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SUBDIFFUSIVE FLUCTUATIONS FOR INTERNAL DIFFUSION LIMITED AGGREGATION[1]

BY GREGORY F. LAWLER

Duke University

Internal diffusion limited aggregation (internal DLA) is a cluster model in $\mathbb{Z}^d$ where new points are added by starting random walkers at the origin and letting them run until they have found a new point to add to the cluster. It has been shown that the limiting shape of internal DLA clusters is spherical. Here we show that for $d \geq 2$ the fluctuations are subdiffusive; in fact, that they are of order at most $n^{1/3}$, at least up to logarithmic corrections. More precisely, we show that for all sufficiently large n the cluster after $m = [\omega_d n^d]$ steps covers all points in the ball of radius $n - n^{1/3}(\ln\ n)^2$ and is contained in the ball of radius $n + n^{1/3}(\ln\ n)^4$.

1. Introduction. Internal diffusion limited aggregation (internal DLA) is a cluster growth model in $\mathbb{Z}^d$ where particles are produced one at a time at the origin and allowed to do random walk until they reach a new point that is then added to the cluster. To be precise, let $S^1(t), S^2(t), \dots$ be independent simple random walks starting at the origin with integer time t. We define $\sigma^1 = 0, A(1) = \{0\}$ and for $j > 1$,

$$\sigma^j = \inf\{t \geq 0\colon\ S^j(t) \notin A(j-1)\},$$

$$A(j) = A(j-1) \cup S^j(\sigma^j).$$

This is a particular case of a model discussed by Diaconis and Fulton [1]. It is closely related to a model called diffusion limited erosion, where random walkers are sent from infinity and erase the first cluster point that they reach. Lawler, Bramson and Griffeath [5] recently showed that the clusters formed by internal DLA have the limiting shape of a sphere. To state the result precisely it is useful to define the inner and outer errors $\delta_I(n)$ and $\delta_O(n)$ by

$$n - \delta_I(n) = \inf\{|z|\colon\ z \notin A([\omega_d n^d])\},$$

$$n + \delta_O(n) = \sup\{|z|\colon\ z \in A([\omega_d n^d])\}.$$

Here ω_d is the volume of the unit ball in $\mathbb{R}^d$ so that $[\omega_d n^d]$ represents the approximate number of lattice points in the ball of radius n. The main result in [5] is that with probability 1,

$$n^{-1}\delta_I(n) \to 0, \qquad n^{-1}\delta_O(n) \to 0.$$

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The proof gave no estimate of the rate of convergence. There has been some nonrigorous work (see, e.g., [2, 3]) that suggests that the error rate should be subdiffusive (of smaller order than $n^{1/2}$). In this paper we will prove that the error is subdiffusive by showing that it is of order at most $n^{1/3}$ at least up to logarithmic corrections. Our theorem is as follows.

THEOREM 1. *If $d \geq 2$, with probability* 1,

$$\lim_{n\to\infty} n^{-1/3}(\ln\ n)^{-2}\delta_I(n) = 0, \tag{1}$$

$$\lim_{n\to\infty} n^{-1/3}(\ln\ n)^{-4}\delta_O(n) = 0. \tag{2}$$

The powers of the logarithm in the theorem are not the best; in fact, one can check in the proof presented that slightly smaller powers will work. However, the methods in this paper are only strong enough to show that $\delta_I(n)$ and $\delta_O(n)$ are $o(n^{1/3}(\ln\ n)^{\alpha})$ for some values of α. A more interesting question, which we are currently unable to answer, is whether or not the errors are $o(n^{\alpha})$ for some $\alpha < 1/3$.

For the remainder of this paper, we assume $d \geq 2$. In Section 2 we prove (1), the result for the inner error. The general idea of the proof is the same as in [5]. We relate the probability that a point $z \in B_n$ is not contained in $A([\omega_d n^d])$ to the probability that a certain random variable (depending on z), $M - \tilde{L}$, equals 0. M and $\tilde{L}$ are far from independent, but we give very sharp estimates for their expectations. If we let $\delta = n - |z|$ we show that

$$\mathbb{E}(M) \approx n\delta, \qquad \mathbb{E}(\tilde{L}) \approx n\delta, \qquad \mathbb{E}(M - \tilde{L}) \approx \delta^2.$$

With high probability one expects that M will differ from $\mathbb{E}(M)$ by an amount of order $\mathbb{E}(M)^{1/2}$ and similarly for $\tilde{L}$. If δ is such that

$$\mathbb{E}(M - \tilde{L}) \gg \mathbb{E}(M)^{1/2} + \mathbb{E}(\tilde{L})^{1/2},$$

then we would expect that $M - \tilde{L} > 0$. This is equivalent to

$$\delta^2 \gg n^{1/2}\delta^{1/2}.$$

Here we see where $\delta = n^{1/3}$ comes in. By throwing in a logarithmic factor, we can use large deviation estimates to make this argument rigorous.

