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A UNIVERSAL ONE-SIDED LAW OF THE ITERATED LOGARITHM[1]

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We prove that the lim inf of suitably normalized sums of i.i.d. nonnegative and nondegenerate random variables can with probability 1 only be a constant between $-2^{1/2}$ and 0. Moreover, we show that each value within this range is attainable by an appropriate choice of the underlying common distribution function.

1. Introduction and main results. Let $X, X_1, X_2, \dots,$ be a sequence of independent and nonnegative random variables with common nondegenerate distribution function F. Denote the inverse or quantile function of F by

$$Q(s) = \inf\{x: F(x) \geq s\}, \qquad 0 < s \leq 1,$$

with $Q(0) = Q(0+)$. Also set

$$\mu(s) = \int_0^{1-s} Q(u)\,du, \qquad 0 < s < 1,$$

and

$$\sigma^2(s) = \int_0^{1-s}\int_0^{1-s} (u \wedge v - uv)\,dQ(u)\,dQ(u),$$

which after some integration by parts equals

$$sQ^2(1-s) + \int_0^{1-s} Q^2(u)\,du - \big(sQ(1-s) + \mu(s)\big)^2.$$

Note for future reference that $\sigma^2(s)$ is a decreasing function of s. [The function $\sigma^2(s)$ is a one-sided version of a function that plays a crucial role in a quantile function approach to the asymptotic distribution of sums of i.i.d. random variables developed in S. Csörgő, Haeusler and Mason (CsHM) (1988a).]

Writing $Lt = \log(t \vee e)$ and $LLt = L(Lt)$, set for $n \geq 1$,

$$b_n = (LLn)/n \quad \text{and} \quad a_n = n^{1/2}\sigma(b_n).$$

Denote the nth partial sum of the X_i's as

$$S_n = X_1 + \cdots + X_n, \qquad n \geq 1.$$

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The main purpose of this paper is to establish the following *universal* one-sided law of the iterated logarithm for sums of i.i.d. nonnegative random variables.

THEOREM 1. *With the above notation and assumptions, there exists a constant* $-2^{1/2} \le K(F) \le 0$ *such that*

$$\liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} = K(F) \quad a.s. \tag{1.1}$$

and, moreover, all values in the interval $[-2^{1/2}, 0]$ *are attainable.*

Our choice of the centering and norming sequences in (1.1) was motivated by those used in Haeusler and Mason (1987) and Haeusler (1993) in their studies of laws of the iterated logarithm for trimmed sums. Indeed, the behavior described in our theorem can be explained roughly by the fact that in general S_n is infinitely often of the same order as its corresponding trimmed sum formed when the top LLn extreme values are deleted, for which the normalization in (1.1) is appropriate for a law of the iterated logarithm.

Recently Einmahl and Mason (1994), as a by-product of a more general investigation into a universal Chung-type law of the iterated logarithm, have shown that if X is nonnegative with a distribution in the Feller class, then the constant in (1.1) is strictly less than 0. To determine the actual value of $K(F)$ in our Theorem 1, one needs more information about the upper tail of F combined with precise large deviation results such as those to be found in Jain and Pruitt (1987). In fact, relying on the work of Jain and Pruitt (1987), Pruitt (1990) proved the following result, which is closely related to our theorem.

THEOREM 1 OF PRUITT (1990). *Assume* $X, X_1, X_2, \dots,$ *are independent and nonnegative random variables with common nondegenerate distribution function* F *in the Feller class. Let* β_n, $n \ge 1$, *be a sequence of positive constants such that* β_n/n *is nondecreasing. Then*

$$P(S_n \le \beta_n \quad i.o.) = 0 \text{ or } 1 \tag{1.2}$$

according to whether the sum

$$\sum_{n=1}^{\infty} n^{-1} p_n \log(3p_n^{-1}) \tag{1.3}$$

is finite or infinite, where $p_n = P(S_n \le \beta_n)$, $n \ge 1$.

