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RADIAL PART OF BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD

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Let ρ_t be the radial part of a Brownian motion in an n-dimensional Riemannian manifold M starting at x and let $T = T_\varepsilon$ be the first time t when $\rho_t = \varepsilon$. We show that $E[\rho_{t\wedge T}^2] = nt - (1/6)S(x)t^2 + o(t^2)$, as $t \downarrow 0$, where $S(x)$ is the scalar curvature. The same formula holds for $E[\rho_t^2]$ under some boundedness condition on M.

1. Introduction and main results. Suppose that M is an n-dimensional Riemannian manifold with $d(\cdot,\cdot)$ as the distance function, X_t is a Brownian motion on M starting from x and $\rho_t = d(x, X_t)$ is its radial part. See, for example, [4] for the definition of Riemannian Brownian motion. If M is a Euclidean space, it is well known that $E[\rho_t^2] = nt$. On a general M, we would like to have an expansion of $E[\rho_t^2]$ in the form $nt + ct^2 + o(t^2)$, where the constant c involves the curvature. However, without any boundedness assumption on the global geometry of M, $E[\rho_t^2]$ may not even be finite (It is known that if the curvature is not bounded from below, Brownian motion may have a finite explosion time [3].) Therefore, if one is only concerned with the local properties, it is reasonable to look at $E[\rho_{t\wedge T}^2]$, where $T = T_\varepsilon$ is the first time that X_t wanders ε distance from x and ε is a fixed small positive number. We will obtain

$$E[\rho_{t\wedge T_\varepsilon}^2] = nt - \tfrac{1}{6}S(x)t^2 + o(t^2) \qquad (t \downarrow 0), \tag{1}$$

where $S(x)$ is the scalar curvature at x and $o(t^2)$ may depend on ε. Since the coefficient of t^2 is $-(1/6)S(x)$, the formula illustrates the now well known phenomenon that Brownian motion dissipates faster on a negatively curved manifold than it does on a positively curved one.

Under some globlal boundedness condition, it is possible to prove

$$E[\rho_t^2] = nt - \tfrac{1}{6}S(x)t^2 + o(t^2), \tag{2}$$

where ε is not involved. We will show that (2) holds under one of the following three conditions:

1. M is compact.
2. M is a complete Riemannian manifold with nonnegative Ricci curvature.
3. The Ricci curvature of M is bounded from below and the exponential map $\exp_x$ at x is a diffeomorphism from T_xM onto M.

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In fact, (2) holds under more general condition. For example, a certain growth condition in [6], which is more general than the above curvature condition, implies (2), as will be noted.

Our result supplements Gray and Pinsky's expansion [2] of mean exit time of Brownian motion from a ball. We also hope that this work will lead us to discuss the possible definition of curvatures (in the sense of Schwarz distribution) on a nonsmooth Riemannian manifold [6].

2. A geometric lemma. Let Δ be the Laplacian on M and let $\rho(y) = d(y, x)$. The following purely geometric lemma does not seem to be well known.

LEMMA 1. *Let $S(x)$ be the scalar curvature at x. Then*

$$\Delta^2 \rho(x) = -\tfrac{4}{3} S(x).$$

PROOF. Let $y_1, y_2, \ldots, y_n$ be normal coordinates about the point x which is identified with $0 = (0, \ldots, 0)$. Let Γ^i_{jk} be the Christoffel symbols. We have $\Gamma^i_{jk}(y) y_j y_k = 0$ for $1 \le i \le n$. Here we used the convention to sum over repeated indices. Note that this holds for any y near 0 because $t \to ty$ is a geodesic.

Differentiate this expression three times and then set $y = 0$ to obtain

$$\partial_j \Gamma^i_{kl}(0) + \partial_k \Gamma^i_{jl}(0) + \partial_l \Gamma^i_{jk}(0) = 0, \qquad 1 \le i, j, k, l \le n. \tag{3}$$

It is well known that $\Gamma^i_{jk}(0) = 0$ and $\partial_l g_{jk}(0) = 0$, where g_{jk} is the metric tensor and g^{jk} is its inverse.

