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EXTINCTION OF CONTACT AND PERCOLATION PROCESSES IN A RANDOM ENVIRONMENT[1]

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We consider the (inhomogeneous) percolation process on $\mathbf{Z}^d \times \mathbf{R}$ defined as follows:

Along each vertical line $\{x\} \times \mathbf{R}$ we put *cuts* at times given by a Poisson point process with intensity $\delta(x)$, and between each pair of adjacent vertical lines $\{x\} \times \mathbf{R}$ and $\{y\} \times \mathbf{R}$ we place *bridges* at times given by a Poisson point process with intensity $\lambda(x, y)$. We say that (x, t) and (y, s) are connected (or in the same cluster) if there is a path from (x, t) to (y, s) made out of *uncut* segments of vertical lines and *bridges*.

If we consider only oriented percolation, we have the graphical representation of the (inhomogeneous) d-dimensional contact process.

We consider these percolation and contact processes in a random environment by taking $\boldsymbol{\delta} = \{\delta(x);\ x \in \mathbf{Z}^d\}$ and $\boldsymbol{\lambda} = \{\lambda(x, y);\ x, y \in \mathbf{Z}^d,\ \|x-y\|_2 = 1\}$ to be independent families of independent identically distributed strictly positive random variables; we use δ and λ for representative random variables.

We prove extinction (i.e., no percolation) of these percolation and contact processes, for almost every $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$, if δ and λ satisfy

$$\mathbf{E}\left\{\left(\log(1+\lambda)\right)^{\beta}\right\} < \infty \quad \text{and} \quad \mathbf{E}\left\{\left(\log\left(1+\frac{1}{\delta}\right)\right)^{\beta}\right\} < \infty$$

for some

$$\beta > 2d^2\left(1+\sqrt{1+\frac{1}{d}}+\frac{1}{2d}\right),$$

and if

$$\mathbf{E}\left\{\left(\log\left(1+\frac{\lambda}{\delta}\right)\right)^{\beta}\right\}$$

is sufficiently small.

1. Introduction. We consider the inhomogeneous (continuous time) percolation process on $\mathbf{Z}^d \times \mathbf{R}$ defined as follows:

Let $\boldsymbol{\delta} = \{\delta(x) > 0;\ x \in \mathbf{Z}^d\}$ and $\boldsymbol{\lambda} = \{\lambda(x,y) > 0; \langle x, y\rangle \in \mathbf{B}(\mathbf{Z}^d)\}$, where $\mathbf{B}(\mathbf{Z}^d)$ denotes the collection of bonds (or edges) $\langle x, y\rangle$ in $\mathbf{Z}^d$, that is, unoriented pairs of sites $x, y \in \mathbf{Z}^d$ with $\|x-y\|_2 = 1$. Along each vertical line $\{x\} \times \mathbf{R}^d$ we put *cuts* at times given by a Poisson point process with intensity $\delta(x)$, and between

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each pair of adjacent vertical lines $\{x\} \times \mathbf{R}$ and $\{y\} \times \mathbf{R}$, $\langle x,y \rangle \in \mathbf{B}(\mathbf{Z}^d)$, we place *bridges* at times given by a Poisson point process with intensity $\lambda(x, y)$. All these Poisson processes are independent of each other.

Given a realization of all these Poisson processes (i.e., a locally finite collection of *cuts* and *bridges*), we consider the subset of $\mathbf{R}^{d+1}$ we obtain by taking $\mathbf{Z}^d \times \mathbf{R}$, removing all *cuts*, and adding all *bridges*, and decompose it into connected components. We call these connected components *clusters*. We say that $(x,t),(y,s) \in \mathbf{Z}^d \times \mathbf{R}$ are connected if they belong to the same cluster; in this case we write $(x,t) \leftrightarrow (y,s)$. Notice that this happens if and only if there is a path from (x,t) to (y,s) made out of *uncut* segments of vertical lines and *bridges*.

More generally, if $\mathbf{W} \subset \mathbf{R}^{d+1}$, we can substitute $\mathbf{W}$ for $\mathbf{R}^{d+1}$ in the above considerations. Thus (x,t) is connected to (y,s) in $\mathbf{W}$, and we write $(x,t) \leftrightarrow_{\mathbf{W}} (y,s)$, if the connection lies in $\mathbf{W}$. Given subsets A and B of $\mathbf{Z}^d \times \mathbf{R}$, we say that $A \leftrightarrow_{\mathbf{W}} B$ if there exist $(x,t) \in A$ and $(y,s) \in B$ such that $(x,t) \leftrightarrow_{\mathbf{W}} (y,s)$. If $\mathbf{W} = \mathbf{R}^{d+1}$ we omit $\mathbf{W}$.

Given an *environment* $\delta, \boldsymbol{\lambda}$, we will denote by $\mathbf{Q} = \mathbf{Q}_{\delta,\lambda}$ the corresponding percolation probability measure, that is, the probability measure on the space of configurations (locally finite collections of *cuts* and *bridges* on $\mathbf{Z}^d \times \mathbf{R}$, which are the realizations of the corresponding Poisson processes). The connectivity function in the region $\mathbf{W} \subset \mathbf{R}^{d+1}$ is then defined by

$$G_{\mathbf{W}}^{\delta,\lambda}\big((x,t),(y,s)\big) = \mathbf{Q}_{\delta,\lambda}\big\{(x,t) \leftrightarrow_{\mathbf{W}} (y,s)\big\}.$$

We may omit $\delta, \boldsymbol{\lambda}$ from the notation. If $\Lambda \subset \mathbf{Z}^d, I \subset \mathbf{Z}$, we will write $G_{\Lambda \times I}$ for $G_{\widetilde{\Lambda} \times I}$, where $\widetilde{\Lambda} = \Lambda \cup \{[x,y]; x,y \in \Lambda, \|x-y\|_2 = 1\}$. We will also write $|(x,t)| = \|(x,t)\|_\infty = \max\{\|x\|_\infty, |t|\}$.

This inhomogeneous percolation process appears as the limit of the percolation processes on $\mathbf{Z}^d \times \mathbf{Z}/n$ studied by Campanino, Klein and Perez (1991). The homogeneous version was independently studied by Bezuidenhout and Grimmett (1991).

The technicalities of defining such a continuous-time percolation process were treated by Bezuidenhout and Grimmett (1991) in the homogeneous case. Their treatment applies to the inhomogeneous case. We will always consider the configuration space with the Skorohod topology they introduced.

If we consider the oriented percolation process we obtain by keeping the *cuts* as above, but replacing the *bridges* by *one-way bridges*, that is, each Poisson process of bridges is replaced by two independent Poisson processes with intensities $\lambda_1(x,y) > 0$ and $\lambda_2(x,y) > 0$, the first giving one-way bridges from $\{x\} \times \mathbf{R}$ to $\{y\} \times \mathbf{R}$, and the second from $\{y\} \times \mathbf{R}$ to $\{x\} \times \mathbf{R}$, and uncut segments can only be traversed in the direction of increasing time, we obtain the graphical representation of the inhomogeneous contact process [see Bezuidenhout and Grimmett (1991) and Liggett (1985)].

Given the inhomogeneous contact process in the environment $\delta, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$, we can consider the percolation process in the environment $\delta, \boldsymbol{\lambda}$, where $\lambda(x,y) = \lambda_1(x,y) + \lambda_2(x,y)$ (i.e., we make all bridges *two-way bridges*). Then clearly no percolation (i.e., no infinite cluster) in the percolation process implies extinction (i.e., no infinite *oriented* cluster) in the contact process, and survival of the contact process implies percolation in the percolation process. We will abuse the language and use extinction and survival also for the percolation process.

We will consider these percolation and contact processes in a random environment by taking $\boldsymbol{\delta} = \{\delta(x);\ x \in \mathbf{Z}^d\}$ and $\boldsymbol{\lambda} = \{\lambda(x,y);\ \langle x,y\rangle \in \mathbf{B}(\mathbf{Z}^d)\}(\boldsymbol{\delta}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ for the contact process) to be independent families of independent identically distributed strictly positive random variables. We will use $\mathbf{P}$ and $\mathbf{E}$ to denote the probability measure and expectation associated with these random variables. We will also use δ and λ for representative random variables.

In a homogeneous environment, that is, $\delta(x) \equiv \delta > 0$, $\lambda(x,y) \equiv \lambda > 0$, the continuous-time percolation process always has a nontrivial phase diagram. The relevant parameter is $\rho = \lambda/\delta$. In any dimension $d \geq 1$ there exists $\rho_c(d)$, with $0 < \rho_c(d) < \infty$, such that if $\rho < \rho_c(d)$ there is no percolation and the connectivity function decays exponentially, that is,

$$G\big((x,t),(y,s)\big) \leq Ce^{-m|(x-y,t-s)|},$$

for some $m > 0$, $C < \infty$, and if $\rho > \rho_c(d)$ there is percolation [Bezuidenhout and Grimmett (1991) and Campanino, Klein and Perez (1991)]. If $d = 1$, we actually have $\rho_c(1) = 1$ [Bezuidenhout and Grimmett (1991)]. There is also a similar picture for the contact process [e.g., Liggett (1985) and Bezuidenhout and Grimmett (1991)].

