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ON THE CRITICAL BEHAVIOR OF THE CONTACT PROCESS IN DETERMINISTIC INHOMOGENEOUS ENVIRONMENTS

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We consider the contact process with inhomogeneous deterministic death rates. We prove the following:

1. Such models may have discontinuous transitions, in the sense of surviving at the critical point.
2. If the death rates are identically 1, except on a set which is small enough in a proper sense and where the death rates take a fixed value smaller than 1, then the critical point is identical to that of the homogeneous system.

Extensions of the results to other $(d+1)$-dimensional systems with d-dimensional deterministic inhomogeneities are also discussed.

1. Introduction. In this paper we are concerned with the critical behavior of the contact process (and related models) in inhomogeneous deterministic environments. The problems that we treat are basically of two kinds:

1. Can the nature of the phase transition change from continuous (for the homogeneous system) to discontinuous (for an inhomogeneous system)?
2. How is the value of the critical point affected by the inhomogeneities?

To describe our results on these problems, we first have to recall the definition of the contact process (with inhomogeneous death rates). It is a continuous-time Markov process on the state space $\{0,1\}^{Z^d}$, whose evolution is governed by the following rules. A 1 at site $x \in Z^d$ flips to 0 at rate $\delta(x) > 0$; we call $\delta(x)$ *the death rate at* x. A 0 at site x flips to 1 at rate $\lambda \times$ (number of nearest neighbors of x which are in state 1). Here the "nearest neighbors of x" are the sites in $\{y \in Z^d\colon \|y-x\|_1 = 1\}$, where $\|\cdot\|_1$ is the l_1-norm on Z^d. We call the parameter $\lambda > 0$ the *infection rate*. This description specifies a unique process; for a proof of this fact and the basic properties of the contact process recalled below, the reader is invited to consult Chapters 3 and 6 in Liggett (1985) and Chapter 4 in Durrett (1988). At the beginning of Section 3, we will recall the graphical construction of this model. The case in which $\delta(x) = 1$ for all x will be called the *homogeneous* case.

From monotonicity considerations it follows that when started from the configuration identically 1, the process converges in distribution to an invariant

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measure which will be denoted by ν. This measure is called the upper invariant measure (it dominates the other invariant measures in a certain sense). The configuration identically 0 is a trap for this process and hence the delta measure concentrated on it, δ_0, is invariant. Monotonicity arguments also imply that if $\nu = \delta_0$ for certain values of $\{\delta(x): x \in Z^d\}$ and λ, then the process is ergodic in the sense that δ_0 is the only invariant measure, and from any initial configuration the process converges to it. Furthermore, once the process is ergodic for particular death and infection rates, it is also ergodic for the same $\{\delta(x)\}$ and any smaller λ. On the other hand, the process is clearly nonergodic if $\nu \neq \delta_0$. We say that the process *survives* if it is nonergodic, and we say that it *dies out* if it is ergodic. These remarks lead to the definition of the critical point

$$\lambda_c = \inf\{\lambda > 0: \delta_0 \neq \nu\}.$$

It is obvious that λ_c is a function of the death rate. In the homogeneous case, it is known that $0 < \lambda_c < \infty$ in every dimension d. Let $(\xi_t^\eta,\ t \geq 0)$ denote the contact process started from the configuration η. For a given $x \in Z^d$, let x^* denote the configuration which is 1 at x and 0 elsewhere. Define

$$\rho_\lambda(x) = P\big(\text{for every } t, \text{ there exists a } y \text{ such that } \xi_t^{x^*}(y) = 1\big),$$

the probability that the configuration is never identically 0. Self-duality implies that

$$\rho_\lambda(x) = \nu\{\eta: \eta(x) = 1\}.$$

It is clear that either $\rho_\lambda(\cdot)$ is identically 0 or else it is nonzero for every x. Therefore

$$\lambda_c = \inf\{\lambda > 0: \rho_\lambda(0) \neq 0\}.$$

In a fundamental paper, Bezuidenhout and Grimmett (1990) proved that in the homogeneous case,

$$\rho_{\lambda_c}(0) = 0 \quad \text{in any dimension,}$$

which is equivalent to the assertion that $\nu = \delta_0$ at $\lambda = \lambda_c$. So the critical homogeneous contact process dies out and the "order parameter" $\rho_\lambda(0)$ is continuous at λ_c.

The first question we address is whether inhomogeneities can make the transition discontinuous.

THEOREM 1.1. *There are inhomogeneous contact processes which have discontinuous transitions, that is*, $\rho_{\lambda_c}(0) > 0$.

REMARK. In each of our examples, the critical point of the inhomogeneous contact process is equal to the critical point of the homogeneous contact process. We chose examples with this feature because they are easier to handle at the critical point. There are examples in which the critical point is smaller than that of the homogeneous system and for which the transition is discontinuous, but they are considerably more complicated.

We now give two explicit examples of inhomogeneous contact processes which exhibit discontinuous phase transitions.

EXAMPLE A. The inhomogeneous contact process in $d = 1$ with

$$\delta(x) = \begin{cases} e^{-i^2}, & \text{if } x = \sum_{j=1}^{i} j^3 \text{ for some } i, \\ 1, & \text{otherwise.} \end{cases}$$

PROPOSITION 1.2. *The process in Example* A *has the same critical point as the homogeneous contact process, but has a discontinuous phase transition.*

This example is one for which we can prove such behavior in the simplest possible way. The reader may nevertheless wonder if the sequence of $\delta(x)$ going to 0 very fast in this example is crucial for such a behavior. The next example and proposition show that this is not the case.

EXAMPLE B. The inhomogeneous contact process in $d = 1$ with

$$\delta(x) = \begin{cases} b, & \text{if } x \text{ is in an interval } [A_i, B_i] \text{ for some } i, \\ 1, & \text{otherwise,} \end{cases}$$

where b is a fixed number in $(0, 1)$, and the sequences A_i and B_i are defined by $A_1 = 0$,

$$B_i = A_i + i^2 \quad \text{and} \quad A_{i+1} = B_i + (i+1)^3.$$

PROPOSITION 1.3. *The process in Example* B *has the same critical point as the homogeneous contact process, but has a discontinuous phase transition.*

In higher dimensions one can also have the same phenomenon, but since the examples are more cumbersome and less explicit, we will omit them.