The necessary information about the asymptotic behavior of the probabilities p_n required to apply Pruitt's theorem can be obtained from the large deviation results of Jain and Pruitt (1987).

Let $K(F)$ be the constant appearing in (1.1) of our Theorem 1 and for any real ε set, for $n \ge 1$,

$$\beta_n = n\Big(\mu_n(b_n) + \big(K(F) + \varepsilon\big)\sigma(b_n)b_n^{1/2}\Big). \tag{1.4}$$

Then, obviously, our Theorem 1 says similarly that the probability in (1.2) is equal to 0 or 1, according as $\varepsilon < 0$ or $\varepsilon > 0$. However, a direct comparison between Pruitt's result and ours is extremely difficult. It is only possible when F is in the Feller class and the sequence β_n in (1.4) is such that $\beta_n > 0$ and the sequence β_n/n is nondecreasing or, at least, is asymptotically equivalent to a sequence with this property. This happens, for instance, when F is in the domain of attraction of a stable law, but, clearly, any in-depth comparison between the two results requires knowledge of the constant $K(F)$ and of the behavior of the functions $\mu(s)$ and $\sigma(s)s^{1/2}$ as $s \downarrow 0$, when, more generally, F is in the Feller class. Even in the stable case, to obtain this information entails a considerable amount of analysis. Our theorem only says that $-2^{1/2} \le K(F) \le 0$ and Pruitt's (1990) Theorem 1 is not concerned with this constant at all.

If one assumes that F is in the domain of attraction of a stable law of index $0 < \alpha \le 2$, written $F \in D(\alpha)$, or F has a slowly varying upper tail, written $F \in D(0)$, then the value of $K(F)$ can be derived from known results by Wichura (1974) for the case $0 < \alpha < 2$, by Klass (1977) for the case $\alpha = 2$ and from our Theorem 1 for the case $\alpha = 0$. Lemma 4, in Section 2, provides the value of $K(F)$ when $F \in D(\alpha)$, $0 \le \alpha \le 2$, thus showing that the choice of $a_n(LLn)^{1/2}$ as our norming sequence is *correct* when $F \in D(\alpha)$ with $0 < \alpha \le 2$. For $F \in D(0)$ it is pointed out in Section 2 that one also has

$$\liminf_{n \to \infty} S_n/\big(a_n(LLn)^{1/2}\big) = 0 \quad \text{a.s.} \tag{1.5}$$

In this case our choice of the norming sequence is optimal in the sense that if we were to replace a_n by the slightly slower sequence

$$a_n(\lambda) = n^{1/2}\sigma(\lambda b_n) \quad \text{with} \quad 1 < \lambda < \infty,\ n \ge 1,$$

then

$$\lim_{n \to \infty} S_n/\big(a_n(\lambda)(LLn)^{1/2}\big) = \infty \quad \text{a.s.} \tag{1.6}$$

The proof of this is deferred to Section 2.

We remark in passing that the lim sup of the normalized sums in (1.1) is in general almost surely infinite, except when $EX^2 < \infty$. As soon as we allow X to have a nondegenerate negative part, our problem can change drastically. Our present norming and centering constants may no longer be appropriate and we then enter into the realm of the one-sided laws of the iterated logarithm of Klass (1976, 1977, 1984), Klass and Teicher (1977) and Pruitt (1981). The norming constants in our Theorem 1 are finite only when $E\{(X^-)^2\} < \infty$. [For a real number x, $x^+ = \max\{0, x\}$ and $x^- = \min\{0, x\}$.] In this case, the conclusion of our theorem remains valid. We state this as a corollary.

