when variances are infinite and it is the regular variation of tail probabilities which is of basic importance. To make this evident and to exhibit the principal differences from the case where $p = 2$, our first results, Theorems 2.1, 2.2 and 2.3, examine the convergence of sums of *jointly p-stable* associated random variables with standard normalization $n^{1/p}$. Next, Theorem 2.8 is a direct analog of Newman's result for variables satisfying a condition like (2) and which satisfy an enhanced condition on p-stable domains of attraction. Theorem 2.13 improves this result for $0 < p < 1$.

The full statements of our results are given in the next section together with explanatory remarks and examples. The proofs of the first three theorems are given in Section 3, and those of Theorems 2.8 and 2.13 are found in Section 4.

2. Theorems. Random variables $X_1, X_2, \dots$ are jointly p-stable if for each $n \in \mathbb{N}$ there exists a finite Borel measure Γ_n on the unit sphere in $\mathbb{R}^n$,

$$\mathscr{S}^{n-1} = \left\{ \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} s_i^2 = 1 \right\}$$

and a vector $\mathbf{b}_n \in \mathbb{R}^n$ such that the characteristic function of $\mathbf{X}_n = (X_1, X_2, \dots, X_n)$ is of the form

$$E \exp i(\mathbf{t}, \mathbf{X}_n) = \exp\left(i(\mathbf{b}_n, \mathbf{t}) + \int_{\mathscr{S}^{n-1}} \int_0^\infty g(\mathbf{t}, \mathbf{s}, r) \frac{p\,dr}{r^{p+1}} \Gamma_n(d\mathbf{s}) \right). \tag{4}$$

Here

$$g(\mathbf{t}, \mathbf{s}, r) = \begin{cases} e^{i(\mathbf{t}, \mathbf{s})r} - 1, & \text{if } 0 < p < 1, \\ e^{i(\mathbf{t}, \mathbf{s})r} - 1 - i(\mathbf{t}, \mathbf{s}) r I(r \le 1), & \text{if } p = 1, \\ e^{i(\mathbf{t}, \mathbf{s})r} - 1 - i(\mathbf{t}, \mathbf{s}) r, & \text{if } 1 < p < 2. \end{cases} \tag{5}$$

That is, for each $n \in \mathbb{N}$ the joint distribution of $X_1, X_2, \ldots, X_n$ is p-stable. Write

$$\mathscr{L}(\mathbf{X}_n) = \gamma_p(\mathbf{b}_n, \Gamma_n).$$

Clearly, if $\{X_1, X_2, \ldots\}$ is also strictly stationary, then, for some $b \in \mathbb{R}^1$,

$$\mathbf{b}_n = \underbrace{(b, \ldots, b)}_{n \text{ times}}, \qquad n \in \mathbb{N}. \tag{6}$$

For one-dimensional stable distributions we have $\mathscr{S}^0 = \{-1, 1\}$, and we shall use the notation

$$\gamma_p(b, \Gamma(\{1\}), \Gamma(\{-1\})) \equiv \gamma_p(b, \Gamma).$$

Jointly stable random variables $X_1, X_2, \ldots, X_n$ are *strictly* p-stable, if either:

(a) $\mathbf{b}_n = 0$, $n \in \mathbb{N}$, when $p \neq 1$, or
(b) $\int_{\mathscr{S}^{n-1}} \mathbf{s}\Gamma_n(d\mathbf{s}) = 0$, $n \in \mathbb{N}$, when $p = 1$.

This holds if, for instance, Γ_n is a symmetric measure on $\mathscr{S}^{n-1}$. As in the case of independent summands, we have separate results for the three cases where $0 < p < 1$, $p = 1$ and $1 < p < 2$.

THEOREM 2.1. *Let $X_1, X_2, \ldots$ be stationary, associated and jointly p-stable, $0 < p < 1$. Then*

$$\frac{S_n}{n^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty, \tag{7}$$

where μ_∞ is a strictly p-stable distribution.

THEOREM 2.2. *Let $X_1, X_2, \ldots$ be stationary, associated and jointly 1-stable. Then there exist constants A_n such that*

$$\frac{S_n}{n} - A_n \sim X_1. \tag{8}$$

In particular, if Γ_n is symmetric for each $n \in \mathbb{N}$, then

$$\frac{S_n}{n} \sim X_1, \qquad n \in \mathbb{N}. \tag{9}$$

THEOREM 2.3. *Let $X_1, X_2, \ldots$ be stationary, associated and jointly p-stable, $1 < p < 2$, with two-dimensional distributions*

$$\mathscr{L}((X_1, X_k)) = \gamma_p((b, b), \Gamma_{\{1, k\}}). \tag{10}$$

If

$$\sum_{k=2}^{\infty} \int_{\mathscr{S}^1} s_1 s_2 \Gamma_{\{1, k\}}(d\mathbf{s}) < +\infty, \tag{11}$$

then

$$\frac{S_n - ES_n}{n^{1/p}} = \frac{S_n - nb}{n^{1/p}} \xrightarrow[\mathscr{D}]{} \mu_\infty, \tag{12}$$

where μ_∞ *is a nondegenerate strictly p-stable distribution.*

The proofs of these results rely on a very simple description of association for jointly p-stable variables due to Lee, Rachev and Samorodnitsky (1990). The measure Γ_n has to be concentrated on the "positive" and "negative" parts of $\mathscr{S}^{n-1}$, that is,

$$\Gamma_n\Big(\mathscr{S}^{n-1} \cap \big\{[0, +\infty)^n \cup (-\infty, 0]^n\big\}^c\Big) = 0. \tag{13}$$

This feature will enable us to characterize the limits in these theorems.

