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BOOK REVIEW

MICHEL LEDOUX AND MICHEL TALAGRAND, *Probability in Banach Spaces: Isoperimetry and Processes*. Springer, New York, 1991, 480 pages $129.00.

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This is a book about the importance of inequalities and the power of isoperimetric methods. The vehicle chosen to illustrate these concepts is the theory of probability in Banach spaces.

From the authors' introduction: "One of the fascinations of the theory of Probability in Banach spaces today is its use of a wide range of rather powerful methods. Since the field is one of the most active contact points between Probability and Analysis, it should be no surprise that many of the techniques are not probabilistic but rather come from Analysis. The book focuses on two connected topics—the use of isoperimetric methods and regularity of random processes—where many of these techniques come into play and which encompass many (although not all) of the main aspects of Probability in Banach spaces. The purpose of this book is to give a modern and, at many places, seemingly definitive account of these topics, from the foundations of the theory to the latest research questions. The book is written so as to require only basic prior knowledge of either Probability or Banach space theory, in order to make it accessible from readers of both fields as well as to non-specialists. It is moreover presented in perspective with the historical developments and strong modern interactions between Measure and Probability theory, Functional Analysis and Geometry of Banach spaces. It is essentially self-contained (with the exception that the proof of a few deep isoperimetric results have not been reproduced), so as to be accessible to anyone starting the subject, including graduate students. Emphasis has been put in bringing forward the ideas we judge important but not on encyclopedic detail."

Talagrand's recent paper [16] on "Concentration of measure and isoperimetric inequalities in product spaces," presented in March 1993 as an IMS Special Invited Lecture at the Second International Symposium on Probability and Its Applications in Bloomington, Indiana, shows that isoperimetry and the associated concentration of measure phenomena have substantial implications for other areas of probability as well. These are typically areas which have problems involving a large number (perhaps infinitely many) of random variables (independent or not). The areas in which profound applications have already been established include stochastic models in physics (such as percolation) and computer science (bin packing, assignment problem, geometric probability). The Ledoux–Talagrand book serves both as a good, convincing illustration of the

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power of the isoperimetric methods and as good preparation for the further developments in [16]. Consequently, the book should prove useful to readers from many diverse areas in pure and applied probability, as well as in functional analysis and the geometry of Banach spaces.

Quotes appearing in the following discussion are from places throughout the book.

The book is divided into three parts. Part 0 concerns the isoperimetric background and generalities on vector-valued random variables. Part I is devoted to Banach-space-valued random variables and their strong limiting properties. Part II focuses on tightness of vector-valued random variables and regularity of random processes.

Part 0. Chapter 1 presents the isoperimetric inequalities and concentration of measure phenomena. "The concentration of measure phenomena, which roughly describes how a well-behaved function is almost a constant on almost all the space, can moreover be seen as the explanation for the two main parts of this work: the first (Part I) deals with 'nice' functions applying isoperimetric inequalities and concentration properties, the second (Part II) tries to determine conditions for a function to be 'nice'." The basic idea of the concentration of measure phenomena may be described as follows: Let (X, ρ, μ) be a (compact) metric space (X, ρ) with a Borel probability measure μ. The concentration function $\alpha(X, r)$, $r > 0$, is defined by

$$\alpha(X, r) = \sup\left\{1 - \mu(A_r) : \mu(A) \geq \tfrac{1}{2}, A \subset X, A \text{ Borel}\right\},$$

where $A_r = \{x \in X : \rho(x, A) < r\}$. Then

$$\mu(A) \geq \tfrac{1}{2} \quad \text{implies} \quad \mu(A_r) \geq 1 - \alpha(X, r).$$

If (X, ρ, μ) is a family for which the concentration function $\alpha(X, r)$ turns out to be extremely small when r increases to infinity, then the concentration of measure phenomena is said to occur. "In the presence of such a property, any nice function is very close to being a constant (median or expectation) on all but a very small set, the smallness of which depends on $\alpha(X, r)$." The concentration of measure phenomena is usually derived from an isoperimetric inequality. Typically, isoperimetric inequalities identify a set H with the property that if $\mu(A) = \mu(H)$, then $\mu(A_r) \geq \mu(H_r)$. Hopefully, H is simple enough that $\mu(H_r)$ is either computable or easily lower bounded. Isoperimetric inequalities are discussed for the sphere, for Gauss space, for the cube and for product measures.

Chapter 2 "collects in rather an informal way some basic facts about processes and infinite dimensional random variables. . .only a few proofs are given." However, the notes and complements supply an adequate set of references for anyone wishing to see the proofs.

Part I. Since the study of Gaussian processes has historically stimulated much of probability in Banach space theory, it is not surprising that Chapter 3 is devoted to Gaussian random variables. "The study of Gaussian random vectors and processes may indeed be considered as one of the fundamental topics of the theory since it inspires many other parts of the field both in the results themselves and in the techniques of investigation." The chapter concentrates on integrability properties, tail behavior and basic comparison properties of Gaussian processes. "This study is a reference for the corresponding results for other types of random variables like Rademacher series (Chapter 4), stable random variables (Chapter 5), and even sums of independent random variables (Chapter 6)." Comparison properties of Gaussian random variables are also discussed and used in studying the regularity properties of the sample paths of Gaussian processes (Chapter 12). (However, Talagrand discovered in [13] that comparison properties can be dispensed with and replaced by the use of concentration of measure. This makes possible all the results he was able to derive in the non-Gaussian case. See also [22].)

Chapter 4 provides one of the nicest discussions that can be found on Rademacher averages. The power of isoperimetric methods yields several interesting new results. Chapter 6 on sums of independent random variables describes "ideas and techniques which go from simple but powerful observations such as symmetrization (randomization) techniques to more elaborate results like those obtained from the isoperimetric inequality for product measures." Both martingale and isoperimetric methods are developed and many of the results are "of basic use in the study of limit theorems later."

