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ON STRASSEN'S LAW OF THE ITERATED LOGARITHM IN BANACH SPACE[1]

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Let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with values in a separable Banach spacc B and set, for each n, $S_n = X_1 + \cdots + X_n$. We give necessary and sufficient conditions in order that

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)} \sum_{i=1}^{n} \|S_i\|^p < \infty \quad \text{a.s.},$$

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)} \sum_{i=0}^{n} \|S_n - S_i\|^p < \infty \quad \text{a.s.},$$

where $p \geq 1$. Furthermore, the exact values of the above lim sup are obtained. Some results are the extensions of Strassen's work to the vector settings and some are new even on the real line. The proofs depend on the construction of an independent sequence with values in $l_p(B)$ and appear as an illustration of the power of the limit law in Banach space.

1. Introduction and statement of the results. Let B be a separable Banach space with norm $\|\cdot\|$, let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with values in B and, as usual, let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i, n \geq 1$.

When $B = R$, Strassen [7] obtained a law of the iterated logarithm which states that if $p \geq 1$ and

$$EX = 0, \qquad \sigma^2 \equiv EX^2 < +\infty, \tag{1.1}$$

then

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)} \sum_{i=1}^{n} |S_i|^p = \frac{2(p+2)^{p/2-1}\sigma^p}{p^{p/2}\left(\int_0^1 (1-t^p)^{-1/2} dt\right)^p} \quad \text{a.s.},$$

where we write $L_2 x$ to denote the function

$$\log \max\{e, \log x\}, \qquad x > 0.$$

A partial purpose of this paper is to extend this result to vector settings. That is, we will establish necessary and sufficient conditions for a B-valued random

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variable X to satisfy

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p < +\infty \quad \text{a.s.},$$

and we will give the exact value of the above limit. Further, if we observe that, for each n,

$$\sum_{i=1}^{n}\|S_i\|^p =_d \sum_{i=0}^{n}\|S_n - S_i\|^p \tag{1.2}$$

(where we use "$=_d$" to denote equality in distribution), it is reasonable for us to conjecture that $\{\sum_{i=0}^{n}\|S_n - S_i\|^p\}_{n\geq 1}$ has strong limit behavior similar to that of $\{\sum_{i=1}^{n}\|S_i\|^p\}_{n\geq 1}$ so that another vector version of the law of Strassen type may hold. Like some known results on strong limits in Banach space, such as the Ledoux–Talagrand law [6] of the iterated logarithm for i.i.d. random variables with Banach values, it should not come as a surprise that our conditions for the conclusions in the infinite-dimensional case are different from those of the original results on the real line.

Before introducing our results, let us give the following notation, which will be used throughout the paper.

For a separable Banach space B, we write B^* and B_1^* to denote the topological dual of B and the unit ball of B^*, respectively. For a B-valued random variable X, write $X \in WM_0^2$ if for all $f \in B^*$ we have $Ef(X) = 0$ and $Ef^2(X) < +\infty$. By Lemma 2.1 of [3], $\sigma^2 \equiv \sup_{f\in B_1^*} Ef^2(X) < \infty$ if $X \in WM_0^2$.

For any $p \geq 1$, we set

$$\Lambda(p) = \frac{2(p+2)^{p/2-1}}{p^{p/2}\left(\int_0^1(1-t^p)^{-1/2}dt\right)^p}.$$

We now state our results.

THEOREM 1. *Let $\{X, X_n;\, n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a separable Banach space B, and let $p \geq 1$ be fixed. In order that*

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_n - S_i\|^p < +\infty \quad a.s., \tag{1.3}$$

it is necessary and sufficient that the following three conditions are fulfilled:

(i) $E\big(\|X\|^2/L_2\|X\|\big) < \infty$;

(ii) $X \in WM_0^2$;

(iii) $\{S_n/\sqrt{2nL_2n}\}_{n\geq 1}$ *is bounded in probability.*

Furthermore,

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=0}^{n}\|S_n - S_i\|^p = \Lambda(p)\sigma^p \quad a.s., \tag{1.4}$$

whenever conditions (i) *and* (ii) *hold and* (iii) *is strengthened to*

(iii′) $S_n/\sqrt{2nL_2n} \to 0$ *in probability.*

THEOREM 2. *Let* $\{X, X_n;\ n \geq 1\}$ *be a sequence of i.i.d. random variables taking values in a separable Banach space* B, *and let* $p \geq 1$ *be fixed. In order that*

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p < +\infty, \tag{1.5}$$

it is necessary and sufficient that the following three conditions are fulfilled:

(i) $E(\|X\|^2/L_2\|X\|) < +\infty$;
(ii) $X \in WM_0^2$;
(iii) $\{S_n/\sqrt{2nL_2n}\}_{n\geq 1}$ *is bounded in probability.*

Furthermore,

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p = \Lambda(p)\sigma^p \quad a.s., \tag{1.6}$$

whenever conditions (i) *and* (ii) *hold and* (iii) *is strengthened to*

(iii′) $S_n/\sqrt{2nL_2n} \to 0$ *in probability.*

Now, let us mention the relation between our results and the classical law of the iterated logarithm due to Ledoux and Talagrand [6], which states that

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{2nL_2n} < +\infty \quad \text{a.s.}, \tag{1.7}$$

if and only if conditions (i)–(iii) are fulfilled. We have the following immediate corollary.