REMARK 2.4. Let X be strictly p-stable, $0 < p < 2$, and define $X_j \equiv X$, $j \in \mathbb{N}$. Then $X_1, X_2, \dots$ are stationary, associated, jointly strictly p-stable and

$$\frac{S_n}{n^{1/p}} = n^{1-(1/p)}X.$$

For $0 < p < 1$ we have $S_n/n^{1/p} \to_{\mathscr{D}} 0$, and the limit μ_∞ in (7) is degenerate. However, for $1 < p < 2$, $(S_n - ES_n)/n^{1/p} = S_n/n^{1/p}$ diverges. Consequently, restrictions like (11) cannot be completely omitted.

REMARK 2.5. Relation (9) is a generalization of the well-known property of Cauchy-distributed random variables [see Feller (1971), page 51], $\mathscr{L}(X + X) = \mathscr{L}(X) * \mathscr{L}(X)$. Both $X, X, \dots$ and the sequence of independent copies of X are associated.

REMARK 2.6. Let $\nu_{\{1,k\}}$ be the Lévy measure of $\mathscr{L}((X_1, X_k))$ and let $a > 0$. Under assumptions of Theorem 2.3, relation (11) is equivalent to

$$\sum_{k=2}^{\infty} \int_{\substack{|x_1| \le a \\ |x_2| \le a}} x_1 x_2 \nu_{\{1,k\}}(dx_1, dx_2) < +\infty. \tag{14}$$

REMARK 2.7. The integrals in (4) and (5) can be evaluated further [see, e.g., Weron (1984)], but the given form seems to be appropriate for associated variables. It provides a formula for the stable Lévy measure [see, e.g., Araujo and Giné (1980)].

Our next results on arbitrary associated sequences will require the concept of a domain of attraction of a jointly stable *sequence*. Let $\{X_j\}_{j \in \mathbb{N}}$ be an arbitrary strictly stationary sequence and let $\{Y_j\}_{j \in \mathbb{N}}$ be a jointly strictly p-stable strictly stationary sequence. We say that $\{X_j\}$ belongs to the domain of strict normal attraction of $\{Y_j\}$ and write $\{X_j\} \in \mathscr{D}_{s,n}(\{Y_j\})$, if for each $N \in \mathbb{N}$, the joint distribution of $\mathbf{Z}_N = (X_1, X_2, \dots, X_N)$ belongs to the domain of strict normal attraction of $\mathscr{L}(\mathbf{W}_N) = \mathscr{L}((Y_1, Y_2, \dots, Y_N))$, that is,

$$\frac{\mathbf{Z}_{N,1} + \mathbf{Z}_{N,2} + \cdots + \mathbf{Z}_{N,n}}{n^{1/p}} \underset{\mathscr{D}}{\to} (Y_1, Y_2, \dots, Y_N) \quad \text{as } n \to +\infty, \tag{15}$$

where $\mathbf{Z}_{N,1}, \mathbf{Z}_{N,2}, \dots, \mathbf{Z}_{N,n}, \dots$ are independent copies of $(X_1, X_2, \dots, X_N)$.

Let us sketch a proof of Newman's central limit theorem ($p = 2$) along the lines we intend to pursue. Suppose that $\{X_j\} \in \mathscr{D}_{s,n}(\{Y_j\})$, and for each $\lambda \in \mathbb{R}^1$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big| E\exp\{i\lambda S_n/n^{1/p}\} - \Big(E\exp\{i\lambda[n/m]^{-1/p}S_m/m^{4p}\}\Big)^{[n/m]}\Big| = 0. \tag{16}$$

By the definition of $\{X_j\} \in \mathscr{D}_{s,n}(\{Y_j\})$,

$$\Big(E\exp\{i\lambda[n/m]^{-1/p}S_m/m^{1/p}\}\Big)^{[n/m]} \to E\exp\{i\lambda T_m/m^{1/p}\} \quad \text{as } n \to +\infty, \tag{17}$$

where $T_m = Y_1 + Y_2 + \cdots + Y_m$. If, in addition,

$$\frac{T_m}{m^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty \quad \text{as } m \to +\infty, \tag{18}$$

then (16) implies

$$\frac{S_n}{n^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty \quad \text{as } n \to +\infty.$$

Note that association of $\{X_j\}$ implies association of $\{Y_j\}$. Thus, to check (18), we can apply Theorems 2.1, 2.2 and 2.3.

