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LIMIT THEOREMS FOR NONLINEAR FUNCTIONALS OF A STATIONARY GAUSSIAN SEQUENCE OF VECTORS[1]

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Limit theorems for functions of stationary mean-zero Gaussian sequences of vectors satisfying long range dependence conditions are considered. Depending on the rate of decay of the coefficients, the limit law can be either Gaussian or the law of a multiple Itô–Wiener integral. We prove the bootstrap of these limit theorems in the case when the limit is normal. A sufficient bracketing condition for these limit theorems to happen uniformly over a class of functions is presented.

1. Introduction. Long range dependence appears in diverse areas such as hydrology, geophysics, economics, meterology and communications; see, for example, Cox (1984). Normal and nonnormal limit theorems for functions of real-valued stationary sequences have been studied by several authors, for example, Rosenblatt (1961), Sun (1963, 1965), Taqqu (1975, 1979), Dobrushin and Major (1979), Major (1981), Breuer and Major (1983), Giraitis and Surgailis (1985). In this paper we will study limit theorems for functions of a vector-valued stationary sequence.

The framework we are going to consider is as follow. Let $\{X_j\}_{j=1}^{\infty}$ be a $\mathbb{R}^d$-valued stationary mean-zero Gaussian sequence, that is, a Gaussian process indexed by $\mathbb{N}$. Set $X_j = (X_j^{(1)}, \dots, X_j^{(d)})$. Given a function $f\colon \mathbb{R}^d \to \mathbb{R}$, we will give conditions on the covariance of the process in order that

$$\left\{ a_n^{-1} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j) \right) \right\}_{n=1}^{\infty} \tag{1.1}$$

converges in distribution for some sequence a_n. This problem was proposed by Sun and Ho [(1986), page 14]. Some partial results for this problem were obtained in Ho and Sun (1990).

In Section 2, we will see that if the covariance matrix of X_1 and X_j goes to zero fast enough, then (1.1) converges in distribution to a normal random variable under the usual normalization $a_n = n^{1/2}$. Precisely, we will show that if

$$\lim_{n \to \infty} n^{-1} \sum_{j,k=1}^{n} r^{(p,q)}(j-k) \tag{1.2}$$

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and

$$\lim_{n \to \infty} n^{-1} \sum_{j,k=1}^{n} \left(r^{(p,q)}(j-k)\right)^2 \tag{1.3}$$

exist for each $1 \le p, q \le d$, where $r^{(p,q)}(k) = E[X_1^{(p)} X_{1+k}^{(q)}]$, then

$$n^{-1/2} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j)\right) \to_d N\left(0, \sigma^2\right), \tag{1.4}$$

where

$$\sigma^2 := 1 + 2\sum_{k=1}^{\infty} E\Big[\left(f(X_1) - Ef(X_1)\right)\left(f(X_{1+k}) - Ef(X_{1+k})\right)\Big]. \tag{1.5}$$

This is a generalization of Theorem 1 in Sun (1965). We also will see that the condition of decay can be weakened if we know the rank of the function f. Suppose that f has rank τ and that

$$\sum_{k=-\infty}^{\infty} \left|r^{(p,q)}(k)\right|^{\tau} < \infty \tag{1.6}$$

for each $1 \le p,\ q \le d$. Then

$$n^{-1/2} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j)\right) \to_d N\left(0, \sigma^2\right), \tag{1.7}$$

where σ^2 is as in (1.5). This a generalization to the multivariate case of Theorem 1 in Breuer and Major (1983).

In Section 3, we will consider the convergence of the process of partial sums

$$\left\{a_n^{-1} \sum_{j=1}^{[nt]} \left(f(X_j) - Ef(X_j)\right) \colon 0 \le t \le 1\right\}_{n=1}^{\infty}. \tag{1.8}$$

In this case we assume that covariance decays as a regularly varying function. Here, the first terms of the Fourier expansion of $f(x)$ in Wick polynomials give the limit. The law limit is the law of a multiple Itô–Wiener integral. Previous work by Taqqu (1975, 1979) and by Dobrushin and Major (1979) is extended. We will follow the approach, via multiple Itô–Wiener integrals, in Dobrushin and Major (1979).

In Section 4, we deal with the bootstrap of the limit theorems presented in the Section 2 in the case when the limit is normal. Efron (1979) introduced an innovative statistical procedure called the bootstrap, which consists of resampling from the sample. Since then, the bootstrap has proved to be a versatile method. It is well known that in the case of dependence, the usual bootstrap fails [see Singh (1981)]. Künsch (1989) [see also Liu and Singh (1992)] introduced a variation of the bootstrap called the moving block bootstrap, which works under dependent data. Several authors [Politis and Romano (1992) and Lahiri (1991)] have considered this moving blocks bootstrap under strong mixing conditions. Lahiri (1993) studied the bootstrap under long range dependence in the case where the covariance is regularly varying [i.e., for the limit theorem in Taqqu (1975, 1979) and Dobrushin and Major (1979)]. He showed that in the case when the limit is not normal, the bootstrap does not work and that in the normal case it works under rescaling. In contrast, we will consider the bootstrap when the limit is normal under the normalization $a_n = n^{1/2}$. Here the moving block bootstrap works just fine. We limit ourselves to the bootstrap of the mean. Via differentiability [see Gill (1989) and Arcones and Giné (1992)], it is possible to get the bootstrap of more estimators.

Finally in Section 5, we will consider the uniform convergence of a class of nonlinear functionals of a stationary Gaussian sequence of vectors. Let $S_n(f) = a_n^{-1}\sum_{j=1}^{n}(f(X_j) - Ef(X_j))$. Suppose that

$$\sup_{f\in\mathcal{F}}|f(X_1)| < \infty \quad \text{a.s.} \quad \text{and} \quad \sup_{f\in\mathcal{F}}\left|E[f(X_1)]\right| < \infty.$$

Then, $\{S_n(f): f \in \mathcal{F}\}$ is a random element with values in $l_\infty(\mathcal{F})$. We say that the random element $\{S_n(f): f \in \mathcal{F}\}$ converges weakly to the process $\{K(f): f \in \mathcal{F}\}$ in $l_\infty(\mathcal{F})$, in the Hoffmann-Jørgensen sense [see Hoffmann-Jørgensen (1984); see also Dudley (1967)] if $\{K(f): f \in \mathcal{F}\}$ has separable support and

$$E^*[H(S_n)] \to E[H(K)]$$

for each continuous and uniformly bounded function H on $l_\infty(\mathcal{F})$. It is known [see, e.g., Theorem 2.12 in Andersen and Dobrić (1987)] that S_n converges weakly to K if and only if the finite dimensional distributions of S_n converge to those of K and there is pseudometric ρ on $\mathcal{F}$ such that $(\mathcal{F}, \rho)$ is totally bounded and

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Pr{}^*\left\{\sup_{\substack{f_1,f_2\in\mathcal{F}\\ \rho(f_1,f_2)\le\delta}}|S_n(f_1) - S_n(f_2)| > \eta\right\} = 0 \tag{1.9}$$

for each $\eta > 0$, where $\Pr^*$ means outer probability. We will give a sufficient bracketing condition on the class of functions $\mathcal{F}$ to S_n to converge weakly to a process K.

All the limit theorems that we will consider are also true for Gaussian fields. We will formulate our results for Gaussian sequences in order to avoid notational mess.

2. Limit theorems for functions of Gaussian vectors. In this section we will study the convergence in distribution of (1.1). A key ingredient in this study will be the diagram formula for expectations of products of Hermite polynomials over a Gaussian vector (see, e.g., page 433 in Breuer and Major (1983)]. A diagram (or a graph) G of order $(l_1, \dots, l_p)$ is a set of points

$\{(j,l): 1 \le j \le p, 1 \le l \le l_j\}$, called vertices, and a set pair of these points $\{((j,l),(k,m)): 1 \le j < k \le p, 1 \le l \le l_j, 1 \le m \le l_k\}$, called edges, such that every vertex is of degree 1. We denote by $\Gamma(l_1,\dots,l_p)$ the set of diagrams of order $(l_1,\dots,l_p)$. Observe that $\Gamma(l_1,\dots,l_p)$ is empty if $l_1+\cdots+l_p$ is an odd number. The set $L_j = \{(j,l): 1 \le l \le l_j\}$ is called the jth level of the graph G. Observe that edges connect vertices of different levels. We will denote the set of edges of the diagram G by $E(G)$. Given an edge $w=((j,l),(k,m))$, let $d_1(w)=j$ and let $d_2(w)=k$. With this notation the diagram formula is:

DIAGRAM FORMULA. Let $(X_1,\dots,X_p)$ be a Gaussian vector with

$$EX_j = 0,\ EX_j^2 = 1 \quad \text{and} \quad E[X_jX_k] = r(j,k) \quad \text{for each } 1 \le j,k \le p.$$

Then

$$E\left[\prod_{j=1}^{p} H_{l_j}(X_j)\right] = \sum_{G\in\Gamma(l_1,\dots,l_p)} \prod_{w\in E(G)} r\big(d_1(w),d_2(w)\big). \tag{2.1}$$

Now, we extend the definition of rank of a function to the multivariate case. Let X be Gaussian vector. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function. If f has a finite second moment, we define the Hermite rank of f with respect to X as

$$\begin{aligned} \operatorname{rank}(f) := \inf\Big\{\tau\colon\ \exists\, l_j \text{ with } \textstyle\sum_{j=1}^d l_j = \tau \\ \text{and } E\left[\big(f(X)-Ef(X)\big)\textstyle\prod_{j=1}^d H_{l_j}\big(X^{(j)}\big)\right] \neq 0\Big\}, \end{aligned} \tag{2.2}$$

where the infimum of the empty set is infinity. Equivalently, it is easy to see that

$$\begin{aligned} \operatorname{rank}(f) = \inf\Big\{\tau\colon\ \exists \text{ polynomial } P \text{ of degree } \tau \\ \text{with } E\left[\big(f(X)-Ef(X)\big)P\big(X^{(1)},\dots,X^{(d)}\big)\right] \neq 0\Big\}. \end{aligned}$$

LEMMA 1. *Let $X=(X^{(1)},\dots,X^{(d)})$ and $Y=(Y^{(1)},\dots,Y^{(d)})$ be two mean-zero Gaussian random vectors on $\mathbb{R}^d$. Assume that*

$$E\big[X^{(j)}X^{(k)}\big] = E\big[Y^{(j)}Y^{(k)}\big] = \delta_{j,k} \tag{2.3}$$

for each $1 \le j,k \le d$. We define

$$r^{(j,k)} := E\big[X^{(j)}Y^{(k)}\big]. \tag{2.4}$$

Let f be a function on $\mathbb{R}^d$ with finite second moment and rank τ, $1 \le \tau < \infty$, with respect to X. Suppose that

$$\psi := \left(\sup_{1\le j\le d}\sum_{k=1}^d \big|r^{(j,k)}\big|\right) \vee \left(\sup_{1\le k\le d}\sum_{j=1}^d \big|r^{(j,k)}\big|\right) \le 1. \tag{2.5}$$

Then

$$\left|E\left[(f(X)-Ef(X))\,(f(Y)-Ef(Y))\right]\right| \le \psi^{\tau} E f^2(X). \tag{2.6}$$

