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ROTATIONAL REPRESENTATIONS OF TRANSITION MATRIX FUNCTIONS

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Let $P(h) = (p_{ij}(h), i, j = 1, 2, \dots, n), h \geq 0, n \geq 1$, be a transition matrix function defining an irreducible recurrent continuous parameter Markov process. Let $(S_i, i = 1, 2, \dots, n)$ be a partition of the circle into sets S_i each consisting of a finite union of arcs $A_{k\ell}$. Let f_t be a rotation of length t of the circle, and denote Lebesgue measure by λ. We generalize and prove for the transition matrix function $P(h)$ a theorem of Cohen ($n = 2$) and Alpern ($n \geq 2$) asserting that every recurrent stochastic $n \times n$ matrix P is given by

$$(*) \qquad p_{ij} = \left(\lambda(S_i \cap f_t^{-1}(S_j))\right) / \lambda(S_i),$$

for some choice of rotation f_t and partition $\{S_i\}$. We prove the existence of a continuous map Φ from the space of $n \times n$ irreducible stochastic matrices into n-partitions of $[0, 1)$, such that every domain matrix P is represented by $(*)$ with $\{S_i\} = \Phi(P)$ and $t = 1/n!$. Furthermore, the representing process $(f_t, \{S_i\})$ has not only the same transition probabilities but also the same probabilistic cycle distribution as the Markov process based on P.

1. Preliminaries. Cohen [2] has proposed a geometric approach to finite order stochastic matrices. Given a natural number n, the set $S = 1, 2, \dots, n$ and the probability space $([0, 1), B, \lambda)$, where B and λ denote the σ-algebra of Borel subsets of $[0, 1)$ and Lebesgue measure, Cohen's idea starts heuristically from the pair $(t, \mathcal{S})$, where $t > 0$ defines the shift transformation $f_t(x) = (x + t) \pmod 1$ on $[0, 1)$, and $\mathcal{S} = (S_i,\ i \in S)$ is a partition of $[0, 1)$ into sets S_i with $\lambda(S_i) > 0$. Then he points out that the expression

$$(1) \qquad p_{ij} = \lambda\left(S_i \cap f_t^{-1}(S_j)\right) / \lambda(S_i), \qquad i, j \in S,$$

defines a stochastic matrix of a recurrent S-state Markov chain having an invariant probability distribution $\pi = (\pi_i, i \in S)$, with $\pi_i = \lambda(S_i)$.

When (1) holds, the stochastic matrix $P = (p_{ij}, i, j \in S)$ is said to have a *rotational representation* symbolized by $(t, \mathcal{S})$. The partition $\mathcal{S}$ is said to have type L if the number of components of S_i is less than or equal to $L, i = 1, \dots, n$. Let $b = b(n)$ denote the least integer such that any $n \times n$ recurrent matrix has a representation $(t, \mathcal{S})$, where $\mathcal{S}$ is of type b.

Conversely, Cohen [2] conjectured that any $n \times n$ irreducible stochastic matrix with $n \geq 2$ has a representation as in (1) such that each set S_i is the union of finitely many intervals $A_{k\ell}$. He proved this conjecture when $n = 2$. Alpern [1]

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showed that this conjecture can be extended from irreducible to recurrent matrices, proving that there exists a rotational representation $(t, \mathcal{S})$, with $t = 1/n!$, and that $b(n)$ belongs to a bounded interval $(\exp(\alpha n^{1/2}), \exp(\beta n))$, where α and β are certain positive numbers. The latter corrects the original false conjecture of Cohen asserting that $b(n) = n - 1$. Haigh [3] has proved that $b(3) = 2$ and that $b(n)$ is nondecreasing. Recently Rodríguez and Valsero [8] improved the Cohen–Alpern rotational representation for reversible stochastic matrices. Moreover, a necessary and sufficient condition for the reversibility in terms of rotational representation is given.

All these solutions to rotational representations are combinatorial approaches that provide decompositions of the stochastic matrices in terms of nonunique cycle distributions.