The above reasoning may be of some value provided there are tools with which we may verify (16). In the case $p = 2$, Newman and Wright (1981) used Newman's inequality to obtain the estimate

$$\begin{aligned} &\Big| E\exp\{i\lambda S_{m\cdot l}/\sqrt{m\cdot l}\} - \Big(E\exp\{i\lambda l^{-1/2}S_m/\sqrt{m}\}\Big)^l \Big| \\ &\quad \le \frac{\lambda^2}{2}\Big(\mathrm{Var}\big(S_{m\cdot l}/\sqrt{m\cdot l}\big) - \mathrm{Var}\big(S_m/\sqrt{m}\big)\Big). \end{aligned} \tag{19}$$

In our case such an inequality cannot be applied to (16) directly, since variances of X_j's are infinite for $p < 2$. However, we will show that a close analog to (19) remains the crucial condition in central limit theorems for associated sequences when $0 < p < 2$.

For an associated strictly stationary sequence $\{X_j\}_{j\in\mathbb{N}}$, define

$$H_{(X_i,X_j)}(x_i, x_j) = P(X_i \le x_i, X_j \le x_j) - P(X_i \le x_i)P(X_j \le x_j). \tag{20}$$

Fix $A > 0$, and define

$$I_p^A(X_i, X_j) = \sup_{a\ge A} a^{p-2}\int_{-a}^{a}\int_{-a}^{a} H_{(X_i,X_j)}(x,y)\,dx\,dy. \tag{21}$$

THEOREM 2.8. *Let $X_1, X_2, \dots$ be stationary and associated. If $\{X_j\} \in \mathscr{D}_{n,s}(\{Y_j\})$, where $\{Y_j\}_{j\in\mathbb{N}}$ is jointly strictly p-stable, $0 < p < 2$ (and Γ_n is symmetric if $p = 1$), and if for some $A > 0$,*

$$\sum_{k=2}^{\infty} I_p^A(X_1, X_k) < +\infty, \tag{22}$$

then there exists a strictly p-stable distribution μ_∞ such that both

$$\frac{X_1 + X_2 + \cdots + X_n}{n^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty \tag{23}$$

and

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{n^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty. \tag{24}$$

The proof of this result uses a generalization of Hoeffding's lemma due to Yu (1993a) and a careful analysis of (16). Moreover, the proof of (16) uses only the portion of (15) which relates X_1 to X_k for each k—that is, set $\mathbf{Z}_k = (X_1, X_k)$ in (15)—and does not require the full force of (15). The bivariate nature of this reduced condition is more in keeping with (2) than is (15), and is easier to verify.

REMARK 2.9. The covariance-like quantity $I_p^A(X_i, X_j)$ satisfies the Cauchy–Schwarz inequality

$$I_p^A(X_i, X_j) \le \sqrt{I_p^A(X_i, X_i)}\,\sqrt{I_p^A(X_j, X_j)}\,.$$

It follows from stationarity that $I_p^A(X_i, X_j)$ is finite if $I_p^A(X_i, X_i) < +\infty$. This is the case if, for example, the law of X_1 belongs to the domain of *normal* attraction of a p-stable law.

PROPOSITION 2.10. *If for some $K > 0$, $P(|X| \ge x) \le Kx^{-p}$, $x > 0$, then $I_p^A(X, X) \le 16(2 - p)^{-2}K < +\infty$.*

PROOF. We have

$$\begin{aligned} H_{(X,X)}(x,y) &= P(X \le x \wedge y)P(X > x \vee y) \\ &\le K \min\{|x \wedge y|^{-p}, |x \vee y|^{-p}\} \le K|x|^{-p/2}|y|^{-p/2}. \end{aligned}$$ □

REMARK 2.11. When additional information is available (e.g., the sequence is ψ-mixing), it may be possible to compute

$$c_p(X,Y) = \sup_{x,y\in\mathbb{R}^1} |x|^{p/2}|y|^{p/2}\left|H_{(X,Y)}(x,y)\right|. \tag{25}$$

Since $H_{(X,Y)}(x,y) = \operatorname{Cov}(I(X \le x), I(Y \le y))$, $c_p(X,Y)$ possesses essentially the same properties as $I_p^A(X, Y)$:

$$c_p(X, Y) \le \sqrt{c_p(X, X)}\sqrt{c_p(Y, Y)},$$

$$c_p(X, X) \le \sup_{x>0} x^p P(|X| \ge x).$$

Further,

$$I_p^A(X, Y) \le 16(2-p)^{-2} c_p(X, Y).$$

Hence in p-stable limit theorems we can replace (22) with

$$\sum_{k=2}^{\infty} c_p(X_1, X_k) < +\infty. \tag{26}$$

REMARK 2.12. One can extend Theorem 2.8 to provide finite-dimensional convergence of the partial sum process indexed by $t \in [0, 1]$. Weak convergence in the function space is a delicate question and can easily fail to hold. Theorem 2 of Avram and Taqqu (1992) shows that for certain associated variables (generated by moving averages of independent and p-stable variables) their partial sum processes converge weakly with respect to the Skorohod M_1 topology on $D[0, 1]$ but not with respect to the J_1 topology.