PROOF. We may assume that $Ef(X)=0$. Let

$$c_{l_1,\dots,l_d} = E\left[f(X)\prod_{j=1}^{d} H_{l_j}(X^{(j)})\right] \tag{2.7}$$

so that

$$f(x) = \sum_{l_1,\dots,l_d=0}^{\infty} \frac{c_{l_1,\dots,l_d}}{\prod_{j=1}^{d} l_j!} \prod_{j=1}^{d} H_{l_j}(x^{(j)}), \tag{2.8}$$

where $x=(x^{(1)},\dots,x^{(d)})$. Observe that

$$c_{l_1,\dots,l_d} = 0 \quad \text{if } \sum_{i=1}^{d} l_i < \tau. \tag{2.9}$$

By the diagram formula and (2.8),

$$\begin{aligned} E[f(X)f(Y)] &= \sum_{\substack{l_1,\dots,l_d=0\\ m_1,\dots,m_d=0}}^{\infty} c_{l_1,\dots,l_d} c_{m_1,\dots,m_d} \left(\prod_{j=1}^{d} (l_j! m_j!)^{-1}\right) \\ &\quad \times E\left[H_{l_1}(X^{(1)})\cdots H_{l_d}(X^{(d)}) H_{m_1}(Y^{(1)})\cdots H_{m_d}(Y^{(d)})\right] \\ &= \sum_{\substack{l_1,\dots,l_d=0\\ m_1,\dots,m_d=0}}^{\infty} c_{l_1,\dots,l_d} c_{m_1,\dots,m_d} \prod_{j=1}^{d} (l_j! m_j!)^{-1} \\ &\quad \times \sum_{G\in\Gamma(l_1,\dots,l_d,m_1,\dots,m_d)} \prod_{w\in E(G)} \alpha(w), \end{aligned} \tag{2.10}$$

where

$$\alpha(w) = \begin{cases} r^{(d_1(w),\, d_2(w)-d)}, & \text{if } d_1(w) \le d < d_2(w), \\ 0, & \text{otherwise.} \end{cases}$$

Let $z_G(p,q)$ be the number of vertices of the graph G joining the levels p and q. If $z_G(p,q) > 0$ for some p,q such that either $1 \le p,q \le d$ or $d+1 \le p,q \le 2d$, then $\prod_{w\in E(G)}\alpha(w) = 0$. So, we only have to consider the graphs G so that $z_G(p,q)=0$ for $1 \le p,q \le d$ and for $d+1 \le p,q \le 2d$. Let $z_G(p,q+d) = a(p,q)$ for $1 \le p,q \le d$. Then

$$\sum_{q=1}^{d} a(p,q) = l_p \tag{2.11}$$

and

$$\sum_{p=1}^{d} a(p,q) = m_q. \tag{2.12}$$

For a graph like that,

$$\prod_{w \in E(G)} \alpha(w) = \prod_{1 \le p,\, q \le d} \left(r^{(p,q)}\right)^{a(p,q)}.$$

Let $(a(p,q))_{1 \le p,q \le d}$ be a matrix of nonnegative numbers satisfying (2.11) and (2.12). The number of graphs such that

$$z_G(p,q) = \begin{cases} 0, & \text{if } 1 \le p,q \le d, \\ 0, & \text{if } d+1 \le p,q \le 2d, \\ a(p,q-d), & \text{if } 1 \le p \le d < q \le 2d \end{cases}$$

is

$$\frac{\prod_{j=1}^{d} l_j! m_j!}{\prod_{p,q=1}^{d} a(p,q)!}.$$

Hence, (2.10) is bounded in absolute value, by

$$\begin{aligned} &\sum_{\substack{l_1,\dots,l_d=0 \\ m_1,\dots,m_d=0}}^{\infty} |c_{l_1,\dots,l_d} c_{m_1,\dots,m_d}| \sum_{(a(p,q))_{p,q=1}^{d} \in \mathcal{A}(\mathbf{l},\mathbf{m})} \prod_{p,q=1}^{d} \left(a(p,q)!\right)^{-1} \left|r^{(p,q)}\right|^{a(p,q)} \\ &\quad \le \sum_{\substack{l_1,\dots,l_d=0 \\ m_1,\dots,m_d=0}}^{\infty} 2^{-1}\left(c_{l_1,\dots,l_d}^2 + c_{m_1,\dots,m_d}^2\right) \sum_{(a(p,q))_{p,q=1}^{d} \in \mathcal{A}(\mathbf{l},\mathbf{m})} \\ &\qquad \times \prod_{p,q=1}^{d} \left(a(p,q)!\right)^{-1} \left|r^{(p,q)}\right|^{a(p,q)}, \end{aligned}$$

where $\mathcal{A}(\mathbf{l},\mathbf{m})$ is the set of all the matrices $d \times d$ of nonnegative numbers satisfying (2.11) and (2.12); $\mathbf{l} = (l_1, \dots, l_d)$ and $\mathbf{m} = (m_1, \dots, m_d)$. We have that

$$\begin{aligned} &\sum_{\substack{l_1,\dots,l_d=0 \\ m_1,\dots,m_d=0}}^{\infty} c_{l_1,\dots,l_d}^2 \sum_{(a(p,q))_{p,q=1}^{d} \in \mathcal{A}(\mathbf{l},\mathbf{m})} \prod_{p,q=1}^{d} \left(a(p,q)!\right)^{-1} \left|r^{(p,q)}\right|^{a(p,q)} \\ &\quad = \sum_{l_1,\dots,l_d=0}^{\infty} c_{l_1,\dots,l_d}^2 \sum_{\substack{\Sigma_{q=1}^{d} a(p,q) = l_p \\ \text{for } 1 \le p \le l}} \prod_{p,q=1}^{d} \left(a(p,q)!\right)^{-1} \left|r^{(p,q)}\right|^{a(p,q)} \\ &\quad = \sum_{l_1,\dots,l_d=0}^{\infty} c_{l_1,\dots,l_d}^2 \prod_{p=1}^{d} \left(\sum_{\Sigma_{q=1}^{d} a(p,q) = l_p} \prod_{q=1}^{d} \left(a(p,q)!\right)^{-1} \left|r^{(p,q)}\right|^{a(p,q)} \right). \end{aligned}$$

First by the multinomial theorem and then by (2.5) and (2.9), the last expression is bounded by

$$\sum_{l_1,\dots,l_d=0}^{\infty} c^2_{l_1,\dots,l_d} \prod_{p=1}^{d} \left((l_p!)^{-1} \left(\sum_{q=1}^{d} \left| r^{(p,q)} \right| \right)^{l_p} \right)$$
$$\le \sum_{l_1,\dots,l_d=0}^{\infty} c^2_{l_1,\dots,l_d} \left(\prod_{p=1}^{d} (l_p!)^{-1} \right) \psi^{\Sigma_{p=1}^{d} l_p}$$
$$\le \sum_{l_1,\dots,l_d=0}^{\infty} c^2_{l_1,\dots,l_d} \left(\prod_{p=1}^{d} (l_p!)^{-1} \right) \psi^{\tau} = E\left[f^2(X)\right]\psi^{\tau}.$$

Similarly,

$$\sum_{\substack{l_1,\dots,l_d=0 \\ m_1,\dots,m_d=0}}^{\infty} c^2_{m_1,\dots,m_d} \sum_{(a(p,q))_{p,q=1}^{d} \in \mathcal{A}(\mathbf{l},\mathbf{m})} \prod_{p,q=1}^{d} \left(a(p,q)!\right)^{-1} \left| r^{(p,q)} \right|^{a(p,q)}$$
$$\le E\left[f^2(X)\right]\psi^{\tau}.$$

Therefore (2.6) follows. □

THEOREM 2. *Let $\{X_j\}_{j=1}^{\infty}$ be a stationary mean-zero Gaussian sequence of $\mathbb{R}^d$-valued vectors. Set $X_j = (X_j^{(1)}, \dots, X_j^{(d)})$. Let f be a function on $\mathbb{R}^d$ with $E[f^2(X_1)] < \infty$. We define*

$$r^{(p,q)}(k) = E\left[X_m^{(p)} X_{m+k}^{(q)}\right] \tag{2.13}$$

for $k \in Z$ and $1 \le p, q \le d$, where m is any integer so large that $m, m+k \ge 1$. Suppose that

$$\lim_{n\to\infty} n^{-1} \sum_{j,k=1}^{n} r^{(p,q)}(j-k) \tag{2.14}$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{j,k=1}^{n} \left(r^{(p,q)}(j-k)\right)^2 \tag{2.15}$$

exist for each $1 \le p, q \le d$. Then

$$n^{-1/2} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j)\right) \to_d N\left(0, \sigma^2\right), \tag{2.16}$$

where

$$\begin{aligned} \sigma^2 &:= E\left[\left(f(X_1) - Ef(X_1)\right)^2\right] \\ &\quad + 2\sum_{k=1}^{\infty} E\left[\left(f(X_1) - Ef(X_1)\right)\left(f(X_{1+k}) - Ef(X_{1+k})\right)\right]. \end{aligned} \tag{2.17}$$

Moreover, there is a constant c that only depends on the sequence of covariances, such that

$$E\left(n^{-1/2}\sum_{j=1}^{n}\left(f(X_j)-Ef(X_j)\right)\right)^2 \le cE\big(f(X_1)-Ef(X_1)\big)^2 \tag{2.18}$$

for each n and each function f with finite second moment.

PROOF. Changing f by $f - Ef(X_1)$, we may assume that

$$Ef(X_1) = 0. \tag{2.19}$$

By a linear transformation, we may assume that

$$E\big[X_1^{(p)}X_1^{(q)}\big] = \delta_{p,q} \tag{2.20}$$

for each $1 \le p, q \le d$. Observe that the rank of $f(x)$ with respect to X is the same as the rank of $f(L^{-1}x)$ with respect to LX for any nondegenerate linear map L from $\mathbb{R}^d$ into itself. Hence, we can expand f as in (2.8) with $c_{l_1,\dots,l_d} = 0$ for $\sum_{j=1}^{d} l_j < \tau$. Let

$$\psi(k) := \left(\sup_{1\le p\le d}\sum_{q=1}^{d}\left|r^{(p,q)}(k)\right|\right) \vee \left(\sup_{1\le q\le d}\sum_{p=1}^{d}\left|r^{(p,q)}(k)\right|\right). \tag{2.21}$$

Let b be a positive number such that $\sup_{j\ge b}\psi(j) \le 1$. Let

$$f_2(x) = \sum_{l_1+\cdots+l_d\ge 2}\frac{c_{l_1\cdots l_d}}{l_1!\cdots l_d!}\prod_{p=1}^{d}H_{l_p}\big(x^{(p)}\big).$$

Then f_2 has rank 2 and by Lemma 1,

$$\begin{aligned}
&\left\|n^{-1/2}\sum_{j=1}^{n}f_2(X_j)\right\|_2 \\
&\quad\le n^{-1/2}\sum_{j=1}^{b}\left\|\sum_{k=1}^{[(n-j+b)/b]} f_2\big(X_{(k-1)b+j}\big)\right\|_2 \\
&\quad= n^{-1/2}\sum_{j=1}^{b}\left(\sum_{k,l=1}^{[(n-j+b)/b]} E\Big[f_2\big(X_{(k-1)b+j}\big)f_2\big(X_{(l-1)b+j}\big)\Big]\right)^{1/2} \qquad (2.22)\\
&\quad\le n^{-1/2}\sum_{j=1}^{b}\left[\frac{n-j+b}{b}\right]^{1/2}\left(\sum_{k=-\infty}^{\infty}\psi^2(kb)\right)^{1/2}\|f_2(X_1)\|_2 \\
&\quad\le b\left(\sum_{k=-\infty}^{\infty}\psi^2(kb)\right)^{1/2}\|f_2(X_1)\|_2.
\end{aligned}$$

Let $\bar{c}_p = c_{0,\dots,1,\dots,0}$, where the only 1 is in the pth place. Hence

$$\begin{aligned}
&\left\| n^{-1/2}\sum_{j=1}^{n} f(X_j)\right\|_2 \\
(2.23)\qquad &\le \sum_{p=1}^{d} |\bar{c}_p| \left\| n^{-1/2}\sum_{j=1}^{n} X_j^{(p)}\right\|_2 + b\left(\sum_{k=-\infty}^{\infty}\psi^2(kb)\right)^{1/2} \|f_2(X_1)\|_2 \\
&\le \left[\sum_{p=1}^{d}\left(n^{-1}\sum_{j,k=1}^{n} r^{(p,p)}(j-k)\right)^{1/2} + b\left(\sum_{k=-\infty}^{\infty}\psi^2(kb)\right)^{1/2}\right] \|f(X_1)\|_2.
\end{aligned}$$

Therefore, (2.18) follows.