COROLLARY 1. *Assume that X is nondegenerate and $E\{(X^-)^2\} < \infty$. Then* (1.1) *remains true.*

2. Proofs. We begin by recording a couple of facts:

$$\limsup_{s \downarrow 0} s^{1/2} Q(1-s)/\sigma(s) \le 1. \tag{2.1}$$

This fact is contained in the proof of Lemma 2.1 of CsHM (1988b). Alternatively, when $0 < \operatorname{Var} X = \sigma^2 < \infty$, then $\sigma(s) \to \sigma$ as $s \downarrow 0$ and $s^{1/2}Q(1-s) \to 0$ as $s \downarrow 0$, implying that the lim sup in (2.1) is equal to zero, whereas, when $EX^2 = \infty$, then it is easy to show that

$$\sigma^2(s) \sim sQ^2(1-s) + \int_0^{1-s} Q^2(u)\,du \quad \text{as } s \downarrow 0,$$

from which (2.1) follows.

For $0 < s < 1$ set

$$\tau^2(s) = \int_0^{1-s} Q^2(u)\,du - \mu^2(s).$$

By considering the two cases $0 < EX^2 < \infty$ and $EX^2 = \infty$, it is readily verified that

$$\limsup_{s \downarrow 0} \tau(s)/\sigma(s) \le 1. \tag{2.2}$$

Let $U_1, U_2, \dots,$ be a sequence of independent uniform (0, 1) random variables and for each integer $n \ge 1$, let $U_{1,n} \le \cdots \le U_{n,n}$ denote the order statistics based on $U_1, \dots, U_n$. The two sequences $X_1, X_2, \dots,$ and $Q(U_1), Q(U_2), \dots,$ are equal in law. Therefore, without loss of generality we shall assume throughout that $X_n = Q(U_n)$ for all $n \ge 1$.

Proof of Theorem 1. We first establish the lower bound:

$$\liminf_{n \to \infty} \frac{S_n - n\mu(b_n)}{a_n (LLn)^{1/2}} \ge -2^{1/2} \quad \text{a.s.} \tag{2.3}$$

Set for $0 < b < 1$ and $n \ge 1$,

$$S_n(b) = \sum_{i=1}^{n} \{Q(U_i) 1(U_i \le 1-b) + Q(1-b) 1(U_i > 1-b)\}.$$

Notice that $S_n(b)$ is an increasing function of b with

$$ES_n(b) = n\mu(b) + nbQ(1-b)$$

and

$$\operatorname{Var} S_n(b) = n\sigma^2(b).$$

Since $S_n \geq S_n(b_n)$, to prove (2.3) it suffices to show that

$$\liminf_{n\to\infty} \frac{S_n(b_n) - n\mu(b_n)}{a_n(LLn)^{1/2}} \geq -2^{1/2} \quad \text{a.s.} \tag{2.4}$$

For this we require two lemmas.

LEMMA 1. *For all* $0 < b < 1$, $0 < \eta \leq 3^{1/2}$ *and* $n \geq 1$,

$$P\big(S_n(b) - n\mu(b) \leq -\eta b^{1/2} n\sigma(b)\big) \leq 2\exp\left(-\frac{\eta^2}{2} nb\right). \tag{2.5}$$

PROOF. Set $\gamma = b^{1/2}Q(1-b)/\sigma(b)$ and choose $0 < \eta \leq 3^{1/2}$. The probability in (2.5) is equal to

$$P\Big(S_n(b) - n\mu(b) - nbQ(1-b) \leq -\eta b^{1/2} n\sigma(b)\{1 + \eta^{-1}\gamma\}\Big),$$

which by Bernstein's inequality [cf. Pollard (1984), page 193] is

$$\leq 2\exp\left[-\frac{\eta^2}{2} nb\{1+\eta^{-1}\gamma\}^2 \Big/ \left(1 + \frac{\gamma}{3}\eta\{1+\eta^{-1}\gamma\}\right)\right].$$

Since $0 < \eta \leq 3^{1/2}$, this last expression is less than or equal to $2\exp(-\eta^2 nb/2)$. □

LEMMA 2. *For all* $\varepsilon > 0$ *there exists a* $\lambda(\varepsilon) > 1$ *such that for all* $1 < \lambda < \lambda(\varepsilon)$ *and all large* k *depending on* λ,

$$\max_{m_k \leq n \leq m_{k+1}} \frac{n\big\{\mu(b_n) - \mu\big(b_{m_k}\big)\big\}}{a_n(LLn)^{1/2}} < \varepsilon, \tag{2.6}$$

where $m_k = [\lambda^k]$, $k \geq 1$, *with* $[x]$ *denoting the integer part of* x.