Although valid for $0 < p < 2$, Theorem 2.8 seems to be most appropriate for the case $1 < p < 2$. The example contained in Remark 2.4 shows that a p-stable limit theorem may hold for $0 < p \le 1$, while

$$\sum_{k=2}^{\infty} I_p^A(X_1, X_k) = \sum_{k=2}^{\infty} I_p^A(X, X) = +\infty.$$

In fact, $\{X_j\} \in \mathscr{D}_{n,s}(\{Y_j\})$ may fail while the p-stable limit theorem continues to hold. Let $Y_1, Y_2, \dots$ be independent identically distributed with $\mathscr{L}(Y_1)$ being strictly p-stable, $0 < p < 1$. Let Z be arbitrary but independent of $\{Y_j\}$. If $X_j = Y_j + Z$, $j \in \mathbb{N}$, then

$$\frac{S_n}{n^{1/p}} \sim Y_1 + n^{1-(1/p)} Z \underset{\mathscr{D}}{\to} Y_1.$$

Clearly, $\mathscr{L}((X_1, X_2, \dots, X_N))$ may have very bad asymptotic properties.

On the other hand, *without* any assumptions on multidimensional distributions the theorem does not hold, even if we impose very restrictive additional mixing properties. This can be seen by the example given in Jakubowski (1993b). The stationary sequence $X_1, X_2, \dots$ is 1-dependent, has infinitely divisible finite-dimensional distributions, is such that $\mathscr{L}(X_1)$ is in the domain of strict normal attraction of some nondegenerate strictly p-stable law μ, $0 < p < 2$ and is such that $\mathscr{L}(X_1 + X_2)$ does not have the immediately preceding property for any μ'. So

$$n\big(P\big(X_1 + X_2 > xn^{1/p}\big) - P\big(X_1 > xn^{1/p}\big)\big) \text{ diverges for some } x \ne 0. \tag{27}$$

Further, for each $N \in \mathbb{N}$, the Lévy measure of $\mathscr{L}((X_1, X_2, \ldots, X_N))$ is concentrated on the positive part of the space, that is, on the set $[0, +\infty)^N$. It follows from Resnick (1988), that $X_1, X_2, \ldots$ are associated.

In summary, there exists a stationary associated sequence satisfying (22) (since it is 1-dependent) and such that (27) holds. However, partial sums of this sequence cannot converge by Kobus (1991). There it is shown that for 1-dependent sequences, and under $\sup_{n \in \mathbb{N}} nP(|X_1| > xn^{1/p}) < +\infty$, $x > 0$, the weak limit of $S_n/n^{1/p}$ exists if and only if we have convergence in (27) for each $x \neq 0$.

That we need more information than only (22) is not surprising. Even for independent sequences, non-Gaussian stable limits involve more detail than Gaussian ones, and the dependent case can involve additional complications [see Samur (1987)]. In search of conditions which are weaker than $\{X_j\} \in \mathscr{D}_{n,s}(\{Y_j\})$, we suggest the following direction, which we prove only for $0 < p < 1$.

THEOREM 2.13. *Let $X_1, X_2, \ldots$ be strictly stationary, associated and such that for some $A \geq 0$ and $0 < p < 1$ condition* (22) *holds. If, for each $m \in \mathbb{N}$,*

$$\mathscr{L}(S_m) \in \mathscr{D}_{s,n}(\mu_m) \tag{28}$$

for some strictly stable laws μ_m, then there exists a strictly p-stable distribution μ_∞ such that

$$\mu_m^{1/m} \Rightarrow \mu_\infty \quad as\ m \to \infty \tag{29}$$

and

$$\frac{S_n}{n^{1/p}} \underset{\mathscr{D}}{\to} \mu_\infty. \tag{30}$$

REMARK 2.14. Jakubowski (1993a) provides *necessary and sufficient* conditions for the convergence $S_n/n^{1/p} \to_{\mathscr{D}} \mu_\infty$, where μ_∞ is a strictly p-stable distribution and the S_n's are partial sums of a stationary sequence (not necessarily associated or strongly mixing). These conditions, however, are more complicated than the conditions of the present paper. We defer the discussion of the more general setting to another place.

3. Proofs in the jointly stable case.

To begin the proofs of Theorems 2.1, 2.2 and 2.3, we observe that a linear transformation of jointly stable random variables is again stable and that we can calculate its characteristic function.

LEMMA 3.1. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be jointly p-stable, with representation* (5). *Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $\mathbf{Y} = A\mathbf{X}$ is again p-stable, with the Lévy measure determined by a measure Γ_A on $\mathscr{S}^{m-1}$ given by*

$$\Gamma_A(B) = \int_{\mathscr{S}^{n-1}} \|A(\mathbf{s})\|^p I\left(\frac{A(\mathbf{s})}{\|A(\mathbf{s})\|} \in B\right)\Gamma_n(d\mathbf{s}), \qquad B \in \mathscr{B}_{\mathscr{S}^{m-1}}. \tag{31}$$

If either $p \neq 1$ or $p = 1$ and Γ_n is symmetric, then the vector in representation (5) of the law of $A\mathbf{X}$ is given by $A\mathbf{b_n}$.