COROLLARY 3. *Let* $\{X, X_n;\ n \geq 1\}$ *be a sequence of i.i.d. random variables with values in a separable Banach space. Then the following three statements are equivalent*:

(a) (1.7) *holds.*
(b) (1.3) *holds for some* (*all*) $p \geq 1$.
(c) (1.5) *holds for some* (*all*) $p \geq 1$.

So far as we know, Theorem 1 is new even on the real line and, since condition (ii) implies (i) and (iii′) [and therefore (iii)] if B is a finite-dimensional space, Theorem 2 extends Strassen's work to random variables with Banach values. Moreover, if B is a type 2 space (see [9] for the definition of type 2 space), by Proposition 3.7 of Ledoux and Talagrand [6], condition (iii′) is implied by (i) and (ii); so Theorems 1 and 2 take the following simpler forms.

COROLLARY 4. *Let* $\{X, X_n;\ n \geq 1\}$ *be a sequence of i.i.d. random variables with values in a separable Banach space of type* 2. *If conditions* (i) *and* (ii) *are* *fulfilled, then, for all* $p \geq 1$,

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=0}^{n}\|S_n - S_i\|^p = \limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p = \Lambda(p)\sigma^p \quad a.s.$$

Otherwise the above limits diverge to infinity with probability 1 *for all* $p \geq 1$.

The proofs we present of Theorems 1 and 2 are quite different from Strassen's approach, which seems to be no longer applicable for our results. Unlike the proofs of some previous laws in Banach space, the typical method via finite-dimensional approximations may not be effective in the proofs of (1.4) and (1.6), since it seems that neither (1.4) nor (1.6) can be treated as a direct corollary of the results on the real line even in the finite-(but multiple) dimensional case. To prove Theorem 1, we will represent the sequence

$$\left\{\left(\sum_{i=0}^{n}\|S_n - S_i\|^p\right)^{1/p}\right\}_{n\geq 1}$$

as a normed sequence of partial sums of independent random variables with values in $l_p(B)$ and apply an earlier result obtained by the author on the law of the iterated logarithm in Banach space to the proof. The proof of Theorem 2 depends mainly on relation (1.2) and the conclusion of Theorem 1.

2. Auxiliary results. In this section we establish two auxiliary results needed in later sections. The purpose of the first lemma is to identify the normalizing numbers in Theorems 1 and 2 with those in [2, Theorem 1.2].

LEMMA 1. *The following assertions hold*:

$$\lim_{n\to\infty} n^{-3}\max\left\{\sum_{i,j=1}^{n}(n-\max(i,j))\lambda_i\lambda_j;\ \lambda_k\in R, 1\leq k\leq n,\ \text{and}\ \max_{k\leq n}|\lambda_k|\leq 1\right\} = \Lambda^2(1), \tag{2.1}$$

$$\lim_{n\to\infty} n^{-1-(2/p)}\max\left\{\sum_{i,j=1}^{n}(n-\max(i,j))\lambda_i\lambda_j;\ \lambda_k\in R, 1\leq k\leq n,\ \text{and}\ \sum_{k=1}^{n}|\lambda_k|^q\leq 1\right\} = (\Lambda(p))^{2/p}, \tag{2.2}$$

where $p > 1$ *and* $1/p + 1/q = 1$.

PROOF. The proof of (2.1) is quite simple. It is easy to verify that the maximum on the left-hand side is obtained at $\lambda_i = 1,\ i = 1,2,\dots,n$, and

$$\sum_{i,j=1}^{n} (n - \max(i,j)) = \tfrac{1}{2}\sum_{i=1}^{n}(n-i)(n+i-1) \sim \tfrac{1}{3}n^3, \qquad n \to \infty.$$

Note that $\Lambda^2(1) = \frac{1}{3}$. Hence (2.1) is valid.

We now prove (2.1). It is obvious that, for each n,

$$B_n \equiv n^{-1-(2/p)} \max\left\{ \sum_{i,j=1}^{n} (n - \max(i,j))\lambda_i\lambda_j;\ \lambda_k \in R, 1 \le k \le n, \text{ and } \sum_{k=1}^{n} |\lambda_k|^q \le 1 \right\}$$
$$= n^{-3} \max\left\{ \sum_{i,j=1}^{n} (n - \max(i,j))\lambda_i\lambda_j;\ \lambda_k \in R, 1 \le k \le n, \text{and } \sum_{k=1}^{n} |\lambda_k|^q \le n \right\}.$$

Let $L_q\,[0,1]$ be the Banach space of all real functions $x(t)$ on $[0,1]$ such that

$$\|x\|_q \equiv \left(\int_0^1 |x(t)|^q dt\right)^{1/q} < +\infty.$$

We show that

$$\text{(2.3)} \qquad \lim_{n\to\infty} B_n = \sup\left\{ \int_0^1 \left(\int_0^t x(s)ds\right)^2 dt;\ x \in L_q[0,1] \text{ and } \|x\|_q \le 1 \right\}.$$

For each n, let $\lambda_1, \dots, \lambda_n \in R$ be such that

$$\sum_{k=1}^{n} |\lambda_k|^q \le n \quad \text{and} \quad B_n = \sum_{i,j=1}^{n} (n - \max(i,j))\lambda_i\lambda_j,$$

and set

$$x_n(t) = \lambda_k, \qquad t \in \left(\frac{k-1}{n}, \frac{k}{n}\right], k = 1,2,\dots,n.$$