In the present paper we prove the existence of a continuous map Φ from the space of $n \times n$ irreducible stochastic matrices P into n-partitions $\mathcal{S} = (S_i, i = 1, \ldots, n)$ of $[0, 1)$ such that the rotational representation process has not only the same transition probabilities but also the same distribution of cycles as the probabilistic cycle distribution of the Markov process ξ on P (Theorem 1).

The difference from Alpern's approach consists in using a canonical decomposition of P in terms of the directed cycles $\hat{c}$ that occur along the sample paths of ξ and of a unique cycle distribution $\delta = (\omega_{\hat{c}})_{\hat{c}}$, where each cycle weight $\omega_{\hat{c}}$ is given a probabilistic interpretation as follows: $\omega_{\hat{c}}$ is the mean number of occurrences of $\hat{c}$ on almost all the trajectories of ξ. For this reason the distribution δ is called a probabilistic cycle distribution. In Theorem 2 this proof is generalized to any semigroup $P(h) = (p_{ij}(h), i, j = 1, \ldots, n), h \geq 0$, of stochastic matrices accepting an invariant probability distribution $\pi = (\pi_i, i = 1, \ldots, n)$ using the canonical cycle decomposition of [6],

$$\pi_i \, p_{ij}(h) = \sum_{\hat{c}} \omega_{\hat{c}}(h) J_c(i, j), \tag{2}$$

where $\hat{c}$ and c are directed cycles and circuits, $(\omega_{\hat{c}}(h))_{\hat{c}}$ is the probabilistic cycle distribution of the h-skeleton chain on $P(h)$, and $J_c(i, j)$ is 1 or 0 according as (i, j) is or is not an edge of the circuit c. A rigorous presentation of the canonical decomposition of stochastic matrices is given in Section 2.

Finally, in Theorem 3 it is shown that the type $L(h)$ of the partition $\Phi(P(h))$ in the rotational representation of each matrix $P(h)$, $h > 0$, is independent of h [then $L(h) = L$ is a semigroup feature].

2. Mapping recurrent stochastic matrices into partitions. Let n be any natural number and $S = \{1, \ldots, n\}$. Consider $P = (p_{ij}, i, j \in S)$, a stochastic matrix defining an S-state homogeneous irreducible Markov chain $\xi = (\xi_m, m \geq 0)$ with the invariant probability distribution $\pi = (\pi_i, \ i \in S)$.

In this section we deal with the existence of a map from $n \times n$ irreducible stochastic matrices P above into a set of n-partitions of $[0, 1)$ and with the study of the continuity properties of this map.

Before this we motivate a basic modification that we introduce into Alpern's algorithm (recalled in subsection 2.2) to ensure the uniqueness of the cycle decomposition on P.

First we notice that the combinatorial device of the Alpern algorithm for obtaining an n-partition depends upon many arbitrary choices, such as the choice of the starting point from where the representative circuits are constructed as well as the choice of the starting circuit of the decomposition (see also [4]). Therefore Alpern's algorithm associates each recurrent matrix P with more than one cycle decomposition that in turn involves many distributions of cycles in the rotational representation process.

On the other hand, when we generalize the rotational representation to transition matrix functions $\{P(h),\ h \geq 0\}$ (cf. Theorem 3), the class $\mathcal{C}(h)$ of representative circuits for $P(h)$ occurring in the Alpern algorithm depends upon h. For the latter it would then be difficult to prove that the type of n-partitions associated with the semigroup $\{P(h),\ h \geq 0\}$ is independent of h—as we shall show in Theorem 3.

In light of this, we must change the combinatorial criterion in the Alpern algorithm to a probabilistic criterion ensuring that the distribution of cycles in the rotational representation matches the unique probabilistic distribution in the Markov process defined by P.

To this end we give in the following subsection a presentation of the probabilistic cycle distribution and of the canonical decomposition of stochastic matrices as studied in [5] and [6]. This will be followed by a subsection recalling Alpern's algorithm.