We apply the lemma to the linear map on $\mathbb{R}^n$ given by

$$(x_1, x_2, \dots, x_n) \mapsto h_n(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n.$$

and obtain that the distribution of S_n is p-stable with the measure Γ_{h_n} given by

$$\begin{aligned} c_n^+ &= \Gamma_{h_n}(\{1\}) = \int_{\mathscr{S}^{n-1}} \left|\sum_{i=1}^n s_i\right|^p I\left(\sum_{i=1}^n s_i > 0\right)\Gamma_n(d\mathbf{s}), \\ c_n^- &= \Gamma_{h_n}(\{-1\}) = \int_{\mathscr{S}^{n-1}} \left|\sum_{i=1}^n s_i\right|^p I\left(\sum_{i=1}^n s_i < 0\right)\Gamma_n(d\mathbf{s}), \end{aligned} \tag{32}$$

where $\mathscr{S}^{n-1} \ni \mathbf{s} = (s_1, s_2, \dots, s_n)$. Consequently, if $X_1, X_2, \dots$ are associated, we have by (13) that

$$\begin{aligned} c_n^+ &= \int_{\mathscr{S}^{n-1}} (s_1 + s_2 + \cdots + s_n)^p I(s_1 \ge 0, s_2 \ge 0, \dots, s_n \ge 0)\Gamma_n(d\mathbf{s}), \\ c_n^- &= \int_{\mathscr{S}^{n-1}} (-s_1 - s_2 - \cdots - s_n)^p I(s_1 \le 0, s_2 \le 0, \dots, s_n \le 0)\Gamma_n(d\mathbf{s}). \end{aligned} \tag{33}$$

Since the function $x \mapsto x^p$ is subadditive, additive and superadditive in nonnegative arguments for $0 < p < 1$, $p = 1$ and $1 < p < 2$, respectively, we have the following key result.

COROLLARY 3.2. *If $X_1, X_2, \dots$ are stationary, associated and jointly p-stable, then the sequences $\{c_n^+\}_{n \in \mathbb{N}}$ and $\{c_n^-\}_{n \in \mathbb{N}}$ are subadditive for $0 < p < 1$, additive for $p = 1$ and superadditive for $1 < p < 2$.*

PROOF OF THEOREM 2.1. Let

$$g_n: \mathbb{R}^n \to \mathbb{R}^1, \qquad g_n(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n^{1/p}}.$$

Applying Lemma 3.1, we see that $\mathscr{L}(S_n/n^{1/p}) = \gamma_p(n^{1-(1/p)}b, c_n^+/n, c_n^-/n)$. Since $\{c_n^+\}$ and $\{c_n^-\}$ are subadditive and nonnegative, they converge, when normalized by n. Finally,

$$\frac{S_n}{n^{1/p}} \xrightarrow[\mathscr{D}]{} \gamma_p\left(0, \inf_{n \in \mathbb{N}} \frac{c_n^+}{n}, \inf_{n \in \mathbb{N}} \frac{c_n^-}{n}\right). \tag{34}$$ □

PROOF OF THEOREM 2.2. Suppose $X_1, X_2, \dots$ are jointly 1-stable with symmetric Γ_n. By Lemma 3.1, Corollary 3.2 and since Γ_n is symmetric, we have, for $c = c_1^+ = c_1^-$,

$$\mathscr{L}(S_n/n) = \gamma_p(b, c, c) = \mathscr{L}(X_1).$$

Since (31) also holds in the general case of jointly 1-stable distributions, for such $X_1, X_2, \dots$ the law of S_n/n may differ from $\gamma_p(0, c_1^+, c_1^-)$ [and hence from $\mathscr{L}(X_1)$] by at most a shift. □

PROOF OF THEOREM 2.3. Exactly as in the case $0 < p < 1$, we have

$$\mathscr{L}\big(S_n/n^{(1/p)}\big) = \gamma_p\big(n^{1-(1/p)}b, c_n^+/n, c_n^-/n\big).$$

Since $1/p < 1$ and the sequences $\{c_n^+\}$ and $\{c_n^-\}$ are superadditive, it is necessary and sufficient for the weak convergence of $S_n/n^{1/p}$ that $b = 0$ (i.e., $EX_1 = 0$) and

$$\sup_{n \in \mathbb{N}} \frac{c_n^+ + c_n^-}{n} < +\infty. \tag{35}$$

We shall prove that (11) provides a bound for (35). Observe that $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathscr{S}^{n-1}$ implies $\max_j |s_j| \le 1$, $\max_{i \ne j}(s_i^2 + s_j^2) \le 1$ but $|s_1| + |s_2| + \cdots + |s_n| \ge 1$. Hence

$$\begin{aligned}
&|s_1 + s_2 + \cdots + s_n|^p \\
&\quad \le \big(|s_1| + |s_2| + \cdots + |s_n|\big)^p \\
&\quad \le \big(|s_1| + |s_2| + \cdots + |s_n|\big)^2 \\
&\quad = \sum_{j=1}^n s_j^2 + \sum_{1 \le i \ne j \le n} |s_i|\,|s_j| \\
&\quad \le \sum_{j=1}^n |s_j|^p + \sum_{1 \le i \ne j \le n} \frac{|s_i|}{\sqrt{s_i^2 + s_j^2}} \frac{|s_j|}{\sqrt{s_i^2 + s_j^2}} \Big(\sqrt{s_i^2 + s_j^2}\Big)^p I\big(s_i^2 + s_j^2 > 0\big).
\end{aligned}$$