PROOF. Notice that for any $m_k \leq n \leq m_{k+1}$,

$$0 \leq \frac{n\{\mu(b_n) - \mu(b_{m_k})\}}{a_n(LLn)^{1/2}} \leq \left(\frac{m_{k+1}}{LLm_k}\right)^{1/2} \int_{1-b_{m_k}}^{1-b_{m_{k+1}}} \frac{Q(u)\,du}{\sigma(1-u)},$$

which by (2.1) is, for all large enough k,

$$\leq 2(m_{k+1}/LLm_k)^{1/2} \int_{1-b_{m_k}}^{1-b_{m_{k+1}}} (1-u)^{-1/2}\,du,$$

which in turn converges to $4(\lambda^{1/2} - 1)$ as $k \to \infty$. Thus by choosing $\lambda(\varepsilon) > 1$ small enough, we have (2.6) for all large enough k depending on $1 < \lambda < \lambda(\varepsilon)$. □

We are now prepared to prove (2.4) and hence (2.3). Choose any $0 < \varepsilon < 1$ and $\lambda(\varepsilon) > 1$ so that (2.6) holds. We see that with $1 < \lambda < \lambda(\varepsilon)$ and $m_k = [\lambda^k]$, $k \geq 1$, for all large enough k,

$$\begin{aligned}
&P\left(\min_{m_k < n \leq m_{k+1}} \frac{S_n(b_n) - n\mu(b_n)}{a_n(LLn)^{1/2}} \leq -\big(2(1+\varepsilon)\big)^{1/2} - \varepsilon\right)\\
&\quad \leq P\left(\min_{m_k < n \leq m_{k+1}} \frac{S_n(b_{m_k}) - n\mu(b_{m_k})}{a_n(LLn)^{1/2}} \leq -\big(2(1+\varepsilon)\big)^{1/2}\right)\\
&\quad \leq P\bigg(\max_{m_k < n \leq m_{k+1}} \Big\{ -S_n(b_{m_k}) + n\mu(b_{m_k}) + nb_{m_k}Q(1-b_{m_k})\Big\}\\
&\qquad\qquad \geq \big(2(1+\varepsilon)LLm_k\big)^{1/2} a_{m_k} + m_k b_{m_k} Q(1-b_{m_k})\bigg),
\end{aligned}$$

which by Lévy's inequality [cf. Loève (1955), page 248)] is less than or equal to

$$\begin{aligned}
2P\bigg(&-S_{m_{k+1}}(b_{m_k}) + m_{k+1}\mu(b_{m_k}) + m_{k+1}b_{m_k}Q(1-b_{m_k})\\
&\geq (2m_{k+1})^{1/2}\sigma(b_{m_k})\left\{\Big[\big((1+\varepsilon)/\lambda\big)LLm_k\Big]^{1/2} - 1\right\} + m_k b_{m_k} Q(1-b_{m_k})\bigg).
\end{aligned}$$

This is easily seen to be, using (2.1), for all large k, less than or equal to

$$2P\Big(S_{m_{k+1}}(b_{m_k}) - m_{k+1}\mu(b_{m_k}) \leq -\Big\{\big((2+\varepsilon)/\lambda^2\big)^{1/2} - 2(1-\lambda^{-1})\Big\} m_{k+1} b_{m_k}^{1/2}\sigma(b_{m_k})\Big).$$

We can assume that $\lambda > 1$ is sufficiently close to 1 so that $2 < (2+\varepsilon)/\lambda^2 - 2(1-\lambda^{-1}) = \eta^2 < 3$. Therefore, by inequality (2.5) this last bound is less than or equal to

$$4\exp\left(-\frac{\eta^2}{2}LLm_k\right).$$

The Borel–Cantelli lemma and the arbitrary choice of $0 < \varepsilon < 1$ finishes the proof of (2.4).