The above results raise the question of how to characterize the environments $\{\delta(x),\ x \in Z^d\}$ for which the transition is discontinuous. In other words, can we find necessary and sufficient conditions for this behavior? Unfortunately, this seems to be a very difficult question. For instance, consider the following example.

EXAMPLE C. The inhomogeneous contact process in dimension d with $\delta(0) = b < 1$ and $\delta(x) = 1$ for $x \neq 0$.

The following result is not hard to show (see Lemma 4.1).

PROPOSITION 1.4. *The process in Example* C *has the same critical point as the homogeneous contact process.*

OPEN PROBLEM 1. We believe that for Example C the phase transition is continuous, but we have not succeeded in proving it. Observe that this claim is stronger than the analogous statement for the homogeneous contact process, which has only been proved recently by Bezuidenhout and Grimmett (1990), using elaborate methods [in part from Barsky, Grimmett and Newman (1991)]. But their approach makes heavy use of translation invariance and seems to be useless in our case.

We turn now to the other type of problem addressed in this paper. When is the critical point of the process different from the critical point of the homogeneous system? We will restrict our attention to the case in which $\delta(x) = 1$ for all x outside of some (small) set S and $\delta(x)$ assumes a constant value $b \in (0, 1)$ on S. In this class of examples we shall ask how small S has to be for the critical point to be unaffected. We will denote by $\lambda_c(S)$ the critical point of such a system, so $\lambda_c(\emptyset)$ is the critical point in the homogeneous case.

Recent methods of Aizenman and Grimmett (1991), who built on an idea of Menshikov (1987), give a sufficient condition for the critical point to be lowered for percolation models. The proof of the next theorem is a straightforward adaptation of their methods and we omit it.

THEOREM 1.5. *If S is such that there is a length R so that each* (*Euclidean*) *ball of radius R in Z^d contains at least one point in S, then $\lambda_c(S) < \lambda_c(\emptyset)$.*

This theorem states that if S is sufficiently dense, then the critical point changes. The next result states that if S is sufficiently thin, then this does not happen.

THEOREM 1.6. *If $S \subset Z^d$ is such that for every l there are only finitely many pairs of points in S a distance not larger than l apart, then $\lambda_c(S) = \lambda_c(\emptyset)$. Moreover, for all $\lambda < \lambda_c(S)$, there are γ and C in $(0, \infty)$ such that for all $x \in Z^d$,*

$$P\big(\xi_t^{x^*}(y) \neq 0 \text{ for some } (y,t) \text{ such that } \|y - x\|_\infty > k \text{ or } t > k\big) \leq Ce^{-\gamma k}.$$

Note that the model in Example B does not satisfy the assumption of Theorem 1.6. For this model we have $\lambda_c(S) = \lambda_c(\emptyset)$ but it is not difficult to see that for λ in $(\lambda_c(\emptyset)b, \lambda_c(\emptyset))$ the exponential decay in Theorem 1.6 does not hold in the time direction.

In $d = 1$, Theorems 1.5 and 1.6 can be combined (leaving aside the exponential decay in Theorem 1.6) in a nice way. In this case S can be described by the doubly infinite sequence $\dots g_{-1}, g_0, g_1 \dots$ of distances between successive points in S (with g_0 indicating the distance between the first two nonnegative sites in S), with the convention that the g_i become eventually $+\infty$ if there is a leftmost or a rightmost site in S. Theorems 1.5 and 1.6 may be combined in $d = 1$ as follows:

(i) If $\limsup_{i\to\infty} g_i < \infty$ and $\limsup_{i\to-\infty} g_i < \infty$, then $\lambda_c(S) < \lambda_c(\emptyset)$.

(ii) If $\lim_{i\to\infty} g_i = \infty$ and $\lim_{i\to-\infty} g_i = \infty$, then $\lambda_c(S) = \lambda_c(\emptyset)$.

Using the observation that the critical point for the half-infinite one-dimensional homogeneous system coincides with $\lambda_c(\emptyset)$ [see Durrett and Griffeath (1983)] together with the techniques in Aizenman and Grimmett (1991), assertion (i) can be strengthened to

(i′) If $\limsup_{i\to\infty} g_i < \infty$ or $\limsup_{i\to-\infty} g_i < \infty$, then $\lambda_c(S) < \lambda_c(\emptyset)$.

OPEN PROBLEM 2. Observe that unfortunately (i′) and (ii) do not say whether the positivity of the density of S on one of the two semi-infinite sets Z_+ and Z_- is also a necessary condition for the critical point to change. In more than one dimension, on the other hand, it is clear that S does not need to be dense for this to happen, since it can simply be a one-dimensional line if b is sufficiently small, or for any $b < 1$ a one-dimensional strip [see Theorem 2 in Bezuidenhout and Grimmett (1990)].

The inhomogeneous contact process in deterministic and random environments (sometimes with site-dependent λ) has been considered in various recent papers. Bramson, Durrett and Schonmann (1991) showed the existence of an intermediate phase in one dimension in an i.i.d. random environment. Liggett (1991, 1992) (see also references therein) has investigated conditions which imply survival and extinction for the process. More recently, Andjel (1992), as well as Aizenman, Klein and Newman (1993) and Klein (1994), obtained results along these lines giving partial answers to questions left open by Liggett. Our approach differs from those in these previous works in that we are concerned with the critical point and specifically how its value and the behavior of the system at this point differ from the homogeneous system. In general, these are rather delicate questions and we are only able to give partial answers.

OPEN PROBLEM 3. Are there contact processes in i.i.d. random environments which have discontinuous transitions, as in Theorem 1.1?

The remainder of the paper is organized as follows. In Section 2 we mention other models for which our results and techniques are valid. In Section 3 we prove Propositions 1.2 and 1.3. Finally, in Section 4 we prove Theorem 1.6.

2. Related models and results. Most of our techniques and results apply to other $(d + 1)$-dimensional systems with d-dimensional inhomogeneities. We chose to state them first and only prove them in detail for the contact process for the sake of clarity of exposition.