Applying Lemma 3.1, we get

$$\begin{aligned}
c_n^+ + c_n^- &= \int_{\mathscr{S}^{n-1}} |s_1 + s_2 + \cdots + s_n|^p \Gamma_n(d\mathbf{s}) \\
&\le n(c_1^+ + c_1^-) + \sum_{1 \le i \ne j \le n} \int_{\mathscr{S}^1} |s_1|\,|s_2| \Gamma_{\{i,j\}}(d\mathbf{s}) \\
&= n(c_1^+ + c_1^-) + \sum_{1 \le i \ne j \le n} \int_{\mathscr{S}^1} s_1 s_2 \Gamma_{\{i,j\}}(d\mathbf{s}).
\end{aligned}$$

The last equality holds by (13). □

4. Proofs for sequences in the domain of attraction of jointly stable sequences.

First, recall (20) and the well-known Hoeffding identity for two random variables X and Y,

$$\operatorname{Cov}(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H_{(X,Y)}(x, y)\, dx\, dy. \tag{36}$$

The Hoeffding identity is a particular case of a fact we will find very useful.

LEMMA 4.1 [Yu (1993a), Lemma 3.1]. *Let f_i, $i = 1, 2$, be absolutely continuous nondecreasing functions on $\mathbb{R}^1$. Then, for any random variables X_1 and X_2, we have*

$$(37)\quad \operatorname{cov}(f_1(X_1), f_2(X_2)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1'(x_1) f_2'(x_2) H_{(X_1, X_2)}(x_1, x_2)\, dx_1\, dx_2$$

if $E|f_1(X_1) f_2(X_2)|$ and $E|f_i(X_i)|$, $i = 1, 2$, are finite.

When we apply Lemma 4.1 to (40), we will get

$$(38)\qquad \operatorname{cov}(f_a(X_i), f_a(X_j)) = \int_{-a}^{a}\int_{-a}^{a} H_{(X_i, X_j)}(x_i, x_j)\, dx_i\, dx_j.$$

PROOF OF THEOREM 2.8. By the remarks preceding Theorem 2.8, we only have to check (16) and (18). Observe that (16) is equivalent to

$$(39)\qquad \begin{aligned} \lim_{m\to\infty} \limsup_{l\to\infty} \Big| E \exp\big\{ i\lambda S_{m\cdot l}/(m\cdot l)^{1/p}\big\} \\ - \big(E\exp\{i\lambda l^{-1/p} S_m / m^{1/p}\}\big)^l \Big| = 0, \qquad \lambda \in \mathbb{R}^1. \end{aligned}$$

Hence in what follows we may assume that $n = l \cdot m$ with m fixed and l tending to infinity.

Fix $a > 0$ and define the function $f_a\colon \mathbb{R}^1 \to \mathbb{R}^1$ by

$$(40)\qquad f_a(x) = \begin{cases} a, & \text{if } x > a, \\ x, & \text{if } |x| \le a, \\ -a, & \text{if } x < -a. \end{cases}$$

Note that $f_a(x/b) = b^{-1} f_{ab}(x)$, that $f_a(x)$ is a nondecreasing function in x and that $\{f_a(X_j)\colon j \ge 1\}$ is again an associated sequence of random variables. Consider the following decomposition:

$$\begin{aligned} n^{-1/p}\sum_{j=1}^{n} X_j &= \sum_{j=1}^{n} f_a\big(n^{-1/p}X_j\big) + \sum_{j=1}^{n}\big(n^{-1/p}X_j - f_a\big(n^{-1/p}X_j\big)\big) \\ &= S_{1,n} + S_{2,n}. \end{aligned}$$

Since $\mathscr{L}(X_1)$ is in the domain of normal attraction of a p-stable law, there is a constant $C > 0$ such that $P(|X_1| > x) \le Cx^{-p}$. If $a = a(X_1, \eta) > (C/\eta)^{1/p}$, then

$$\begin{aligned} P(S_{2,n} \ne 0) &\le P\big(\exists 1 \le j \le n\colon |X_j| > an^{1/p}\big) \\ &\le nP\big(|X_1| > an^{1/p}\big) \\ &\le Ca^{-p} < \eta. \end{aligned}$$

Consequently,

$$(41)\qquad \big| E\exp\{i\lambda S_n/n^{1/p}\} - E\exp\{i\lambda S_{n,1}\}\big| < 2\eta.$$

A similar reasoning also shows that

$$(42)\quad \left| \big(E\exp\{i\lambda l^{-1/p} S_m/m^{1/p}\}\big)^l - \left(E\exp\left\{\sum_{j=1}^{m} f_a\big(n^{-1/p}X_j\big)\right\}\right)^l \right| < 2\eta.$$