Then, $\|x_n\|_q \le 1$ and

$$B_n = \iint_D (1 - \max(s,t))x_n(s)x_n(t)ds\,dt + o(1), \qquad n \to \infty,$$

where $D = \{(s,t);\ 0 \le s \le 1, 0 \le t \le 1\}$. Using the Fubini theorem and partial integration yields

$$\text{(2.4)} \iint_D (1 - \max(s,t))x(s)x(t)\,ds\,dt = \int_0^1 \left(\int_0^t x(s)\,ds\right)^2 dt, \qquad x \in L_q[0,1].$$

Therefore,

$$(2.5)\quad \limsup_{n\to\infty} B_n \le \sup\left\{ \int_0^1 \left(\int_0^t x(s)\,ds \right)^2 dt;\ x \in L_q[0,1] \text{ and } \|x\|_q \le 1 \right\}.$$

On the other hand, let $x \in L_q[0,1]$ and $\|x\|_q \le 1$, and, without loss of generality, assume $x(t)$ is continuous on $[0,1]$. If we set

$$\lambda_k = \inf\left\{ |x(t)|;\ t \in \left(\frac{k-1}{n}, \frac{k}{n} \right] \right\}, \qquad k = 1, 2, \dots, n,$$

then, for each n,

$$\begin{aligned} B_n &\ge n^{-3} \sum_{i,j=1} (n - \max(i,j)) \lambda_i \lambda_j \\ &= \iint_D (1 - \max(s,t)) x(s) x(t)\,ds\,dt + o(1) \\ &= \int_0^1 \left(\int_0^t x(s)\,ds \right)^2 dt + o(1), \qquad \text{as } n \to \infty. \end{aligned}$$

Hence we have proven

$$(2.6)\quad \liminf_{n\to\infty} B_n \ge \sup\left\{ \int_0^1 \left(\int_0^t x(s)\,ds \right)^2 dt;\ x \in L_q[0,1] \text{ and } \|x\|_q \le 1 \right\}.$$

Thus (2.3) follows from (2.5) and (2.6).

Writing $x(t)$ in terms of $y(t) = x(1-t)$ yields that the right-hand side in (2.3) is equal to

$$\sup\left\{ \int_0^1 \left(\int_t^1 y(s) ds \right)^2 dt;\ y \in L_q[0,1] \text{ and } \|y\|_q \le 1 \right\}.$$

By (5) of [7],

$$(2.7)\qquad \sup_{x \in K} \int_0^1 |x(t)|^p\,dt = \Lambda(p),$$

where

$$K = \left\{ x;\ x(0) = 0, x(t) \text{ is absolutely continuous on } [0,1] \text{ and } \int_0^1 |\dot{x}(t)|^2 dt \le 1 \right\}.$$

If we observe that the left-hand side in (2.7) is equal to

$$\sup\left\{ \int_0^1 \left| \int_0^t x(s)\,ds \right|^p dt;\ x \in L_2[0,1] \text{ and } \|x\|_2 \le 1 \right\},$$

then, in order to complete our proof, we need only show that

$$(2.8)\quad \begin{aligned} &\sup\left\{\left(\int_0^1\left(\int_t^1 y(s)\,ds\right)^2 dt\right)^{1/2};\, y\in L_q[0,1] \text{ and } \|y\|_q\le 1\right\}\\ &=\sup\left\{\left(\int_0^1\left|\int_0^t x(s)\,ds\right|^p dt\right)^{1/p};\, x\in L_2[0,1] \text{ and } \|x\|_2\le 1\right\}.\end{aligned}$$

Define the linear continuous operators T_1 and T_2 as

$$T_1\colon L_2[0,1]\to L_p[0,1],\qquad (T_1x)(t)=\int_0^t x(s)\,ds,\qquad x\in L_2[0,1],$$

$$T_2\colon L_q[0,1]\to L_2[0,1],\qquad (T_2y)(t)=\int_t^1 y(s)\,ds,\qquad y\in L_q[0,1],$$

and note that

$$(L_p[0,1])^*=L_q[0,1],\qquad (L_2[0,1])^*=L_2[0,1]$$

and, for any $x\in L_2[0,1]$ and $y\in L_q[0,1]$,

$$(T_1x,y)=\int_0^1 y(t)\left(\int_0^t x(s)\,ds\right)dt=\int_0^1 x(t)\left(\int_t^1 y(s)\,ds\right)dt=(x,T_2y),$$

where the second equality follows from partial integration. Therefore, T_2 is the dual operator of T_1, so

$$\|T_2\|=\|T_1\|.$$

From the definitions of T_1 and T_2, this is equivalent to (2.8). □

For the development of results in this paper, we need some concepts from [8]. Let $\{b_n\}$ be a sequence of real numbers such that

$$(2.9)\qquad 0<b_1\le b_2\le\cdots,\ b_n\uparrow\infty.$$

An increasing sequence $\{n_k\}$ of positive integers will be called admissible if and only if

$$b_{n_{k+1}}\ge\lambda b_{n_k}\quad\text{for some } \lambda>1 \text{ and all } k\ge 1.$$

The next lemma is somewhat like the results in [8], and the proof is based on a technique used in [5].