The other systems for which one can extend most of the results discussed in Section 1 are ordinary bond or site percolation [see, for instance, Grimmett (1989) or Durrett (1988)], ferromagnetic nearest neighbor Ising models [see Chapter 4 in Liggett (1985)] and hence, via a path space representation, a quantum lattice system called the Ising model with transverse field [see Campanino, Klein and Perez (1991) or Aizenman, Klein and Newman (1993)]. In the case of the quantum system mentioned above, if one considers the d-dimensional version, with purely local inhomogeneities (i.e., d-dimensional) in the strength of the transverse field, then its path space representation corresponds to a (classical) $(d+1)$-dimensional system with d-dimensional inhomogeneities.

Consider ordinary percolation on Z^{d+1}. In the case of bond percolation, each pair of neighboring sites in Z^{d+1} is thought of as defining a bond. Each bond $\{x, y\}$ is open with probability $p_{\{x,y\}}$ independently from bond to bond. The homogeneous case corresponds to all $p_{\{x,y\}}$ being identical to a unique parameter p. One defines the clusters as the maximal sets of sites in Z^{d+1} which can be joined by continuous paths of open bonds. In the homogeneous case, it is well known that when $d+1 \geq 2$ there is a critical point $p_c \in (0,1)$ such that the probability of the occurrence of an infinite cluster is 0 for $p < p_c$ and 1 for $p > p_c$. In $d+1=2$ or for large enough d [Hara and Slade (1990)], it is also known that this probability is 0 at p_c, and this is believed to be true in any dimension.

In analogy with the inhomogeneous contact process, one can think of a percolation model, in dimension $d+1$, for which $p_{\{y,z\}} = p_x$ if $y = (x, v)$ and $z = (x, v+1)$, where $x \in Z^d$ and $v \in Z$, and $p_{\{y,z\}} = p$ when y and z are neighbors but not of the preceding form. We think of p as the parameter which plays the role of λ. Such models were, for instance, considered in Campanino and Klein (1991). One can easily adapt Examples A, B and C to this context. A very small $\delta(x)$ in the contact process corresponds to p_x very close to 1 in percolation, while $\delta(x) = 1$ corresponds to $p_x = p$. Hence for the analogue of Example A one can take $d+1=2$ and

$$p_x = \begin{cases} 1 - e^{-i^2}, & \text{if } x = \sum_{j=1}^{i} j^3 \text{ for some } i, \\ p, & \text{otherwise.} \end{cases}$$

The analogues of Examples B and C should be clear now and we omit them.

All the results for the contact process presented in this paper extend with the same proofs to this class of inhomogeneous percolation models. In fact, there are some simplifications in the proofs, due to the discreteness of the lattice and also to better (and more elementary) lower bounds on the connectivities at the critical point. As a consequence one can easily find examples analogous to Examples A and B in higher dimensions for which the proof that the transition is discontinuous applies with no extra difficulty (contrary to the present situation for the contact process).

OPEN PROBLEM 4. The question of whether the percolation analogue of Example C (the model with $p_x = b > p_c$ for $x = 0$ and $p_x = p$ otherwise) percolates or not at its critical point $p = p_c$ is also open. [While revising this paper, we learned that Zhang (1994) proved that there is no percolation in this case when $d + 1 = 2$.]

Two known facts related to our results for ordinary percolation should be mentioned. Chayes, Chayes and Durrett (1987) proved that certain inhomogeneous percolation models percolate at their critical point. In Chayes and Chayes (1986) examples are also given of percolation models in subsets of Z^d (wedges) where the transitions are discontinuous. In our case, the fact that we are dealing with $(d + 1)$-dimensional systems with d-dimensional inhomogeneities imposes an extra restriction on the choices of environments and requires new arguments.

We limit ourselves now to a very brief description of the extensions to the other two types of models. Complete definitions are omitted, but we use standard notation and terminology, and we give references where detailed descriptions can be found.

Ising models in $(d + 1)$ dimensions with d-dimensional inhomogeneities were studied by McCoy and Wu (1968), McCoy (1970), Shankar and Murty (1987) and Campanino and Klein (1991). In all these papers the inhomogeneity is random, in contrast with the present paper. As in the case of ordinary percolation, one can consider analogues of the contact process as follows: let the interaction $J_{\{y,z\}}$ for the bond $\{y,z\}$ be given by $J_{\{y,z\}} = J_x$ if $y = (x,v)$ and $z = (x, v + 1)$, where $x \in Z^d$ and $v \in Z$, and $J_{\{y,z\}} = J$ when y and z are neighbors but not of this form. The existence of models of this type with discontinuous transition (i.e., such that the spontaneous magnetization is positive at the critical point) can be proven by combining the techniques that we use for the contact process with the Fortuin–Kasteleyn representation of these models in terms of dependent percolation models [see Aizenman, Chayes, Chayes and Newman (1988)]. To each bond $\{x, y\}$ in this dependent percolation model there corresponds a parameter $p_{\{x,y\}} = 1 - e^{-J_{\{x,y\}}}$ which plays essentially the role of the probability that the bond is occupied. One can easily adapt the examples with discontinuous transitions from percolation to Ising models. The proofs of discontinuity are basically the same as those given in Section 3, since the Harris–FKG inequalities also hold for the dependent percolation models involved. In the place of the independence used for events related to disjoint parts of Z^{d+1}, one has to bound the probabilities of the intersection of such events by products of the probabilities of the corresponding events for modified versions of the system, in which these disjoint regions are separated by layers of spins, all frozen as $+1$ or all frozen as -1 and regarded as boundary conditions.

Unfortunately, we are not yet able to extend Theorem 1.6 to this type of model nor to those discussed in the next paragraph.