Therefore, we may concentrate on the analysis of $S_{1,n}$ alone. For notational convenience, set

$$U_{n,j} = U_{n,j}^{(a)} = f_a\left(n^{-1/p} X_j\right).$$

Take $\lambda \in \mathbb{R}^1$ and recall that $n = l \cdot m$. By (19),

$$\begin{aligned}
&\left| E \exp\left\{ i\lambda \sum_{j=1}^{n} U_{n,j} \right\} - \left(E \exp\left\{ i\lambda \sum_{j=1}^{m} U_{n,j} \right\} \right)^{l} \right| \\
&= \left| E \exp\left\{ i(\lambda\sqrt{n}) \frac{1}{\sqrt{n}} \sum_{j=1}^{n} U_{n,j} \right\} \right. \\
&\qquad \left. - \left(E \exp\left\{ i(\lambda\sqrt{n}) \frac{1}{\sqrt{m}} \frac{1}{\sqrt{l}} \sum_{j=1}^{m} U_{n,j} \right\} \right)^{l} \right| \\
&\le \frac{\lambda^2 n}{2} \left(\operatorname{Var}\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} U_{n,j} \right) - \operatorname{Var}\left(\frac{1}{\sqrt{m}} \sum_{j=1}^{m} U_{n,j} \right) \right) \\
&= 2\lambda^2 \left(\sum_{j=2}^{m} \left(\frac{1}{m} - \frac{1}{n} \right) (j-1) \left(n \operatorname{Cov}(U_{n,1}, U_{n,j}) \right) \right. \\
&\qquad \left. + \sum_{j=m+1}^{n} \left(1 - \frac{j-1}{n} \right) \left(n \operatorname{Cov}(U_{n,1}, U_{n,j}) \right) \right) \\
&\le 2\lambda^2 \left(\frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(n \operatorname{Cov}(U_{n,1}, U_{n,j}) \right) \right. \\
&\qquad \left. + \frac{1}{m} \sum_{i=1}^{m} \sum_{j=m+1}^{n} \left(n \operatorname{Cov}(U_{n,1}, U_{n,j}) \right) \right) \\
&= 2\lambda^2 \frac{1}{m} \sum_{i=1}^{m} \sum_{j=i}^{n} \left(n \operatorname{Cov}(U_{n,1}, U_{n,j}) \right) \\
&= 2\lambda^2 \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(n^{1-(2/p)} \operatorname{Cov}\left(f_{a \cdot n^{1/p}}(X_1), f_{a \cdot n^{1/p}}(X_j) \right) \right).
\end{aligned}$$

By definition (21), if $a \cdot n^{1/p} \ge A$ we have

$$\begin{aligned}
&n^{1-(2/p)} \operatorname{Cov}\left(f_{a \cdot n^{1/p}}(X_1), f_{a \cdot n^{1/p}}(X_j) \right) \\
&\le n^{1-(2/p)} \left(a \cdot n^{1/p} \right)^{2-p} I_p^A(X_1, X_j) \\
&= a^{2-p} I_p^A(X_1, X_j).
\end{aligned}$$

Combining the two last estimates, we get

$$\left| E\exp\left\{ i\lambda \sum_{j=1}^{n} U_{n,j} \right\} - \left(E\exp\left\{ i\lambda \sum_{j=1}^{m} U_{n,j} \right\} \right)^{l} \right|$$

$$\le 2\lambda^2 a^{2-p} \frac{1}{m} \sum_{i=1}^{m} \sum_{j=i}^{l\cdot m} I_p^A(X_1, X_j)$$

$$\le 2\lambda^2 a^{2-p} \frac{1}{m} \sum_{i=1}^{m} \sum_{j=i}^{\infty} I_p^A(X_1, X_j) \to 0 \quad \text{as } m \to \infty.$$

We have established (16). For $0 < p \le 1$ nothing remains to be proved since (18) is satisfied automatically by Theorems 2.1 and 2.2. For $1 < p < 2$ we have to check (11).

We know by our condition on domains of attraction, (15), that for each $k \in \mathbb{N}$,

$$\frac{\mathbf{V}_{k,1} + \mathbf{V}_{k,2} + \cdots + \mathbf{V}_{k,n}}{n^{1/p}} \xrightarrow[\mathscr{D}]{} (Y_1, Y_k) \quad \text{as } n \to +\infty.$$

Here $\mathbf{V}_{k,1}, \mathbf{V}_{k,2}, \dots, \mathbf{V}_{k,n}, \dots$ are independent copies of (X_1, X_k). In particular, it follows from Jacod and Shiryaev [(1987), page 362, Theorem 2.35] that for each $a > 0$ as $n \to +\infty$,

$$(43) \quad n\,\mathrm{Cov}\big(f_a(n^{-1/p}X_1), f_a(n^{-1/p}X_k)\big) \to \int_{\mathbb{R}^2} f_a(x_1) f_a(x_2) \nu_{\{1,k\}}(dx_1, dx_2).$$

As stated previously, $\nu_{\{1,k\}}$ is the Lévy measure of $\mathscr{L}((Y_1, Y_k))$. Hence

$$\int_{\mathscr{S}^1} s_1 s_2 \Gamma_{\{1,k\}}(d\mathbf{s}) = (2-p)\int_{x_1^2+x_2^2\le 1} x_1 x_2 \nu_{\{1,k\}}(dx_1, dx_2)$$

$$\le (2-p)\int_{\mathbb{R}^2} f_1(x_1) f_1(x_2) \nu_{\{1,k\}}(dx_1, dx_2)$$

$$\le (2-p) I_p^A(X_1, X_k).$$

The last inequality follows from (43). The series in (11) therefore converges by (22). □

Proof of Theorem 2.13. Observe that (16) holds since its proof is based solely on (22) and properties of one-dimensional distributions. We shall check (29). For each $m \in \mathbb{N}$ we have

$$\mu_m^{1/m} = \gamma_p(0, c_m^+/m, c_m^-/m),$$

and similarly as in the proof of Theorem 2.1 it is enough to prove that both $\{c_m^+\}_{m\in\mathbb{N}}$ and $\{c_m^-\}_{m\in\mathbb{N}}$ are *subadditive* sequences.