Life is not so simple in an inhomogeneous environment $\boldsymbol{\delta}, \boldsymbol{\lambda}$. Let

$$\underline{\rho}(x) = \frac{1}{\delta(x)} \min_{y;\ \|x-y\|_2 = 1} \lambda(x,y),$$

$$\overline{\rho}(x) = \frac{1}{\delta(x)} \max_{y;\ \|x-y\|_2 = 1} \lambda(x,y).$$

It follows from the monotonicity properties of $G((x,t),(y,s))$ with respect to each $\delta(x)$ and $\lambda(x,y)$, that if $\sup_x \overline{\rho}(x) < \rho_c$ we have exponential decay of the connectivity function and no percolation, and if $\inf_x \underline{\rho}(x) > \rho_c$ we have percolation. The interesting nontrivial cases are those where the above conditions are not satisfied, in particular, when we can find sites where $\overline{\rho}(x) < \rho_c$ and sites where $\underline{\rho}(x) > \rho_c$, so the system exhibits phenomena similar to Griffiths' singularities [Griffths (1969)]. This typically happens in *random environments*.

The one-dimensional ($d = 1$) contact process in a random environment was studied by Liggett (1991, 1992), who gave conditions on the probability distributions of δ and λ for extinction and survival. Another one-dimensional survival result is due to Bramson, Durrett and Schonmann (1991). Andjel (1993) exhibited examples of survival in two or more dimensions. Aizenman, Klein and Newman (1993) found probability distributions under which the continuous-time percolation process (any dimension) always survives.

Campanino, Klein and Perez (1991) (see their Theorem 4.1) gave the first proof of extinction for the multidimensional continuous-time percolation process in a random environment, thus also proving extinction for the multidimensional contact process in a random environment. They also proved survival for the continuous-time percolation process in a random environment for

$d \geq 2$. Aizenman, Klein and Newman (1993) gave a proof of survival in the one-dimensional case.

Campanino, Klein and Perez studied the ground state behavior of a d-dimensional Ising model with a transverse field in a random environment. They rewrote the two-point correlation function as the limit of two-point functions of $(d + 1)$-dimensional classical Ising models with d-dimensional disorder, which were studied by Campanino and Klein (1991). Using the Fortuin–Kasteleyn representation and comparison theorems, Campanino and Klein bounded these two-point functions by the connectivity functions of the corresponding independent percolation processes, for which they obtained estimates that proved extinction in a random environment. Campanino, Klein and Perez refined their methods to obtain estimates uniform in the approximation step. Since these discrete-time percolation processes converge weakly to the continuous-time percolation process, their results yield a proof of extinction for the latter.

A review of results on continuous-time percolation, contact processes and related quantum spin systems in disordered environments is given in Klein (1993).

In this article we extend the Campanino, Klein and Perez (1991) proof of extinction to a larger class of probability distributions. We also perform the proof directly in the continuous-time percolation process.

Our main result is given in Theorem 1.1. The above-mentioned results are discussed in more detail in Remarks 1.3 to 1.9. The motivation and justification behind our conditions for extinction [see (1.1) to (1.3) in Theorem 1.1] are given in Remarks 1.5 to 1.8. A discussion of the proof of Theorem 1.1 is given after the remarks.

THEOREM 1.1. *Let $d \geq 1$ and consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ in a random environment $\boldsymbol{\delta}, \boldsymbol{\lambda}$. Let*

$$\beta > 2d^2\left(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}}\right) \tag{1.1}$$

and suppose

$$\Gamma = \max\left\{\mathbf{E}\left\{\left(\log(1+\lambda)\right)^{\beta}\right\}, \mathbf{E}\left\{\left(\log\left(1+\frac{1}{\delta}\right)\right)^{\beta}\right\}\right\} < \infty. \tag{1.2}$$

Then there exists $\varepsilon = \varepsilon(d, \beta, \Gamma) > 0$ such that we have no percolation for $\mathbf{P}$-almost every $\boldsymbol{\delta}, \boldsymbol{\lambda}$ if

$$\mathbf{E}\left\{\left(\log\left(1+\frac{\lambda}{\delta}\right)\right)^{\beta}\right\} < \varepsilon. \tag{1.3}$$

Furthermore, there exists $q(\beta, d) > 1$ with the property that given $q \in (1, q(\beta, d))$ and $m > 0$ we can find $\varepsilon(d, \beta, \Gamma, m, q) > 0$ such that if (1.3) *holds with*

$\varepsilon = \varepsilon(d, \beta, \Gamma, m, q)$, then for every $x \in \mathbf{Z}^d$ we have

$$G((x,t),(y,s)) \le C_{x,\delta,\lambda} \exp\left\{-m\left|\left(x-y, \left(\log\left(1+|t-s|\right)\right)^q\right)\right|\right\} \tag{1.4}$$

for all $y \in \mathbf{Z}^d$ and $t, s \in \mathbf{R}$, where $C_{x,\delta,\lambda} < \infty$ for $\mathbf{P}$-almost every δ, λ.

COROLLARY 1.2. *Let $d \ge 1$ and consider the d-dimensional contact process in a random environment $\delta, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$. Let β satisfy (1.1) and $\delta, \boldsymbol{\lambda}$ satisfy (1.2), where $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$. Then there exists $\varepsilon = \varepsilon(d, \beta, \Gamma) > 0$ such that the contact process becomes extinct for $\mathbf{P}$-almost every $\delta, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ if (1.3) holds.*

REMARK 1.3. Theorem 1.1 was proved by Campanino, Klein and Perez (1991) (see their Theorem 4.1) under the assumption

$$\Gamma' = \max\left\{\mathbf{E}\left(\frac{1}{\delta^\kappa}\right), \mathbf{E}\left(e^{\kappa\lambda}\right)\right\} < \infty$$

for some $\kappa > 0$. In this case they proved that given $q > 1$ and $m > 0$ there exists $\varepsilon = \varepsilon(d, \kappa, \Gamma', m, q) > 0$ such that if

$$\mathbf{E}\left(\left(\frac{\lambda}{\delta}\right)^k\right) < \varepsilon,$$

then (1.4) holds with $C_{x,\delta,\boldsymbol{\lambda}} < \infty$ for $\mathbf{P}$-almost every $\delta, \boldsymbol{\lambda}$.

Campanino, Klein and Perez (1991) actually studied the independent percolation process on $\mathbf{Z}^d \times \mathbf{Z}/n$ with occupation probabilities

$$q^n_{(x,y),(y,s)} = \begin{cases} 1 - e^{-2[J(x,y)/n]}, & \text{if } t = s,\ \|x-y\|_2 = 1, \\ 1 - e^{-2K_n(x)}, & \text{if } x = y,\ |t-s| = \frac{1}{n}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{h} = \{h(x) > 0,\ x \in \mathbf{Z}^d\}, \mathbf{J} = J(x,y) > 0, \langle x,y\rangle \in \mathbf{B}(\mathbf{Z}^d)\}$ are given, and $\exp[-2K_n(x)] = \tanh[h(x)/n]$ [they have $J(x,y) \equiv J$, but their article is based on the work of Campanino and Klein (1991), who study the same model without the dependence on n, so their considerations apply also to inhomogeneous $\{J(x,y)\}$]. Let $\mathbf{Q}^{(n)} = \mathbf{Q}^{(n)}_{\mathbf{h},\mathbf{J}}$ denote the corresponding percolation probability measure, which can be considered as a probability measure on the space of configurations of the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$. The $\mathbf{Q}^{(n)}_{\mathbf{h},\mathbf{J}}$ converge weakly to $\mathbf{Q}_{\delta,\boldsymbol{\lambda}}$, with $\delta(x) = h(x)$, $\lambda(x,y) = 2J(x,y)$ [the technicalities of such convergence are discussed in Bezuidenhout and Grimmett (1991)]. Thus for appropriate events A we have $\mathbf{Q}^{(n)}_{\mathbf{h},\mathbf{J}}(A) \to \mathbf{Q}_{\delta,\boldsymbol{\lambda}}(A)$. Since the results in Campanino, Klein and Perez (1991) hold uniformly in n for n large, their results also hold for the continuous-time percolation process.

REMARK 1.4. Another way of stating Theorem 1.1 is to introduce a coupling constant $\gamma > 0$, that is, to consider the random environment $\boldsymbol{\delta}, \gamma\boldsymbol{\lambda}$. Then if (1.2) holds for $\boldsymbol{\delta}, \boldsymbol{\lambda}$, there exists $\gamma_1 = \gamma_1(d, \beta, \Gamma, m, q)$ such that (1.4) holds in the random environment $\boldsymbol{\delta}, \gamma\boldsymbol{\lambda}$ if $\gamma < \gamma_1$. The results in Campanino, Klein and Perez (1991) are stated this way.

REMARK 1.5. The scaling $(\boldsymbol{\delta}, \boldsymbol{\lambda}) \to (\gamma\boldsymbol{\delta}, \gamma\boldsymbol{\lambda})$, $\gamma > 0$, of the random environment corresponds to simply rescaling the time variable. (1.3) is clearly invariant under such scaling, so Γ can be replaced by

$$\widetilde{\Gamma} = \min_{\gamma > 0} \max \left\{ \mathbf{E}\left\{ \left(\log(1 + \gamma\lambda) \right)^{\beta} \right\}, \mathbf{E}\left\{ \left(\log\left(1 + \frac{1}{\gamma^{\delta}} \right) \right)^{\beta} \right\} \right\}.$$

Notice that the minimum is attained at some $\widetilde{\gamma} > 0$. Notice also that

$$\mathbf{E}\left\{ \left(\log\left(1 + \frac{\lambda}{\delta} \right) \right)^{\beta} \right\} \leq 2^{\beta + 1}\widetilde{\Gamma},$$

so $\widetilde{\Gamma}$ small enough suffices for extinction.

REMARK 1.6. Andjel (1993) gave counterexamples to the conclusions of Theorem 1.1 and Corollary 1.2 when $\beta < d$, $d \geq 2$. He took $\delta \equiv 1$ and λ to be a Bernoulli random variable and showed that the process may survive although $\mathbf{E}\{(\log(1 + \lambda))^{\beta}\}$ is as close to 0 as one wishes.