Next we will show that for any $t < \infty$,

$$(2.24)\qquad n^{-1/2}\sum_{j=1}^{n}\sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d}\prod_{i=1}^{d}\left((l_i)^{-1}H_{l_i}\left(X_j^{(i)}\right)\right) \to_d N\left(0,\sigma_t^2\right),$$

where

$$\sigma_t^2 = \sum_{\substack{l_1,\dots,l_d=1\\ m_1,\dots,m_d=1}}^{t} c_{l_1,\dots,l_d} c_{m_1,\dots,m_d}\beta(l_1,\dots,l_d,m_1,\dots,m_d)$$

and

$$\begin{aligned}
&\beta(l_1,\dots,l_d,m_1,\dots,m_d) \\
(2.25)\qquad &= \sum_{(a(p,q))_{p,q=1}^{d}\in\mathcal{A}(\mathbf{l},\mathbf{m})} \lim_{n\to\infty} n^{-1}\sum_{j,k=1}^{n}\prod_{p,q=1}^{d}\left(a(p,q)!\right)^{-1} \\
&\qquad\qquad \left(r^{(p,q)}(j-k)\right)^{a(p,q)}
\end{aligned}$$

Observe that $\beta(l_1,\dots,l_d,m_1,\dots,m_d) = 0$ if $\sum_{p=1}^{d} l_p \ne \sum_{p=1}^{d} m_p$ and that (2.14) and (2.15) imply that the limit in (2.25) exists.

In order to show (2.24), it suffices to show convergence of moments, that is, that

$$(2.26)\qquad E\left(n^{-1/2}\sum_{j=1}^{n}\sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d}\prod_{i=1}^{d}(l_i!)^{-1}H_{l_i}\left(X_j^{(i)}\right)\right)^{\mu} \to 0$$

if μ is an odd positive integer and

$$(2.27)\qquad E\left(n^{-1/2}\sum_{j=1}^{n}\sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d}\prod_{i=1}^{d}(l_i!)^{-1}H_{l_i}\left(X_j^{(i)}\right)\right)^{\mu} \to \frac{\sigma_t^{\mu}\mu!}{(\mu/2)!2^{\mu/2}}$$

if μ is an even positive integer.

In the case when μ is an even positive integer, that is, $\mu = 2\nu$, where ν is a positive integer, by the diagram formula, we have that

$$
\begin{aligned}
&E\left(n^{-1/2}\sum_{j=1}^{n}\sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d}\prod_{p=1}^{d}(l_p!)^{-1}H_{l_p}\left(X_j^{(p)}\right)\right)^{\mu}\\
&\quad= n^{-\nu}\sum_{j_1,\dots,j_{2\nu}=1}^{n}\ \sum_{\substack{0\le l_{p,q}\le t\\ 1\le p\le d,\ 1\le q\le 2\nu}}\ \prod_{p=1}^{2\nu}\left(c_{l_{p,1},\dots,l_{p,d}}\prod_{q=1}^{d}(l_{p,q}!)^{-1}\right)\\
&\qquad\times\sum_{G\in\Gamma(l_{1,1},\dots,l_{1,d},\dots,l_{2\nu,1},\dots,l_{2\nu,d})}\ \prod_{w\in E(G)}\alpha(w,j_1,j_2,\dots,j_{2\nu}),
\end{aligned}
\tag{2.28}
$$

where $\alpha(w, j_1, j_2, \dots, j_{2\nu}) = r^{(p,q)}(j_b - j_a)$ if w joins the levels $d(a-1)+p$ and $d(b-1)+q$, where $1 \le a, b \le 2\nu$ and $1 \le p, q \le d$.

Similarly, if μ is an odd positive integer, that is, $\mu = 2\nu+1$, where $\nu = 0, 1, \dots,$ we have that

$$
\begin{aligned}
&E\left(n^{-1/2}\sum_{j=1}^{n}\sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d}\prod_{p=1}^{d}(l_p!)^{-1}H_{l_p}\left(X_j^{(p)}\right)\right)^{\mu}\\
&\quad= n^{-(2\nu+1)/2}\sum_{j_1,\dots,j_{2\nu+1}=1}^{n}\ \sum_{\substack{0\le l_{p,q}\le t\\ 1\le q\le d,\ 1\le p\le 2\nu+1}}\ \prod_{p=1}^{2\nu+1}\frac{c_{l_{p,1},\dots,l_{p,d}}}{l_{p,1}!\cdots l_{p,d}!}\\
&\qquad\times\sum_{G\in\Gamma(l_{1,1},\dots,l_{1,d},\dots,l_{2\nu+1,1},\dots,l_{2\nu+1,d})}\ \prod_{w\in E(G)}\alpha(w,j_1,j_2,\dots,j_{2\nu+1}).
\end{aligned}
\tag{2.29}
$$

In any case, we have to study the limit of

$$
n^{-\gamma/2}\sum_{j_1,\dots,j_\gamma=1}^{n}\ \prod_{w\in E(G)}\alpha(w,j_1,j_2,\dots,j_\gamma)
\tag{2.30}
$$

for each graph $G \in \Gamma(l_{1,1}, \dots, l_{1,d}, \dots, l_{\gamma,1}, \dots, l_{\gamma,d})$, where γ is a positive integer.

We say that a graph G is null if there is an edge w joining the levels $d(a-1)+p$ and $d(a-1)+q$ for some $1 \le a \le \gamma$ and $1 \le p, q \le d$. By (2.20), if G is a null graph, $\prod_{w\in E(G)}\alpha(w, j_1, j_2, \dots, j_\gamma) = 0$. So in this case, the limit of (2.30) is zero.

Let G^* be a graph of $\Gamma(\sum_{j=1}^{d} l_{1,j}, \dots, \sum_{j=1}^{d} l_{\gamma,j})$ such that the number of vertices joining the levels a and b is equal,

$$
\sum_{p,q=1}^{d} z_G\big(d(a-1)+p, d(b-1)q\big),
$$

where $z_G(d(a-1)+p, d(b-1)q)$ is the number of edges of G that join the levels $d(a-1)+p$ and $d(b-1)q$. Observe that G^* is the graph obtained from G grouping every d levels in one. As, in the one dimensional case, we say that G^* is a regular graph if it is possible to divide the levels $1, \dots, \gamma$ into pairs so that the edges of G^* only join levels between each of these pairs. We will say that the graph G is regular if the corresponding graph G^* is regular. We claim that if G is not a regular graph, the limit of (2.30) is zero. Because $\sum_{k=-\infty}^{\infty}(r^{(p,q)}(k))^2 < \infty$, for each $1 \le p, q \le d$, we may define the following function in $L_2[-\pi, \pi]$:

$$h^{(p,q)}(\lambda) = \sum_{k=-\infty}^{\infty} \exp(-ik\lambda) r^{(p,q)}(k),$$

where the series is converging in $L_2[-\pi, \pi]$. Observe that $h^{(p,q)}(\lambda)$ is a complex function, $h^{(p,q)}(-\lambda) = h^{(q,p)}(\lambda) = \overline{h^{(p,q)}(\lambda)}$ and $h^{(p,p)}(\lambda) \ge 0$. We have that

$$r^{(p,q)}(k) = (2\pi)^{-1} \int_{-\pi}^{\pi} \exp(ik\lambda) h^{(p,q)}(\lambda)\, d\lambda.$$

Let $s = 2^{-1}\sum_{p=1}^{\gamma}\sum_{q=1}^{d} l_{p,q}$. Suppose that $E(G)$ consists of the edges $w_1, \dots, w_s$, where $d_1(w_t) = d(m_{2t-1} - 1) + p_{2t-1}$ and $d_2(w_t) = d(m_{2t} - 1) + p_{2t}$, for $1 \le t \le s$. Then

$$\begin{aligned}
&n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{w \in E(G)} \alpha(w, j_1, \dots, j_\gamma) \\
&= n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{t=1}^{s} r^{(p_{2t-1}, p_{2t})}\left(j_{m_{2t}} - j_{m_{2t-1}}\right) \\
&= n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} (2\pi)^{-s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{t=1}^{s} \exp\Big(i\lambda_t\big(j_{m_{2t}} - j_{m_{2t-1}}\big)\Big) \\
&\quad \times h^{(p_{2t-1}, p_{2t})}(\lambda_t)\, d(\lambda_1) \cdots d(\lambda_s) \\
&= n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} (2\pi)^{-s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(\prod_{k=1}^{\gamma} \exp\left(i j_k \sum_{p=1}^{\bar{l}_k} U_{k,p}\right)\right) \\
&\quad \times \prod_{t=1}^{s} h^{(p_{2t-1}, p_{2t})}(\lambda_t) d(\lambda_1) \cdots d(\lambda_s),
\end{aligned}$$

where $\bar{l}_k = \sum_{p=1}^{d} l_{k,p}$ and each $U_{k,p}$ is one of $\pm\lambda_l$. Observe that the set $\{U_{k,p}\colon 1 \le k \le \gamma,\ 1 \le p \le \bar{l}_k\}$ is exactly $\{\pm\lambda_l\colon 1 \le l \le s\}$. Because $\sum_{j=1}^{n} \exp(ija) = \exp(i2^{-1}(n+1)a)(\sin(2^{-1}na)/\sin(2^{-1}a))$,

$$\begin{aligned}
&n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{w \in E(G)} \alpha(w, j_1, \dots, j_\gamma) \\
&= n^{-\gamma/2}(2\pi)^{-s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(\prod_{k=1}^{\gamma} \frac{\sin\big(2^{-1} n \sum_{p=1}^{\bar{l}_k} U_{k,p}\big)}{\sin\big(2^{-1} \sum_{p=1}^{\bar{l}_k} U_{k,p}\big)}\right) \\
&\quad \times \prod_{t=1}^{s} h^{(p_{2t-1}, p_{2t})}(\lambda_t) d(\lambda_1) \cdots d(\lambda_s).
\end{aligned}$$

We claim that

$$(2.31)\qquad \left|h^{(p,q)}(\lambda)\right|^2 \le h^{(p,p)}(\lambda)h^{(q,q)}(\lambda) \quad \text{a.s.}$$

with respect to the Lebesgue measure in $[-\pi,\pi]$. Given a, $b \in \mathbb{C}$, the complex-valued Gaussian process $\{Y_k := aX_k^{(p)} + bX_k^{(q)}\}_{k=1}^{\infty}$ has by covariance sequence

$$E\left[Y_1\overline{Y}_{1+k}\right] = |a|^2 r^{(p,p)}(k) + a\overline{b}r^{(p,q)}(k) + \overline{a}br^{(q,p)}(k) + |b|^2 r^{(q,q)}(k).$$

The spectral density of this Gaussian process is

$$|a|^2 h^{(p,p)}(\lambda) + a\overline{b}h^{(p,q)}(\lambda) + \overline{a}bh^{(q,p)}(\lambda) + |b|^2 h^{(q,q)}(\lambda)$$

for each a, $b \in \mathbb{C}$. By the Herglotz's lemma [see, e.g., Breiman (1992), Lemma 11.19],

$$|a|^2 h^{(p,p)}(\lambda) + a\overline{b}h^{(p,q)}(\lambda) + \overline{a}bh^{(q,p)}(\lambda) + |b|^2 h^{(q,q)}(\lambda) \ge 0$$

for each a, $b \in \mathbb{C}$, so, by the Sylvester's conditions for a symmetric Hermitian bilinear form to be positive definite, (2.31) holds.

Let $h(\lambda) = \sum_{1 \le p \le d} h^{(p,p)}(\lambda)$. We have that $\int_{-\pi}^{\pi} h^2(\lambda)d(\lambda) < \infty$ and

$$(2.32)\qquad \begin{aligned} &\left|n^{-\gamma/2} \sum_{j_1,\ldots,j_\gamma=1}^{n} \prod_{w \in E(G)} \alpha(w, j_1, \ldots, j_\gamma)\right| \\ &\quad \le n^{-\gamma/2}(2\pi)^{-s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{k=1}^{\gamma} \left| \frac{\sin\left(2^{-1}n\sum_{p=1}^{l_k} U_{k,p}\right)}{\sin\left(2^{-1}\sum_{p=1}^{l_k} U_{k,p}\right)} \right| \\ &\quad \times \prod_{t=1}^{s} h(\lambda_t)d(\lambda_1)\cdots d(\lambda_s). \end{aligned}$$

Now, the argument in Lemma 2.5 in Sun (1963) implies that (2.32) goes to zero for a nonregular graph G^*. Therefore, (2.30) goes to zero for a nonregular graph G.