The investigations of the strong law of large numbers (Chapter 7) and the law of the iterated logarithm (Chapter 8) are clear examples of where "the isoperimetric approach proves to be an efficient tool." A main feature of both investigations is that, under moment conditions similar to the scalar case, the *almost sure* statement reduces to an *in probability* statement. Among the results included are infinite-dimensional versions of the Kolmogorov and Prokhorov strong laws of large numbers as well as Kolmogorov's LIL and the Hartman–Wintner–Strassen form of the LIL in Banach spaces. The subsection titled "On the identification of the limits" presents many of the most recent results on the cluster set.

Part II. Starting with Chapter 9 (Type and Cotype of Banach Spaces), the book studies "The possibility of a control in probability, or in the weak topology, of probability distributions of sums of independent random variables." The extension to infinite dimensions does not generally hold, and conditions must be placed on the Banach spaces to ensure the existence and tightness of some probability measures. One classification of Banach spaces is based on the concept of type and cotype (see Chapter 9). These conditions are further used in the study of the central limit theorem (Chapter 10), in which the development focuses on normal limits and $\sqrt{n}$ normalizations. However, this "framework is rich enough to analyze the main questions."

In Chapter 11 "another approach is taken to the existence and tightness of certain measures in the framework of random functions and processes." The book investigates "sufficient conditions for the almost sure boundedness or continuity of the sample paths of $X = (X_t)_{t \in T}$ in terms of the 'geometry' (in the metric sense) of T. By geometry, we mean some metric entropy or majorizing measure condition which estimates the size of T as a function of some parameters related to X." Majorizing measure conditions are more precise than metric entropy conditions because they take more into account the local geometry of the index set. Chapter 11 focuses on sufficient conditions for sample boundedness and continuity of random processes. The main concern of Chapter 12 is necessity. In particular, the sufficient majorizing measure condition for a Gaussian process to be almost surely bounded or continuous is shown to be also necessary. This characterization thus provides a complete understanding of the regularity properties of Gaussian paths. The arguments of proof rely heavily on the basic ultrametric structure which lies behind a majorizing measure condition. The results of Chapters 11 and 12 "lead to a definitive treatment" of random Fourier series with applications to harmonic analysis in Chapter 13. Paper [14] corrects a mistake appearing in this chapter.

The purpose of Chapter 14 "is to present applications of the random process techniques to infinite dimensional limit theorems, and in particular to the central limit theorem." The major focus is on processes with continuous sample paths. Since $C(T)$, the continuous functions on a compact metric space T, "is not well behaved with respect to the type or cotype 2 properties, we will rather have to seek nice classes of random variables in $C(T)$ for which a central limit property can be established. This point of view leads to enlarge this framework and to investigate limit theorems for empirical measures or processes. Random geometric descriptions of the CLT may then be produced via this approach, as well as complete descriptions for nice classes of functions (indicator functions of some sets) on which the empirical processes are indexed. While these random geometric descriptions do not solve the central limit problem in infinite dimensions. . .they clearly describe the main difficulties inherent to the problem from the empirical point of view."

The final chapter (Chapter 15) gives a sample of "applications of isoperimetric methods and the process techniques of Probability in Banach spaces to the local theory of Banach spaces."

The authors should be commended on the readability—and hence the accessibility—of this book, which includes not only new proofs of many known results, but also a large number of new results. Additionally, for the reader's ease, a diagram on page 4 "describes some of the interactions between the two main parts of the book and the natural connections between the various chapters." Finally, each chapter is accompanied by a fairly complete set of notes and references.

Although a number of the topics of this book overlap with those in other books, such as those by Araujo and Giné [1] and Linde [9], the approach and flavor are substantially different. The main emphasis in the Ledoux–Talagrand book is on proving good inequalities with the infinite-dimensional aspect secondary. Thus, the potential audience transcends those whose main interest is probability in Banach spaces. The brief treatment of empirical processes given here will soon be complemented by two forthcoming books—one due to R.M. Dudley and the other due to J. A. Wellner and A. van der Vaart.

If the impact of a book is gauged by the research it stimulates, this book has already been a success. It has stimulated substantial progress on the two main themes, isoperimetry and processes. The sampling below, now prefaced with a few comments, gives a partial sense of this impact. Papers [7] and [8] complement the book's development with a semigroup approach to isoperimetry and concentration of measures. The importance of [16] on broadening the areas of potential impact of isoperimetric methods cannot be overstated. Papers [2], [3], [5] and [6] provide applications of the isoperimetric methods to large and moderate deviations. Paper [10] shows how the concentration of measure phenomena for Gaussian processes can ultimately be useful in the study of the sample path properties of local times. The definitive results on sample continuity of random Fourier series, appearing in [15], are exactly in the spirit of Chapter 13 of the book. Papers [20] and [21] are connected with problems of evaluation of tails, and essentially optimal answers are given by elaborating on isoperimetric methods. Paper [18] contains everything Talagrand claims he currently knows about majorizing measures. Lemma 10.15, which "may be considered as some vector valued extension of a strong law of large numbers for squares," has been used in [4] in bootstrapping of empirical measures.

In summary, the power and beauty of this book is that it represents a shift in perspective in probability in Banach spaces: Inequalities are of fundamental importance and isoperimetric methods are a powerful means of establishing them. These insights and the results they have yielded thus far are likely to stimulate more research in many different directions.

The following reference list includes work on isoperimetry stimulated by the book ([7], [8], [11], [12] and [16]) and work on random processes stimulated by the book ([2]–[6], [10], [13]–[15] and [17]–[22]), along with other references ([1] and [9]).

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