2.1. A directed circuit c in a finite set S is any periodic function $c: Z \to S$, where Z is the set of integers. The smallest integer $p = p(c) \geq 1$ for which $c(n + p) = c(n)$ for all $n \in Z$ is called the period of c. Cyclic permutations are avoided if we redefine a circuit to be any equivalence class of periodic functions c with respect to the following equivalence relation: $c \sim c'$ iff $c(n) = c'(n + i)$ for some $i \in Z$ and all $n \in Z$. Then a circuit c of period s is defined by taking any sequence of s consecutive images, that is, $c = (c(n), c(n + 1), \ldots, c(n + s - 1), c(n)), n \in Z$.

Let $c = (c(n), c(n + 1), \ldots, c(n + s - 1), c(n))$ be a directed (class-) circuit with distinct points $c(n), \ldots, c(n + s - 1)$. Then $\widehat{c} = (c(n), c(n + 1), \ldots, c(n + s - 1))$ is called a directed class-cycle (or, for short, a cycle) associated with the circuit c.

Let $P = (p_{ij}, i, j \in S)$, and let ξ be the irreducible matrix and the Markov process mentioned at the beginning of this section. Then, according to [5], on each trajectory ω of ξ infinitely many cycles successively occur. For example, if the values $(\xi_n(\omega))_{n \geq 0}$ of ξ are $(1, 4, 2, 3, 2, 6, 7, 6, 1, \ldots)$, then the sequence of cycles occurring on this trajectory is given by $(2, 3), (6, 7), (4, 2, 6, 1), \ldots$.

This allows us to define a new Markov chain $y = (y_n(\omega))_{n \geq 0}$ whose value at time n is the track of the remaining states in sequence, after discarding the cycles formed up to n along $(\xi_n(\omega))_{n \geq 0}$ [see [4] and [7]).

In Table 1 we give the trajectories $(\xi_n(\omega))_n$ and $(y_n(\omega))_n$ as well as the cycles occurring along $(\xi_n(\omega))_n$.

Note that a cycle $\widehat{c} = (i_1, \ldots, i_r)$ of ω is closed by the edge (i_r, i_1) which appears on ω either one time after $\widehat{c}$ or many times before, as $(1, 4)$ in the cycle $(4, 2, 6, 1)$

TABLE 1

n	0	1	2	3	4
$\xi_n(\omega)$	1	4	2	3	2
$y_n(\omega)$	[1]	[1,4]	[1,4,2]	[1,4,2,3]	[1,4,2]
Cycles					(2,3)
n	5	6	7	8	...
$\xi_n(\omega)$	6	7	6	1	...
$y_n(\omega)$	[1,4,2,6]	[1,4,2,6,7]	[1,4,2,6]	[1]	...
Cycles			(6,7)	(4,2,6,1)	...

above, where the time-unit is the jump time of $(\xi_n(\omega))$ (see [5] for more details).

Let ω be an arbitrary trajectory of ξ, and let $\widehat{c}$ be a directed cycle appearing in ω.

Associate with $\widehat{c}$ the number $w_{c,n}(\omega)$ of occurrences of $\widehat{c}$ up to time n on ω. Then

$$w_{c,n}(\omega) = \sum_{m=1}^{n} 1_{\{\text{cycle } \hat{c} \text{ occurs}\}}(\omega).$$

Let

(3) $$\sigma_n(\omega;\, i, j) = (1/n)\operatorname{card}\{m \le n\colon\ \xi_{m-1}(\omega) = i,\ \xi_m(\omega) = j\}.$$

Following [5] we may write

(3′) $$\sigma_n(\omega;\, i, j) = (1/n) \sum_{\hat{c}\, \in\, \mathcal{C}_n(\omega)} w_{c,n}(\omega) J_c(i, j) + \varepsilon_n(\omega;\, i,j)/n,$$

where $\mathcal{C}_n(\omega)$ is the set of all the directed cycles occurring until time n along the sample path $(\xi_n(\omega))$,

$$J_c(i, j) = \begin{cases} 1, & \text{if } (i, j) \text{ is an edge of } c, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\varepsilon_n(\omega;\, i, j) = 1_{A_{ij}}(\omega),$$

with

$$A_{ij} \equiv A_{ij}(n) = \{\text{the last occurrence of } (i, j) \text{ does not happen together with the occurrence of a cycle of } \mathcal{C}_n(\omega)\}.$$