Because every graph $G \in \Gamma(l_{1,1}, \ldots, l_{1,d}, \ldots, l_{2\nu+1,1}, \ldots, l_{2\nu+1,d})$ is either null or nonregular, the limit of (2.29) is zero. Therefore, (2.26) follows.

If a graph $G \in \Gamma(l_{1,1}, \ldots, l_{1,d}, \ldots, l_{2\nu,1}, \ldots, l_{2\nu,d})$ is a nonnull regular graph, then there are pairs $(b_1, b_2), \ldots, (b_{2\nu-1}, b_{2\nu})$ such that G^* only has edges joining the levels in the same pair. Hence the edges of G only join points in the same group, being the groups of levels

$$\begin{aligned} &\left\{(b_1-1)d+i\colon 1 \le i \le d\right\} \cup \left\{(b_2-1)d+i\colon 1 \le i \le d\right\} \\ &\quad \cdots \left\{(b_{2\nu-1}-1)d+i\colon 1 \le i \le d\right\} \cup \left\{(b_{2\nu}-1)d+i\colon 1 \le i \le d\right\}. \end{aligned}$$

We also must have that no edge joins levels u and v such that $[(u-1)/d] = [(v-1)/d]$. Define

$$a_i(p,q) = z_G\left((b_{2i-1}-1)d+p, (b_{2i}-1)d+q\right).$$

Observe that

$$\sum_{q=1}^{d} a_i(p,q) = l_{b_{2i-1},p}$$

and

$$\sum_{p=1}^{d} a_i(p,q) = l_{b_{2i},q}.$$

We have that for this graph

$$\begin{aligned}
&n^{-\nu} \sum_{j_1,\dots,j_{2\nu}=1}^{n} \prod_{w\in E(G)} \alpha(w, j_1, j_2, \dots, j_{2\nu}) \\
&= n^{-\nu} \sum_{j_1,\dots,j_{2\nu}=1}^{n} \prod_{i=1}^{\nu} \prod_{p,q=1}^{d} \left(r^{(p,q)}\left(j_{b_{2i-1}} - j_{b_{2i}}\right)\right)^{a_i(p,q)} \\
&= \prod_{i=1}^{\nu} \left(n^{-1} \sum_{j,k=1}^{n} \prod_{p,q=1}^{d} \left(r^{(p,q)}(k-j)\right)^{a_i(p,q)} \right) \\
&\to \prod_{i=1}^{\nu} \left(\lim_{n\to\infty} n^{-1} \sum_{j,k=1}^{n} \prod_{p,q=1}^{d} \left(r^{(p,q)}(k-j)\right)^{a_i(p,q)} \right).
\end{aligned} \tag{2.33}$$

For a fixed partition

$$Q = \left\{ \left(l_{b_1,1},\dots,l_{b_1,d},l_{b_2,1},\dots,l_{b_2,d}\right),\dots,\left(l_{b_{2\nu-1},1},\dots,l_{b_{2\nu-1},d},l_{b_{2\nu},1},\dots,l_{b_{2\nu},d}\right)\right\},$$

we have that

$$\begin{aligned}
&\sum_{G \text{ determining } Q} \sum_{j_1,\dots,j_{2\nu}=1}^{n} n^{-\nu} \prod_{k=1}^{2\nu} \frac{c_{l_{b_k,1},\dots,l_{b_k,d}}}{l_{b_k,1}!,\dots,l_{l_{b_k},d}!} \\
&\times \prod_{w\in E(G)} \alpha(w, j_1, j_2, \dots, j_{2\nu}) \to \prod_{i=1}^{\nu} c_{l_{2b_i-1,1},\dots,l_{2b_i-1,d}} c_{l_{2b_i,1},\dots,l_{2b_i,d}} \\
&\times \beta\left(l_{2b_i-1,1},\dots,l_{2b_i-1,d},l_{2b_i,1},\dots,l_{2b_i,d}\right).
\end{aligned} \tag{2.34}$$

Because there are $(2\nu)!/\nu!2^\nu$ possible divisions of $l_{1,1},\dots,l_{1,d},\dots,l_{2\nu,1},\dots,l_{2\nu,d}$ such groups, the limit of (2.28) is

$$\begin{aligned}
&\frac{(2\nu)!}{\nu!2^\nu} \sum_{\substack{0\le l_{k,i}\le t \\ 1\le i\le d,\, 1\le k\le 2\nu}} \prod_{k=1}^{\nu} c_{l_{2k-1,1},\dots,l_{2k-1,d}} c_{l_{2k,1},\dots,l_{2k,d}} \\
&\times \beta(l_{2k-1,1},\dots,l_{2k-1,d},l_{2k,1},\dots,l_{2k,d}) = \frac{(2\nu)!}{\nu!2^\nu}\sigma_t^{2\nu}.
\end{aligned}$$

Therefore, (2.27) follows and (2.24) holds. Finally notice that (2.18) and (2.24) imply (2.16). □

REMARK 3. Conditions (2.14) and (2.15) are necessary conditions in Theorem 2 in the sense that if

$$\left\{n^{-1/2}\sum_{j=1}^{n} f(X_j)\right\}_{n=1}^{\infty} \tag{2.35}$$

converges in distribution for each function f with $E[f(X)] = 0$ and $E[f^2(X)] < \infty$, then (2.14) and (2.15) hold. In fact, it is enough to assume the convergence in distribution for each polynomial of degree less than 2. If we take $f(x) = \sum_{p=1}^{d} a_p x_j^{(p)}$, (2.35) says that

$$\left\{g_n := n^{-1/2}\sum_{j=1}^{n}\sum_{p=1}^{d} a_p X_j^{(p)}\right\}_{n=1}^{\infty}$$

converges in distribution. However, g_n is a Gaussian r.v. with mean zero, so it converges in distribution if and only if its variance does, that is, if and only if

$$n^{-1}\sum_{j,k=1}^{n}\sum_{p,q=1}^{d} a_p a_q r^{(p,q)}(k-j)$$

converges. This expression converges for each $a_1, \dots, a_d$ if and only if (2.14) holds for each $1 \le p, q \le d$. Next, suppose that

$$\left\{K_n := n^{-1/2}\sum_{j=1}^{n}\sum_{p=1}^{d} a_p H_2\left(X_j^{(p)}\right)\right\}_{n=1}^{\infty} \tag{2.36}$$

converges in distribution. We will see that the convergence in distribution of this sequence implies convergence of second moments. We need to introduce some notation. Let $\mathcal{H}$ be a linear space of mean-zero Gaussian random variables. Let $\mathcal{P}_m(\mathcal{H})$ be the closure in L_2 of the linear span of the set of random variables $\{g_1 \cdots g_r : r \le m, g_j \in \mathcal{H}\}$. The chaos space of order m associated with $\mathcal{H}$ is the orthogonal complement of $\mathcal{P}_{m-1}(\mathcal{H})$ in $\mathcal{P}_m(\mathcal{H})$. Then, for any $h \in \mathcal{K}_m(\mathcal{H})$,

$$\|h\|_p \le \left(\frac{p-1}{q-1}\right)^{m/2} \|h\|_q \tag{2.37}$$

for any $1 < q < p < \infty$, where c only depends on m [see Theorem 3 in Nelson (1973); see also Lemma 3.2 in Arcones and Giné (1993)]. In our case, for each n, by the Gram–Schmidt orthogonalization process, there are orthogonal Gaussian r.v.'s $Y_1, \dots, Y_{nd}$ such that $\{X_j^{(p)}\}_{1 \le j \le n, 1 \le p \le d}$ is in the linear span of $\{Y_j\}_{1 \le j \le nd}$. It is well known that for $\sum_{i=1}^{d} (\alpha^{(i)})^2 = 1$,

$$H_\tau\left(\sum_{i=1}^{d} \alpha^{(i)} x^{(i)}\right) = \sum_{j_1+\cdots+j_d = m\tau} \tau! \prod_{i=1}^{d} (j_i!)^{-1} \left(\alpha^{(i)}\right)^{j_i} H_{j_i}\left(x^{(i)}\right). \tag{2.38}$$

So we have that K_n is a linear combination of the functions $H_1(Y_j)H_1(Y_k)$, $1 \le j \ne k \le d$, and $H_2(Y_j)$, $1 \le j \le d$, that is, K_n is an element of the chaos of order 2 of the Gaussian space generated by $\{Y_j\}_{1 \le j \le nd}$. Hence, (2.37) holds for the sequence $\{K_n\}_{n=1}^{\infty}$. This and the Paley–Zygmund inequality ($E|Z|^q \le \lambda^q + \|Z\|_p^q(\Pr\{|Z| \le \lambda\})^{(p-q)/p}$) imply that $E[|K_n|^p]$ converges for each $0 < p < \infty$. In particular,

$$E\left[|K_n|^2\right] = n^{-1} \sum_{j,k=1}^{n} \sum_{p,q=1}^{d} a_p a_q \left(r^{(p,q)}(k-j)\right)^2$$

converges for each $a_1, \dots, a_p$, so condition (2.15) holds for each $1 \le p, q \le d$.

As in the one dimensional case [Breuer and Major (1983)], if we restrict to functions with a determined rank in order to get the convergence of (2.16), we require a weaker condition.

THEOREM 4. *Let $\{X_j\}_{j=1}^{\infty}$ be a stationary mean-zero Gaussian sequence of $\mathbb{R}^d$-valued vectors. Set $X_j = (X_j^{(1)}, \dots, X_j^{(d)})$. Let f be a function on $\mathbb{R}^d$ with rank τ, $1 \le \tau < \infty$. We define*

$$r^{(p,q)}(k) = E\left[X_m^{(p)} X_{m+k}^{(q)}\right] \tag{2.39}$$

for $k \in Z$, where m is any large enough number such that $m, m+k \ge 1$. Suppose that

$$\sum_{k=-\infty}^{\infty} \left|r^{(p,q)}(k)\right|^{\tau} < \infty \tag{2.40}$$

for each $1 \le p, q \le d$. Then

$$n^{-1/2} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j)\right) \to_d N\left(0, \sigma^2\right), \tag{2.41}$$

where

$$\begin{aligned} \sigma^2 := {} & E\left[\left(f(X_1) - Ef(X_1)\right)^2\right] \\ & + 2 \sum_{k=1}^{\infty} E\left[\left(f(X_1) - Ef(X_1)\right)\left(f(X_{1+k}) - Ef(X_{1+k})\right)\right]. \end{aligned} \tag{2.42}$$

Moreover, there is a finite constant c, depending only on the sequence of covariances, such that

$$E\left(n^{-1/2} \sum_{j=1}^{n} \left(f(X_j) - Ef(X_j)\right)\right)^2 \le cE\left(f(X_1) - Ef(X_1)\right)^2 \tag{2.43}$$

for each n and each function f with second moment finite with rank τ.