LEMMA 2. *Let $\{X_n\}$ be a sequence of independent (not necessarily identically distributed) random variables with values in a separable Banach space, let $\{b_n\}$ satisfy* (2.9) *and let $\{n_k\}$ be admissible. Set $S_n=\sum_{i=1}^n X_i$, $n\ge 1$, and assume*

$$(2.10)\qquad \sum_k P\left\{\|S_{n_k}\|>Mb_{n_k}\right\}<+\infty\quad\text{for some } M>0.$$

Then, for any $r > 0$,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{b_n} \geq r \quad a.s., \tag{2.11}$$

whenever

$$\sum_k P\Big\{\|S_{n_k}\| > rb_{n_k}\Big\} = +\infty. \tag{2.12}$$

PROOF. For given $\varepsilon > 0$, choose a positive integer s such that

$$\frac{b_{n_{k+1}}}{b_{n_{k+s}}} M < \varepsilon \quad \text{for all } k,$$

and define the stopping time

$$\tau_k = \min\{n;\, n \geq n_k, \|S_n\| > rb_n\},$$

where we take $\tau = \infty$ if $\|S_n\| \leq rb_n$ for all $n \geq n_k$. Hence we have

$$\begin{aligned} D_k &\equiv \big\{\|S_n\| \leq rb_n \text{ for all } n \geq n_{k+s}, \tau_k \in [n_k, n_{k+1})\big\} \\ &\supset \big\{\|S_n\| \leq rb_n \text{ for all } n \geq n_{k+s}, \|S_{n_{k+1}}\| \leq Mb_{n_{k+1}}, \tau_k \in [n_k, n_{k+1})\big\} \\ &\supset \big\{\|S_n - S_{n_{k+1}}\| \leq (r-\varepsilon)b_n \text{ for all } n \geq n_{k+s}, \\ &\qquad\qquad \|S_{n_{k+1}}\| \leq Mb_{n_{k+1}}, \tau_k \in [n_k, n_{k+1})\big\}. \end{aligned}$$

Thus, for each k,

$$\begin{aligned} &P(D_k) + P\big\{\|S_{n_{k+1}}\| > Mb_{n_k}\big\} \\ &\quad \geq P\big\{\|S_n - S_{n_{k+1}}\| \leq (r-\varepsilon)b_n \text{ for all } n \geq n_{k+s}\big\} P\big\{\tau_k \in [n_k, n_{k+1})\big\}. \end{aligned}$$

Given a positive integer N, we have for any $n_{k+1} \geq N$ that

$$\begin{aligned} &\big\{\|S_n - S_{n_{k+1}}\| \leq (r-\varepsilon)b_n \text{ for all } n \geq n_{k+s}\big\} \\ &\quad \supset \big\{\|S_n - S_{n_{k+1}}\| \leq (r-\varepsilon)b_n \text{ for all } n \geq n_{k+s}, \|S_m\| \leq (r-2\varepsilon)b_m \\ &\qquad\quad \text{for all } m \geq N\big\} \\ &\quad = \big\{\|S_m\| \leq (r-2\varepsilon)b_m \text{ for all } m \geq N\big\}, \end{aligned}$$

since $r \leq M$ and therefore $\|S_{n_{k+1}}\| \leq rb_{n_{k+1}}$ implies $\|S_{n_{k+1}}\| \leq \varepsilon b_{n_{k+s}}$.

From the above, for any $n_{k+1} \geq N$,

$$\begin{aligned} &P(D_k) + P\big\{\|S_{n_{k+1}}\| \geq Mb_{n_{k+1}}\big\} \\ &\quad \geq P\big\{\|S_m\| \leq (r-2\varepsilon)b_m \text{ for all } m \geq N\big\} P\big\{\tau_k \in [n_k, n_{k+1})\big\}. \end{aligned}$$

From the definition of D_k we easily see that at most s of the events D_k can occur at a single time, so

$$\sum_k P(D_k) = E(\#D_k \text{ which occur }) \leq s.$$

Using assumption (2.10) yields

$$\begin{aligned}\infty &> \sum_{n_{k+1}\ge N}\Big(P(D_k)+P\{\|S_{n_{k+1}}\|>Mb_{n_{k+1}}\}\Big)\\ &\ge P\{\|S_m\|\le(r-2\varepsilon)b_m \text{ for all } m\ge N\}\sum_{n_{k+1}\ge N}P\{\tau_k\in[n_k,n_{k+1})\}.\end{aligned}$$

Hence it follows from (2.12) that

$$P\{\|S_m\|\le(r-2\varepsilon)b_m \text{ for all } m\ge N\}=0.$$

Thus

$$\limsup_{n\to\infty}\frac{\|S_n\|}{b_n}\ge r-2\varepsilon \quad \text{a.s.},$$

and since $\varepsilon>0$ is arbitrary we have proven (2.11). □

3. Proof of Theorem 1. The following notation will be used throughout this and the next section. Write

$$b_n=\sqrt{2n^{1+2/p}L_2n}, \qquad n=1,2,\dots,$$

and, for fixed $p\ge 1$, set

$$l_p(B)=\Big\{\{x_n\}_{n\ge1};\, x_n\in B, n\ge 1 \text{ and } \sum_n\|x_n\|^P<+\infty\Big\}.$$

The norm on $l_p(B)$ is defined by

$$|||\bar{x}|||=\Big\{\sum_n\|x_n\|^p\Big\}^{1/p},$$

where $\bar{x}=\{x_n\}\in l_p(B)$. We easily see that $l_p(B)$ is a separable Banach space.