The last model we will consider in this section is the (quantum) Ising model with transverse field. It is defined by the formal Hamiltonian

$$H = -J \sum_{x,y \in Z^d:\, \|x-y\|_1 = 1} \sigma_3(x)\sigma_3(y) - \sum_{x \in Z^d} h(x)\sigma_1(x),$$

where $\sigma_3(x)$ and $\sigma_1(x)$ are Pauli spin 1/2 matrices, $J > 0$ is a free parameter (i.e., it plays the role of λ in the contact process) and $h(x)$ are the site-dependent transverse fields. As discussed in Campanino, Klein and Perez (1991) and Aizenman, Klein and Newman (1993), this model has a "path space representation" in terms of a $(d+1)$-dimensional Ising model with d-dimensional inhomogeneities. So, via the Fortuin–Kasteleyn representation, one can prove that this model has a discontinuous transition for appropriate choices of $\{h(x): x \in Z^d\}$.

3. Proofs of discontinuous transitions. First, we recall the graphical construction of the contact process and some related basic notation. We associate each site $x \in Z^d$ with $2d + 1$ independent Poisson processes, one with rate $\delta(x)$ and $2d$ others with rate λ. Make these Poisson processes also independent from site to site. For each x, let $\{T_n^{x,k}: n \geq 1\}$, $k = 0, 1, 2, \dots, 2d$, be the arrival times of these $2d + 1$ processes, respectively. For each x and $n \geq 1$, we write a δ mark at the point $(x, T_n^{x,0})$ and draw arrows from $(x, T_n^{x,k})$ to $(x + e_k, T_n^{x,k})$ for $k = 1, \dots, 2d$, where e_i, $i = 1, \dots, 2d$, are the unit vectors of Z^d. We say that there is a path from (x, s) to (y, t) if there is a sequence of times $s_0 = s < s_1 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \dots, x_n = y$ so that for $i = 1, 2, \dots, n$ there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ for $i = 0, 1, \dots, n$ do not contain any δ. We will use the notation $\{V \to W\}$, where V and W are subsets of $R^d \times R_+$, to denote the event that there is a path from some point in V to some point in W. For $U \subset R^d \times R_+$, we write $\{V \to^U W\}$ for the event that such a path can be found inside U, except perhaps for the arrows in the path and the two endpoints of the path. If we replace $\to$ by $\not\to$, we indicate the complement of the previously defined event. Observe that if $A \subset Z^d$ and we define ξ_t^A to be the indicator function of the set $\{x \in Z^d: A \times \{0\} \to (x, t)\}$, then $(\xi_t^A, t \geq 0)$ is a version of the inhomogeneous contact process. We will denote by $P(\cdot)$ the probability law associated with the graphical construction described above. Sometimes, for comparison, we will also refer to the similar construction in the homogeneous case; $P_H(\cdot)$ will denote the corresponding law.

We shall use the definition of an increasing event that was given by Bezuidenhout and Grimmett (1991). Briefly, an event E is said to be *increasing* if the following holds: for any realization of the graphical construction that is in E, every other realization obtained from it by the addition of arrows or the suppression of δ marks is also in E. The complement of an increasing event is said to be *decreasing*. The Harris–FKG inequality says that if E and F are either both increasing or both decreasing events, then

$$P(E \cap F) \geq P(E)P(F).$$

The l^∞-norm in R^d or R^{d+1} will be denoted by $\|\cdot\|$; distances will be measured with respect to this norm except when explicitly stated otherwise.

In this section λ_c will denote the critical value of the homogeneous model. We shall write $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. $C, c, C_1, C_2, \dots$ will denote positive and finite constants, whose exact value is irrelevant and may even change from appearance to appearance. When referring to sets with a single element, we will simplify the notation, representing such a set by its unique element.

Our main task in this section is to prove Proposition 1.2. To clarify the significance of our choice of parameters in the process of Example A, we shall work with the following more general class of inhomogeneous models.

EXAMPLE D. The inhomogeneous contact process in $d = 1$ with $\delta(x) = 1$ except at the sites $x_1, x_2, \dots$ is given by

$$x_1 = 1 \quad \text{and} \quad x_{i+1} = x_i + \lfloor (i+1)^q \rfloor,$$

where the death rates are

$$\delta(x_i) = \exp(-i^r)$$

for some choice of positive parameters q and r. (Recall that Example A had $q = 3$ and $r = 2$.)

A basic ingredient for the proof of Proposition 1.2 is the qualitative difference between the decay of connectivities for the homogeneous system in the critical and subcritical regimes. Bezuidenhout and Grimmett (1991) proved that for any $\lambda < \lambda_c$, there is a $\gamma > 0$ such that for all x in Z^d and for all $t > 0$,

$$P_H\big((0,0) \to \{x\} \times R\big) \le e^{-\gamma \|x\|} \text{ and } P_H\big((0,0) \to Z^d \times \{t\}\big) \le e^{-\gamma t}. \tag{3.1}$$

From (3.1) it is easy to obtain an exponential bound for the probability of connection between two distant sets. Indeed, applying the Harris–FKG inequality to the intersection of the event that there is no death mark in $\{0\} \times [0,1]$ with $\{\{0\} \times [0,1] \to \{(x,t) : \|(x,t)\| > l\}$, one obtains

$$P_H\Big(\{0\} \times [0,1] \to \big\{(x,t) : \|(x,t)\| > l\big\}\Big) \le C e^{-\gamma l} \tag{3.2}$$

This gives the following bound for the probability of connection between two sets for the subcritical contact process on Z^d:

$$P_H(S \to T) \le \alpha(S) e^{-\gamma D(S,T)}, \tag{3.3}$$

where $\alpha(S)$ is the number of sets of the form $\{x\} \times [i, i+1)$ $(x \in Z^d,\ i = 0, 1, \dots)$ which have a nonempty intersection with S, and

$$D(S,T) = \inf\big\{\|(x,t) - (y,u)\| : (x,t) \in S, (y,u) \in T\big\}$$

is the distance between S and T.

The following lemma will be used several times.

LEMMA 3.1. (a) *Suppose* $E_1, E_2, \dots$ *are all increasing* (*or all decreasing*) *events such that* $P(E_i) > 0$ *for all* i. *If* $\sum_{i=1}^{\infty} P(E_i^c) < \infty$, *then* $P(\bigcap_{i=1}^{\infty} E_i) > 0$.