PROOF. As in Theorem 2, we may assume that $Ef(X_1) = 0$ and $E[X_1^{(p)}X_1^{(q)}] = \delta_{j,k}$ for each $1 \le p, q \le d$. Let $\psi(k)$ be as in (2.21). Let b be a positive number such that $\sup_{j \ge b} \psi(j) \le 1$. By the argument in (2.22),

$$\left\| n^{-1/2} \sum_{j=1}^{n} f(X_j) \right\|_2 \le b \left(\sum_{k=-\infty}^{\infty} \psi^{\tau}(kb) \right)^{1/2} \|f(X_1)\|_2. \tag{2.44}$$

Therefore, (2.43) follows. Hence, it suffices to show that for any $t < \infty$,

$$n^{-1/2} \sum_{j=1}^{n} \sum_{l_1,\dots,l_d=0}^{t} c_{l_1,\dots,l_d} \prod_{i=1}^{d} \left((l_i)^{-1} H_{l_i}\left(X_j^{(i)}\right) \right) \to_d N\left(0, \sigma_t^2\right), \tag{2.45}$$

where

$$\sigma_t^2 = \sum\nolimits^t_{\substack{l_1,\dots,l_d=1\\ m_1,\dots,m_d=1}} c_{l_1,\dots,l_d} c_{m_1,\dots,m_d} \beta(l_1,\dots,l_d,m_1,\dots,m_d)$$

and

$$\beta(l_1,\dots,l_d,m_1,\dots,m_d) = \sum_{(a(p,q))_{p,q=1}^d \in \mathcal{A}(\mathbf{l},\mathbf{m})} \sum_{k=-\infty}^{\infty} \prod_{p,q=1}^{d} (a(p,q)!)^{-1} \left(r^{(p,q)}(k)\right)^{a(p,q)}.$$

Observe that $\beta(l_1,\dots,l_d,m_1,\dots,m_d) = 0$ if $\sum_{p=1}^d l_p \ne \sum_{p=1}^d m_p$. By the method in Theorem 2 [(2.33) and (2.34)], we have convergence of

$$n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{w \in E(G)} \alpha(w, j_1, j_2, \dots, j_\gamma) \tag{2.46}$$

for a regular graph G. Observe that (2.40) implies that

$$n^{-1} \sum_{j,k=1}^{n} \prod_{p,q=1}^{d} \left(r^{(p,q)}(k-j)\right)^{a(p,q)}.$$

converges for any matrix $(a(p,q))_{1 \le p,q \le d}$ satisfying (2.11) and (2.12), with $\sum_{j=1}^d l_j = \sum_{j=1}^d m_j \ge \tau$. Hence, it suffices to show that (2.46) converges to zero, for each graph $G \in \Gamma(l_{1,1},\dots,l_{1,d},\dots,l_{\gamma,1},\dots,l_{\gamma,d})$ that is not regular. As in the proof of Theorem 2, let G^* be a graph of $\Gamma(\sum_{j=1}^{d} l_{1,j}, \dots, \sum_{j=1}^{d} l_{\gamma,j})$ such that the number of vertices joining the levels a and b is

$$\sum_{p,q=1}^{d} z_G\big(d(a-1)+p, d(b-1)+q\big).$$

We have that

$$\left|\prod_{w \in E(G)} \alpha(w, j_1, j_2, \dots, j_\gamma)\right| \le \prod_{w \in E(G^*)} \psi\big(j_{d_2(w)} - j_{d_1(w)}\big). \tag{2.47}$$

By the argument in Breuer and Major [(1983), page 433], if G^* is not a regular diagram and $\sum_{k=-\infty}^{\infty} \phi(k)^\tau$, then

$$n^{-\gamma/2} \sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{w \in E(G)} \phi\big(j_{d_2(w)} - j_{d_1(w)}\big) \to 0 \tag{2.48}$$

as n tends to infinity. Therefore, if G is not a regular graph,

$$n^{-\gamma/2}\left|\sum_{j_1,\dots,j_\gamma=1}^{n} \prod_{w \in E(G)} \alpha(w, j_1, j_2, \dots, j_\gamma)\right| \to 0$$

and the result follows. □

REMARK 5. Condition (2.36) is a necessary condition in Theorem 2 in some situations. Suppose that $\{X_j\}_{j=1}^{\infty}$ is a stationary mean-zero Gaussian sequence of $\mathbb{R}^d$-valued vectors such that for any polynomial of degree and rank τ, the sequence in (2.41) converges in distribution. By a linear transformation, we may assume that (2.20) holds. In this case, we know that for each function f of the form $f(x^{(1)}, \dots, x^{(d)}) = \sum_{p=1}^{d} a_p H_\tau(x^{(p)})$, the sequence in (2.41) converges in distribution. By the argument in Remark 3, we get that

$$n^{-1} \sum_{j,k=1}^{n} \sum_{p,q=1}^{d} a_p a_q \big(r^{(p,q)}(k-j)\big)^\tau$$

converges for each $a_1, \dots, a_d$. If τ is an even number, this implies (2.36). If $r^{(p,q)}(j)$ is eventually either nonnegative or nonpositive for each $1 \le p, q \le d$, then (2.36) holds.

3. Non-central-limit theorems for functions of a sequence of Gaussian vectors. Several authors [Taqqu (1975, 1979), Dobrushin and Major (1979), etc.] have considered the convergence in distribution in the case where the covariance goes to zero, but not fast enough. Assuming that the decay is of the order of a slowly varying function, the limit is, in general, non-Gaussian.

Our approach is via multiple Wiener–Itô integrals, as in Dobrushin and Major (1979), precisely in the version in Ho and Sun (1990), where the weak convergence is considered jointly for several Gaussian sequences. First we describe the results of these authors that we will need in the proof of the main theorem of this section. Let $\{X_j\}_{j=1}^{\infty}$ be a stationary Gaussian sequence of vectors. Let $r^{(p,q)}(k)$ be as in (2.13). Suppose that there is a slowly varying function $L(k)$ and a number $0 < \alpha < 1$ such that

$$\lim_{k\to\infty} \frac{k^{\alpha} r^{(p,q)}(k)}{L(k)} = b_{p,q} \tag{3.1}$$

for each $1 \le p, q \le d$, where $b_{p,q}$ is a finite constant. Then it is known [see, e.g., Theorem 3A, page 14 of Major (1981) and Ho and Sun (1990)] that there are measures $G^{(p,q)}$, for $1 \le p, q \le d$, such that

$$\int_{-\pi}^{\pi} \exp(ikx)\, dG^{(p,q)}(x) = r^{(p,q)}(k) \tag{3.2}$$

for each integer k and for each $1 \le p,\, q \le d$. There is also a joint random spectral measure $(Z_{G^{(1,1)}}(dx_1), \dots, Z_{G^{(d,d)}}(dx_d))$ such that

$$\int_{-\pi}^{\pi} \exp(ikx)\, dZ_{G^{(p,p)}}(x) = X_k^{(p)} \tag{3.3}$$

for each positive integer k and each $1 \le p \le d$, and

$$E\left[\int_{-\pi}^{\pi} f(x)\, dZ_{G^{(p,p)}}(x) \overline{\int_{-\pi}^{\pi} g(y)\, dZ_{G^{(q,q)}}(y)}\right] = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dG^{(p,q)}(x) \tag{3.4}$$

for any measurable complex functions f and g such that

$$\int_{-\pi}^{\pi} |f(x)|^2\, dG^{(p,p)}(x) < \infty \quad \text{and} \quad \int_{-\pi}^{\pi} |g(y)|^2\, dG^{(q,q)}(y) < \infty.$$

Let $(Z_{G_0^{(1,1)}}, \dots, Z_{G_0^{(d,d)}})$ be the joint random spectral measure that is the limit of

$$\left\{\left(n^{\alpha/2} L^{-1/2}(n) Z_{G^{(1,1)}}(A_1), \dots, n^{\alpha/2} L^{-1/2}(n) Z_{G^{(d,d)}}(A_d)\right) : A_i \in \mathcal{B}[-\pi, \pi]\right\}. \tag{3.5}$$

If $\alpha < 1/\tau$, we have that

$$\left\{\left(a_n^{-1} \sum_{j=1}^{[nt]} H_\tau(X_j^{(1)}), \dots, a_n^{-1} \sum_{j=1}^{[nt]} H_\tau(X_j^{(d)})\right) : 0 \le t \le 1\right\} \tag{3.6}$$

converges in distribution to

$$\Bigg\{\Bigg(\int_{[-\pi,\pi]^\tau} \frac{\exp\big(it(x_1+\cdots+x_\tau)\big)-1}{i(x_1+\cdots+x_\tau)}\, dZ_{G_0^{(1,1)}}(x_1)\cdots dZ_{G_0^{(1,1)}}(x_\tau),\dots,$$

$$\int_{[-\pi,\pi]^\tau} \frac{\exp\big(it(x_1+\cdots+x_\tau)\big)-1}{i(x_1+\cdots+x_\tau)} \tag{3.7}$$

$$\times\, dZ_{G_0^{(d,d)}}(x_1)\cdots dZ_{G_0^{(d,d)}}(x_\tau)\Bigg)\colon 0\le t\le 1\Bigg\},$$

where $a_n = n^{(2-\tau\alpha)/2}(L(n))^{\tau/2}$. Next we present a non-central-limit theorem for functions of random vectors.

THEOREM 6. *Let $\{X_j\}_{j=1}^\infty$ be a stationary mean-zero Gaussian sequence of $\mathbb{R}^d$-valued vectors satisfying* (3.1) *for some $0<\alpha<1$. Let f be a function on $\mathbb{R}^d$ with rank τ, $1\le\tau<1/\alpha$. Then*

$$\Bigg\{a_n^{-1}\sum_{j=1}^{[nt]}\big(f(X_j)-Ef(X_j)\big)\colon 0\le t\le 1\Bigg\} \tag{3.8}$$

converges in distribution to

$$\Bigg\{\sum_{l_1,\dots,l_\tau=1}^{d} e_{l_1,\dots,l_\tau}\int_{[-\pi,\pi]^\tau} \frac{\exp\big(it(x_1+\cdots+x_\tau)\big)-1}{i(x_1+\cdots+x_\tau)} \tag{3.9}$$

$$\times\, dZ_{G_0^{(l_1,l_1)}}(x_1)\cdots dZ_{G_0^{(l_\tau,l_\tau)}}(x_\tau)\colon 0\le t\le 1\Bigg\},$$

where

$$e_{l_1,\dots,l_\tau} = (\tau!)^{-1}E\Bigg[f\big(X_1^{(1)},\dots,X_1^{(d)}\big)\prod_{p=1}^{d}H_{p(l_1,\dots,l_\tau)}\big(X_1^{(p)}\big)\Bigg],$$

$p(l_1,\dots,l_\tau)$ is the number of $l_1,\dots,l_\tau$ that are equal to p and $a_n = n^{(2-\tau\alpha)/2}L^{\tau/2}(n)$. Moreover, there is a finite constant c, depending only on the covariance sequence, such that

$$E\Bigg(a_n^{-1}\sum_{j=1}^{n}\big(f(X_j)-Ef(X_j)\big)\Bigg)^2 \le cE\big(f(X_1)-Ef(X_1)\big)^2 \tag{3.10}$$

for each n and each function f with finite second moment with rank τ.