Assume (i), (ii) and (iii) [(iii′)] hold. We now prove (1.3) [(1.4)]. Before the technical details, let us outline the main idea of the proof. As is well known, condition (i) is equivalent to

$$\sum_n P\Big\{\|X\|>\sqrt{2nL_2n}\Big\}<+\infty.$$

Then, if we do the following truncation,

$$\begin{aligned}Y_n&=X_nI_{\{\|X_n\|\le\sqrt{2nL_2n}\}}-E\Big(X_nI_{\{\|X_n\|\le\sqrt{2nL_2n}\}}\Big),\\ Z_n&=X_nI_{\{\|X_n\|>\sqrt{2nL_2n}\}}-E\Big(X_nI_{\{\|X_n\|>\sqrt{2nL_2n}\}}\Big),\\ S_n'&=\sum_{i=1}^nY_i \quad\text{and}\quad S_n''=\sum_{i=1}^nZ_i, \qquad n=1,2,\dots,\end{aligned}$$

it follows immediately from the Borel–Cantelli lemma that

$$\lim_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)} \sum_{i=0}^{n} \|S_n'' - S_i''\|^p = 0 \quad \text{a.s.}$$

Hence, (1.3) and (1.4) are equivalent to, respectively,

$$\limsup_{n\to\infty} \frac{1}{b_n}\left\{\sum_{i=0}^{n} \|S_n' - S_i'\|^p\right\}^{1/p} < +\infty \quad \text{a.s.,} \tag{3.1}$$

$$\limsup_{n\to\infty} \frac{1}{b_n}\left\{\sum_{i=0}^{n} \|S_n' - S_i'\|^p\right\}^{1/p} = \left(\Lambda(p)\right)^{1/p}\sigma \quad \text{a.s.} \tag{3.2}$$

Recall Theorem 1.2 of Chen [2], which states that if $\{\xi_n\}$ is a sequence of independent random variables with values in a separable Banach space E with norm $|||\cdot|||$ such that $\xi_n \in WM_0^2$, $n \geq 1$, and

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{s_{n+1}}{s_n} < +\infty, \tag{3.3}$$

where

$$s_n \equiv \sup_{\alpha\in E_1^*} E\alpha^2\left(\sum_{i=1}^{n} \xi_i\right), \qquad n = 1, 2, \dots, \tag{3.4a}$$

$$\sum_n \frac{E|||\xi_n|||}{\left(2s_n^2 L_2 s_n^2\right)^{\lambda/2}} < +\infty \quad \text{for some } \lambda > 2 \tag{3.4b}$$

and

$$\left\{\sum_{i=1}^{n} \xi_i \Big/ \sqrt{2s_n^2 L_2 s_n^2}\right\}_{n\geq 1} \quad \text{is bounded in probability,} \tag{3.5}$$

then

$$\limsup_{n\to\infty} \left|\left|\left|\sum_{i=1}^{n} \xi_i\right|\right|\right| \Big/ \sqrt{2s_n^2 L_2 s_n^2} < +\infty \quad \text{a.s.} \tag{3.6}$$

Furthermore,

$$\limsup_{n\to\infty} \left|\left|\left|\sum_{i=1}^{n} \xi_i\right|\right|\right| \Big/ \sqrt{2s_n^2 L_2 s_n^2} = 1 \quad \text{a.s.,} \tag{3.7}$$

whenever (3.5) is strengthened to

$$\sum_{i=1}^{n} \xi_i \Big/ \sqrt{2s_n^2 L_2 s_n^2} \to 0 \quad \text{in probability.} \tag{3.8}$$

Now, take $E = l_p(B)$ and define a sequence $\{\xi_n\}$ of independent random variables with values in $l_p(B)$ as follows:

$$\begin{aligned}\xi_1 &= \{Y_1, 0, 0, 0, \ldots\},\\ \xi_2 &= \{Y_2, Y_2, 0, 0, \ldots\},\\ \xi_3 &= \{Y_3, Y_3, Y_3, 0, \ldots\},\\ &\vdots\end{aligned}$$

We see that $\xi_n \in WM_0^2, n \geq 1$, and

$$\left|\left|\left| \sum_{i=1}^{n} \xi_i \right|\right|\right| = \left\{ \sum_{i=1}^{n} \|S_n' - S_i'\|^p \right\}^{1/p}, \qquad n = 1, 2, \ldots. \tag{3.9}$$

What we intend below is to verify (3.3), (3.4b) and (3.5) [(3.8)] for such $\{\xi_n\}$ and to identify (3.1) and (3.2) with (3.6) and (3.7), respectively.

Let $\{s_n\}$ be defined by (3.4a). It is easy to verify that, when $p = 1$,

$$s_n^2 = \sup_{f_0, \ldots, f_n \in B_1^*} \max \left\{ E\left(\sum_{i=0}^{n} \lambda_i f_i (S_n' - S_i') \right)^2 ; \max_{i \leq n} |\lambda_i| \leq 1 \right\}$$

and, when $p > 1$,

$$s_n^2 = \sup_{f_0, \ldots, f_n \in B_1^*} \max \left\{ E\left(\sum_{i=0}^{n} \lambda_i f_i (S_n' - S_i') \right)^2 ; \sum_{i=0}^{n} |\lambda_i|^q \leq 1 \right\},$$

where $1/p + 1/q = 1$.