Then Theorem 3 of [5] asserts that, for any initial distribution p of ξ, the sequence $(\mathcal{C}_n(\omega), w_{c,n}(\omega)/n)$ of sample weighted cycles associated with the chain ξ converges almost surely to a class $(\mathcal{C}_\infty, \omega_{\hat{c}})$, that is,

$$(4)\qquad \begin{aligned} \mathcal{C}_\infty &= \lim_{n\to\infty} \mathcal{C}_n(\omega), && \mathbb{P}_p\text{-a.s.} \\ \omega_c &= \lim_{n\to\infty} \big(w_{c,n}(\omega)/n\big), && \mathbb{P}_p\text{-a.s.} \end{aligned}$$

That ω_c is well defined follows from [7].

Consider $\mathcal{C}_\infty = \{\hat{c}_1, \ldots, \hat{c}_N\}$. If we let $n \to \infty$ in (3′), Theorem 5 of [5] asserts that each irreducible matrix P has the following representation:

$$(5)\qquad \pi_i p_{ij} = \sum_{k=1}^{N} \omega_{\hat{c}_k} J_{c_k}(i, j),$$

where, according to (4) and the definition of $w_{c,n}$, the cycle weights ω_c are uniquely determined with the probabilistic interpretation of being the mean number of occurrences of the cycle $\hat{c}$ along almost all the sample paths of ξ. The latter is in good agreement with the unicity of $\pi_i p_{ij}$ as a limit of $\sigma_n(\omega; i, j)$ defined by (3).

DEFINITION. The collection $\delta = (\omega_{\hat{c}},\ \hat{c} \in \mathcal{C}_\infty)$ defined by (4) is called a probabilistic cycle distribution associated to P, while equation (5) is called a canonical decomposition of P.

We point out that the probabilistic cycle distribution has an exact expression in terms of P and its powers by equations

$$(6)\qquad \begin{aligned} \omega_{\hat{c}} &= \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{s-1} i_s} p_{i_s i_1} \\ &\quad \times N(i_2, i_2 \mid i_1) N(i_3, i_3 \mid i_1, i_2) \cdots N(i_s, i_s \mid i_1, \ldots, i_{s-1}), \end{aligned}$$

where $\hat{c} = (i_1, \ldots, i_s)$ and where

$$\begin{aligned} &N(i_k, i_k \mid i_1, \ldots, i_{k-1}) \\ &\quad = \sum_{n=0}^{\infty} \Pr(\xi_n = i_k,\ \xi_m \neq i_1, \ldots, i_{k-1},\ \text{for } 1 \le m \le n \mid \xi_0 = i_k) \end{aligned}$$

is the taboo Green function (see [4] and [5] for more details).

2.2. In this subsection we recall part of Alpern's algorithm for rotational representation that we shall need in our further investigations. Alpern starts from the decomposition

$$(7)\qquad \pi_i p_{ij} = \sum_{k=1}^{m} w_{\tilde{c}_k} C^k(i, j), \qquad i, j \in S,$$

where $\widetilde{c}_k$ are ordered sequences $(i_1, \ldots, i_s, i_1)$ of distinct elements $i_1, \ldots, i_s \in S$, which together with the positive numbers $w_{\widetilde{c}_k}$ are nonunique solutions to algebraic equations, while the matrix $C^k = (C^k(i,j),\ i,j \in S)$ is associated with each $\widetilde{c}_k$ and has entries defined as

$$C^k(i,j) = \begin{cases} s^{-1}, & \text{if } (i,j) \in \{(i_1,i_2), \ldots, (i_s,i_1)\}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Noting that $\widetilde{c}_k$ is an element of the class-circuit c_k as defined in the previous subsection, we have

$$J_{c_k} = C^k + \cdots + C^k, \tag{9}$$

where C^k is repeated s times.