(b) *Suppose that $S \subset Z^d \times R_+$ is the union of a finite or countable collection of sets S_l such that $\alpha(S_l) = 1$ for every l. Then*

$$P_H(S \not\to T) \geq \prod_l \Big[1 - \exp\big(-\gamma D(S_l, T)\big)\Big].$$

PROOF. (a) By the Harris–FKG inequality, for every J we have

$$P\left(\bigcap_{i=1}^{J} E_i\right) \geq \prod_{i=1}^{J} P(E_i),$$

so upon letting $J \to \infty$ we obtain

$$P\left(\bigcap_{i=1}^{\infty} E_i\right) \geq \prod_{i=1}^{\infty} \Big[1 - P(E_i^c)\Big] > 0.$$

(b) From the Harris–FKG inequality and (3.3), we obtain

$$\begin{aligned} P_H(S \not\to T) &= P_H\left(\bigcap_l \{S_l \not\to T\}\right) \\ &\geq \prod_l P_H(S_l \not\to T) \\ &\geq \prod_l \Big[1 - \exp\big(-\gamma D(S_l, T)\big)\Big]. \end{aligned}$$

□

We now discuss the case of $\lambda = \lambda_c$ in one dimension. The proof of (5.4) in Durrett, Schonmann and Tanaka (1989) implies

$$P_H\big((0,0) \to^U \{x\} \times R_+\big) \geq x^{-3} \quad \text{for } x \text{ large enough},$$

where $U = \{-x, \dots, x\} \times [0, x^5]$. This inequality implies the following one, which will be used in the next lemma:

$$P_H\big(\{0\} \times R_+ \to^V \{x\} \times R_+\big) \geq x^{-3} \quad \text{for } x \text{ large enough}, \tag{3.4}$$

where $V = \{0, \dots, x\} \times [0, x^5]$.

LEMMA 3.2. *Assume that $q > 0$ and $r > 0$. Then the process in Example* D *survives when $\lambda = \lambda_c$.*

PROOF. Set $h_0 = 0$ and $h_i = \delta(x_i)^{-1/2} = \exp(i^r/2)$ for $i \geq 1$. Define the events

$$\begin{aligned} A_i^1 &= \big\{\text{no death mark in } \{x_i\} \times [0, h_i]\big\}, \\ A_i^2 &= \big\{\{x_i\} \times [h_{i-1}, h_i] \to \{x_{i+1}\} \times [h_{i-1}, h_i]\big\}, \\ A_i &= A_i^1 \cap A_i^2. \end{aligned}$$

Divide $\{x_i, \dots, x_{i+1}\} \times [h_{i-1}, h_i]$ into parts of the form $\{x_i, \dots, x_{i+1}\} \times [k(i+1)^{5q}, (k+1)(i+1)^{5q}]$, for $k = \lfloor h_{i-1}/(i+1)^{5q} \rfloor + 1, \dots, \lfloor h_i/(i+1)^{5q} \rfloor - 1$. Using (3.4) and the independence of the graphical construction in disjoint space–time regions, we have for large i,

$$P\Big(\big(A_i^2\big)^c\Big) \le \big(1 - (i+1)^{-3q}\big)^{\lfloor h_i/(i+1)^{5q} \rfloor - \lfloor h_{i-1}/(i+1)^{5q} \rfloor - 1}.$$

The mean value theorem implies that $h_i - h_{i-1} \ge r h_{i-1}/(2i)$, for $i \ge 2$, so it follows that

$$\sum_{i=1}^{\infty} P(A_i^c) \le \sum_{i=1}^{\infty} P\Big(\big(A_i^1\big)^c\Big) + \sum_{i=1}^{\infty} P\Big(\big(A_i^2\big)^c\Big) < \infty.$$

Lemma 3.1(a) implies that

$$\rho_{\lambda_c}(1) \ge P\left(\bigcap_{i=1}^{\infty} A_i\right) > 0,$$

and so $\rho_{\lambda_c}(0) > 0$. □

We now have to prove that the critical value of the inhomogeneous model is equal to λ_c. Our method of proof requires us to assume that $q > r$ in Example D, and we shall do so for the rest of the section. We will also need later to assume that $r > 1$ (see Lemma 3.5), and this is the reason we chose, somewhat arbitrarily, $r = 2$ and $q = 3$ in Example A. The next lemma tells us that the sites with small $\delta(x)$ are too far apart to permit propagation of the process over the intervening distances in a reasonable length of time.

LEMMA 3.3. *Assume that $q > r$ in Example* D. *Fix s such that $q > s > r$ and let*

$$g_i = i^s$$

and

$$H_i = (g_i)^2/\delta(x_i) = i^{2s}\exp(i^r)$$

for $i \ge 1$. Also define the sets

$$V_i = [x_i - g_i, x_i + g_i] \times [0, H_i]$$

and the events

$$E_i = \big\{\{x_i\} \times [0, H_i/2] \not\to (V_i)^c\big\}.$$

Then, for every $\lambda < \lambda_c$, $\sum_{i=1}^{\infty} P(E_i^c) < \infty$.

PROOF. For fixed i and for each $k = 1, 2, \dots$, consider the event

$$B_k^1 = \Big\{\{x_i\} \times \big[kg_i, (k+1)g_i\big] \not\to^{U_k} \big(R\backslash\{x_i\}\big) \times \big\{(k+1)g_i\big\}\Big\},$$

where $U_k = [x_i - g_i, x_i + g_i] \times [kg_i, (k+1)g_i]$, R is the set of real numbers and $A \backslash B$ is the set $A \cap B^c$. Let

$$B_k^2 = \Big\{\text{there is a death mark in } \{x_i\} \times \big[(k+1)g_i - 1, (k+1)g_i\big] \\ \text{and no arrow starts or ends in this set}\Big\}.$$

When i is large, the interval $[x_i - g_i, x_i + g_i]$ does not contain any site x_j with $j \neq i$. Applying Lemma 3.1(b) with the S_l's taken to be subintervals of $\{x_i\} \times [kg_i, (k+1)g_i]$ of length 1, we obtain

$$P(B_k^1) > C_1 > 0, \tag{3.6}$$

where C_1 does not depend on i or k. Also,

$$P(B_k^2) \geq C_2 \delta(x_i). \tag{3.7}$$

Define $B_k = B_k^1 \cap B_k^2$ and

$$D_i = \bigcup_{k = \lfloor H_i/(2g_i) \rfloor + 1}^{\lfloor H_i/g_i \rfloor - 1} B_k.$$