LEMMA 4.2. *If* $X_1, X_2, \dots$ *are associated and* (28) *holds, then, for each* $m \in \mathbb{N}$ *and* $x > 0$,

$$nP\left(\sum_{j=1}^{m} X_j > xn^{1/p}\right) - nP\left(\sum_{j=1}^{m} X_j I(X_j > 0) > xn^{1/p}\right) \longrightarrow 0, \tag{44}$$

$$nP\left(\sum_{j=1}^{m} X_j < -xn^{1/p}\right) - nP\left(\sum_{j=1}^{m} X_j I(X_j < 0) < -xn^{1/p}\right) \longrightarrow 0 \tag{45}$$

as $n \to \infty$.

PROOF. Consider (44). First, we shall prove a slightly weaker statement: For each $\delta > 0$ as $n \to +\infty$,

$$nP\left(\sum_{j=1}^{m} X_j > xn^{1/p}\right) - nP\left(\sum_{j=1}^{m} X_j I\left(X_j > -\delta n^{1/p}\right) > xn^{1/p}\right) \longrightarrow 0. \tag{46}$$

To see this, set

$$A_n = \left\{\sum_{j=1}^{m} X_j > xn^{1/p}\right\}, \qquad B_n = \left\{\sum_{j=1}^{m} X_j I\left(X_j > -\delta n^{1/p}\right) > xn^{1/p}\right\}.$$

Then $A_n \subset B_n$ and

$$\begin{aligned} nP(B_n \setminus A_n) &\le n \sum_{1 \le i \ne j \le m} P\left(X_i \le -\delta n^{1/p},\, X_j > xn^{1/p}/m\right) \\ &\le \sum_{1 \le i \ne j \le m} P\left(X_i \le -\delta n^{1/p}\right)\left\{nP\left(X_j > xn^{1/p}/m\right)\right\} \\ &\longrightarrow 0 \quad \text{as } n \to +\infty, \end{aligned}$$

where the last inequality holds by association.

By (28) the limit of $nP(\sum_{j=1}^{m} X_j > xn^{1/p})$ is $c_m^+ x^{-p}$. Hence by choosing $\delta > 0$ small enough we can make

$$\lim_{n \to \infty} nP\left[xn^{1/p} > \sum_{j=1}^{m} X_j I\left(X_j > -\delta n^{1/p}\right) > (x - m\delta)n^{1/p}\right]$$

as small as desired. It remains to observe that

$$\begin{aligned} &\left\{\sum_{j=1}^{m} X_j I\left(X_j > -\delta n^{1/p}\right) > xn^{1/p}\right\} \\ &\qquad \subset \left\{\sum_{j=1}^{m} X_j I(X_j > 0) > xn^{1/p}\right\} \\ &\qquad \subset \left\{\sum_{j=1}^{m} X_j I\left(X_j > -\delta n^{1/p}\right) > (x - m\delta)n^{1/p}\right\}. \end{aligned}$$

Relation (45) can be proved the same way. □

By (28), (44) and (45) as $n \to +\infty$,

$$\left(E\exp\left\{-\lambda n^{-1/p}\sum_{j=1}^{m} X_j I(X_j > 0)\right\}\right)^n \to \exp(c_m^+ h_\lambda), \tag{47}$$

$$\left(E\exp\left\{-\lambda n^{-1/p}\left(-\sum_{j=1}^{m} X_j I(X_j < 0)\right)\right\}\right)^n \to \exp(c_m^- h_\lambda), \tag{48}$$

where $h_\lambda = \int_0^\infty (e^{-\lambda x} - 1)/x^{p+1}\,dx$. Notice that $-\exp\{-C\sum_{j=1}^{k} X_j I(X_j > 0)\}$ and $-\exp\{-C\sum_{j=k+1}^{k+l} X_j I(X_j > 0)\}$ are associated when $C > 0$ is a constant. Thus

$$E\exp\left\{\frac{-\lambda}{n^{1/p}}\sum_{j=1}^{k+l} X_j I(X_j > 0)\right\}$$
$$\geq E\exp\left\{\frac{-\lambda}{n^{1/p}}\sum_{j=1}^{k} X_j I(X_j > 0)\right\}\cdot E\exp\left\{\frac{-\lambda}{n^{1/p}}\sum_{j=1}^{l} X_j I(X_j > 0)\right\}.$$

Hence

$$\exp(c_{k+l}^+ h_\lambda) \geq \exp(c_k^+ h_\lambda)\exp(c_l^+ h_\lambda),$$

or $c_{k+l}^+ \leq c_k^+ + c_l^+$, that is, subadditivity of $\{c_m^+\}_{m\in\mathbb{N}}$ holds.

The reasoning for $\{c_m^-\}_{m\in\mathbb{N}}$ is similar, with the only exception that in this case different random variables, namely $\exp\{C\sum_{j=1}^{k} X_j I(X_j < 0)\}$ and $\exp\{C\sum_{j=k+1}^{k+l} X_j I(X_j < 0)\}$, are associated $(C > 0)$. □

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