Define f_t with $t = 1/M$, where $M = n!$, and let $(A_k,\ k = 1, \ldots, m)$ be any partition of $A = [0, 1/M)$ such that the relative distribution $(\lambda(A_k)/\lambda(A),\ k = 1, \ldots, m)$ is given by $(w_{\widetilde{c}_1}, \ldots, w_{\widetilde{c}_m})$. Define $A_{k\ell} = f_t^{\ell-1}(A_k)$ and $U_k = \bigcup_{\ell=1}^{M} A_{k\ell}$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, M$. Define the sets

$$S_i = \bigcup A_{k\ell}, \qquad i = 1, \ldots, n, \tag{10}$$

where the label (k, ℓ) in the union is defined as follows:

(11)
(i) k is the index of a chosen representative c_k of a class-circuit which passes i and which occurs in the decomposition (7);
(ii) ℓ denotes all the points identical to i of the $M/p(c_k)$ repetitions of the cycle $\widehat{c}_k$ associated to the c_k chosen in part (i).

Then $\mathcal{S} = (S_i,\ i = 1, \ldots, n)$ is a partition of $[0, 1)$. Accordingly, the Lebesgue measure of $A_{k\ell}$ is given by

$$\lambda(A_{k\ell}) = (1/M) w_{\widetilde{c}_k}. \tag{12}$$

Finally,

$$\lambda\big(S_i \cap f_t^{-1}(S_j)\big) = \pi_i p_{ij},$$

and then $(1/n!, \mathcal{S})$ is a rotational representation of P.

2.3. Let us recall that we defined a cycle $\widehat{c} = (i_1, \ldots, i_s), s > 1$, as an equivalence class whose elements are all the cyclic permutations of $(i_1, \ldots, i_s)$, that is, $\widehat{c} = \{(i_1, \ldots, i_s), (i_2, \ldots, i_s, i_1), \ldots, (i_s, i_1, \ldots, i_{s-1})\}$. Then choosing a representative of a cycle $\widehat{c}$ amounts to a choice of a starting point for $\widehat{c}$.

We are now prepared to prove the main result of this section.

THEOREM 1. *Given $n \geq 2$, for each ordering providing all the possible cycles in $S = \{1, \ldots, n\}$ and for each choice of the representatives of these cycles there* *exists a map Φ from the space of $n \times n$ irreducible stochastic matrices P into n-partitions $\mathcal{S} = \{S_i,\ i = 1, \dots, n\}$ of $[0, 1)$ such that the rotational representation process $(f_t, \{S_i\})$ with $t = 1/n!$ and $\{S_i\} = \Phi(P)$ has the same transition probabilities and the same distribution of cycles as the probabilistic cycle distribution of the Markov process on P.*

If the measures of the component sets of S_i converge, then the sequence of partitions converges in the metric d defined as

$$d(\mathcal{S}, \mathcal{S}') = \sum_i \lambda(S_i + S_i'), \tag{13}$$

where + denotes symmetric difference.

PROOF. We first appeal to Theorem 5 of [5], according to which any irreducible stochastic matrix P admits a canonical decomposition given by equation (5). Replacing the function J_{c_k} by C^k according to (9), we may write the canonical decomposition (5) in the following way:

$$\pi_i p_{ij} = \sum_{k=1}^{N} \big(p(c_k)\omega_{\hat{c}_k}\big) C^k(i, j), \qquad i, j = 1, \dots, n. \tag{14}$$

In order to assign P to an n-partition $\{S_i,\ i = 1, \dots, n\}$ of a rotational representation we shall apply Alpern's approach to the decomposition (14) instead of (7). Before this, we must locate the starting points of all the cycles, which are originally indexed as $c_1, c_2, \dots$. Accordingly, the first N cycles are those occurring in the canonical decomposition (14). The choice of an ordering for the cycles and of their starting points is unimportant for (14), but is essential for the uniqueness of the partition $\{S_i\}$ that we have to find.