To complete the proof of our Theorem 1, we need only show that

$$\liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} \leq 0 \quad \text{a.s.,} \tag{2.7}$$

since (2.3) and (2.7) combined with the Hewitt–Savage 0-1 law imply the existence of a constant $-2^{1/2} \leq K(F) \leq 0$ such that (1.1) holds almost surely.

Set $n_k = k^{2k}$, $k \geq 1$. To establish (2.7) it obviously suffices to prove that

$$\liminf_{k\to\infty} \frac{S_{n_k} - n_k\mu(b_{n_k})}{a_{n_k}(LLn_k)^{1/2}} \leq 0 \quad \text{a.s.} \tag{2.8}$$

Assertion (2.8) will be a consequence of the following lemma combined with a result due to Kiefer (1972).

LEMMA 3. *For all* $\varepsilon > 0$,

$$P\left(\frac{S_{n_k} - n_k\mu(b_{n_k})}{a_{n_k}(LLn_k)^{1/2}} \geq \varepsilon,\ U_{n_k,n_k} < 1 - b_{n_k} \quad i.o.\right) = 0. \tag{2.9}$$

PROOF. Obviously,

$$\frac{Q(U_{n_k,n_k})}{a_{n_k}(LLn_k)^{1/2}} 1(U_{n_k,n_k} < 1 - b_{n_k}) \leq b_{n_k}^{1/2} Q(1 - b_{n_k}) \Big/ \left(\sigma(b_{n_k}) LLn_k\right),$$

which by (2.1) converges to zero as $k \to \infty$. Note that (2.1) also implies that

$$\mu(b_{n_k}) / \left(a_{n_k}(LLn_k)^{1/2}\right) \to 0 \quad \text{as } k \to \infty.$$

Thus to prove (2.9) it is enough to show that for all $\varepsilon > 0$,

$$P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q(U_{i,n_k}) - \mu(b_{n_k})\right\}}{a_{n_k}(LLn_k)^{1/2}} \geq \varepsilon,\ U_{n_k,n_k} < 1 - b_{n_k} \quad \text{i.o.}\right) = 0. \tag{2.10}$$

Now

$$\begin{aligned} P_k &:= \left(\sum_{i=1}^{n_k-1} \frac{\left\{Q(U_{i,n_k}) - \mu(b_{n_k})\right\}}{a_{n_k}(LLn_k)^{1/2}} \geq \varepsilon,\ U_{n_k,n_k} < 1 - b_{n_k}\right) \\ &\leq F_{n_k}(1 - 2b_{n_k}) \\ &\quad + \int_{1-2b_{n_k}}^{1-b_{n_k}} P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q(U_{i,n_k}) - \mu(b_{n_k})\right\}}{a_{n_k}(LLn_k)^{1/2}} \geq \varepsilon \,\Big|\, U_{n_k,n_k} = x\right) dF_{n_k}(x) \\ &:= P_k^{(1)} + P_k^{(2)}, \end{aligned}$$

where F_{n_k} denotes the distribution function of U_{n_k,n_k}.