REMARK 1.7. Aizenman, Klein and Newman (1993) gave counterexamples to the conclusions of Theorem 1.1 when $\beta < d$, which also hold if the theorem is stated with a coupling constant as in Remark 1.4. For example, if $\lambda \equiv 1$ and

$$u^d \mathbf{P}\left\{ \log \frac{1}{\delta} > u \right\} \to \infty, \qquad d \geq 2, \tag{1.5}$$

or

$$\frac{u}{\log u} \mathbf{P}\left\{ \log \frac{1}{\delta} > u \right\} \to \infty, \qquad d = 1,$$

they prove that the process always survives. Notice that (1.5) is ruled out by (1.2) if $\beta > d$, but it is compatible with (1.2) if $\beta < d$.

REMARK 1.8. The natural conjecture that follows from the previous two remarks is that Theorem 1.1 and Corollary 1.2 should hold if $\beta > d$. Schonmann has given a heuristic argument to that effect.

REMARK 1.9. Companino, Klein and Perez (1991) showed that in any random environment $\boldsymbol{\delta}, \gamma\boldsymbol{\lambda}$, we always have survival for γ large enough. If $d = 1$, Liggett (1992) proved survival of the contact process if λ_1 and λ_2 are identically distributed and

$$2\mathbf{E}(\delta)\mathbf{E}\left(\frac{1}{\lambda_1}\right) + (\mathbf{E}\delta^2)\left(\mathbf{E}\left(\frac{1}{\lambda_1}\right)\right)^2 < 1.$$

Bramson, Durrett and Schonmann (1991) proved survival of the contact process for $\lambda_1 = \lambda_2 \equiv 1$ and δ having a Bernoulli distribution. Aizenman, Klein and Newman (1993) proved survival for the one-dimensional continuous-time percolation process under the conditions of Theorem 1.1 with δ and λ interchanged. They actually proved that summability of the connectivity function in the dual process implies the existence of an infinite cluster in the original process. Since the dual process is the same continuous-time percolation process but with λ and δ interchanged, survival then follows from Theorem 1.1.

REMARK 1.10. Jitomirskaya and Klein (1993) considered continuous-time percolation and contact processes in a quasi-periodic environment and obtained results similar to Theorem 1.1.

To prove Theorem 1.1, we *optimize* the Campanino, Klein and Perez proof. The proof involves a multiscale analysis of the type used in the theory of random Schrödinger operators by von Dreifus (1987), Spencer (1988) and von Dreifus and Klein (1989). An exposition of the technique in the context of percolation processes is given in Klein (1994).

The multiscale analysis is needed to control the Griffiths-type singularities introduced by the random environment. To see that, let $\rho_\Lambda = \sup_{x\in\Lambda}\overline{\rho}(x)$ and $\delta_\Lambda = \inf_{x\in\Lambda}\delta(x)$ for $\Lambda \subset \mathbf{Z}^d$. It was shown by Campanino, Klein and Perez (1991)—see their formula (3.11)—that for $\mathbf{W} = \Lambda\times\mathbf{R}$ we have

$$G_{\mathbf{W}}\big((x,t),(y,s)\big) \le \left(1-\frac{8d\rho_\Lambda}{1-\theta}\right)^{-1} e^{-\theta\delta_\Lambda|t-s|}\left(\frac{8d\rho_\Lambda}{1-\theta}\right)^{|x-y|} \tag{1.6}$$

for all $x,y\in\Lambda$, $t,s\in\mathbf{R}$ and any $0<\theta<1$, as long as

$$8d\rho_\Lambda < 1-\theta. \tag{1.7}$$

Under the hypotheses of Theorem 1.1, (1.7) will be satisfied with high probability if Λ is a cube with side of a certain *fixed* length L_0 [depending only on the parameters in (1.1) to (1.3)]. But if $\mathbf{P}\{\underline{\rho}(x) > \rho_c\} > 0$, then, with probability 1 (with respect to $\mathbf{P}$), for any length scale L we will always find infinitely many cubes in $\mathbf{Z}^d$ with sides of length L in which all $\underline{\rho}(x) > \rho_c$. If Λ is such a cube, the system will try to form large clusters in $\Lambda\times\mathbf{R}$. Thus in almost all environments we will find infinitely many regions of all sizes in which (1.7) does not hold and the usual expansions [like the one used in Campanino, Klein and Perez (1991) to get (1.6)] do not converge. A multiscale analysis is used to control the effect of these singular regions. One must control not only the size and frequency of singular regions but also the behavior of the system inside a singular region.

The difficulty of the latter is enhanced in the case of the systems discussed in this article, since our singular regions are cylinders infinitely extended in the time direction. Moreover, in this situation the connectivity function cannot have exponential decay in the time direction, since given any exponential rate of decay we can find, with probability 1, regions inside which the connectivity function exhibits a slower exponential rate of decay in the time direction than the given rate, thus the slower than exponential decay in the time direction in (1.4).

In the remainder of this article, we will restrict ourselves to a continuous-time percolation process in a random environment $\delta, \boldsymbol{\lambda}$. In the next section we introduce the basic tools we will need for our expansions: the Harris–FKG, van den Berg–Kesten and Hammersley–Simon–Lieb inequalities. We also give a proof of no percolation and decay for homogeneous continuous-time percolation processes (see Proposition 2.1 and Corollary 2.2), which will serve as a prototype for our proof of Theorem 1.1. In Section 3 we state (and explain) the result of our multiscale analysis (Theorem 3.2) and show how Theorem 1.1 can be derived from it (see Theorem 3.3 and the following discussion). We complete the proof of Theorem 1.1 by proving Theorem 3.2 in Section 4.

2. Inequalities. Configurations in the continuous-time percolation process are locally finite collections of *cuts* and *bridges* on $\mathbf{Z}^d \times \mathbf{R}$. In this picture we think of $\mathbf{Z}^d \times \mathbf{R}$ as a collection of vertical lines in $\mathbf{R}^{d+1}$ indexed by points in $\mathbf{Z}^d$; a configuration is a collection of *cuts* (points) on these vertical lines and *bridges* (horizontal line segments) connecting nearest neighbor vertical lines, such that any bounded subset of $\mathbf{R}^{d+1}$ contains a *finite* number of *cuts* and *bridges*. We will denote by Ω the space of configurations, equipped with the Skorohod topology.

The space Ω has a natural partial order: if $w, w' \in \Omega$, we have $w \le w'$ if w' contains all *bridges* in w and w contains all *cuts* in w'. Functions on Ω which are nondecreasing (nonincreasing) with respect to this partial order are called positive (negative); events are positive (negative) if their characteristic functions are positive (negative).

An event will be called local if it is determined by configurations inside a bounded region of $\mathbf{R}^{d+1}$.

Let us fix an environment $\delta, \boldsymbol{\lambda}$. We will use several inequalities.

The Harris–FKG inequality. *Let A and B be local positive* (*negative*) *events, with* $\mathbf{Q}(\partial A) = \mathbf{Q}(\partial B) = 0$. *Then*

$$\mathbf{Q}(A \cap B) \ge \mathbf{Q}(A)\mathbf{Q}(B). \tag{2.1}$$

The van den Berg–Kesten (BK) inequality. *Let A be an event,* $\mathbf{W} \subset \mathbf{R}^{d+1}$. *We set*

$$A|_{\mathbf{W}} = \{w \in A;\ w' \in A \text{ if } w' \cap \mathbf{W} = w \cap \mathbf{W}\}.$$

If A and B are events, we let $A \circ B$ be the event of A and B occurring disjointly, that is, $A \circ B = \{w \in A \cap B;$ *there exist* $\mathbf{W}_1, \mathbf{W}_2 \subset \mathbf{R}^{d+1}, \mathbf{W}_1 \cap \mathbf{W}_2 = \emptyset,$ *with* $w \in A|_{\mathbf{W}_1} \cap B|_{\mathbf{W}_2}\}$. *The van den Berg–Kesten (BK) inequality says that if both A and B are local positive (negative) events, with*

$$\mathbf{Q}(\partial A) = \mathbf{Q}(\partial B) = \mathbf{Q}\big(\partial(A \circ B)\big) = 0,$$

then

$$\mathbf{Q}(A \circ B) \leq \mathbf{Q}(A)\mathbf{Q}(B). \tag{2.2}$$

Both the Harris–FKG and BK inequalities are well known for the discrete-time percolation processes studied by Campanino, Klein and Perez (1991) [see the discussion and references in Campanino and Klein (1991)]. By the weak convergence of the probability measures, the inequalities hold as stated for the continuous-time percolation process [see the discussion in Bezuidenhout and Grimmett (1991)]. We also have the Harris–FKG and BK inequalities if A and B are finite intersections of events of the form $\{U \leftrightarrow_{\mathbf{W}} V\}$, where U and V are closed subsets of $\mathbf{Z}^d \times \mathbf{R}$, and $\mathbf{W}$ is a closed subset of $\mathbf{R}^{d+1}$, or if they are complements of such events [Bezuidenhout and Grimmett (1991)].