From (3.6), (3.7), the Harris–FKG inequality and the independence of the B_k for different k, it follows that

$$P(D_i^c) \leq \big(1 - C_3 \delta(x_i)\big)^{\lfloor g_i/\delta(x_i) \rfloor - \lfloor g_i/2\delta(x_i) \rfloor - 1} \leq e^{-C_4 g_i} \tag{3.8}$$

Observe now that

$$\begin{aligned} &P(E_i^c \cap D_i) \\ &\quad \leq P\big(\text{there are } (x,t), (y,s) \in V_i \text{ such that } |x-y| \geq g_i \\ &\qquad \text{or } |t-s| \geq g_i \text{ and } (x,t) \to^H (y,s)\big), \end{aligned} \tag{3.9}$$

where

$$H = \big\{x \colon \delta(x) = 1\big\} \times R_+.$$

Using (3.2), we see that the right-hand side of (3.9) is less than $\alpha(V_i) e^{-\gamma g_i} \leq (2g_i + 1) H_i e^{-\gamma g_i}$, which is a summable sequence. This together with (3.8) proves Lemma 3.3. □

Using the assumptions and notation of the preceding lemma, we now consider the boxes

$$\Lambda_i = [-x_i + 1, x_i - 1] \times [0, 2H_{i-1}]$$

and let F_i be the event that Λ_i is crossed from bottom to top by a path:

$$F_i = \big\{[-x_i + 1, x_i - 1] \times \{0\} \to^{\Lambda_i} [-x_i + 1, x_i - 1] \times \{2H_{i-1}\}\big\}.$$

LEMMA 3.4. *Assume that $q > r$ in Example* D. *For every $\lambda < \lambda_c$ there exists an $\varepsilon > 0$ such that $P(F_i) \le 1 - \varepsilon$ for all i large enough.*

PROOF. From Lemmas 3.1(a) and (3.3),

$$P\left(\bigcap_{j=1}^{i-1} E_j\right) \ge \left(\bigcap_{j=1}^{\infty} E_j\right) = C_5 > 0.$$

Hence

$$P(F_i^c) \ge P\left(F_i^c \,\middle|\, \bigcap_{j=1}^{i-1} E_j\right) C_5. \tag{3.10}$$

But $H_j \le 2H_{i-1}$ for $j \le i-1$, and hence on $\bigcap_{j=1}^{i-1} E_j$, F_i can only happen if $[-x_i+1, x_i-1] \times \{0\}$ is connected to the set

$$T_i := [-x_i+1, x_i-1] \times \{2H_{i-1}\} \cup \left(\bigcap_{j=1}^{i-1} \{x_j\} \times [H_j/2, 2H_{i-1}]\right),$$

using only the part of the space where $\delta(x) = 1$. Therefore

$$P\left(\left(\bigcap_{j=1}^{i-1} E_j\right) \cap F_i\right) \le P\left(\left(\bigcap_{j=1}^{i-1} E_j\right) \cap \{[-x_i+1, x_i-1] \times \{0\} \to^H T_i\}\right).$$

Using the Harris–FKG inequality for one increasing and one decreasing event, we get

$$P\left(\left(\bigcap_{j=1}^{i-1} E_j\right) \cap F_i\right) \le P\left(\bigcap_{j=1}^{i-1} E_j\right) P([-x_i+1, x_i-1] \times \{0\} \to^H T_i). \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$P(F_i^c) \ge C_5 P_H([-x_i+1, x_i-1] \times \{0\} \not\to T_i). \tag{3.12}$$

Now Lemma 3.4 follows easily from (3.12) and an application of Lemma 3.1(b) with $S_l = \{(l, 0)\}$. □

We are finally ready to complete the proof of Proposition 1.2. It will be an immediate consequence of the next lemma and Lemma 3.2.

LEMMA 3.5. *Assume that $q > r > 1$ in Example* D. *Then the process dies out for every $\lambda < \lambda_c$.*

PROOF. We use the notation of the previous lemmas. Consider the boxes

$$\Omega_i = [-x_i+1, x_i-1] \times [0, H_i]$$

and observe that

$$\begin{aligned} P\big((0,0) \to \Omega_i^c\big) \\ \le P\big((0,0) \to^{\Omega_i} \{-x_i+1, x_i-1\} \times [0, H_i]\big) \\ + P\big((0,0) \to^{\Omega_i} [-x_i+1, x_i-1] \times \{H_i\}\big), \end{aligned}$$

and so

$$\begin{aligned} &P\big((0,0) \to \Omega_i^c\big) \\ &\quad \le P\big(\text{there are } (x,t),(y,s) \in \Omega_i \text{ such that } |x-y| > i^q \text{ and } (x,t) \to^H (y,s)\big) \\ &\quad\quad + P(F_i)^{\lfloor H_i/(2H_{i-1}) \rfloor}. \end{aligned}$$

The first term goes to 0 by an application of (3.2) [since $\alpha(\Omega_i)$ is of the order of $2i^{q+1}i^{2s}e^{i^r}$], while the second term goes to 0 by Lemma 3.4 and the fact that $H_i/(2H_{i-1})$ goes to ∞ as $i \to \infty$ (since $r > 1$). This proves the result. □

It is clear that Example B was constructed so that the stretches with $\delta(x) = b$ mimic the effect in Example A of the sites x_i. The proof of Proposition 1.3 is a straightforward adaptation of that of Proposition 1.2, once one recalls the following fact proved in Durrett and Schonmann (1988). If one considers a contact process on the finite set $\{1, 2, \dots, n\}$, so that the sites 1 and n have only one neighbor each and $\delta(x) = b < 1$ for all x, then if $\lambda > b\lambda_c$, there is a positive γ such that

$$\lim_{n \to \infty} P\big(\{1, \dots, n\} \times \{0\} \to \{1, \dots, n\} \times \{e^{\alpha n}\}\big) = 1 \quad \text{if } \alpha < \gamma$$

and the limit is 0 if $\alpha > \gamma$. In reality, to prove Proposition 1.3, one only needs the much more elementary result that states that the limit is 1 for sufficiently small α and is 0 for sufficiently large α.