First note that

$$P_k^{(1)} = \left(1 - 2b_{n_k}\right)^{n_k} \leq \exp(-2LLn_k).$$

Next, since conditioned on $U_{n_k,n_k} = x$,

$$\sum_{i=1}^{n_k-1} Q(U_{i,n_k}) =_{\mathcal{D}} \sum_{i=1}^{n_k-1} Q(V_i(x)),$$

where $V_1(x), \dots, V_{n_k-1}(x)$ are independent uniform $(0, x)$ random variables, we see that

$$P_k^{(2)} = \int_{1-2b_{n_k}}^{1-b_{n_k}} P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q(V_i(x)) - \mu(b_{n_k})\right\}}{a_{n_k}(LLn_k)^{1/2}} \geq \varepsilon\right) dF_{n_k}(x),$$

which, since $\mu(b_{n_k}) \ge \mu(1-x)$ for $1-2b_{n_k} \le x \le 1-b_{n_k}$, is less than or equal to

$$\int_{1-2b_{n_k}}^{1-b_{n_k}} P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q\big(V_i(x)\big) - \mu(1-x)\right\}}{a_{n_k}(LLn_k)^{1/2}} \ge \varepsilon\right) dF_{n_k}(x).$$

Observe that for all $1-2b_{n_k} \le x \le 1-b_{n_k}$ and all large k,

$$(n_k-1)\{\mu(1-x)/x - \mu(1-x)\}/\big(a_{n_k}(LLn_k)^{1/2}\big) \le 4b_{n_k}^{1/2}\mu(b_{n_k})/\sigma(b_{n_k}).$$

Noting that the right side of this last inequality converges to zero, we obtain the bound valid for all large k:

$$P_k^{(2)} \le \int_{1-2b_{n_k}}^{1-b_{n_k}} P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q\big(V_i(x)\big) - \mu(1-x)/x\right\}}{a_{n_k}(LLn_k)^{1/2}} \ge \frac{\varepsilon}{2}\right) dF_{n_k}(x).$$

Since $EQ(V_1(x)) = \mu(1-x)/x$, we can apply Chebyshev's inequality to get, for each $1-2b_{n_k} \le x \le 1-b_{n_k}$,

$$P\left(\sum_{i=1}^{n_k-1} \frac{\left\{Q\big(V_i(x)\big) - \mu(1-x)/x\right\}}{a_{n_k}(LLn_k)^{1/2}} \ge \frac{\varepsilon}{2}\right) \le \frac{4EQ^2\big(V_1(x)\big)}{\varepsilon^2 LLn_k \sigma^2\big(b_{n_k}\big)},$$

which, by (2.2) and the fact that F is nondegenerate implies $\mu(s) = O\big(\sigma(s)\big)$ as $s \downarrow 0$, for all large k uniformly in $1-2b_{n_k} \le x \le 1-b_{n_k}$, is less than or equal to

$$8\varepsilon^{-2}\int_0^{1-b_{n_k}} Q^2(s)\,ds\Big/\Big(\sigma^2\big(b_{n_k}\big)LLn_k\Big) \le C/LLn_k$$

for some constant $0 < C < \infty$. Therefore, for all large k,

$$P_k^{(2)} \le CF_{n_k}\big(1-b_{n_k}\big)/LLn_k \le C\exp(-LLn_k)/LLn_k.$$

From our bounds on $P_k^{(1)}$ and $P_k^{(2)}$ it is clear that for all large k

$$P_k \le 2C\exp(-LLn_k)/LLn_k.$$

Since

$$\sum_{k=1}^{\infty} \exp(-LLn_k)/LLn_k < \infty,$$

we infer from the Borel–Cantelli lemma that (2.9) is true for all $\varepsilon > 0$. □

To finish the proof of (2.8) we need a special case of a result due to Kiefer (1972), namely,

(2.11) $$P\big(1 - U_{n_k,n_k} > b_{n_k} \text{ i.o.}\big) = 1.$$

Since $\{1 - U_{n,n}\}_{n\geq 1}$ is equal in law to $\{U_{1,n}\}_{n\geq 1}$, (2.11) is contained in the proof of Theorem 2 of Kiefer (1972).

By (2.11) and Lemma 3, for almost every ω there exists a subsequence $\{r_k\} \subset \{n_k\}$ such that $U_{r_k,r_k} < 1 - b_{r_k}$ for $k \geq 1$ and

$$\liminf_{k\to\infty} \frac{S_{r_k} - r_k\mu(b_{r_k})}{a_{r_k}(LLr_k)^{1/2}} \leq 0,$$

which of course implies (2.8). This completes the proof of (1.1).