For any $f_0, \ldots, f_n \in B_1^*$ and $\lambda_0, \ldots, \lambda_n \in R$,

$$\begin{aligned} E\left(\sum_{i=0}^{n} \lambda_i f_i (S_n' - S_i') \right)^2 &\leq E\left(\sum_{i=0}^{n} \lambda_i f_i (S_n - S_i) \right)^2 \\ &= \sum_{i,j=0}^{n} \lambda_i \lambda_j \big(n - \max(i,j)\big) E\big(f_i(X) f_j(X)\big) \\ &\leq \sigma^2 \sum_{i,j=0}^{n} \lambda_i \lambda_j \big(n - \max(i,j)\big). \end{aligned}$$

Hence, by Lemma 1 we obtain

$$\limsup_{n \to \infty} \frac{s_n^2}{n^{1+(2/p)}} \leq \big(\Lambda(p)\big)^{2/p} \sigma^2. \tag{3.10}$$

On the other hand, given $f \in B_1^*$

$$s_n^2 \geq \max\left\{E\left(\sum_{i=0}^{n} \lambda_i f(S_n' - S_i')\right)^2; (\lambda_0, \dots, \lambda_n) \in C(p)\right\}$$

$$= \max\left\{\sum_{i,j=0}^{n} \lambda_i \lambda_j \sum_{k=\max(i,j)+1}^{n} Ef^2(Y_k); (\lambda_0, \dots, \lambda_n) \in C(p)\right\},$$

where

$$C(p) = \begin{cases} \left\{(\lambda_0, \dots, \lambda_n); \sum_{i=0}^{n} |\lambda_i|^q \leq 1\right\}, & \text{if } p > 1, \\ \left\{(\lambda_0, \dots, \lambda_n); \max_{i \leq n} |\lambda_i| \leq 1\right\}, & \text{if } p = 1. \end{cases}$$

From the definition of Y_k it follows that

$$\lim_{k \to \infty} Ef^2(Y_k) = Ef^2(X).$$

By the Toeplitz lemma, uniformly for $1 \leq i, j < n$,

$$\lim_{n \to \infty} \frac{1}{n - \max(i,j)} \sum_{k=\max(i,j)+1}^{n} Ef^2(Y_k) = Ef^2(X).$$

Using Lemma 1 and the Toeplitz lemma again yields

$$\liminf_{n \to \infty} \frac{s_n^2}{n^{1+(2/p)}} \geq (\Lambda(p))^{2/p} Ef^2(X).$$

and since $f \in B_1^*$ is arbitrary we have

$$\liminf_{n \to \infty} \frac{s_n^2}{n^{1+(2/p)}} \geq (\Lambda(p))^{2/p} \cdot \sigma^2 \tag{3.11}$$

Combining (3.10) and (3.11), we have proven

$$s_n^2 \sim n^{1+(2/p)} (\Lambda(p))^{2/p} \sigma^2, \qquad \text{as } n \to \infty. \tag{3.12}$$

Hence

$$\sqrt{2 s_n^2 L_2 s_n^2} \sim b_n (\Lambda(p))^{1/p} \sigma, \qquad \text{as } n \to \infty. \tag{3.13}$$

Therefore, by (3.9), we need only show (3.6) [(3.7)] holds. To do this, by Theorem 1.2 of Chen [2] it suffices to verify that (3.3), (3.4b) and (3.5) [(3.8)] hold.

From (3.12), (3.3) is obvious. As is well known, condition (i) implies

$$\sum_n \frac{E\|Y_n\|^3}{(2n L_2 n)^{3/2}} < +\infty,$$

and from the definition of $\{\xi_n\}$,

$$E|||\xi_n|||^3 = n^{3/p}E\|Y_n\|^3, \qquad n = 1, 2, \ldots.$$

It follows that

$$\sum_n \frac{E|||\xi_n|||^3}{b_n^3} < +\infty.$$

By (3.13), (3.4b) holds in particular for $\lambda = 3$.

At last, we prove (3.5) [(3.8)]. By (3.9) and Jensen's inequality, for each n,

$$E\left|\left|\left|\sum_{i=1}^{n} \xi_i\right|\right|\right|^p = \sum_{i=0}^{n} E\|S_n' - S_i'\|^p \le nE\|S_n'\|^p. \tag{3.14}$$

By following a standard procedure via symmetrization and the Hoffmann–Jørgensen inequality [4], we see that

$$\left\{E\|S_n'\|^p/(2nL_2n)^{p/2}\right\}_{n\ge 1} \quad \text{is bounded}$$

if (iii) holds, and

$$\lim_{n\to\infty} E\|S_n'\|^p/(2nL_2n)^{p/2} = 0$$

if (iii′) holds. Therefore, the desired conclusion follows from (3.13) and (3.14).

Conversely, suppose (1.3) holds and define $\{\eta_n\}$ as follows:

$$\begin{aligned}\eta_1 &= \{X_1, 0, 0, 0, \ldots\}\\ \eta_2 &= \{X_2, X_2, 0, 0, \ldots\}\\ \eta_3 &= \{X_3, X_3, X_3, 0, \ldots\}\\ &\vdots\end{aligned}$$

Then, for each n,

$$\left|\left|\left|\sum_{i=1}^{n} \eta_i\right|\right|\right| = \left\{\sum_{i=0}^{n} \|S_n - S_i\|^p\right\}^{1/p}, \tag{3.15}$$

so (1.3) implies that

$$\limsup_{n\to\infty} \left|\left|\left|\sum_{i=1}^{n} \eta_i\right|\right|\right| \Big/ b_n \le K \quad \text{a.s. for some } K > 0.$$

Hence

$$\limsup_{n\to\infty} |||\eta_n|||/b_n \le 2K \quad \text{a.s.}$$

By the Borel–Cantelli lemma

$$\sum_n P\{|||\eta_n||| > 2Kb_n\} < +\infty.$$

From the definition of $\{\eta_n\}$, for each n,

$$|||\eta_n||| = n^{1/p}\|X_n\|.$$

Thus

$$\sum_n P\Big\{\|X\| > 2K\sqrt{2n\,L_2 n}\Big\} < +\infty.$$

This implies condition (i)