The outer error is analyzed in Section 3 using a somewhat different argument than that in [5]. By the estimate of the inner error, in $[\omega_d n^d]$ steps we expect at most $O(n^{d-(2/3)}(\ln\ n)^2)$ of the walkers to reach the sphere of radius n before adding onto the cluster. This alone is not sufficient to show that they all add on soon. It is necessary to show that the walkers that do reach the sphere of radius n are spread out relatively evenly around the sphere. An estimate of this type (Corollary 10) is made is this section. With this estimate, the remainder of the proof can be made by a fairly straightforward estimate of the expected number of walkers that get to a certain distance without adding to the cluster.

It is convenient for the proof to define the random walks $S^1, S^2, \dots$ in terms of two other collections of random walks. Suppose we have a collection of independent simple random walks on $\mathbb{Z}^d$,

$$\{X^i(t)\colon\ i = 1, 2, \dots;\ Y^x(t),\ x \in \mathbb{Z}^d\},$$

with $X^i(0) = 0$ for all i and $Y^x(0) = x$. Define as before $\sigma^1 = 0$, $A(1) = \{0\}$ and for $j > 1$,

$$\sigma^j = \inf\{t \geq 0\colon\ X^j(t) \notin A(j-1)\},$$

$$A(j) = A(j-1) \cup \{X^j(\sigma^j)\}.$$

Let

$$S^j(t) = \begin{cases} X^j(t), & t \leq \sigma^j, \\ Y^{X^j(\sigma^j)}(t - \sigma^j), & t \geq \sigma^j. \end{cases}$$

Then it is easy to verify that $S^1, S^2, \dots$ are independent simple random walks and as before,

$$\sigma^j = \inf\{t \geq 0\colon\ S^j \notin A(j-1)\}.$$

We will write B_n for the discrete ball of radius n,

$$B_n = \{z \in \mathbb{Z}^d\colon\ |z| < n\}$$

and

$$\partial B_n = \{z \in \mathbb{Z}^d \setminus B_n\colon\ |z - y| = 1 \text{ for some } y \in B_n\}.$$

If $S(t)$ is a simple random walk, we write

$$\xi_n = \inf\{t > 0\colon\ S(t) \in \partial B_n\}.$$

Similarly, we write

$$\xi_n^j = \inf\{t > 0\colon\ S^j(t) \in \partial B_n\},$$

$$\xi_n^x = \inf\{t > 0\colon\ Y^x(t) \in \partial B_n\}.$$

If $z \in \mathbb{Z}^d$ or $A \subset \mathbb{Z}^d$, we let

$$\tau_z = \inf\{t > 0\colon\ S(t) = z\}, \qquad \tau_A = \inf\{t > 0\colon\ S(t) \in A\},$$

and similarly we define $\tau_z^j, \tau_z^x, \tau_A^j, \tau_A^x$. We will introduce one other notation. We say that a sequence of numbers γ_n converges to zero quickly, written

$$\gamma_n \xrightarrow{q} 0,$$

if it goes to zero faster than any power of n, that is, if for all $\beta < \infty$,

$$\lim_{n \to \infty} n^\beta \gamma_n = 0.$$

One example of such a sequence is $\gamma_n = \exp\{-c(\ln\ n)^\alpha\}$ for any $c > 0$, $\alpha > 1$. Note that if $\gamma_n \xrightarrow{q} 0$ and p is any polynomial, then $p(n)\gamma_n \xrightarrow{q} 0$. We will also use the convention that c, c_1, c_2 will denote positive constants that depend only on the dimension d; however, the exact value of the constants may change from line to line.

2. Inner error. In this section we prove (1). Let

$$D(n) = \{z \in \mathbb{Z}^d\colon\ n - 2n^{1/3}(\ln\ n)^2 \leq |z| \leq n - n^{1/3}(\ln\ n)^2\}.$$

We will prove that there exists $\gamma_n \xrightarrow{q} 0$ such that

$$P\{D(n) \subset A([\omega_d n^d])\} \geq 1 - \gamma_n. \tag{3}$$

It then follows immediately from the Borel–Cantelli lemma that with probability 1, for all n sufficiently large,

$$D(n) \subset A([\omega_d n^d]).$$

Because all but a finite number of x are in $D(n)$ for some $n = n(x)$, this will prove (1). To prove (3) it clearly suffices to show that there exists $\gamma_n \xrightarrow{q} 0$ such that for all $z \in D(n)$,

$$P\{z \notin A([\omega_d n^d])\} \leq \gamma_n. \tag{4}$$

This is the estimate we will prove.