The Hammersley–Simon–Lieb (HSL) inequality. Let $\Lambda, \Lambda' \subset \mathbf{Z}^d$, I, I' closed intervals in $\mathbf{R}$, $\mathbf{W} = \Lambda \times I$, $\mathbf{W}' = \Lambda' \times I'$ (if $\mathbf{W}' = \mathbf{Z}^d \times \mathbf{R}$ it will be omitted from the notation). We set

$$\begin{aligned}
\partial_H(\mathbf{W}, \mathbf{W}') &= \Lambda \cap \Lambda' \times \Big(\big(\partial(I' \backslash I)\big) \cap I\Big),\\
\partial(\Lambda, \Lambda') &= \big\{(x,y) \in (\mathbf{Z}^d)^2; x \in \Lambda \cap \Lambda', y \in \Lambda' \backslash \Lambda, \|x-y\|_2 = 1\big\},\\
\partial_V(\mathbf{W}, \mathbf{W}') &= \partial(\Lambda, \Lambda') \times (I \cap I'),\\
\partial(\mathbf{W}, \mathbf{W}') &= \partial_H(\mathbf{W}, \mathbf{W}') \cup \Big(\big\{x \in \mathbf{Z}^d; (x,y) \in \partial(\Lambda, \Lambda') \text{ for some } y\big\} \times (I \cap I')\Big),\\
\partial^e(\mathbf{W}, \mathbf{W}') &= \partial_H(\mathbf{W}, \mathbf{W}') \cup \Big(\big\{y \in \mathbf{Z}^d; (x,y) \in \partial(\Lambda, \Lambda') \text{ for some } y\big\} \times (I \cap I')\Big).
\end{aligned}$$

The form of the HSL inequality we will use says that

$$\begin{aligned}
G_{\mathbf{W}'}(X,Y) \leq{}& \sum_{Z \in \partial_H(\mathbf{W},\mathbf{W}')} G_{\mathbf{W}\cap\mathbf{W}'}(X,Z) G_{\mathbf{W}'}(Z,Y)\\
&+ \sum_{(z,z') \in \partial(\Lambda,\Lambda')} \lambda(z,z') \int_{I \cap I'} G_{\mathbf{W}\cap\mathbf{W}'}\big(X,(z,t)\big) G_{\mathbf{W}}\big((z',t),Y\big)\,dt
\end{aligned} \tag{2.3}$$

for all $X \in \mathbf{W} \cap \mathbf{W}', Y \in \mathbf{W}' \backslash \mathbf{W}$.

This inequality can be obtained either directly from the BK inequality or from the similar inequality for the approximating discrete-time percolation processes stated by Campanino, Klein and Perez (1991).

We will actually use the following consequence of (2.3): if $X \in \mathbf{W}$, let

$$G_{\mathbf{W}}(X, \partial) = \sum_{Z \in \partial_H \mathbf{W}} G_{\mathbf{W}}(X,Z) + \sum_{(z,z') \in \partial(\Lambda, \mathbf{Z}^d)} \lambda(z,z') \int_I G_{\mathbf{W}}\big(X,(z,t)\big)\,dt.$$

Since $G_{\mathbf{W}'}(Z, Y)$ is continuous in Z and $I \cap I'$ is closed, it follows from (2.3) that if $x \in \mathbf{W} \cap \mathbf{W}'$, $Y \in \mathbf{W}' \backslash \mathbf{W}$, we have

$$G_{\mathbf{W}'}(X, Y) \leq G_{\mathbf{W}}(X, \partial) G_{\mathbf{W}'}(Z_1, Y) \tag{2.4}$$

for some $Z_1 \in \partial^e(\mathbf{W}, \mathbf{W}')$.

We will refer to (2.4) as the HSL inequality.

The HSL inequality gives global decay out of local decay. This can be seen as follows:

For $L > 0$, $x \in \mathbf{Z}^d$, let

$$\Lambda_L(x) = \{ y \in \mathbf{Z}^d;\ \|x - y\|_\infty < L \}.$$

For $X = (x, t) \in \mathbf{Z}^d \times \mathbf{R}$, $L > 0$, $T > 0$, let

$$B_{L,T}(X) = \Lambda_L(x) \times [t - T, t + T].$$

PROPOSITION 2.1. *Suppose there exist* $L > 0$, $T > 0$ *and* $m > 0$ *such that*

$$G_{B_{L,T}(X,0)}\big((x, 0), \partial\big) \leq e^{-m} \tag{2.5}$$

for all $x \in \mathbf{Z}^d$. *Then*

$$G\big((x, t), (y, s)\big) \leq \exp\left\{ -m \left(\max\left\{ \frac{\|x - y\|_\infty}{L + 1}, \frac{|t - s|}{T} \right\} - 1 \right) \right\} \tag{2.6}$$

for all $(x, t), (y, s) \in \mathbf{Z}^d \times \mathbf{R}$.

PROOF. By the stationarity in the t-direction of the percolation probabilities, we have

$$G_{B_{L,T}(X)}(X, \partial) \leq e^{-m} \tag{2.7}$$

for all $X \in \mathbf{Z}^d \times \mathbf{R}$.

Let us fix $X = (x, t)$. If $Y \notin B_{L,T}(X)$, it follows from (2.4) and (2.7) that

$$G(X, Y) \leq e^{-m} G(Z_1, Y)$$

for some $Z_1 \in \partial^e B_{L,T}(X)$.

This procedure can be performed n times, yielding points $Z_0 = X_1, \ldots, Z_n$, with $Z_j \in \partial^e B_{L,T}(Z_{j-1})$, $j = 1, \ldots, n$, and

$$G(X, Y) \leq e^{-nm} G(Z_n, Y),$$

as long as $Y \notin B_{L,T}(Z_j)$ for $j = 0, 1, \ldots, n - 1$. But this is always true as long as

$$n < \max\left\{ \frac{\|x - y\|_\infty}{L + 1}, \frac{|t - s|}{T} \right\}.$$

Since $G(Z_n, Y) \leq 1$ and n is a positive integer, we obtain (2.6). □

Proposition 2.1 gives a one-scale proof of exponential decay of the connectivity function and no percolation for homogeneous continuous-time percolation processes, as follows:

COROLLARY 2.2. *Let us consider the homogeneous continuous-time percolation process on* $\mathbf{Z}^d \times \mathbf{R}$ [i.e., $\delta(x) \equiv \delta$, $\lambda(x,y) \equiv \lambda$). *There is no percolation if*

$$\rho \equiv \frac{\lambda}{\delta} < \frac{1}{2d}. \tag{2.8}$$

Furthermore, in this case there exist $m = m(\rho, \delta) > 0$ *and* $C = C(\rho, \delta) > \infty$, *such that*

$$G\big((x,t),(y,s)\big) \leq C\exp\{-m|(x-y,t-s)|\} \tag{2.9}$$

for all $(x,t),(y,s) \in \mathbf{Z}^d \times \mathbf{R}$.

PROOF. By the definition of a homogeneous continuous-time percolation process, we have

$$G_{B_{1,T}(x,0)}\big((x,0),\partial\big) \leq 2e^{-2\delta T} + (2d)\lambda(2T) = 2\big(e^{-2\xi} + d\rho\xi\big), \tag{2.10}$$

where $\xi = 2\delta T$.

Given $\rho < 1/(2d)$, a bit of calculus shows that we can choose $\xi = \xi(\rho) > 0$ such that

$$2\big(e^{-2\xi} + d\rho\xi\big) < 1,$$

so the corollary follows immediately from Proposition 2.1. □

Corollary 2.2 has a straightforward extension to an inhomogeneous continuous-time percolation process.

COROLLARY 2.3. *Let us consider the continuous-time percolation process on* $\mathbf{Z}^d \times \mathbf{R}$ *in an inhomogeneous environment* $\boldsymbol{\delta}, \boldsymbol{\lambda}$. *Suppose*

$$\lambda_{\mathbf{Z}^d} \equiv \sup_{\langle x,y\rangle} \lambda(x,y) < \infty \quad \textit{and} \quad \delta_{\mathbf{Z}^d} \equiv \inf_x \delta(x) > 0. \tag{2.11}$$

Then there is no percolation if

$$\frac{\lambda_{\mathbf{Z}^d}}{\delta_{\mathbf{Z}^d}} < \frac{1}{2d}. \tag{2.12}$$

Furthermore, in this case there exist $m = m(\lambda_{\mathbf{Z}^d}, \delta_{\mathbf{Z}^d}) > 0$ *and* $C = C(\lambda_{\mathbf{Z}^d}, \delta_{\mathbf{Z}^d}) < \infty$, *such that*

$$G\big((x,t),(y,s)\big) \leq C\exp\{-m|(x-y,t-s)|\} \tag{2.13}$$

for all $(x,t),(y,s) \in \mathbf{Z}^d \times \mathbf{R}$.

REMARK 2.4. Notice that if follows from (1.6) that Corollary 2.3 is still valid with (2.11) and (2.12) replaced by $\rho_{\mathbf{Z}^d} < 1/(8d)$ and $\delta_{\mathbf{Z}^d} > 0$.

3. The multiscale analysis. Theorem 1.1 is the extension of Corollary 2.3 to a random environment, with conditions (1.1) and (1.2) replacing (2.11) and (1.3) replacing (2.12). Unfortunately, the proof of Theorem 1.1 is not a straightforward extension of the proof of Corollary 2.3. Conditions (1.1) to (1.3), or similar probabilistic conditions, imply conditions similar to (2.11) and (2.12) inside cubes with side of a certain fixed length L_0, but only with high probability. In other words, given such an initial scale L_0 and the parameters β and Γ in (1.1) and (1.2), we can pick ε in (1.3) to obtain conditions similar to (2.11) and (2.12) and decay as in (2.13), with the desired probability, only inside cubes with side of length L_0. To go to higher scales, we use a multiscale analysis. We will introduce an increasing sequence of length scales. We will characterize sites in $\mathbf{Z}^d$ as regular or singular in a given scale, in such a way that the process will have the desired decay properties inside boxes with side given by the length scale and centered at regular (in that scale) sites. The next step is to show, by induction on the scale index, that if the probability of a site to be regular in a given scale is good enough (for that scale, in a sense to be made precise), then the probability of a site to be regular in the next scale is also good enough. This induction step is the crux of the argument; a site which is regular in a given scale is surrounded by sites which are mostly regular in the previous scale, but there will be some nearby sites that are singular in the previous scale. We will show that under hypotheses (1.1) and (1.2), the probability that the divergences introduced by the singular sites are not too bad and can be controlled by the decay we get from the surrounding regular sites is good enough for the given scale. Finally, we use an argument based on the Borel–Cantelli lemma to derive Theorem 1.1 from the multiscale analysis.