We now prove (ii). It is obvious that (1.3) implies $Ef(X) = 0$ for all $f \in B^*$. [Otherwise it follows from the law of large numbers that, with probability 1, $\|S_n\| \sim Cn$ for some $C > 0$, which leads to a contradiction with (1.3).] Without loss of generality we may assume X is symmetric. Let $C > 0$ be arbitrary, but fixed, and define

$$X_n' = X_n I_{\{|X_n|\le C\}} - X_n I_{\{|X_n|>C\}}, \qquad \overline{X}_n = X_n I_{\{|X_n|\le C\}},$$
$$\overline{S}_n = \sum_{i=1}^n \overline{X}_i \quad\text{and}\quad T_n = \sum_{i=1}^n X_i', \qquad n = 1,2,\dots.$$

By symmetry, $\{X_n'\}$ has the same distribution as $\{X_n\}$. Since by the 0–1 law (1.3) implies

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(1/p)}\sum_{i=0}^n \|S_n - S_i\|^p = M \quad \text{a.s.},$$

for some finite number M, it follows from

$$\sum_{i=0}^n \|\overline{S}_n - \overline{S}_i\|^p \le \frac{1}{2}\left\{\sum_{i=0}^n \|S_n - S_i\|^p + \sum_{i=0}^n \|T_n - T_i\|^p\right\}$$

that

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)}\sum_{i=0}^n \|\overline{S}_n - \overline{S}_i\|^p \le 2^{p-1}M \quad \text{a.s.},$$

For each $f \in B^*$, specializing (1.4) to the case $B = R$ yields

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2 n)^{-(p/2)}\sum_{i=0}^n |f(\overline{S}_n - \overline{S}_i)|^p$$
$$= \Big(Ef^2(XI_{\{|X|\le C\}})\Big)^{p/2}\Lambda(p) \quad \text{a.s.}$$

Hence we have proven that, for each $f \in B^*$,

$$\left(Ef^2(XI_{\{|X|\le C\}})\right)^{p/2}\Lambda(p) \le \frac{1}{2}M\|f\|^p,$$

from which the desired conclusion follows when $C \to \infty$.

To prove (iii), simply note that, for each n,

$$n\|S_n\|^p \le 2^{p-1}\left\{\sum_{i=1}^{n}\|S_i\|^p + \sum_{i=0}^{n}\|S_n - S_i\|^p\right\}.$$

By (1.2) this implies, for each n and any $M > 0$,

$$\begin{aligned} &P\left\{\|S_n\| > M\sqrt{2nL_2n}\right\} \\ &\le 2P\left\{\sum_{i=0}^{n}\|S_n - S_i\|^p > \frac{M^p}{2}n^{1+(p/2)}(2L_2n)^{p/2}\right\}. \end{aligned} \tag{3.16}$$

Since (1.3) clearly implies

$$\left\{n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=0}^{n}\|S_n - S_i\|^p\right\}_{n\ge1} \quad \text{is bounded in probability,}$$

we can see that (iii) follows from (3.16).

4. Proof of Theorem 2. The proof of Theorem 2 depends mainly on the conclusion of Theorem 1. Assume that conditions (i), (ii) and (iii) hold. By Theorem 1, there exists a constant $\Gamma > 0$ such that

$$\limsup_{n\to\infty}\left|\left|\left|\sum_{i=1}^{n}\eta_i\right|\right|\right| \Big/ b_n = \Gamma \quad \text{a.s.,} \tag{4.1}$$

where $\{\eta_n\}$ is defined in Section 3 and $\Gamma = (\Lambda(p))^{1/p}\cdot\sigma$ whenever (iii′) holds. Note that, for any $f \in B_1^*$, Strassen's result [7] implies

$$\begin{aligned} &\limsup_{n\to\infty}\ n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p \\ &\qquad \ge \limsup_{n\to\infty}, n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}|f(S_i)|^p = \Lambda(p)\left\{Ef^2(X)\right\}^{p/2} \quad \text{a.s.} \end{aligned}$$

Thus

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2L_2n)^{-(p/2)}\sum_{i=1}^{n}\|S_i\|^p \ge \Lambda(p)\sigma^p \quad \text{a.s.}$$

Therefore, in order to prove (1.5) [(1.6)], it suffices to show that

$$\limsup_{n\to\infty}\frac{1}{b_n}\left\{\sum_{i=1}^{n}\|S_i\|^p\right\}^{1/p} \le \Gamma \quad \text{a.s.}$$

By the 0–1 law, the left-hand side is equal to, with probability 1, either some finite constant or ∞. Therefore, it suffices to show that, for any constant r such that

$$\limsup_{n\to\infty} \frac{1}{b_n}\left\{\sum_{i=1}^{n} \|S_i\|^p\right\}^{1/p} > r \quad \text{a.s.},$$

we have

$$r \leq \Gamma. \tag{4.2}$$

Let r be fixed; take $\lambda > 1$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n}\left\{\sum_{i=1}^{n} \|S_i\|^p\right\}^{1/p} > \lambda^{1/2+1/p} r \quad \text{a.s.},$$

and set $n_k = [\lambda^k], k = 1, 2, \ldots$. Since $\{(\sum_{i=1}^{n} \|S_i\|^p)^{1/p}\}_{n\geq 1}$ is nondecreasing and

$$\lim_{k\to\infty} b_{n_k}/b_{n_{k+1}} = \lambda^{-(1/p)-1/2},$$

we easily see that

$$\limsup_{k\to\infty} \frac{1}{b_n}\left\{\sum_{i=1}^{n_k} \|S_i\|^p\right\}^{1/p} > r \quad \text{a.s.}$$