Specifically, we apply Alpern's algorithm with labeling (11), according to which there exists a rotational representation process $(f_t, \{S_i\})$, where $t = 1/n!$ and the partition $\{S_i = \bigcup A_{k\ell},\ i = 1, \dots, n\}$ of $[0, 1)$ is given by (10) and (11) such that the distribution $(\lambda(A_k)/\lambda(A),\ k = 1, \dots, N)$ with $A = [0, 1/n!)$ matches the probability cycle distribution $(p(c_k)\omega_{\hat{c}_k},\ k = 1, \dots, N)$. From (12) and (14) we have

$$\lambda(A_{k\ell}) = (1/M)p(c_k)\omega_{\hat{c}_k}. \tag{15}$$

Hence the uniqueness of the probabilistic cycle weights $\omega_{\hat{c}_k}$ given by (4) implies that of the measures of the $A_{k\ell}$. In turn, for a fixed ordering of the cycles in S and for a choice of the cycle-representatives, the measures of the $A_{k\ell}$ uniquely determine the partition $\{S_i\}$ above.

Accordingly, for any fixed $n \geq 2$ there exists a map Φ assigning to each $n \times n$ irreducible stochastic matrix P a partition $\mathcal{S} = \{S_i,\ i = 1, \dots, n\}$ of $[0, 1)$ such that the rotational representation process $(f_t, \{S_i\})$ with $t = 1/n!$ has the same transition probabilities and the same distribution of cycles as the probabilistic cycle distribution of the Markov process on P.

Finally, if we endow the space of all partitions of $[0, 1)$ with the metric d given by (13), then the covergence in metric d of a sequence of partitions follows from the very definition of S_i, $i = 1, \dots, n$. □

3. Rotational representation of transition matrix functions. The approach to rotational representations of stochastic matrices presented in the previous section inspires investigation of certain generalizations. It is the generalization of the Cohen–Alpern theorem for the semigroups of matrices that we prove in this section.

Let $n \geq 1$, $S = \{1, \dots, n\}$ and $P = (P(h),\ h \geq 0)$ be an honest standard transition probability function, that is, $P(h) = (p_{ij}(h),\ i, j \in S)$ is a stochastic matrix for all $h \geq 0$ satisfying the Kolmogorov–Chapman equations $P(h)P(s) = P(h + s)$ for all $h, s \geq 0$, and $p_{ij}(h) \to \delta_{ij}$, as $h \to 0^+$, for all $i, j \in S$, where δ denotes Kronecker's symbol. Assume that there exists an invariant probability distribution $\pi = (\pi_i,\ i \in S)$, that is, $\pi_i > 0$, $i \in S$, and $\pi' P(h) = \pi$, $h \geq 0$.

When such an eigenvector exists we call P a recurrent transition matrix function.

As we know, the previous hypotheses imply the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and of an S-state continuous parameter Markov process $\xi = (\xi_h, h \geq 0)$ with the standard transition matrix function P above. Assume ξ is irreducible. Assign to each h the discrete skeleton chain $\Xi_h = (\xi_{nh}, n \geq 0)$ with parameter scale h. Then Ξ_h is an aperiodic, irreducible Markov chain with transition matrix $P(h)$.

We now prove the Cohen–Alpern theorem for the transition matrix function P above.

THEOREM 2. *A standard transition matrix function $P = (P(h),\ h \geq 0)$ is recurrent if and only if for each $h > 0$ there exists a rotational representation $(t, \mathcal{S}(h))$ for $P(h)$, that is,*

$$p_{ij}(h) = \lambda\big(S_i \cap f_t^{-1}(S_j)\big)/\lambda(S_i), \qquad i, j \in S,\ h > 0, \tag{16}$$

where $t = 1/n!$ and $\mathcal{S}(h) = (S_i(h), i = 1, \dots, n)$ is a partition of $[0, 1)$ with $\lambda(S_i(h)) = \pi_i$ for all h.

Moreover, there exists a map Φ defined by Theorem 1 which, for any irreducible standard transition matrix function $(P(h),\ h > 0)$, assigns to each $P(h),\ h > 0$ an n-partition $\mathcal{S}(h) = (S_i(h),\ i = 1, \dots, n)$ of $[0, 1)$ such that for all h the $S_i(h)$ have the same labels for their component intervals, $i = 1, \dots, n$.