PROOF. We may assume that (2.19) and (2.20) hold. We have that

$$f\big(x^{(1)},\dots,x^{(d)}\big) = \sum_{l_1,\dots,l_d=0}^{\infty} c_{l_1,\dots,l_d}\prod_{j=1}^{d}(l_j!)^{-1}H_{l_j}\big(x^{(j)}\big),$$

where $c_{l_1,\dots,l_d} = E[f(X_1)\prod_{j=1}^d H_{l_j}(X_1^{(j)})]$. Observe that $c_{l_1,\dots,l_d} = 0$ if $\sum_{i=1}^d l_j < \tau$. Let

$$g(x^{(1)},\dots,x^{(d)}) = \sum_{l_1+\cdots+l_d \geq \tau+1}^{\infty} c_{l_1,\dots,l_d} \prod_{j=1}^{d} (l_j!)^{-1} H_{l_j}(x^{(j)}). \tag{3.11}$$

By a computation similar to (2.22),

$$\begin{aligned} &E\left(a_n^{-1}\sum_{j=1}^n g(X_j)\right)^2 \\ &\quad \leq n^{-2+\tau\alpha}(a(n))^{-\tau} b^2 \sum_{j,k=1}^n \psi^{\tau+1}(b(k-j)) Eg^2(X_1) \to 0. \end{aligned} \tag{3.12}$$

Hence, for proving (3.8), it is enough to show the convergence in distribution of

$$\left\{ a_n^{-1} \sum_{j=1}^{[nt]} \sum_{l_1+\cdots+l_d=\tau} c_{l_1,\dots,l_d} \prod_{p=1}^{d} (l_p!)^{-1} H_{l_p}(X_j^{(p)}) \colon 0 \leq t \leq 1 \right\}. \tag{3.13}$$

Consider the polynomials in $(y^{(1)},\dots,y^{(d)})$, $\{\prod_{p=1}^d (y^{(p)})^{j_p}\}_{j_1+\cdots+j_d=\tau}$. Because these polynomials are linearly independent, there are $\alpha_{k_1,\dots,k_d}^{(p)}$, for $k_1+\cdots+k_d = \tau$ and $1 \leq p \leq d$ such that the matrix

$$\left((\alpha_{k_1,\dots,k_d}^{(1)})^{j_1} \cdots (\alpha_{k_1,\dots,k_d}^{(d)})^{j_d} \right)_{\substack{j_1+\cdots+j_d=\tau \\ k_1+\cdots+k_d=\tau}} \tag{3.14}$$

is nondegenerate. We may suppose that

$$\sum_{p=1}^{d} (\alpha_{k_1,\dots,k_d}^{(p)})^2 = 1$$

for each $k_1,\dots,k_d$. Therefore, there is a matrix

$$(\beta(k_1,\dots,k_d,l_1,\dots,l_d))_{\substack{k_1+\cdots+k_d=\tau \\ l_1+\cdots+l_d=\tau}}$$

such that

$$\begin{aligned} &\sum_{k_1+\cdots+k_d=\tau} \beta(k_1,\dots,k_d,l_1,\dots,l_d)(\alpha_{k_1,\dots,k_d}^{(1)})^{j_1} \cdots (\alpha_{k_1,\dots,k_d}^{(d)})^{j_d} \\ &\quad = \begin{cases} (\tau!)^{-1} \prod_{p=1}^{d} l_p!, & \text{if } (j_1,\dots,j_d) = (l_1,\dots,l_d), \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.15}$$

By this and (2.38),

$$\sum_{k_1+\cdots+k_d=\tau}\sum_{l_1+\cdots+l_d=\tau} c_{l_1,\dots,l_d}\left(\prod_{p=1}^{d}(l_p!)^{-1}\right)$$
$$\times\beta(k_1,\dots,k_d,l_1,\dots,l_d)H_\tau\left(\sum_{p=1}^{d}\alpha^{(p)}_{k_1,\dots,k_d}x^{(p)}\right)$$
$$=\sum_{k_1+\cdots+k_d=\tau}\sum_{l_1+\cdots+l_d=\tau}\sum_{j_1+\cdots+j_d=\tau} c_{l_1,\dots,l_d}\beta(k_1,\dots,k_d,l_1,\dots,l_d)\tau!$$
$$\times\prod_{p=1}^{d}(l_p!\cdot j_p!)^{-1}\left(\alpha^{(p)}_{k_1,\dots,k_d}\right)^{j_p}H_{j_p}\left(x^{(p)}\right)$$
$$=\sum_{l_1+\cdots+l_d=\tau} c_{l_1,\dots,l_d}\prod_{p=1}^{d}(l_p!)^{-1}H_{l_p}\left(x^{(p)}\right).$$

Let

$$\gamma(k_1,\dots,k_d)=\sum_{l_1+\cdots+l_d=\tau} c_{l_1,\dots,l_d}\beta(k_1,\dots,k_d,l_1,\dots,l_d)\prod_{p=1}^{d}(l_p!)^{-1}.$$

Then

$$\begin{aligned}(3.16)\qquad & a_n^{-1}\sum_{j=1}^{[nt]}\sum_{l_1+\cdots+l_d=\tau} c_{l_1,\dots,l_d}\prod_{p=1}^{d}(l_p!)^{-1}H_{l_p}\left(X_j^{(p)}\right)\\ &=a_n^{-1}\sum_{j=1}^{[nt]}\sum_{k_1+\cdots+k_d=\tau}\gamma(k_1,\dots,k_d)H_\tau\left(\sum_{p=1}^{d}\alpha^{(p)}_{k_1,\dots,k_d}X_j^{(p)}\right).\end{aligned}$$

Let $Y_j^{(k_1,\dots,k_d)}=\sum_{p=1}^{d}\alpha^{(i)}_{k_1,\dots,k_d}X_j^{(p)}$. Let $(Z_{G^{(1,1)}}(dx_1),\dots,Z_{G^{(d,d)}}(dx_d))$ be the joint random spectral measure of $\{X_j\}$ as described before. Then the joint random measure of $\{Y_j^{(k_1,\dots,k_d)}\}_{j\ge 1,k_1+\cdots+k_d=\tau}$ is $\{\sum_{p=1}^{d}\alpha^{(p)}_{k_1,\dots,k_d}Z_{G^{(p,p)}}\}_{k_1+\cdots+k_d=\tau}$. Hence

$$\left\{a_n^{-1}\sum_{j=1}^{[nt]}H_\tau\left(Y_j^{(k_1,\dots,k_d)}\right): k_1+\cdots+k_d=\tau,\ 0\le t\le 1\right\}$$

converges in distribution to

$$\left\{\sum_{j_1,\dots,j_\tau=1}^{d}\alpha^{(j_1)}_{k_1,\dots,k_d}\cdots\alpha^{(j_\tau)}_{k_1,\dots,k_d}M_{j_1,\dots,j_\tau}(t): k_1+\cdots+k_d=\tau,\ 0\le t\le 1\right\},$$

where

$$M_{j_1,\dots,j_\tau}(t):=\int_{[-\pi,\pi]^\tau}\frac{\exp\left(it(x_1+\cdots+x_\tau)\right)-1}{i(x_1+\cdots+x_\tau)}\,dZ_{G_0^{(j_1,j_1)}}(x_1)\cdots dZ_{G_0^{(j_\tau,j_\tau)}}(x_\tau).$$

Therefore,

$$\left\{a_n^{-1}\sum_{j=1}^{[nt]}\sum_{k_1+\cdots+k_d=\tau}\gamma(k_1,\ldots,k_d)H_\tau\big(Y_j^{(k_1,\ldots,k_d)}\big)\colon 0\le t\le 1\right\}$$

$$\to_d\left\{\sum_{j_1,\ldots,j_\tau=1}^{d}\sum_{k_1+\cdots+k_d=\tau}\gamma(k_1,\ldots,k_d)\alpha_{k_1,\ldots,k_d}^{(j_1)}\cdots\alpha_{k_1,\ldots,k_d}^{(j_\tau)}\times M_{j_1,\ldots,j_\tau}(t)\colon 0\le t\le 1\right\}. \tag{3.17}$$

We have that

$$\begin{aligned}&\sum_{j_1,\ldots,j_\tau=1}^{d}\sum_{k_1+\cdots+k_d=\tau}\gamma(k_1,\ldots,k_d)\alpha_{k_1,\ldots,k_d}^{(j_1)}\cdots\alpha_{k_1,\ldots,k_d}^{(j_\tau)}M_{j_1,\ldots,j_\tau}(t)\\&\quad=\sum_{j_1,\ldots,j_\tau=1}^{d}\sum_{k_1+\cdots+k_d=\tau}\sum_{l_1+\cdots+l_d=\tau}\frac{c_{l_1,\ldots,l_d}}{\prod_{p=1}^{d}l_p!}\beta(k_1,\ldots,k_d,l_1,\ldots,l_d)\\&\qquad\times\alpha_{k_1,\ldots,k_d}^{(j_1)}\cdots\alpha_{k_1,\ldots,k_d}^{(j_\tau)}M_{j_1,\ldots,j_\tau}(t).\end{aligned}$$

By (3.15),

$$\begin{aligned}&\sum_{k_1+\cdots+k_d=\tau}\beta(k_1,\ldots,k_d,l_1,\ldots,l_d)\frac{c_{l_1,\ldots,l_d}}{\prod_{p=1}^{d}l_p!}\beta(k_1,\ldots,k_d,l_1,\ldots,l_d)\\&\qquad\times\alpha_{k_1,\ldots,k_d}^{(j_1)}\cdots\alpha_{k_1,\ldots,k_d}^{(j_\tau)}M_{j_1,\ldots,j_\tau}(t)\\&\quad=\begin{cases}(\tau!)^{-1}c_{l_1,\ldots,l_d}M_{j_1,\ldots,j_\tau}(t), & \text{if } l_p=p(j_1,\ldots,j_\tau)\text{ for }1\le p\le d,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

From the last observation, (3.15) and the fact that $\tau!\,e_{j_1,\ldots,j_\tau}=c_{l_1,\ldots,l_d}$ if $l_p=p(j_1,\ldots,j_\tau)$ for $1\le p\le d$, it follows that the distributional limit of (3.8) is (3.9).

As far as (3.10), we have, as in (2.22),

$$E\left(a_n^{-1}\sum_{j=1}^{n}f(X_j)\right)^2\le b^2a_n^{-2}\sum_{j,k=1}^{n}\psi^\tau\big(b(j-k)\big)Ef^2(X_1).$$

Because $n^{\tau\alpha-2}(L(n))^{-\tau}\sum_{k=-n}^{n}(n-|k|)\psi^\tau(k)$ is a bounded sequence, (3.10) follows. □

REMARK 7. Let

$$X(t):=\sum_{l_1,\ldots,l_\tau=1}^{d}e_{l_1,\ldots,l_\tau}\int_{[-\pi,\pi]^\tau}\frac{\exp\big(it(x_1+\cdots+x_\tau)\big)-1}{i(x_1+\cdots+x_\tau)}\times dZ_{G_0^{(l_1,l_1)}}(x_1)\cdots dZ_{G_0^{(l_\tau,l_\tau)}}(x_\tau).$$

It is well know that there is a vector space $\mathcal{H}$ of Gaussian r.v.'s such that $X(t)$ belongs to the homogeneous chaos of order τ of $\mathcal{H}$. It is easy to see that $\{X(t)\colon 0 \le t \le 1\}$ has a version with bounded and continuous paths. So, by Theorem 2.3 in Arcones (1994), there is an ortho-Gaussian sequence $\{g_p\}_{p=1}^{\infty}$ of r.v.'s in $\mathcal{H}$ such that

$$\{X(t)\colon 0 \le t \le 1\} = \left\{ \sum_{j_1,\dots,j_\tau=1}^{\infty} a_{j_1,\dots,j_\tau}(t) \prod_{p=1}^{\infty} H_{p(j_1,\dots,j_\tau)}(g_p)\colon 0 \le t \le 1 \right\}, \tag{3.18}$$

where $a_{j_1,\dots,j_\tau}(t) = (\tau!)^{-1} E[X(t)\prod_{p=1}^{\infty} H_{p(j_1,\dots,j_\tau)}(g_p)]$ and the series in (3.18) converges a.s. uniformly in $0 \le t \le 1$. So (3.9) is a $C_u([0,1],d)$-valued chaos, where $d(s,t) = |s-t|$.

4. A bootstrap central-limit theorem for dependent data. In this section we consider the bootstrap of the limit theorem in Section 2. Because σ^2 in (2.16) is not the variance of $f(X)$, the usual bootstrap does not work. We will show that the moving bootstrap does. This procedure consists of the following: Given the sample $f(X_1),\dots,f(X_n)$, we make blocks of size b_n,

$$B_i = \left\{ f(X_i),\dots,f\left(X_{i+b_n-1}\right) \right\} \tag{4.1}$$

for $i = 1,\dots,n-b_n+1$. Let

$$Y_{n,i} = b_n^{-1} \sum_{j=i}^{b_n+i-1} f(X_j), \tag{4.2}$$

for $i = 1,\dots,n-b_n+1$, be the averages of each block. Now, we take bootstrap random variables $Z_{n,i}$ for $i = 1,\dots,m_n$, which are i.i.d. r.v.'s with common law

$$(n-b_n+1)^{-1} \sum_{i=1}^{n-b_n+1} \delta_{Y_{n,i}}. \tag{4.3}$$

Let

$$\overline{Y}_n = (n-b_n+1)^{-1} \sum_{i=1}^{n-b_n+1} Y_{n,i}. \tag{4.4}$$

We will show that

$$(b_n/m_n)^{1/2} \sum_{i=1}^{m_n} (Z_{n,i} - \overline{Y}_n) \to_d N(0,\sigma^2) \tag{4.5}$$

in probability, where σ is as in (2.17), meaning that

$$\sup_t \left| \Pr^{\text{bot}} \left\{ (b_n/m_n)^{1/2} \sum_{i=1}^{m_n} (Z_{n,i} - \overline{Y}_n) \le t \right\} - \Phi(t/\sigma) \right| \to_{\Pr} 0$$

where $\Pr^{\mathrm{bot}}$ stands for the bootstrap probability, that is, for the probability with respect to the random variable $Z_{n,i}$ given $f(X_1),\dots,f(X_n)$, and Φ is the cumulative distribution of a standard normal random variable.