By the Borel–Cantelli lemma

$$\sum_k P\left\{\left(\sum_{i=1}^{n_k} \|S_i\|^p\right)^{1/p} > r b_{n_k}\right\} = \infty,$$

so it follows from (1.2) and (3.15) that

$$\sum_k P\left\{\left|\left|\left|\sum_{i=1}^{n_k} \eta_i\right|\right|\right| > r b_{n_k}\right\} = \infty. \tag{4.3}$$

In order to apply Lemma 2 to (4.3), we now prove

$$\sum_k P\left\{\left|\left|\left|\sum_{i=1}^{n_k} \eta_i\right|\right|\right| > M b_{n_k}\right\} < \infty \quad \text{for some } M > 0. \tag{4.4}$$

Set $m_k = n_1 + \cdots + n_k, k = 1, 2, \ldots$. It follows from (4.1) that

$$\lim_{k\to\infty} \frac{1}{b_{m_{k+1}}}\left|\left|\left|\sum_{i=m_k+1}^{m_{k+1}} \eta_i\right|\right|\right| \leq 2\Gamma \quad \text{a.s.}$$

Hence

$$\sum_k P\left\{\left|\left|\left|\sum_{i=m_k+1}^{m_{k+1}} \eta_i\right|\right|\right| > 2\Gamma b_{m_{k+1}}\right\} < +\infty.$$

Note that, for each k,

$$\begin{aligned}\left|\left|\left|\sum_{i=m_k+1}^{m_{k+1}} \eta_i\right|\right|\right| &= \left\{\sum_{i=m_k}^{m_{k+1}} \|S_{m_{k+1}} - S_i\|^p + (m_{k+1} - m_k - 1)\|S_{m_{k+1}} - S_{m_k}\|^p\right\}^{1/p} \\ &\geq \left\{\sum_{i=m_k}^{m_{k+1}} \|S_{m_{k+1}} - S_i\|^p\right\}^{1/p} =_d \left|\left|\left|\sum_{i=1}^{n_{k+1}} \eta_i\right|\right|\right|.\end{aligned}$$

Therefore

$$\sum_k P\left\{\left|\left|\left|\sum_{i=1}^{n_{k+1}} \eta_i\right|\right|\right| > 2\Gamma b_{m_{k+1}}\right\} < +\infty.$$

From the definitions of $\{n_k\}$ and $\{m_k\}$, one can find a constant $A > 0$ such that

$$b_{m_k} \leq A b_{n_k} \quad \text{for all } k.$$

So (4.4) holds for $M \equiv 2\Gamma A$.

By Lemma 2, (4.3) implies (4.2).

Conversely, assume (1.5) holds. This implies $\{n^{-1-(p/2)}(2L_2 n)^{-(p/2)} \sum_{i=1}^n \|S_i\|^p\}_{n\geq 1}$ is bounded in probability. By (1.2), condition (iii) follows from (3.16) so there exists a constant $M > 0$ such that

$$P\left\{\|S_n\| < M\sqrt{2n L_2 n}\right\} \geq \tfrac{1}{2} \quad \text{for all } n.$$

Set

$$K = \limsup_{n\to\infty} \frac{1}{b_n}\left(\sum_{i=1}^n \|S_i\|^p\right)^{1/p}.$$

We now prove

$$\sum_m P\left\{\left(\sum_{i=2^m+1}^{2^{m+1}} \|S_i - S_{2^m}\|^p\right)^{1/p} > (K+M) b_{2^{m+1}}\right\} < +\infty. \tag{4.5}$$

Suppose this is not the case, and set

$$\begin{aligned} A_m &= \left\{\left(\sum_{i=2^m+1}^{2^{m+1}} \|S_i - S_{2^m}\|^p\right)^{1/p} > (K+M) b_{2^{m+1}}\right\} \\ B_m &= \left\{\|S_{2^m}\| < M\sqrt{2^{m+1} L_2 2^m}\right\}, \qquad m = 1, 2, \ldots.\end{aligned}$$

Then,

$$\sum_m P(A_m) = \infty \quad \text{and} \quad P(B_m) > \tfrac{1}{2}, \qquad m \geq 1.$$

By Lemma 1 of [1],

$$P\{A_m B_m \text{ i.o. }\} \geq \tfrac{1}{2}.$$

It is easily seen how this implies

$$P\left\{ \left(\sum_{i=1}^{2^{m+1}} \|S_i\|^p \right)^{1/p} > K b_{2^{m+1}} \qquad \text{i.o.} \right\} \geq \tfrac{1}{2}.$$

By the 0–1 law

$$\limsup_{m\to\infty} \frac{1}{b_{2^{m+1}}} \left(\sum_{i=1}^{2^{m+1}} \|S_i\|^p \right)^{1/p} > K \quad \text{a.s.}$$

This contradicts the definition of K.

By (3.15), (4.5) implies

$$\sum_m P\left\{ \left\| \sum_{i=1}^{2^m} \eta_i \right\| > (K+M) b_{2^{m+1}} \right\} < \infty.$$

Hence (1.3) holds. By the necessity part of Theorem 1, we also obtain (i) and (ii).

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