The following lemma shows that all values in the interval $[-2^{1/2}, 0]$ are attainable.

LEMMA 4. *If* $F \in D(\alpha)$, $0 \leq \alpha \leq 2$, *then*

$$\liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} = K(\alpha) \quad a.s., \tag{2.12}$$

where

$$K(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0, \\ \left\{\left(\Gamma(2-\alpha)\right)^{1/\alpha} - 1\right\}\alpha(1-\alpha)^{-1}\left((2-\alpha)/2\right)^{1/2}, & \\ & \text{if } 0 < \alpha < 2,\ \alpha \neq 1, \\ -2^{1/2}, & \text{if } \alpha = 2, \\ -\kappa/2^{1/2}, & \text{if } \alpha = 1, \end{cases} \tag{2.13}$$

with κ *being the Euler constant.*

[Since the function $K(\alpha)$ is continuous and strictly decreasing on $[0, 2]$, taking all values from 0 to $-2^{1/2}$, Lemma 4 completes the proof of our Theorem 1.]

PROOF. First assume that $F \in D(0)$. Now $1 - F(x)$ is slowly varying at infinity if and only if $Q(1-s)$ is rapidly varying at zero, that is, for all $\lambda > 1$,

$$Q(1-s)/Q(1-s\lambda) \to \infty \quad \text{as } s \downarrow 0$$

[cf. Corollary 1.2.1.5 in de Haan (1975)]. This allows us to apply Theorem 1.3.2 in de Haan (1975) to obtain

$$\int_0^{1-s} Q(u)\,du = o\big(sQ(1-s)\big) \quad \text{as } s \downarrow 0 \tag{2.14}$$

and

$$\int_0^{1-s} Q^2(u)\,du = o\big(sQ^2(1-s)\big) \quad \text{as } s \downarrow 0. \tag{2.15}$$

Limit relations (2.14) and (2.15) together imply that

$$\sigma^2(s) \sim sQ^2(1-s) \quad \text{as } s \downarrow 0. \tag{2.16}$$

From (2.14) and (2.16) we get

$$n\mu(b_n)/\left(a_n(LLn)^{1/2}\right) \to 0 \quad \text{as } n \to \infty,$$

which in combination with (1.1) yields

$$\liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} = \liminf_{n\to\infty} S_n/\left(a_n(LLn)^{1/2}\right) \le 0 \quad \text{a.s.} \tag{2.17}$$

This obviously says that the lim inf's in (2.17) are equal to zero almost surely.

Next assume that $F \in D(2)$. It can be inferred from Corollary 1 of CsHM (1988a) that when $F(0-) = 0$, $F \in D(2)$ if and only if

$$s^{1/2}Q(1 - s\lambda)/\sigma(s) \to 0 \quad \text{as } s \downarrow 0 \text{ for all } \quad 0 < \lambda < \infty \tag{2.18}$$

if and only if

$$\sigma(s) \text{ is slowly varying at zero.} \tag{2.19}$$

From (2.18) and (2.19) it is simple to argue that

$$n\int_{1-b_n}^{1} Q(u)\,du/\left(a_n(LLn)^{1/2}\right) \to 0 \quad \text{as } n \to \infty, \tag{2.20}$$

which in turn implies that

$$\liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} = \liminf_{n\to\infty} \frac{S_n - n\mu}{a_n(LLn)^{1/2}}, \tag{2.21}$$

where $\mu := \mu(0+)$ is the mean of X. By our Theorem 1, the lim inf's in (2.21) are greater than or equal to $-2^{1/2}$ almost surely. Next, an application of (2.6) of Theorem 1 of Kuelbs (1985) shows that

$$\liminf_{n\to\infty} \frac{S_n - n\mu}{a_n(LLn)^{1/2}} \le -2^{1/2} \quad \text{a.s.}$$

So putting everything together we get

$$\liminf_{n\to\infty} \frac{S_n - n\mu}{a_n(LLn)^{1/2}} = \liminf_{n\to\infty} \frac{S_n - n\mu(b_n)}{a_n(LLn)^{1/2}} = -2^{1/2} \quad \text{a.s.} \tag{2.22}$$

Alternatively, we could have applied statement (2.15) after Theorem 2.6 of Klass (1977) to derive (2.22) from (2.21).