THEOREM 8. *Let $\{X_j\}_{j=1}^{\infty}$ be a $\mathbb{R}^d$-valued stationary mean-zero Gaussian sequence satisfying*

$$\sum_{k=-\infty}^{\infty} \left|r^{(p,q)}(k)\right| < \infty \tag{4.6}$$

for each $1 \le p, q \le d$. Let f be a measurable function on $\mathbb{R}^d$ such that

$$E\left[f^4(X)\right] < \infty. \tag{4.7}$$

Conditionally on the sample, take $\{Z_{n,i}\}_{i=1}^{m_n}$ i.i.d. r.v.'s *with the law in* (4.3). *Suppose that*

$$m_n, b_n \to \infty, \qquad b_n^3 n^{-1} \to 0 \quad and \quad \limsup_{n\to\infty} m_n^{-1} b_n n^{1/2} < \infty. \tag{4.8}$$

Then (4.5) *holds.*

PROOF. We may assume (2.19) and (2.20). By the c.l.t. for triangular arrays [see, e.g., Araujo and Giné (1980), page 63], if $\{\xi_{n,j}\}_{j=1}^{k_n}$ is a triangular array such that:

(i) $\sum_{j=1}^{k_n} E\xi_{n,j}^2 \to 1$,
(ii) $\sum_{j=1}^{k_n} E\xi_{n,j}^2 I_{|\xi_{n,j}|>\varepsilon} \to 0$ for each $\varepsilon > 0$,
(iii) $E\xi_{n,j} = 0$ for each $1 \le j \le k_n$ and each n,

then

$$\sum_{j=1}^{k_n} \xi_{n,j} \to_d N(0,1).$$

This implies that in order to prove (4.5), it is enough to check that

$$\begin{aligned} &\mathrm{Var}^{\mathrm{bot}}\left(\left(b_n/m_n\right)^{1/2}\sum_{i=1}^{m_n}(Z_{n,i}-\overline{Y}_n)\right)\\ &\quad = b_n(n-b_n+1)^{-1}\sum_{i=1}^{n-b_n+1} Y_{n,i}^2\\ &\qquad - \left(b_n^{1/2}(n-b_n+1)^{-1}\sum_{i=1}^{n-b_n+1} Y_{n,i}\right)^2 \to_{\Pr} \sigma^2 \end{aligned} \tag{4.9}$$

and that

$$(4.10)\qquad (b_n m_n)^{-1/2} \max_{1\le j\le n-b_n+1} \left| \sum_{i=j}^{b_n+j-1} f(X_i) \right| \to_{\Pr} 0.$$

We have that

$$(4.11)\qquad \begin{aligned} &b_n^{1/2}(n-b_n+1)^{-1} \sum_{i=1}^{n-b_n+1} Y_{n,i} \\ &= b_n^{-1/2}(n-b_n+1)^{-1} \sum_{i=1}^{n-b_n+1} \sum_{j=i}^{b_n+i-1} f(X_j) \\ &= b_n^{1/2}(n-b_n+1)^{-1} \sum_{j=1}^{n} f(X_j) + b_n^{-1/2}(n-b_n+1)^{-1} \sum_{j=1}^{b_n-1} (j-b_n) f(X_j) \\ &\quad + b_n^{-1/2}(n-b_n+1)^{-1} \sum_{j=n-b_n+2}^{n} (n+1-j-b_n) f(X_j). \end{aligned}$$

Because $n^{-1/2}\sum_{j=1}^{n} f(X_j)$ converges in distribution and $b_n n^{-1} \to 0$,

$$(4.12)\qquad b_n^{1/2}(n-b_n+1)^{-1} \sum_{j=1}^{n} f(X_j) \to_{\Pr} 0.$$

Observe that the method in proving (2.22) gives that

$$(4.13)\qquad n^{-1} E\left(\sum_{j=1}^{n} a_j f(X_j) \right)^2 \le b^2 \max_{1\le j\le n} a_j^2 \left(\sum_{k=-\infty}^{\infty} \psi(kb) \right) E f^2(X_1).$$

Hence

$$(4.14)\qquad b_n^{-1/2}(n-b_n+1)^{-1} \sum_{j=1}^{b_n-1} (j-b_n) f(X_j) \to_{\Pr} 0$$

and

$$(4.15)\qquad b_n^{-1/2}(n-b_n+1)^{-1} \sum_{j=n-b_n+2}^{n} (n+1-j-b_n) f(X_j) \to_{\Pr} 0.$$

Observe that (4.11)–(4.15) imply that

$$(4.16)\qquad b_n^{1/2}(n-b_n+1)^{-1} \sum_{i=1}^{n-b_n+1} Y_{n,i} \to_{\Pr} 0.$$

We have that

$$\begin{aligned} b_n n^{-1} \sum_{i=1}^{n-b_n+1} Y_{n,i}^2 &= n^{-1} \sum_{j=1}^{n} f^2(X_j) + 2n^{-1} \sum_{i=1}^{n-b_n+1} \sum_{j=1}^{b_n-1} f(X_i)f(X_{i+j}) \\ &\quad + (nb_n)^{-1} \sum_{j=1}^{b_n-1} (j-b_n) f^2(X_j) + (nb_n)^{-1} \\ &\quad \times \sum_{j=n-b_n+2}^{n} (n+1-j-b_n) f^2(X_j) \\ &\quad - 2(nb_n)^{-1} \sum_{i=1}^{n-b_n+1} \sum_{j=1}^{b_n-1} j f(X_i)f(X_{i+j}) \\ &\quad + 2(nb_n)^{-1} \sum_{i=1}^{b_n-2} \sum_{j=1}^{b_n-1-i} (i+j-b_n) f(X_i)f(X_{i+j}). \end{aligned} \tag{4.17}$$

By the ergodic theorem,

$$n^{-1} \sum_{j=1}^{n} f^2(X_j) \to E[f^2(X_1)] \quad \text{a.s.} \tag{4.18}$$

We have that

$$\begin{aligned} E\left[2n^{-1} \sum_{i=1}^{n-b_n+1} \sum_{j=1}^{b_n-1} f(X_i)f(X_{i+j})\right] &= 2n^{-1}(n-b_n+1) \sum_{j=1}^{b_n-1} E[f(X_1)f(X_{1+j})] \\ &\to 2 \sum_{j=1}^{\infty} E[f(X_1)f(X_{1+j})]. \end{aligned} \tag{4.19}$$

We also have that

$$\begin{aligned} &\left\| n^{-1} \sum_{i=1}^{n-b_n+1} \sum_{j=1}^{b_n-1} \left(f(X_i)f(X_{i+j}) - E[f(X_i)f(X_{i+j})]\right) \right\|_2 \\ &\le n^{-1} \sum_{j=1}^{b_n-1} \sum_{k=1}^{2b_n} \left\| \sum_{i=1}^{[(n+1-b_n)/2b_n]} \left(f\left(X_{2b_n(i-1)+k}\right) f\left(X_{2b_n(i-1)+k+j}\right) \right.\right. \\ &\qquad \left.\left. - E\left[f\left(X_{2b_n(i-1)+k}\right) f\left(X_{2b_n(i-1)+k+j}\right) \right] \right) \right\|_2 \\ &\quad + \left\| n^{-1} \sum_{i=2b_n[(n+1-b_n)/2b_n]+1}^{n-b_n+1} \sum_{j=1}^{b_n-1} \left(f(X_i)f(X_{i+j}) - E[f(X_i)f(X_{i+j})]\right) \right\|_2. \end{aligned} \tag{4.20}$$

We want to bound the last expression using the method in (2.22). Fix j and k and let

$$\left(Y_i^{(1)},\dots,Y_i^{(2d)}\right) = \left(X^{(1)}_{2b_n(i-1)+k},\dots,X^{(d)}_{2b_n(i-1)+k},\right.$$
$$\left. X^{(1)}_{2b_n(i-1)+k+j},\dots,X^{(d)}_{2b_n(i-1)+k+j}\right).$$

Let d_j be the rank of the covariance matrix of $(Y_i^{(1)},\dots,Y_i^{(2d)})$. Then there is a matrix $(a_{p,q}(j))_{1\le p\le d_j,\,1\le q\le 2d}$ such that if

$$Z_i^{(p)} = \sum_{q=1}^{2d} a_{p,q}(j)Y_i^{(q)},$$

then $(Z_i^{(1)},\dots,Z_i^{(d_j)})$ is a standard Gaussian random vector. For $i<k$,

$$\left|E\left[Z_i^{(p)}Z_k^{(q)}\right]\right| \le 4d^2 \sup_{\substack{1\le p\le d_j,\\ 1\le q\le 2d}} \left(a_{p,q}(j)\right)^2\psi\left(2b_n(k-i)\right).$$

As $j\to\infty$, the covariance matrix of $(Y_1^{(1)},\dots,Y_1^{(2d)},Y_j^{(1)},\dots,Y_j^{(2d)})$ converges to the identity matrix $(2d)\times(2d)$. Hence

$$\sup_{j\ge 1}\ \sup_{1\le p\le d_j,\,1\le q\le 2d} \left(a_{p,q}(j)\right)^2 =: r<\infty.$$

If n is so large that $4d^3 r\psi(b_n)\le 1$, the sequence $\{Z := (Z_k^{(1)},\dots,Z_k^{(d_j)})\}_{k=1}^{\infty}$ satisfies (2.5). Hence we can apply the Lemma 1 to (4.20) to get

$$\left\| n^{-1}\sum_{i=1}^{n-b_n+1}\sum_{j=1}^{b_n-1}\left(f(X_i)f(X_{i+j}) - E\left[f(X_i)f(X_{i+j})\right]\right)\right\|_2 \le cn^{-1}b_n^2\left(n/b_n\right)^{1/2}\to 0,$$

where c is a finite constant. Similarly, we can deal with all the other terms in (4.17). The details are omitted. So, we get that (4.9) holds.

As to (4.10), because $E[f^4(X)]<\infty$, $n^{-1/4}|f(X_n)|\to 0$ a.s. From this and the fact that $\limsup_{n\to\infty} m_n^{-1}b_n n^{1/2}<\infty$, it follows that (4.10) goes to zero a.s. □

5. The central-limit theorem for a class of functions of Gaussian vectors. In this section, we will consider the limit theorems in Sections 2 and 3 uniformly over class functions. First, we observe that by the usual Cramér and Wold device, we get a finite dimensional version of these theorems. For example, suppose that $f_1,\dots,f_m$ are functions on $\mathbb{R}^d$ with finite second moment and rank τ or greater, $1\le\tau<\infty$, and $\sum_{k=-\infty}^{\infty}|r^{(p,q)}(k)|^{\tau}<\infty$ for each $1\le p,\ q\le d$. Then by Theorem 4,

$$n^{-1/2}\sum_{j=1}^{n}\sum_{p=1}^{m} a_p\left(f_p(X_j) - Ef_p(X_j)\right) \to_d N\left(0,\ \sum_{p,q=1}^{m} a_p a_q\langle f_p,f_q\rangle\right),$$

for each real numbers $a_1, \dots, a_m$, where

$$\langle f_p, f_q \rangle := \operatorname{Cov}(f_p(X_1), f_q(X_1)) + \sum_{k=1}^{\infty} \Big(\operatorname{Cov}(f_p(X_1), f_q(X_{k+1})) + \operatorname{Cov}(f_p(X_{k+1}), f_q(X_1)) \Big).$$

This fact and the Cramér and Wold device [see, e.g., Theorem 7.7 in Billingsley (1969)] imply that

$$\left\{ \left(n^{-1/2} \sum_{j=1}^{n} (f_1(X_j) - Ef_1(X_j)), \dots, n^{-1/2} \sum_{j=1}^{n} (f_m(X_j) - Ef_m(X_j)) \right) \right\}_{n=1}^{\infty}$$

converges in distribution to a mean-zero Gaussian vector $(g_1, \dots, g_m)$ with covariance given by $E[g_p g_q] = \langle f_p, f_q \rangle$. Similarly, it is possible to get finite dimensional versions of Theorems 2 and 6.