We start with some definitions. We fix ν such that $0 < \nu < 1$ and set

$$B_L(X) = B_{L,\, e^{L^\nu}}(X)$$

for any $X \in \mathbf{Z}^d \times \mathbf{R},\ L > 0$.

DEFINITION. Let $m > 0,\ L > 1$. A site $x \in \mathbf{Z}^d$ is said to be (m,L)-regular if

$$G_{B_L((x,0))}\big((x,0),\partial\big) \le e^{-mL}.$$

Otherwise we call x (m,L)-singular. A set $\Lambda \subset \mathbf{Z}^d$ is (m,L)-regular if every $x \in \Lambda$ is (m,L)-regular; otherwise it is (m,L)-singular.

We have the following modification of Proposition 2.1, which is proved in the same way.

LEMMA 3.1. *Let* Λ *be a* (m,L)*-regular region*, $\mathbf{W} \subset \Lambda \times \mathbf{R}$. *Then, if* $(x,t),(y,s)$

$\in \mathbf{W}$, *we have*

$$G_{\mathbf{W}}\big((x,t),(y,s)\big) \le \exp\left\{-mL\left(\left|\left(\frac{x-y}{L+1},(t-s)e^{-L^{\nu}}\right)\right|-1\right)\right\}. \tag{3.1}$$

Our multiscale analysis will proceed in the following way: given increasing length scales L_k, $k = 0,1,2,\dots$, a decay rate $m_0 > 0$ and $p > d$, we will assume (and prove later) tht we can arrange the parameters in (1.1) to (1.3) so that

$$\mathbf{P}\{0 \text{ is } (m_0, L_0)\text{-regular}\} \ge 1 - \frac{1}{L_0^p}.$$

We will then show, by induction on k, that this probabilistic estimate holds in all scales, but with a slightly smaller m. Of course, the length scales must be appropriately defined, the original parameters must satisfy certain relations and we will need to pick our initial scale L_0 sufficiently large [so ε in (1.3) will have to be sufficiently small]. The main (and precise) result of the multiscale analysis is given by the following theorem.

THEOREM 3.2. *Let $d \ge 1$ and consider the continuous-time percolation process on $\mathbf{Z}^d \times R$ in the random environment $\delta, \boldsymbol{\lambda}$. Let*

$$\beta > 2d^2\left(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}}\right), \tag{3.2}$$

and suppose (1.2) *holds. Set*

$$\alpha = d + \sqrt{d^2 + d},$$

and choose ν and p such that

$$\begin{aligned} &\frac{\alpha d(\alpha+\beta+1)}{\beta(\alpha - d + \alpha d)} < \nu < 1,\\ &\alpha d < p < \frac{\beta(\nu(\alpha - d + \alpha d) - \alpha d) - \alpha d}{\alpha}. \end{aligned} \tag{3.3}$$

Let m_0 and m_∞ be given, with $0 < m_\infty < m_0$. There exists $\overline{L} = \overline{L}(d,\beta,\Gamma,\nu,p,m_0, m_\infty) < \infty$ such that, if for some $L_0 > \overline{L}$ we have

$$\mathbf{P}\{0 \textit{ is } (m_0, L_0)\textit{-regular}\} \ge 1 - \frac{1}{L_0^p}, \tag{3.4}$$

then, setting $L_{k+1} = (L_k)^\alpha$, $k = 0,1,\dots$, we also have

$$\mathbf{P}\{0 \textit{ is } (m_\infty, L_k)\textit{-regular}\} \ge 1 - \frac{1}{L_k^p} \tag{3.5}$$

for all $k = 0, 1, 2, \dots$.

Notice that (3.3) can be satisfied because of (3.2). Indeed, let

$$f(\theta) = \frac{\theta d(\theta + 1)}{\theta - d} \quad \text{for } \theta > d.$$

It is easy to see that $f(\theta)$ attains its minimum at $\theta = \alpha \equiv d + \sqrt{d^2 + d}$, and $f(\alpha) = 2d^2\left(1 + \sqrt{1 + 1/d} + 1/(2d)\right)$. Thus (3.2) just says that $\beta > f(\alpha)$, and (3.3) says that we picked $0 < \nu < 1$ and $p > \alpha d$ such that

$$\beta > \frac{\alpha(p + d)}{\alpha\nu - d(\alpha(1 - \nu) + \nu)} = \frac{\alpha(p + d)}{\nu(\alpha - d + \alpha d) - \alpha d}.$$

Theorem 3.2 will be proven in Section 4. Its importance is justified by the following theorem, which is some sort of extension of Proposition 2.1 to a random environment.

THEOREM 3.3. *Let $d \geq 1$ and consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ in the random environment $\delta, \boldsymbol{\lambda}$. Let $\nu, \alpha, p, m_\infty, L_0$ be such that $0 < \nu < 1, \alpha > 1$, $p > \alpha d, m_\infty > 0$, $L_0 > 1$. Set $L_{k+1} = L_k^\alpha$, $k = 0, 1, 2, \dots$. Suppose*

$$\mathbf{P}\{0 \text{ is } (m_\infty, L_k)\text{-regular}\} \geq 1 - \frac{1}{L_k^p} \tag{3.6}$$

for all $k = 0, 1, 2, \dots$. Then, for any m such that $0 < m < m_\infty$, we have that for every $x \in \mathbf{Z}^d$,

$$G\big((x, t), (y, s)\big) \leq C_x \exp\left\{-m\left|\left(x - y, \left(\log\left(1 + |t - s|\right)\right)^{1/\nu}\right)\right|\right\}$$

for all $y \in \mathbf{Z}^d$ and $t, s \in \mathbf{R}$, with $C_x = C_x(\delta, \boldsymbol{\lambda}, m) < \infty$ for $\mathbf{P}$-almost every $\delta, \boldsymbol{\lambda}$.

PROOF. The proof of Theorem 3.3 is the same as the proof of Corollary 3.2 in Campanino and Klein (1991). We will sketch the proof for completeness.

Let $b > 2$ (to be chosen later) and let us fix $x \in \mathbf{Z}^d$. We have from (3.6) that

$$\mathbf{P}\{\Lambda_{bL_{k+1}}(x) \text{ is not a } (m_\infty, L_k)\text{-regular region}\} \leq \frac{(2bL_{k+1})^d}{L_k^p} = \frac{(2b)^d}{L_k^{p - \alpha d}}.$$

Since $p > \alpha d$ the above probabilities are summable. We can now use the Borel–Cantelli lemma to conclude that, with probability 1, we can find $k_1 = k_1(x, b, \delta, \boldsymbol{\lambda}) < \infty$, such that $\Lambda_{bL_{k+1}}(x)$ is a (m_∞, L_k)-regular region for all $k \geq k_1$.

Let $Y = (y, s) \in \mathbf{Z}^d \times \mathbf{R}$. Except for a bounded set of Y's we can always choose $k \geq k_1$, such that

$$bL_k \leq \left|\left(x - y, \left(\log\left(1 + |s|\right)\right)^{1/\nu}\right)\right| < bL_{k+1}.$$

In this case $y \in \Lambda_{bL_{k+1}}(x)$, which is a (m_∞, L_k)-regular region. It now follows as in Lemma 3.1 that

$$G((x,0),Y) \le e^{-m_\infty L_k n},$$

with n a positive integer such that

$$n \le \min\left\{\frac{bL_{k+1}}{L_k+1}, \max\left\{\frac{\|x-y\|_\infty}{L_k+1}, |s|e^{-L_k^\nu}\right\}\right\}.$$

If $\|x-y\|_\infty \ge (\log(1+|s|))^{1/\nu}$, we take n such that

$$\frac{\|x-y\|_\infty}{L_k+1} \ge n \ge \frac{\|x-y\|_\infty}{L_k+1} - 1.$$

If not, we must have

$$bL_k < \left(\log\left(1+|s|\right)\right)^{1/\nu},$$

so

$$|s|e^{-L_k^\nu} \ge e^{(b^\nu-1)L_k^\nu} - e^{-L_k^\nu} \ge \frac{bL_{k+1}}{L_k+1} \ge \frac{\|x-y\|_\infty}{L_k+1}$$

for k sufficiently large. In this case we take

$$\frac{bL_{k+1}}{L_k+1} \ge n \ge \frac{bL_{k+1}}{L_k+1} - 1 \ge \frac{\left(\log\left(1+|s|\right)\right)^{1/\nu}}{L_k+1} - 1.$$

Thus we get

$$\begin{aligned} G((x,0),Y) &\le \exp\left\{-m_\infty L_k\left(\left|\left(\frac{x-y}{L_k+1}, \frac{\left(\log\left(1+|s|\right)\right)^{1/\nu}}{L_k+1}\right)\right| - 1\right)\right\} \\ &\le \exp\left\{-m_\infty\left(\frac{L_k}{L_k+1} - \frac{1}{b}\right)\left|\left(x-y, \left(\log\left(1+|s|\right)\right)^{1/\nu}\right)\right|\right\} \\ &\le \exp\left\{-m\left|\left(x-y, \left(\log\left(1+|s|\right)\right)^{1/\nu}\right)\right|\right\} \end{aligned}$$

if b is taken large enough and k is large enough. □

Theorem 1.1 follows from Theorems 3.2 and 3.3. To see that, it suffices to prove that, given (1.1) and (1.2), ν, p satisfying (3.3) and $m_0 > 0$, there exists $\varepsilon = \varepsilon(d, \beta, \Gamma, m_0, \nu, p) > 0$ such that if (1.3) holds then (3.4) is satisfied for some L_0 sufficiently large.