Finally, we make some comments about the existence of models with discontinuous transitions in $d > 1$. In the proof of Proposition 1.2, the basic ingredient is the difference between the critical and subcritical behaviors of the connectivity probabilities in the homogeneous case. The exponential decay below λ_c has been proved for the homogeneous contact process in any dimension by Bezuidenhout and Grimmett (1991), so this part presents no problem. It is also widely believed that at the critical point the connectivity probabilities in the time and space direction decay only as powers of the distance, in a fashion which would allow us to extend Examples A and B and Propositions 1.2 and 1.3 to higher dimensions immediately. But unfortunately the rigorous results in the form of lower bounds for the connectivity probability in the space direction (which concerns us) are rather inadequate at present. We are nevertheless able to construct examples in $d > 1$ which are (not so straightforward) adaptations of Examples A and B for which we can prove that the transition is discontinuous. But we omit our examples because this seems a marginal issue and because we expect that improved knowledge of the decay of connectivities at the critical point will trivialize this point in the near future. It is interesting to remark, however, that our examples in $d > 1$ rely only on the fact that the exponential decay rates γ in (3.1) tend to 0 as λ increases to λ_c.

4. Proof of Theorem 1.6. Given a set V in $R^d \times R_+$, the cluster of V is defined to be the set

$$C(V) = \{(x,t) \in Z^d \times R_+ \colon V \to (x,t)\}.$$

For technical reasons it will be convenient to "discretize the time." With this in mind, we recall the following definition. For each set $V \subset Z^d \times R_+, \alpha(V)$ is the number of sets of the form $\{x\} \times [i, i+1)$ $(x \in Z^d,\ i = 0, 1 \dots)$ which have a nonempty intersection with V. One can think of $\alpha(V)$ as a (convenient) measure of the volume of V. We will use the abbreviation

$$\overline{C}(x,t) := C(\{x\} \times [t, t+1]).$$

Given a site z and a length k, define the box

$$B(z,k) := (z + [-k,k]^d) \times [0,k] = \{(w,s) \in R^d \times R_+ \colon \|(w,s) - (z,0)\| \le k\}$$

and let

$$\partial B(z,k) := \{(w,s) \in R^d \times R_+ \colon \|(w,s) - (z,0)\| = k\}$$

be its top and lateral boundary.

In this section we are considering models such that $\delta(x) = 1$ for $x \notin S$ and $\delta(x) = b$ for $x \in S$, where b is a fixed number in $(0,1)$. Example C is the particular case $S = \{0\}$. The first lemma below is a technical strengthening of Proposition 1.4.

LEMMA 4.1. *For the model in Example* C, *for any* $\lambda < \lambda_c(\emptyset)$ *there is a constant* $c > 0$ *such that* $P(\{0\} \times [0,1] \to \partial B(0,n)) < \exp(-cn^{1/3})$ *for large* n.

PROOF. For each n consider the slabs

$$R_j = Z^d \times (jn^{1/2}, (j+1)n^{1/2}],$$

$j = 0, 1, \dots$. Define the events

$$E_j = \left\{\{0\} \times (jn^{1/2}, (j+1)n^{1/2} - 1] \not\to Z^d \times \{(j+1)n^{1/2}\}\right\}.$$

These events are independent because the R_j are disjoint. From Lemma 3.1(b) it follows, as for (3.6), that $P(E_j) > p > 0$, where p does not depend on j or n.

Now, again using (3.2.),

$$\begin{aligned}
&P(\{0\} \times [0,1] \to \partial B(0,n)) \\
&\quad \le P\left(\bigcup_{j=0}^{\lfloor n^{1/2} \rfloor - 1} (E_j)^c \right) + P_H(\exists\, (x,t),(y,s) \in B(0,n) \text{ a distance at least} \\
&\qquad\qquad\qquad\qquad n^{1/2} - 1 \text{ apart and connected by a path}) \\
&\quad \le (1-p)^{n^{1/2}} + (2n+1)^d n C e^{-\gamma(n^{1/2}-1)} \\
&\quad < \exp(-cn^{1/3})
\end{aligned}$$

for large enough n, where $c > 0$. □

The next proposition contains the main arguments of the proof of Theorem 1.6 and it is also of intrinsic interest. Define

$$l(S) = \inf\{\|x-y\|:\ x,y \in S,\ x \ne y\}.$$

PROPOSITION 4.2. *For fixed b and $\lambda < \lambda_c(\emptyset)$, there exists a finite l_0 such that if $l(S) > l_0$, then the process dies out, and moreover there exist C and γ in $(0,\infty)$ such that for any $x \in Z^d$,*

$$P((x,0) \to \partial B(x,n)) \le Ce^{-\gamma n}.$$

PROOF. We will show that there is a k_0 such that if $l(S) > 4k_0$, then there is an $\eta \in (0,1)$ with the property that for all $z \in Z^d$,

$$E\Big(\alpha\big(\overline{C}(z,0) \cap \partial B(z,2k_0)\big)\Big) < \eta. \tag{4.1}$$

The proposition will follow then, with $l_0 = 4k_0$, from an argument which is a straightforward adaptation of a well-known procedure to prove exponential decay of connectivities [e.g., the proof of Theorem (5.1) on page 84 of Grimmett (1991), with the homogeneity there replaced here by the uniformity in z in (4.1)]. For the sake of completeness we show now how to obtain the exponential decay of connectivities from (4.1).