Finally, we consider the case when $F \in D(\alpha)$, $0 < \alpha < 2$. The appropriate lim inf statement in this case can be derived, after some lengthy but routine analysis using standard properties of regularly varying functions, from the work of Wichura (1974). For the sake of brevity we omit these details. This completes both the proof of Lemma 4 and of our Theorem 1. □

PROOF OF ASSERTION (1.6). Applying Theorem 6 of Kiefer (1972) [also refer to page 134 of Deheuvels (1986)], we have for any choice of $1 < \lambda' < \lambda < \infty$,

$$\limsup_{n \to \infty} n\big(1 - U_{n-k_n, n}\big)/LLn < \lambda' \quad \text{a.s.,} \tag{2.23}$$

where $k_n = [cLLn]$ and $c > 0$ is suitably small. Now for any such $1 < \lambda' < \lambda < \infty$, obviously by (2.16) and (2.23), we get

$$\liminf_{n \to \infty} S_n/\big(a_n(\lambda)(LLn)^{1/2}\big) \geq c\lambda^{-1/2} \liminf_{n \to \infty} Q(1 - \lambda' b_n)/Q(1 - \lambda b_n) \quad \text{a.s.}$$

Since $Q(1-s)$ is rapidly varying at zero, this says that (1.6) holds for all $\lambda > 1$. □

PROOF OF COROLLARY 1. First assume that $E\{(X^+)^2\} < \infty$. In this case, (1.1) is an easy consequence of the usual law of the iterated logarithm because $\sigma(s)^2 \to \sigma^2 = \operatorname{Var} X < \infty$ as $s \downarrow 0$ and it is readily verified that $n\{\mu - \mu(b_n)\} = o(n^{1/2})$ as $n \to \infty$.

Now assume that $E\{(X^+)^2\} = \infty$. Set $Q^+ = \max\{Q, 0\}$ and $Q^- = \min\{Q, 0\}$. Further, let $\mu_+(s)$, $\mu_-(s)$, $\sigma_+(s)$ and $\sigma_-(s)$ be defined as $\mu(s)$ and $\sigma(s)$ with Q^+ and Q^- replacing Q, in the respective formula. Let, for each $n \geq 1$,

$$S_n^+ = Q^+(U_1) + \cdots + Q^+(U_n) \quad \text{and} \quad S_n^- = Q^-(U_1) + \cdots + Q^-(U_n).$$

Since we are also assuming that $E\{(X^-)^2\} < \infty$, it is straightforward to show that $\sigma_+(b_n)/\sigma(b_n) \to 1$, $\sigma_-(b_n)/\sigma(b_n) \to 0$ and $n\{E(X^-) - \mu_-(b_n)\} = o(n^{1/2})$ as $n \to \infty$. Thus by the usual law of the iterated logarithm,

$$\lim_{n \to \infty} \frac{S_n^- - n\mu_-(b_n)}{a_n(LLn)^{1/2}} = 0 \quad \text{a.s.}$$

Applying our Theorem 1, we have

$$\liminf_{n \to \infty} \frac{S_n^+ - n\mu_+(b_n)}{a_n(LLn)^{1/2}} = K \quad \text{a.s.,}$$

where $-2^{1/2} \leq K \leq 0$. Noting that $\mu(s) = \mu_+(s) + \mu_-(s)$, we see that the proof is complete. □

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