The curvature tensor R^i_{jkl} is given by

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^p_{lj} \Gamma^i_{kp} - \Gamma^p_{kj} \Gamma^i_{lp}.$$

It follows that

$$R^i_{jkl}(0) = \partial_k \Gamma^i_{lj}(0) - \partial_l \Gamma^i_{kj}(0), \tag{4}$$

$$R^i_{kjl}(0) = \partial_j \Gamma^i_{lk}(0) - \partial_l \Gamma^i_{kj}(0). \tag{5}$$

From (4) + (5) − (3), we obtain

$$R^i_{jkl}(0) + R^i_{kjl}(0) = -3\, \partial_l \Gamma^i_{jk}(0).$$

Hence,

$$\partial_l \Gamma^i_{jk}(0) = -(1/3)[R^i_{jkl}(0) + R^i_{kjl}(0)]. \tag{6}$$

Since $\partial_i g_{jk} = \Gamma^p_{ij} g_{pk} + \Gamma^p_{ik} g_{pj}$, it follows that

$$\partial_l\, \partial_i\, g_{jk}(0) = -(1/3)[R^k_{ijl}(0) + R^k_{jil}(0) + R^j_{ikl}(0) + R^j_{kil}(0)]. \tag{7}$$

The scalar curvature is $S = R^p_{qpq}$. Since $R^i_{jkl} = -R^i_{jlk}$ and $R^i_{jkl} = -R^j_{ikl}$, we have

$$R^i_{jll} = 0 \quad \text{and} \quad R^j_{llj} = -S.$$

A direct computation using (7) yields

$$\partial_j \, \partial_l g_{jl}(0) = (1/3)S,$$

$$\Delta \rho^2 = g^{jk}\partial_j \, \partial_k \rho^2 - g^{jk}\Gamma^i_{jk} \, \partial_i \rho^2 = 2g^{jk}\delta_{jk} - 2g^{jk}\Gamma^i_{jk} y_i,$$

$$\Delta^2 \rho^2(0) = 2\partial_q \, \partial_q g^{pp}(0) - 4 \, \partial_i (g^{jk}\Gamma^i_{jk})(0) = 2\partial_q \, \partial_q g^{pp}(0) - 4 \, \partial_q \Gamma^q_{pp}(0).$$

One can show that $\partial_q \, \partial_q g^{ij}(0) = -\partial_q \partial_q g_{ij}(0)$ (by differentiating $g^{ip} g_{pj} = \delta_{ij}$ twice) and

$$\partial_q \Gamma^q_{pp}(0) = \partial_p \, \partial_q g_{pq}(0) - (1/2)\partial_q \, \partial_q g_{pp}(0).$$

So, $\Delta^2 \rho^2(0) = -4\partial_p \, \partial_q g_{pq}(0) = -(4/3)S$. □

3. Proof of (1). Now let us apply Itô's formula to $\rho_t = \rho(X_t, x)$:

$$\begin{aligned} \rho^2(X_{t\wedge T_\varepsilon}, x) &= M_{t\wedge T_\varepsilon} + \tfrac{1}{2} \int_0^{t\wedge T_\varepsilon} \Delta \rho^2(X_s, x) \, ds \\ &= M_{t\wedge T_\varepsilon} + \tfrac{1}{2} \int_0^t I_{[s<T_\varepsilon]} \Delta \rho^2(X_{s\wedge T_\varepsilon}, x) \, ds, \end{aligned}$$

where M_t is a martingale. So

(8) $$E[\rho^2(X_{t\wedge T_\varepsilon}, x)] = \tfrac{1}{2} E\left[\int_0^t I_{[s<T_\varepsilon]} \Delta \rho^2(X_{s\wedge T_\varepsilon}, x) \, ds\right].$$

By Lemma 1,

$$\Delta^2 \rho^2(X_s, x) = -\tfrac{4}{3} S(x) + O(\rho(X_s, x)).$$

Also by Itô's formula,

(9) $$E[\Delta \rho^2(X_{s\wedge T_\varepsilon}, x)] = 2n + \tfrac{1}{2} E\left\{\int_0^{s\wedge T_\varepsilon} [-\tfrac{4}{3} S(x) + O(\rho(X_u, x))] \, du\right\}.$$

We can write

$$I_{[s<T_\varepsilon]} \Delta \rho^2(X_{s\wedge T_\varepsilon}, x) = \Delta \rho^2(X_{s\wedge T_\varepsilon}, x) - I_{[s\geq T_\varepsilon]} \Delta \rho^2(X_{T_\varepsilon}, x).$$

So from (8) and (9),

(10) $$\begin{aligned} &E[\rho^2(X_{t\wedge T_\varepsilon}, x)] \\ &\quad = \tfrac{1}{2} \left\{\int_0^t E[\Delta \rho^2(X_{s\wedge T_\varepsilon}, x)] \, ds - E((t - t\wedge T_\varepsilon) \Delta \rho^2(X_{T_\varepsilon}, x))\right\} \\ &\quad = nt - \tfrac{1}{3} S(x) E\left[\int_0^t (s \wedge T_\varepsilon) \, ds\right] + R_1, \end{aligned}$$

where the remainder R_1 is given by

$$R_1 = \int_0^t \int_0^s E[O(\rho(X_u, x));\ u < T_\varepsilon]\, du\, ds - \tfrac{1}{2} E[(t - t \wedge T_\varepsilon)\Delta\rho^2(X_{T_\varepsilon}, x)].$$