But this follows from (1.6): We always have

$$G_{B_L((0,0))}(X,Y) \le G_{\Lambda_L(0)\times\mathbf{R}}(X,Y).$$

Let $\rho_L = \rho_{\Lambda_L(0)}$, $\delta_L = \delta_{\Lambda_L(0)}$, $\lambda_L = \max_{(z,z')\in\partial(\Lambda_L(0),\mathbf{Z}^d)} \lambda(z,z')$. If $8d\rho_L < 1$, we pick θ such that $8d\rho_L < 1-\theta$. It follows that

$$\begin{aligned} G_{B_L((0,0))}\big((0,0),\partial\big) &\le (1-8d\rho_L)^{-1}(8d\rho_L)^{L-1}cL^{d-1}2e^{L^\nu}\lambda_L \\ &\quad + 2\left(1-\frac{8d\rho_L}{1-\theta}\right)^{-1} e^{-\theta\delta_L e^{L^\nu}}(2L)^d, \end{aligned} \tag{3.7}$$

where $c = c(d)$ is a constant such that $|\partial(\Lambda_L(0), \mathbf{Z}^d)| \le cL^{d-1}$.

Suppose

$$8d\rho_L < e^{-2m_0}, \tag{3.8}$$

$$\delta_L > e^{-L^\nu/2}, \tag{3.9}$$

$$\lambda_L < e^{L^\nu} \tag{3.10}$$

and take $\theta = (1-e^{-2m_0})/2$. Then (3.7) gives

$$\begin{aligned} G_{B_L((0,0))}\big((0,0),\partial\big) &\le 2ce^{2m_0}\big(1-e^{-2m_0}\big)^{-1}L^{d-1}e^{-2m_0L+2L^\nu} \\ &\quad + 2^{d+1}\big(1-e^{-2m_0}\big)^{-1}\big(1+e^{-2m_0}\big)L^d \\ &\quad \times \exp\left\{-\frac{\big(1+e^{-2m_0}\big)}{2}e^{L^\nu/2}\right\} \le e^{-m_0L} \end{aligned}$$

for all L sufficiently large.

Using Chebyshev's inequality, we get

$$\begin{aligned} \mathbf{P}\{\delta_L \le e^{-L^\nu/2}\} &\le (2L)^d\mathbf{P}\{\delta \le e^{-L^\nu/2}\} \\ &= (2L)^dP\left\{\log\left(1+\frac{1}{\delta}\right) \ge \log\big(1+e^{L^\nu/2}\big)\right\} \\ &\le 2^{d+\beta}L^{d-\beta\nu}E\left(\left(\log\left(1+\frac{1}{\delta}\right)\right)^\beta\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathbf{P}\{\lambda_L \ge e^{L^\nu}\} &\le cL^{d-1}\mathbf{P}\{\lambda \ge e^{L^\nu}\} \\ &\le cL^{d-1}\mathbf{P}\left\{\log(1+\lambda) \ge \log\big(1+e^{L^\nu}\big)\right\} \\ &\le cL^{d-1-\beta\nu}\mathbf{E}\Big(\big(\log(1+\lambda)\big)^\beta\Big), \end{aligned} \tag{3.12}$$

$$\begin{aligned}
\mathbf{P}\{8d\rho_L \geq e^{-2m_0}\} &\leq (2L)^d \mathbf{P}\left\{\rho \geq \frac{e^{-2m_0}}{8d}\right\} \\
&\leq (2L)^d 2d \mathbf{P}\left\{\frac{\lambda}{\delta} \geq \frac{e^{-2m_0}}{8d}\right\} \\
&= (2d)(2L)^d \mathbf{P}\left\{\log\left(1+\frac{\lambda}{\delta}\right) \geq \log\left(1+\frac{e^{-2m_0}}{8d}\right)\right\} \\
&\leq 2^{d+1}\left(\log\left(1+\frac{e^{-2m_0}}{8d}\right)\right)^{-\beta} L^d \mathbf{E}\left\{\left(\log\left(1+\frac{\lambda}{\delta}\right)\right)^{\beta}\right\}.
\end{aligned} \tag{3.13}$$

To finish the proof, we need only to show that, given (1.1), (1.2), ν, p satisfying by (3.3) and $m_0 > 0$, then for all L sufficiently large there exists $\varepsilon(d, \beta, \Gamma, m_0, \nu, L, p) > 0$, such that if (1.3) holds with $\varepsilon = \varepsilon(d, \beta, \Gamma, m_0, \nu, L, p) > 0$ we have

$$\mathbf{P}\{(3.8), (3.9) \text{ and } (3.10) \text{ hold}\} \geq 1 - \frac{1}{L^p}. \tag{3.14}$$

Since $p + d < \beta\nu$ by (3.3), (3.14) follows from (3.11), (3.12) and (3.13) if L is sufficiently large and $\mathbf{E}\{(\log(1 + \lambda/\delta))^{\beta}\}$ is sufficiently small. □

4. The proof of Theorem 3.2. The proof of Theorem 3.2 proceeds by induction in k. The case $k = 0$ is just the hypothesis (3.5). The induction step is given by the following lemma:

LEMMA 4.1. *Let $d, \beta, \Gamma, \alpha, \nu, p$ be as in Theorem 3.2, set $\theta_0 = \min\{\alpha(1-\nu), 1\}$ and pick m_0 and θ with $m_0 > 0$ and $0 < \theta < \theta_0$. There exist $l_1 = l_1(d, \beta, \Gamma, \nu, p, \theta) < \infty$ and $a = a(d, \beta, \nu, m_0) > 0$, such that for any l and m with $l \geq l_1$ and $m_0 \geq m \geq 1/l^{\theta}$, if*

$$\mathbf{P}\{0 \textit{ is } (m, l)\textit{-regular}\} \geq 1 - \frac{1}{l^p},$$

then we also have

$$\mathbf{P}\{0 \textit{ is } (M, L)\textit{-regular}\} \geq 1 - \frac{1}{L^p},$$

with $L = l^{\alpha}$ and

$$M = m - \frac{a}{l_0^{\theta}} \geq \frac{1}{L^{\theta}}.$$

PROOF. Since $\alpha d < p$, we can pick a positive integer R such that

$$\alpha < \frac{(R+1)p}{p + (R+1)d}.$$

Two boxes $\Lambda_{l_1}(x_1), \Lambda_{l_2}(x_2)$ will be called nonoverlapping if for all $y_1 \in \Lambda_{l_1}(x_1), y_2 \in \Lambda_{l_2}(x_2)$ we have $\|y_1 - y_2\|_2 > 1$. We will say that $x_1, x_2 \in \mathbf{Z}^d$ are l-nonoverlapping if $\Lambda_l(x_1)$ and $\Lambda_l(x_2)$ are nonoverlapping. Notice that if $\|x_1 - x_2\|_\infty > 2l + 1$, then x_1 and x_2 are l-nonoverlapping.

If x_1 and x_2 are l-nonoverlapping, the events $\{x_i \text{ is } (m,l)\text{-regular}\}$, $i = 1,2$, are independent. It follows that

$$\mathbf{P}\{\text{there exist } x_1,\dots,x_{R+1} \in \Lambda_L(0) l\text{-nonoverlapping which are } (m,l)\text{-singular}\} \le \frac{(2L)^{d(R+1)}}{l^{p(R+1)}} < \frac{1}{2L^p}$$

for l sufficiently large by our choice of R. Thus

$$\begin{aligned} &\mathbf{P}\{\text{there exist } x_1,\dots,x_R \in \Lambda_L(0) \text{ such that } \Lambda_L(0)\setminus \cup_{j=0}^{R} \Lambda_{2l+1}(x_j) \\ &\qquad \text{is a } (m,l)\text{-regular region}\} \ge 1 - \frac{1}{2L^p}. \end{aligned} \tag{4.1}$$

We can thus restrict ourselves to the case when the event described in (4.1) occurs.

We want to estimate $G_{B_L(0)}(0,Y)$ for $Y \in \partial B_L(0)$. There are two distinct cases, $Y \in \partial_H B_L(0)$ or $Y \in \partial_V B_L(0)$, where

$$\partial_H B_L(0) = \Lambda_L(0) \times \{-e^{L^\nu}, e^{L^\nu}\},$$
$$\partial_V B_L(0) = \partial\Lambda_L(0) \times [-e^{L^\nu}, e^{L^\nu}].$$

SUBLEMMA 4.2. *Suppose the event described in* (4.1) *occurs. Then there exist* $l_2 = l_2(d, m_0, \nu, R, \theta) < \infty$ *and* $a_1 = a_1(d,\nu,m_0,R) > 0$ *such that for* $l > l_2$ *we have*

$$G_{B_L(0)}(0,Y) \le e^{-M_1 L}$$

for all $Y \in \partial_V B_L(0)$ *with*

$$M_1 = m - \frac{a_1}{l^{\theta_0}} \ge \frac{1}{L^\theta}.$$

PROOF. We can find $y_1,\dots,y_{R'} \in \Lambda_L(0)$, with $R' \le R$, $n_1,\dots,n_{R'} \in \{1,2,\dots,R\}$ with $n_1 + n_2 + \cdots + n_{r'} \le R$, such that the $\Lambda_{2n_i(l+1)}(y_i)$, $i = 1,\dots,R'$, are nonoverlapping, and $\cup_{j=1}^{R}\Lambda_{2l+1}(x_j) \subset \cup_{i=1}^{R'}\Lambda_{2n_i(l+1)}(y_i)$. It follows that $\Lambda' = \Lambda_L(0)\setminus \cup_{i=1}^{R'} \Lambda_{2n_i(l+1)}(y_i)$ is a (m,l)-regular region.