Let $\mathcal{F}$ be a class of functions on $\mathbb{R}^d$. In this section we will give a bracketing condition for the weak convergence of

$$\left\{ S_n(f) := a_n^{-1} \sum_{j=1}^{n} (f(X_j) - Ef(X_j)) : f \in \mathcal{F} \right\}.$$

The bracketing number corresponding to the p-norm is defined as

$$\begin{aligned} N_{[\,]}^{(p)}(\varepsilon, \mathcal{F}) = \min \{ r \colon & \exists \text{ measurable function } f_1, \dots, f_r \text{ and } \Delta_1, \dots, \Delta_r \text{ such that} \\ & E\Delta_i^p(X) \le \varepsilon^p \text{ for each } i \le r \text{ and for each } f \in \mathcal{F}, \\ & \exists i \le r \text{ such that } |f_i - f| \le \Delta_i \}. \end{aligned}$$

For a class $\mathcal{F}$ we define the rank of the class as $\operatorname{rank}(\mathcal{F}) := \inf\{\operatorname{rank}(f) \colon f \in \mathcal{F}\}$.

THEOREM 9. *Let $\{X_j\}_{j=1}^{\infty}$ be a stationary mean-zero Gaussian sequence of $\mathbb{R}^d$-valued vectors. Let $\mathcal{F}$ be a class of measurable functions on $\mathbb{R}^d$ such that* $\operatorname{rank}(\mathcal{F}) = \tau$, *for some $1 \le \tau < \infty$. Suppose that one of the following conditions is satisfied*:

(i′) $\lim_{n \to \infty} n^{-1} \sum_{j,k=1}^{n} r^{(p,q)}(j-k)$ *and* $\lim_{n \to \infty} n^{-1} \sum_{j,k=1}^{n} (r^{(p,q)}(j-k))^2$ *exist for each $1 \le p, q \le d$ and $\tau = 1$.*

(i″) $\sum_{k=-\infty}^{\infty} |r^{(p,q)}(k)|^{\tau} < \infty$ *for each $1 \le p, q \le d$.*

(i‴) *There is a slowly varying function $L(k)$ and a number $0 < \alpha < 1/\tau$ such that* $\lim_{k \to \infty} (k^{\alpha} r^{(p,q)}(k)/L(k)) = b_{p,q}$, *for each $1 \le p, q \le d$, where $b_{p,q}$ is a finite constant.*

(ii) *Suppose also that*

$$\int_0^{\infty} \left(N_{[\,]}^{(2)}(\varepsilon, \mathcal{F}) \right)^{1/2} d\varepsilon < \infty. \tag{5.1}$$

Then the process

$$\left\{a_n^{-1}\sum_{j=1}^{n}\left(f(X_j)-Ef(X_j)\right): f\in\mathcal{F}\right\} \tag{5.2}$$

converges weakly to a process $\{K(f): f\in\mathcal{F}\}$ *that has a separable support in* $l_\infty(\mathcal{F})$, *where* $a_n = n^{1/2}$ *if either hypothesis* (i′) *or* (i″) *holds and* $a_n = n^{(2-\tau\alpha)/2}(L(n))^{\tau/2}$ *if hypothesis* (i‴) *holds. Moreover, if either condition* (i′) *or condition* (i″) *is satisfied*, $\{K(f): f\in\mathcal{F}\}$ *is a mean-zero Gaussian process with covariance given by* $E[K(f_1)K(f_2)] = \langle f_1, f_2\rangle$. *If condition* (i‴) *is satisfied*,

$$K(f) = \sum_{l_1,\dots,l_\tau=1}^{d} e_{l_1,\dots,l_\tau}(f)\int_{[-\pi,\pi]^\tau}\frac{\exp\left(it(x_1+\cdots+x_\tau)\right)-1}{i(x_1+\cdots+x_\tau)} \times dZ_{G_0^{(l_1,l_1)}}(x_1)\cdots dZ_{G_0^{(l_\tau,l_\tau)}}(x_\tau),$$

where

$$e_{l_1,\dots,l_\tau}(f) = (\tau!)^{-1}E\left[f\left(X_1^{(1)},\dots,X_1^{(1)}\right)\prod_{p=1}^{d}H_{p(l_1,\dots,l_\tau)}\left(X_1^{(p)}\right)\right].$$

PROOF. We apply the mentioned Theorem 2.12 in Andersen and Dobrić (1987) with the $\|\cdot\|_2$-distance. As we saw before, we have convergence of the finite dimensional distributions. So, we need to show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Pr{}^*\left\{\sup_{\substack{f_1,f_2\in\mathcal{F}\\ \|f_1-f_2\|_2\le\delta}}|S_n(f_1)-S_n(f_2)|>\eta\right\}=0 \tag{5.3}$$

for each $\eta > 0$. By hypothesis, there are maps

$$\pi_q: \mathcal{F}\to\mathcal{F}, \qquad \Delta_q: \mathcal{F}\to\mathcal{F}$$

such that

$$|f-\pi_q f|\le\Delta_q f, \qquad \|\Delta_q f\|_2\le 2^{-q}, \qquad \#\pi_q\mathcal{F},\ \#\Delta_q\mathcal{F}\le N_q := N_{[\,]}^{(2)}(2^{-q}).$$

We may assume that $\pi_q f\in\mathcal{F}$. Either Theorem 2, 4 or 6 implies the convergence of the finite dimensional distributions of $\{S_n(f): f\in\mathcal{F}\}$ to $\{K(f): f\in\mathcal{F}\}$. Take $p<q$. Let $\tau_j = \pi_j\circ\cdots\circ\pi_q$. Observe that $\tau_j = \pi_j\circ\tau_{j+1}$. By the triangular inequality,

$$E\left[\sup_{f\in\mathcal{F}}|S_n(f-\tau_p f)|\right] \le E\left[\sup_{f\in\mathcal{F}}|S_n(f-\tau_q f)|\right] + \sum_{j=p+1}^{q}E\left[\sup_f|S_n(\tau_j f-\tau_{j-1}f)|\right]. \tag{5.4}$$

We have that

$$
\begin{aligned}
E\left[\sup_{f\in\mathcal{F}}|S_n(f-\tau_q f)|\right] &\le a_n^{-1}E\left[\sup_{f\in\mathcal{F}}\left(\sum_{j=1}^{n}\Delta_q f(X_j)+\Delta_q f(X_j)\right)\right]\\
&\le a_n^{-1}E\left[\sup_{f\in\mathcal{F}}\sum_{j=1}^{n}\Delta_q f(X_j)\right]+na_n^{-1}2^{-q}\\
&\le E\left[\sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n}a_n^{-1}(\Delta_q f(X_j)-E\,\Delta_q f(X_j))\right|\right]+2na_n^{-1}2^{-q} \qquad (5.5)\\
&\le \left(E\left[\sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n}a_n^{-1}(\Delta_q f(X_j)-E\,\Delta_q f(X_j))\right|^2\right]\right)^{1/2}\\
&\quad+2na_n^{-1}2^{-q}\\
&\le cN_q^{1/2}2^{-q}+2na_n^{-1}2^{-q}
\end{aligned}
$$

because the cardinality of $\{\Delta_q f: f\in\mathcal{F}\}$ is less than N_q and $\|S_n(f)\|_2\le c\|f\|_2$. We also have that

$$
(5.6)\quad E\left[\sup_{f}|S_n(\tau_j f-\tau_{j-1}f)|\right]\le E\left[\sup_{f\in\tau_j\mathcal{F}}|S_n(\pi_{j-1}f-f)|\right]\le cN_j^{1/2}2^{-(j-1)}.
$$

From (5.4)–(5.6) we get that

$$
(5.7)\qquad E\left[\sup_{f\in\mathcal{F}}|S_n(f-\tau_p f)|\right]\le 2na_n^{-1}2^{-q}+3c\sum_{j=p+1}^{q}N_j^{1/2}2^{-j}.
$$

Now τ_p determines a partition of $\mathcal{F}$ in N_p pieces. Let us call them $E_1,\dots,E_{N_p}$. Let

$$
\begin{aligned}
\Lambda=\{(i,j)\colon 1\le i\le j\le N_p \text{ such that there exist } f\in E_i\\
\text{and } g\in E_j \text{ such that } \|f-g\|_2\le\delta\}.
\end{aligned}
$$

For each $(i,j)\in\Lambda$, take $\phi_{(i,j)}\in E_i$ and $\varphi_{(i,j)}\in E_j$ such that $\|\phi_{(i,j)}-\varphi_{(i,j)}\|_2\le\delta$. If $\|f-g\|_2\le\delta$, then there are $1\le i\le j\le N_p$ such that $f\in E_i$ and $g\in E_j$. Then

$$
\begin{aligned}
|S_n(f)-S_n(g)| &\le |S_n(f-\tau_p f)|+\left|S_n\left(\phi_{(i,j)}-\tau_p\phi_{(i,j)}\right)\right|+\left|S_n\left(\phi_{(i,j)}-\varphi_{(i,j)}\right)\right|\\
&\quad+\left|S_n\left(\varphi_{(i,j)}-\tau_p\varphi_{(i,j)}\right)\right|+|S_n(g-\tau_p g)|\\
&\le 4\sup_f|S_n(f-\tau_p f)|+\sup_{(i,j)\in\Lambda}\left|S_n\left(\phi_{(i,j)}-\varphi_{(i,j)}\right)\right|.
\end{aligned}
$$

From this and (5.7),

$$E\left[\sup_{\|f-g\|_2\le\delta}|S_n(f-g)|\right]$$
$$\le 4E\left[\sup_f|S_n(f-\tau_p f)|\right]+E\left[\sup_{(i,j)\in\Lambda}\left|S_n\left(\phi_{(i,j)}-\varphi_{(i,j)}\right)\right|\right]$$
$$\le 8na_n^{-1}2^{-q}+12c\sum_{j=p}^{q}N_j^{1/2}2^{-j}+c\delta N_p^2.$$

Letting $q\to\infty$,

$$E\left[\sup_{\|f-g\|_2\le\delta}|S_n(f-g)|\right]\le 12c\sum_{j=p}^{\infty}N_j^{1/2}2^{-j}+c\delta N_p^2.$$

Therefore,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}E\left[\sup_{\|f-g\|_2\le\delta}|S_n(f-g)|\right]\le 12c\sum_{j=p}^{\infty}N_j^{1/2}2^{-j}$$

for any p. Therefore (5.3) follows. □

From Theorem 9, it is possible to get the weak convergence for some classes of functions. For example, if H: $\mathbb{R}^d\to\mathbb{R}$ is a function with finite second moment and the Gaussian sequence satisfies any of the three conditions in Theorem 9, then

$$\left\{a_n^{-1}\sum_{j=1}^{n}\Big(\exp\big(itH(X_j)\big)-E\exp\big(itH(X_j)\big)\Big):\ |t|\le T\right\}$$

converges weakly to a Gaussian process for each $T<\infty$. Observe that

$$E\left|\exp\big(itH(X_1)\big)-\exp\big(isH(X_1)\big)\right|^2\le(t-s)^2EH^2(X_1).$$

Hence $N_{[\,]}^{(2)}(\varepsilon,\mathcal{F}_T)\le(2(EH^2(X_1))^{1/2}T/\varepsilon)+1$, where $\mathcal{F}_T=\{\exp(itH(x)):\ |t|\le T\}$.

In Arcones and Yu (1994) there is a result on empirical processes of partial sums of Gaussian fields under long range dependence. Theorem 9 can be applied to the study of the asymptotics of M-estimators [see Theorem 2 in Arcones and Yu (1994)].

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