It is clear that if a path starts in $\{z\} \times [0,1]$ and leaves the box $B(z, m_1 + m_2)$, then it must contain two subpaths, with only one endpoint in common, one connecting $\{z\} \times [0,1]$ to a point in $\partial B(z,m_1)$, and the other connecting this point to $\partial B(z, m_1+m_2)$. Using the van den Berg–Kesten inequality [see Bezuidenhout and Grimmett (1991) for a version for contact processes] and translation invariance in the time direction, one obtains [after summing over all sets of the form $\{w\} \times [i, i+1)$, $w \in Z^d$, $i = 0,1,2,\dots$, which intersect $\partial B(z,m_1)$]

$$\begin{aligned}
&P\big(\overline{C}(z,0) \cap \partial B(z, m_1+m_2) \ne \emptyset\big) \\
&\quad \le E\Big(\alpha\big(\overline{C}(z,0) \cap \partial B(z,m_1)\big)\Big) \max_{w \in Z^d} P\big(\overline{C}(w,0) \cap \partial B(w,m_2) \ne \emptyset\big).
\end{aligned}$$

Iteration of this inequality leads now readily to the desired exponential decay:

$$P\big(\overline{C}(z,0)\cap\partial B(z,n)\neq\emptyset\big)\leq\Big(E\Big(\alpha\big(\overline{C}(z,0)\cap\partial B(z,2k_0)\big)\Big)\Big)^{\lfloor n/2k_0\rfloor}$$
$$\leq(\eta)^{-1}\big((\eta)^{1/2k_0}\big)^n.$$

We turn now to the proof of (4.1). If $l(S)>4k$, then $[-2k,2k]^d+z$ can intersect S in at most one site. If the intersection is not empty, then let y be this site. (If the intersection is empty, the argument below becomes trivial.) Divide the set $A:=\overline{C}(z,0)\cap\partial B(z,2k)$ into two parts: let A_1 denote the set of points in A which are reached from $\{z\}\times[0,1]$ by paths which do not intersect $\{y\}\times[0,\infty)\cap B(z,k)$ (observe that $A_1=A$ if $\|y-z\|>k$), and let $A_2=A\backslash A_1$. Clearly,

$$E\big(\alpha(A)\big)\leq E\big(\alpha(A_1)\big)+E\big(\alpha(A_2)\big). \tag{4.2}$$

To bound the first term on the right-hand side above, observe that

$$E\big(\alpha(A_1)\big)\leq\alpha\big(\partial B(z,2k)\big)P_H\big(\{z\}\times[0,1]\rightarrow\partial B(z,k)\big).$$

But for some finite constant C,

$$\alpha\big(\partial B(z,2k)\big)\leq Ck^d, \tag{4.3}$$

so that translation invariance of the homogeneous system and (3.2) imply that for large enough k,

$$E\big(\alpha(A_1)\big)<1/3. \tag{4.4}$$

The second term on the right-hand side of (4.2) is 0 if $\|y-z\|>k$, and we will use Lemma 4.1 to bound this term in case $\|y-z\|\leq k$. Denote by $\widetilde{P}(\cdot)$ the law of the model defined in Example C and considered in Lemma 4.1. Observe that if $\|y-z\|\leq k$ and there is a path connecting a point (y,s) in $\{y\}\times[0,k]$ to a point in $\partial B(z,2k)$, then this path must exit the box of radius k centered at (y,s). Using translation invariance in the time direction and the fact that y is the only site in $([-2k,2k]^d+z)\cap S$, we obtain

$$E\big(\alpha(A_2)\big)\leq\alpha\big(\partial B(z,2k)\big)k\widetilde{P}\big(\{0\}\times[0,1]\rightarrow\partial B(0,k)\big).$$

By (4.3) and Lemma 4.1, it follows now that for large enough k,

$$E\big(\alpha(A_2)\big)<1/3. \tag{4.5}$$

Inequalities (4.2), (4.4) and (4.5) imply (4.1), completing the proof of the proposition. □

With the help of Proposition 4.2 it is easy to prove Theorem 1.6. Recall that $b\in(0,1)$ and S are fixed. It is clear that $\lambda_c(S)\leq\lambda_c(\emptyset)$, so we only need to show that the inhomogeneous system dies out for every $\lambda<\lambda_c(\emptyset)$. Given such a λ, let l_0 be the number given by Proposition 4.2 (for the fixed value of b that we have). By the assumption on S, there is a finite N such that if we define $S_0 = S \backslash [-N, N]^d$, then S_0 has the property

$$l(S_0) > l_0.$$

Hence the conclusion of Proposition 4.2 applies to the system with S_0 replacing S. We will present now a simple way to exploit this fact in order to show that the system with inhomogeneities in S dies out. In order to be able to use some elementary ergodic theory below, we extend the graphical construction (in a standard fashion) to $Z^d \times R$. This is done by considering the corresponding Poisson processes which give rise to δ marks and arrows on R instead of R_+.

We say that there is a *cut* at time $t \in Z$ if the following two events both happen:

1. For each site $x \in [-N, N]^d$, there is a death mark in $\{x\} \times [t-1, t]$ and no arrow leaves or enters this set.
2. $[-N, N]^d \times (-\infty, t-1] \not\to_U (Z^d \backslash [-N, N]^d) \times \{t\}$, where $U = (Z^d \backslash [-N, N]^d) \times R$.

Using the exponential decay of the connectivity probability in Proposition 4.2 for the event in (ii) and the Harris–FKG inequality, it follows that

$$P(\text{there is a cut at time } t) = C > 0,$$

where C does not depend on t. From the ergodicity of the underlying Poisson processes, we obtain

$$P(\text{there are positive cut times}) = 1.$$

But if there is a positive cut time, then the cluster of $(0, 0)$ must be finite. This proves that the system dies out and finishes the proof that $\lambda_c(S) = \lambda_c(\emptyset)$.

The exponential decay of connectivities claimed in Theorem 1.6 can be proved with a little more effort. The argument basically relies on the same reasoning used to prove Proposition 4.2, so we merely sketch it and leave the details to the reader. Observe that in the proof of Proposition 4.2 (including the proof of Lemma 4.1) the only fact about the homogeneous system that was used was the exponential decay of connectivities, from (3.1) and (3.2). Think of the system with inhomogeneities in S as resulting from a modification in the finite region $[-N, +N]^d$ of the system with inhomogeneities in S_0. Keep in mind the exponential decay of connectivities proved in Proposition 4.2 for the system defined by S_0. Now it should be clear that the system defined by S has a structure similar to that of the model introduced in Example C (with the background being the system defined by S_0 rather than the homogeneous system, and the modification affecting the finite region $[-N, +N]^d$ rather than just the origin), and one can adapt the proof of Proposition 4.2 to this setting to complete the proof of Theorem 1.6.

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