We set

$$B = B_L(0), \qquad B_i = B_{2n_i(l+1),e^{L^\nu}}((y_i,0)),$$
$$\partial^e B_i = \partial^e(B_i,B),\ i = 1,\dots,R', \qquad B' = B\setminus \bigcup_{i=1}^{R'} B_i.$$

If $0 \in B'$, we set $\partial^e B_0 = \{0\}$; otherwise $0 \in B_{i'}$ for some i' and we set $\partial^e B_0 = \partial^e B_{i'}$. Similarly, if $y \in B'$, $\partial^e B_{R+1} = \{Y\}$; otherwise $\partial^e B_{R+1} = \partial^e B_{i''}$ for some i'' with $Y \in B_{i''}$.

We have

$$\{0 \leftrightarrow_B Y\} = \bigcup_{r=0}^{R'} \bigcup_{\{i_1,\dots,i_r\} \subset \{1,\dots,R'\}} \{\partial^e B_0 \leftrightarrow_{B'} \partial^e B_{i_1}\} \\ \circ \{\partial^e B_{i_1} \leftrightarrow_{B'} \partial^e B_{i_2}\} \circ \cdots \circ \{\partial^e B_{i_r} \leftrightarrow_{B'} \partial^e B_{R'+1}\}.$$

By Lemma 3.1,

$$\mathbf{Q}\{\partial^e B_{j_1} \leftrightarrow_{B'} \partial^e B_{j_2}\} \le \Big(c\big(2R(l+1)\big)\Big)^{d-1} 2e^{L^\nu}\Big)^2 \exp\left\{-ml\left(\frac{D_{j_1,j_2}}{l+1} - 1\right)\right\},$$

where $D_{j_1,j_2} = \min\{\|x_1 - x_2\|_\infty;\ (x_k, t) \in \partial^e B_{jk} \text{ for some } t, k = 1, 2\}$.

It thus follows from (4.2) and the BK inequality that

$$G_B(0,Y) \le R!\Big(c\big(2R(l+1)\big)^{d-1} 2e^{L^\nu}\Big)^{2(R+1)} \\ \times \exp\left\{-mL\left(\frac{L-1-4R(l+1)}{l+1} - (R+1)\right)\right\} \le e^{-M_1' L},$$

where

$$M_1' = m - \frac{c_1 m_0}{l} - \frac{c_2}{l^{\alpha(1-\nu)}}$$

for some fixed constants $c_1, c_2 > 0$ depending only on d, ν and R.

Thus we have

$$M_1' \ge M_1 = m - \frac{a_1}{l^{\theta_0}} \ge \frac{1}{L^\theta}$$

for l sufficiently large, with

$$a_1 = a_1(d, \nu, m_0, R) > 0. \qquad \square$$

We must now estimate $G_{B_L(0)}(0, Y)$ for $Y \in \partial_H B_L(0)$. Here we must control the behavior of the process inside the singular regions. This will be done under the occurrence of events that again we will show to have the desired probability.

Notice that it follows from (3.3) that

$$1 < \alpha(1-\nu) + \nu < \frac{\alpha\nu}{d}.$$

SUBLEMMA 4.3. *Suppose the event described in* (4.1) *occurs. Let* κ, b, γ *and* τ *be such that*

$$\alpha(1-\nu) + \nu < \kappa < \frac{b}{d} < \frac{\alpha\nu}{d}, \qquad 0 < \gamma < b - \kappa d,\ \nu < \tau < \kappa - \alpha(1-\nu),$$

let

$$\widetilde{\Lambda} = \left(\bigcup_{j=1}^{R} \Lambda_{l^\kappa}(x_j) \right) \cap \Lambda_L(0),$$

and suppose

$$(4.2) \qquad e^{-l^\gamma} \sum_{\langle x,y \rangle \subset \tilde{\Lambda}} \lambda(x,y) - \sum_{x \in \tilde{\Lambda}} \log\left(1 - \exp\left(-\delta(x) e^{-l^\gamma}\right)\right) \le l^b.$$

Then there exists $l_2 = l(d, m_0, \nu, \beta, \theta, \kappa, b, \tau) < \infty$, *such that for* $l > l_2$ *we have*

$$G_{B_L(0)}(0, Y) \le \exp\{-M_2 e^{l^\tau/8}\}$$

for all $Y \in \partial_H B_L(0)$, *with* $M_2 = m - e^{-l^\tau/6} > 1/L^\theta$.

PROOF. We take $Y = (y, e^{L^\nu})$, $y \in \Lambda_L(0)$, the case $Y = (y, -e^{L^\nu})$ being similar. Let $N = [e^{L^\nu - l^\tau}]$, where $[a]$ denotes the largest integer less than or equal to a. Notice $\tau < \alpha\nu$. We set

$$S_j = B_{L, e^{l^\tau}/2}\left(\left(0, (j - \tfrac{1}{2}) e^{l^\tau}\right)\right), \qquad j = 1, 2, \ldots, N.$$

Events in different S_j's are independent.

The main difficulty in the proof is how to control the percolation inside the cylinders based on singular regions. To do so, given $s \in \mathbf{R}$, we introduce the event D_s based on

$$H_s = \widetilde{\Lambda} \times \left[s - \tfrac{1}{2} e^{-l^\gamma}, s + \tfrac{1}{2} e^{-l^\gamma}\right]$$

given by

$$D_s = \Big\{ \text{there are no } \textit{bridges} \text{ in } H_s \text{ and for each } x \in \widetilde{\Lambda} \text{ the line segment } \{x\} \times \left[s - \tfrac{1}{2} e^{-l^\gamma}, s + \tfrac{1}{2} e^{-l^\gamma}\right] \text{ has at least one } \textit{cut} \Big\}.$$

Clearly,

$$D_s \subset \left\{ \widetilde{\Lambda} \times \left\{s - \tfrac{1}{2} e^{-l^\gamma}\right\} \leftrightarrow_{H_s} \widetilde{\Lambda} \times \left\{s + \tfrac{1}{2} e^{-l^\gamma}\right\} \right\}^c$$

and

$$\mathbf{Q}(D_s) = \exp\left\{ - \sum_{\langle x,y \rangle \subset \tilde{\Lambda}} \lambda(x,y) e^{-l^\gamma} \right\} \prod_{x \in \tilde{\Lambda}} \left(1 - e^{-\delta(x) e^{-l^\gamma}}\right) \ge e^{-l^b}$$

by (4.2).

We will write $B_\Lambda = \Lambda \times [-e^{L^\nu}, e^{L^\nu}]$ for any $\Lambda \subset \mathbf{Z}^d$. Let

$$\tilde{\Lambda} = \left(\bigcup_{j=1}^{R} \Lambda_{2l+1}(x_j) \right) \cap \Lambda_L(0)$$

and let

$$F_j = \left\{ \partial^e (B_{\hat{\Lambda}}, B_L(0)) \cap S_j \leftrightarrow_{S_j \setminus B_{\hat{\Lambda}}} B_{\Lambda_L(0)\setminus\tilde{\Lambda}} \cap S_j \right\}^c,$$

that is, F_j is the event that there is no connection inside $S_j \setminus B_{\hat{\Lambda}}$ from the exterior boundary of $B_{\hat{\Lambda}}$ to $B_{\Lambda_L(0)\setminus\tilde{\Lambda}}$. Since $S_j \setminus B_{\tilde{\Lambda}} \subset B_{\Lambda'}$ and Λ' is a (m,l)-regular region, it follows from Lemma 3.1 that

$$\mathbf{Q}(F_j^c) \le \left[\left(c\big(2(2l+2)\big)^{d-1} e^{l^\tau} \right) \left(c(2l^\kappa)^{d-1} e^{l^\tau} \right) \right]^R \exp\left\{ -ml \left(\frac{l^\kappa - (2l+2)}{l+1} - 1 \right) \right\}$$
$$\le e^{-c_3 l^{\kappa-\theta}} \le e^{-c_3 l^{\kappa - \alpha(1-\nu)}},$$

for some $c_3 = c_3(d, \tau, \kappa, R, \theta) > 0$ and l sufficiently large, since $m \ge 1/l^\theta, \theta \le \alpha(1-\nu)$ and $\tau < \kappa - \alpha(1-\nu)$.

We now set

$$A_j = F_j \cap D_{(j-1/2)e^{l^\tau}}, \qquad j = 1, 2, \dots, N.$$

Since both F_j and $D_{(j-1/2)e^{l^\tau}}$ are local negative events, the Harris–FKG inequality gives

$$\mathbf{Q}(A_j) \ge \mathbf{Q}(F_j)\mathbf{Q}(D_{(j-1/2)e^{l^\tau}}) \ge \left(1 - e^{-c_3 l^{\kappa-\alpha(1-\nu)}}\right) e^{-l^b} \ge e^{-2l^b}$$

for l sufficiently large.

Let $A = \cup_{j=1}^N A_j$; since the A_j's are independent identically distributed events, we have

$$\mathbf{Q}(A^c) = \prod_{J=1}^{N} \left(1 - \mathbf{Q}(A_j)\right) = \left(1 - \mathbf{Q}(A_1)\right)^N \le (1 - e^{-2l^b})^N \le e^{-Ne^{-2l^b}}.$$

But $N = [e^{l^{\alpha\nu} - l^\tau}] \ge e^{l^{\alpha\nu}/2}$ for l large. Since $b < \alpha\nu$, we conclude

$$\mathbf{Q}(A^c) \le \exp\{-e^{l^{\alpha\nu}/2 - 2l^b}\} \le \exp\{-e^{l^{\alpha\nu}/4}\} \tag{4.3}$$

for l sufficiently large.