PROOF. Applying Theorem 9 of [6], there exists a class $(\mathcal{C}, \omega_{\hat{c}}(h))$ of weighted cycles such that we have the following canonical decomposition;

$$\pi_i p_{ij}(h) = \sum_{\hat{c} \in \mathcal{C}} \omega_{\hat{c}}(h) J_c(i, j), \tag{17}$$

where $(\omega_{\hat{c}}(h),\ \hat{c} \in \mathcal{C})$ is the probabilistic cycle distribution of the discrete skeleton Ξ_h on $P(h)$. Then the statement of the theorem follows from the course of the proof of Theorem 1 applied to each stochastic matrix $P(h),\ h > 0$.

That for each i the sets $S_i(h), h > 0$, have the same labels for their component intervals $A_{k\ell}(h)$ results from the very definition (11) and from the same Theorem 9 of [6], according to which the cycles in the canonical representation (17) do not depend on h. (The latter expresses in fact, as proved in [6], the well known theorem of Lévy, Austin and Ornstein concerning the positivity of transition probabilities $p_{ij}(h), h > 0, i, j \in S$.) The proof is complete. □

We call (16) a *rotational representation of the transition matrix function P* and denote it by $(t, \mathcal{S}(\cdot))$. The partition $\mathcal{S}(h), h > 0$, is said to be of type $L(h)$ if the number of components $A_{k\ell}(h)$ of $S_i(h)$ is less than or equal to $L(h), i = 1, \ldots, n$.

Let $b(n)$ be the least integer such that every $n \times n$ recurrent transition matrix function has a rotational representation of type b, that is, all the corresponding discrete-skeleton transition matrices $P(h), h > 0$, have a rotational representation $(t, \mathcal{S}(h))$, where the type of $\mathcal{S}(h)$ is equal to b. We have the following theorem.

THEOREM 3.

(i) *If $(t, \mathcal{S}(\cdot))$ is a rotational representation of a transition matrix function P, then the type $L(h)$ of the partition $\mathcal{S}(h)$ is independent of h.*

(ii) *There exist positive constants α and β such that* $\exp(\alpha n^{1/2}) < b(n) < \exp(\beta n)$.

PROOF. (i) For any $i = 1, \ldots, n$ we have $S_i(h) = \bigcup A_{k\ell}(h)$, where the union is taken over all the pairs (k, ℓ) such that the ℓ-th vertex of the cycle $\widehat{c}_k$ occurring in the canonical decomposition (17) is exactly the element i. Applying Theorem 2, the number of component intervals $A_{k\ell}(h)$ defining the set $S_i(h)$ is time invariant, $i = 1, \ldots, n$, and in turn this happens for $L(h)$ as well.

(ii) The bounds for $b(n)$ follow from Theorem 2 of Alpern [1]. □

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REFERENCES

[1] ALPERN, S. (1983). Rotational representations of stochastic matrices. *Ann. Probab.* **11** 789–794.

[2] COHEN, J. E. (1981). A geometric representation of stochastic matrices; theorem and conjecture. *Ann. Probab.* **9** 899–901.

[3] HAIGH, J. (1985). Rotational representations of stochastic matrices. *Ann. Probab.* **13** 1024–1027.

[4] KALPAZIDOU, S. (1995). *Cycle Representations of Markov Processes*. Springer, New York.

[5] KALPAZIDOU, S. (1990). Asymptotic behaviour of sample weighted circuits representing recurrent Markov chains. *J. Appl. Probab.* **27** 545–556.

[6] KALPAZIDOU, S. (1991). Continuous parameter circuit processes with finite state space. *Stochastic Process. Appl.* **39** 301–325.

[7] QIAN, M. P. and QIAN M. (1979). The decomposition into a detailed balance part and a circulation part of an irreversible stationary Markov chain. *Sci. Sinica* **2** 69–79. (Special issue.)

[8] RODRÍGUEZ DEL TÍO, P. and VALSERO BLANCO, M. C. (1991). A characterization of reversible Markov chains by a rotational representation. *Ann. Probab.* **19** 605–608.

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