Since $E[O(\rho(X_u, x));\ u < T_\varepsilon] \to 0$ as $u \to 0$, the first term of R_1 is $o(t^2)$. Let us estimate the second term of R_1. From Stroock [5] or [6], there is a constant $C > 0$ such that when ε and t are small enough,

$$\text{(11)} \qquad P[T_\varepsilon < t] = P[\sup_{s \le t} \rho(X_s, x) \ge \varepsilon] \le C \exp\left\{-\frac{\varepsilon^2}{Ct}\right\},$$

which is $o(t^2)$ for fixed ε. Since

$$E[t - t \wedge T_\varepsilon] \le tP(T_\varepsilon < t) = o(t^2) \qquad (t \downarrow 0),$$

we have $|R_1| = o(t^2)$. Moreover,

$$\begin{aligned} E\left[\int_0^t (s \wedge T_\varepsilon)\, ds\right] &= (1/2)t^2 P(t < T_\varepsilon) + E[tT_\varepsilon - (1/2)T_\varepsilon^2;\ T_\varepsilon < t] \\ &= (1/2)t^2 + E[tT_\varepsilon^2 - (1/2)(T_\varepsilon^2 + t^2);\ T_\varepsilon < t] \\ &= (1/2)t^2 + o(t^2) \qquad (t \downarrow 0). \end{aligned}$$

Combining with (10), we have proved (1). □

4. Proof of (2). Let us first assume that M is compact. Then ρ^2 is bounded. We have

$$\begin{aligned} E[\rho^2(X_t, x)] &= E[\rho^2(X_{t\wedge T_\varepsilon}, x);\ t \le T_\varepsilon] + E[\rho^2(X_t, x);\ T_\varepsilon < t] \\ &= E[\rho^2(X_{t\wedge T_\varepsilon}, x)] + E[-\rho^2(X_{t\wedge T_\varepsilon}, x) + \rho^2(X_t, x);\ T_\varepsilon < t] \\ &= E[\rho^2(X_{t\wedge T_\varepsilon}, x)] + o(t^2) \qquad (t \downarrow 0). \end{aligned}$$

We obtain (2) from (1).

Next we assume that M is a complete Riemannian manifold with nonnegative Ricci curvature. Then it is well known (e.g., see page 173 in [1]) that the heat kernel $p_t(x, y)$ on M satisfies the estimates

$$\text{(12)} \quad ct^{-n/2} \exp(-\rho(x, y)^2/ct) \le p_t(x, y) \le C|B(x, t^{1/2})|^{-1} \exp(-\rho(x, y)^2/Ct),$$

where $t > 0$, $x, y \in M$ are arbitrary, c and C are some fixed positive constants and $|B(x, \delta)|$ is the volume of a ball of radius δ centered at x. For small t, $|B(x, t^{1/2})| \approx c_1 t^{n/2}$ for some constant c_1.

Using the above heat kernel estimates, one can show (see Lemma II.1.2 in [5]) that the inequality (11) holds for all $\varepsilon > 0$ and sufficiently small $t > 0$.

Let $(\rho^2)^*_t = \sup_{s\le t} \rho^2(X_s, x)$. We have from (11)

$$\begin{aligned} E[(\rho^2)^*_t;\, T_\varepsilon < t] &= \int_0^\infty P((\rho^2)^*_t > u,\, T_\varepsilon < t)\, du \\ &= \int_0^\infty P(T_u < t, T_\varepsilon < t)\, du = \int_0^\infty P(T_{u\vee\varepsilon} < t)\, du \\ &\le \int_0^\infty C \exp\left\{-\frac{(u\vee\varepsilon)^2}{Ct}\right\} du. \end{aligned}$$

The above expression is essentially $C\exp(-\varepsilon^2/Ct)$. The computation for the compact case can be carried over to prove (2).

Now we assume that the Ricci curvature of M is bounded from below, say by $-(n-1)c$, and the exponential map $\exp_x$ is a diffeomorphism from $T_x M$ onto M. By Theorem 5.7.2 in [1], the heat kernel on the n-dimensional hyperbolic space M_c of constant negative curvature $-c$ satisfies (12); therefore, (2) holds on M_c by the above proof. By Theorem 5.1, Chapter 6 in [4], $\rho_t \le \rho^c_t$ holds on a suitable probability space, where ρ^c_t is the radial component of the Bownian motion in M_c. It follows immediately that (2) holds also on M. □

Finally, we note that by a result in [6], if the volume growth on M is controlled by the exponential of the square distance (Grigor'yan–Karp–Li condition) and if the bounded geometry radius is bounded below by the negative exponential of the square distance, then the inequality (11) holds for all $\varepsilon > 0$ and small $t > 0$ (see Theorem 3 in [6] for details), which implies (2) as our proof has shown.

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