We have

$$G_{B_L(0)}(0, Y) \le \mathbf{Q}\{\{0 \leftrightarrow_{B_L(0)} Y\} \cap A\} + \mathbf{Q}(A^c). \tag{4.4}$$

By the definition of the event A, we have

$$\{0 \leftrightarrow_{B_L(0)} Y\} \cap A \subset C,$$

where C is the event that there exists a connection in $B_{\Lambda_{L(0)}\setminus\hat{\Lambda}}$ of vertical length greater than or equal to e^{l^τ}, so

$$C \subset \bigcup \{(y_1, s_1) \leftrightarrow_{B\Lambda_L(0)\setminus\hat{\Lambda}} (y_2, s_2)\},$$

the union being taken over all $(y_1,s_1),(y_2,s_2) \in B_{\Lambda_L(0)\setminus\hat{\Lambda}}$ with $|s_1 - s_2| \geq e^{l^\nu}$.

Again, applying Lemma 3.1, we get

$$\begin{aligned}\mathbf{Q}(C) &\leq c\big((2L)^d 2e^{L^\nu}\big)^2 \exp\{-ml(e^{l^\tau - l^\nu} - 1)\} \\ &\leq e^{l^\tau/2} \exp\Big\{-\big(m - e^{-l^\tau/3}\big)\Big\}\end{aligned} \tag{4.5}$$

for l sufficiently large.

Inequalities (4.3)–(4.5) give us

$$G_{B_L(0)}(0, Y) \leq \exp\{-M_2 e^{e^{l^\tau}/8}\},$$

with $M_2 = m - e^{-l^\tau/6}$ for l sufficiently large.

We can now complete the proof of Lemma 4.1, modulo an estimate on the probability of the event described by (4.2). Let us assume the events described in (4.1) and (4.2) occur. Let us also assume the event $\lambda_L \leq e^{L^\nu}$. We have, by Sublemmas 4.2 and 4.3,

$$G_{B_L(0)}(0, \partial) \leq cL^{d-1} 2e^{L^\nu} e^{-M_1 L} e^{L^\nu} + 2(2L)^d \exp\{-M_2 e^{l^\tau/8}\} \leq e^{-ML},$$

with

$$M = m - \frac{a}{l^{\theta_0}} \geq \frac{1}{L^\theta}$$

for some $a = c(d, v, m_0, R, \beta, \theta) > 0$, if l is sufficiently large.

By (3.12) we have

$$\mathbf{P}\{\lambda_L \geq c^{L^\nu}\} < \frac{1}{4L^p}$$

for l sufficiently large. Thus it only remains to show that

$$\mathbf{P}\{(4.2) \text{ holds}\} \geq 1 - \frac{1}{4L^p}$$

for l sufficiently large.

Let

$$\zeta(x) = e^{-l^\gamma} \sum_{\langle y, y'\rangle \subset \Lambda_{l^\kappa}(x)} \lambda(y, y') - \sum_{y \in \Lambda_{l^\kappa}(x)} \log\big(1 - e^{-\delta(y)e^{-l^\gamma}}\big).$$

It suffices to show

$$\mathbf{P}\Big\{\zeta(x) \leq \frac{l^b}{R} \text{ for all } x \in \Lambda_L(0)\Big\} \geq 1 - \frac{1}{4L^p}. \tag{4.6}$$

We have

$$\mathbf{P}\left\{\zeta(x) > \frac{l^b}{2dR}\right\} \le \mathbf{P}\left\{e^{-l^\gamma} \sum_{\langle y,y'\rangle \subset \Lambda_{l^\kappa}(x)} \lambda(y,y') > \frac{l^b}{4dR}\right\} + \mathbf{P}\left\{-\sum_{y \in \Lambda_{l^\kappa}(x)} \log\left(1 - e^{-\delta(y)e^{-l^\gamma}}\right) > \frac{l^b}{4dR}\right\}. \tag{4.7}$$

So we will estimate both terms in (4.7). We start with the second:

$$\begin{aligned}
&\mathbf{P}\left\{-\sum_{y \in \Lambda_{l^\kappa}(x)} \log\left(1 - e^{-\delta(y)e^{-l^\gamma}}\right) > \frac{l^b}{4dR}\right\} \\
&\le (2l^\kappa)^d \mathbf{P}\left\{-\log\left(1 - e^{-\delta e^{-l^\gamma}}\right) > \frac{l^{b-\kappa d}}{2^{d+1}dR}\right\} \\
&= 2^d l^{\kappa d} \mathbf{P}\left\{\delta < -e^{l^\gamma} \log\left(1 - \exp\left\{-\frac{l^{b-\kappa d}}{2^{d+1}dR}\right\}\right)\right\} \\
&\le 2^d l^{\kappa d} \mathbf{P}\left\{\delta < 2\exp\left\{l^\gamma - \frac{l^{b-\kappa d}}{2^{d+1}dR}\right\}\right\} \\
&\le 2^d l^{\kappa d} \mathbf{P}\left\{\delta < \exp\left\{-\frac{l^{b-\kappa d}}{2^{d+2}dR}\right\}\right\} \\
&= 2^d l^{\kappa d} \mathbf{P}\left\{\log\left(1 + \frac{1}{\delta}\right) > \log\left(1 + \exp\left\{\frac{l^{b-\kappa d}}{2^{d+2}dR}\right\}\right)\right\} \\
&\le 2^d l^{\kappa d} \mathbf{P}\left\{\log\left(1 + \frac{1}{\delta}\right) > \frac{l^{b-\kappa d}}{2^{d+2}dR}\right\} \\
&\le 2^d l^{\kappa d} l^{-\beta\gamma} \mathbf{E}\left(\left(\log\left(1 + \frac{1}{\delta}\right)\right)^\beta\right)
\end{aligned} \tag{4.8}$$

for l sufficiently large, since $0 < \gamma < b - \kappa d$.

For the first term in (4.7), we have

$$\begin{aligned}
\mathbf{P}\left\{e^{-l^\gamma} \sum_{\langle y,y'\rangle \subset \Lambda_{l^\kappa}(x)} \lambda(y,y') > \frac{l^b}{4dR}\right\} &\le (2l^\kappa)^d \mathbf{P}\left\{\lambda > e^{l^\gamma} \frac{l^{b-\kappa d}}{2^{d+1}dR}\right\} \\
&\le 2^d l^{\kappa d} \mathbf{P}\{\lambda > e^{l^\gamma/2}\} \\
&\le 2^d l^{\kappa d} \mathbf{P}\left\{\log(1+\lambda) > \frac{1}{2} l^\gamma\right\} \\
&\le 2^{d+\beta} l^{\kappa d} l^{-\beta\gamma} \mathbf{E}\left(\left(\log(1+\lambda)\right)^\beta\right)
\end{aligned} \tag{4.9}$$

for l sufficiently large. It follows from (4.7) to (4.9) and (1.2) that

$$\mathbf{P}\left\{\zeta(x) > \frac{l^b}{2dR}\right\} < \frac{2^{d+\beta+1}\Gamma}{l^{\beta\gamma-\kappa d}} \tag{4.10}$$

for l sufficiently large.

For a number t let $\langle t \rangle$ = largest integer less than t. Let $\mathcal{L} = (\langle l^\kappa \rangle \mathbf{Z}^d) \cap \Lambda_L(0)$. We have

$$\Lambda_L(0) \subset \bigcup_{x \in \mathcal{L}} \Lambda_{l^\kappa}(x)$$

and

$$|\mathcal{L}| \leq \left(\frac{3L}{l^\kappa}\right)^d = 3l^{(\alpha-\kappa)d}.$$

From (4.10) we have

$$\text{(4.11)} \qquad \mathbf{P}\left\{\text{there exists } x \in \mathcal{L} \text{ for which } \zeta(x) > \frac{l^b}{2dR}\right\} \leq \frac{3^d \cdot 2^{d+\beta+1}\Gamma}{l^{\beta\gamma-\alpha d}}.$$

For $x \in \Lambda_L(0)$, let

$$\mathcal{L}_x = \left\{y \in \mathcal{L};\ \Lambda_{l^\kappa}(x) \cap \Lambda_{l^\kappa}(y) \neq \emptyset\right\}.$$

Notice that $|\mathcal{L}_x| \leq 2d$.

Clearly,

$$\zeta(x) \leq \sum_{y \in \mathcal{L}_x} \zeta(y).$$

Thus it follows from (4.11) that

$$\mathbf{P}\left\{\text{there exists } x \in \Lambda_L(0) \text{ for which } \zeta(x) > \frac{l^b}{R}\right\} \leq \frac{3^d \cdot 2^{d+\beta+1}\Gamma}{l^{\beta\gamma-\alpha d}} < \frac{1}{4L^p}$$

for l sufficiently large since $\beta\gamma > \alpha(p+d)$.

This completes the proof of Lemma 4.1. □

We can now complete the proof of Theorem 3.2. Given $m_0 > 0$, if L_0 is sufficiently large we can apply Lemma 4.1 to conclude that

$$\mathbf{P}\{0 \text{ is } (m_k, L_k)\text{-regular}\} \geq 1 - \frac{1}{L_k^p}$$

for all $k = 0, 1, 2, \ldots$, with $m_{k+1} \geq m_k - a/L_k^{\theta_0}$. It thus suffices to show $m_k \geq m_\infty$ for all $k = 0, 1, 2, \ldots$.

But

$$m_k \geq m_0 - a\sum_{j=0}^{k-1} \frac{1}{L_j^{\theta_0}}.$$

Thus it suffices to choose L_0 sufficiently large so that

$$a\sum_{j=0}^{\infty} \frac{1}{L_j^{\theta_0}} = a\sum_{j=0}^{\infty} \frac{1}{L_j^{\theta_0 \alpha^j}} < m_0 - m_\infty. \qquad \square$$

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