For any n, m and $z \in B_n$, define

$$\begin{aligned} N = N(n,m,z) &= \text{\# of first } m \text{ walks that first visit } z \text{ before either} \\ &\qquad \text{adding to the cluster or leaving } B_n \\ &= \sum_{j=1}^{m} I\{\tau_z^j < \sigma^j \wedge \xi_n^j\}, \\ M = M(n,m,z) &= \text{\# of first } m \text{ walks that visit } z \text{ before leaving } B_n \\ &= \sum_{j=1}^{m} I\{\tau_z^j < \xi_n^j\}, \\ L = L(n,m,z) &= \text{\# of first } m \text{ walks that visit } z \text{ some time} \\ &\qquad \text{between } \sigma^j \text{ and } \xi_n^j \\ &= \sum_{j=1}^{m} I\{\sigma^j \leq \tau_z^j < \xi_n^j\}. \end{aligned}$$

Note that $N \geq M - L$ and

$$\{z \in A(m)\} \supset \{N > 0\}.$$

For any $z \in B_n$, let

$$\tilde{L} = \tilde{L}(n, z) = \sum_{y \in B_n} I\{\tau_z^y < \xi_n^y\}.$$

Note that for any n, m and $z \in B_n$,

$$\tilde{L}(n, z) \geq L(n, m, z).$$

Therefore, for any $a > 0$, $z \in B_n$,

$$P\{z \notin A(m)\} \leq P\{M - L = 0\} \leq P\{M(n, m, z) \leq a\} + P\{\tilde{L}(n, z) \geq a\}. \tag{5}$$

For the remainder of this section, we let $m = m(n) = [\omega_d n^d]$. We will estimate $\mathbb{E}(M(n, m, z))$ and $\mathbb{E}(\tilde{L}(n, z))$. Note that for $z \in B_n$,

$$\mathbb{E}(M(n, m, z)) = [\omega_d n^d] P\{\tau_z < \xi_n\},$$

$$\mathbb{E}(\tilde{L}(n, z)) = \sum_{y \in B_n} P^y\{\tau_z < \xi_n\}.$$

[Here we write P^x to indicate probabilities assuming $S(0) = x$ with the understanding that $S(0) = 0$ if no x is written.] Let G_n denote the Green's function for the ball B_n:

$$G_n(x, y) = \mathbb{E}^x \sum_{t=0}^{\xi_n - 1} I\{S(t) = y\}, \qquad x, y \in B_n.$$

It is standard that for $x, y \in B_n$,

$$G_n(x, y) = P^x\{\tau_y < \xi_n\} G_n(y, y).$$

For $d \geq 3$, we let G be the unrestricted Green's function,

$$G(x, y) = \mathbb{E}^x \sum_{t=0}^{\infty} I\{S(t) = y\},$$

and for $d = 2$, we let a be the potential kernel

$$a(x, y) = \lim_{s \to \infty} \sum_{t=0}^{s} [P^x\{S(t) = x\} - P^x\{S(t) = y\}].$$

Note that G_n, G and a are all symmetric functions. We write $G(x) = G(0, x)$ and $a(x) = a(0, x)$. For $x, y \in B_n$, the unrestricted Green's function and the potential kernel are related to G_n by the formulas (see [4], Propositions 1.5.8 and 1.6.3)

$$G_n(x, y) = G(x, y) - \mathbb{E}^x[G(S(\xi_n), y)], \quad d \geq 3, \tag{6}$$

$$G_n(x, y) = \mathbb{E}^x[a(S(\xi_n), y)] - a(x, y), \quad d = 2. \tag{7}$$

For any $x \in B_n$, let

$$b(x) = b(x, n) = \mathbb{E}^x[|S(\xi_n)| - n].$$

Note that $0 \le b(x) \le 1$. In the lemmas that follow all $o(\cdot)$ and $O(\cdot)$ terms will represent error terms as n goes to infinity that are uniformly bounded in z.

LEMMA 2. *For $z \in D(n)$, $|z| = n - \delta$,*

$$\omega_d n^d G_n(0,z) = 2n\delta + 2b(z)n + (d-1)\delta^2 + o(n^{1/8}).$$

PROOF. We first assume $d \ge 3$. By (6),

$$G_n(0,z) = G_n(z,0) = G(z) - \mathbb{E}^z[G(S(\xi_n))].$$

For $|x| \ge n/2$, it follows from [4], Theorem 1.5.4, that

$$G(x) = \frac{2}{(d-2)\omega_d}|x|^{2-d} + o(n^{-d+(1/8)}).$$

Therefore, if $\varepsilon = d - 1/8$,

$$\begin{aligned}\omega_d \mathbb{E}^z[G(S(\xi_n))] &= \frac{2}{d-2}\mathbb{E}^z[|S(\xi_n)|^{2-d}] + o(n^{-\varepsilon})\\ &= \frac{2}{d-2}\mathbb{E}^z[n^{2-d}(1 + (2-d)n^{-1}(|S(\xi_n)| - n) + O(n^{-2}))]\\ &\quad + o(n^{-\varepsilon})\\ &= \frac{2}{d-2}n^{2-d} - 2b(z)n^{1-d} + o(n^{-\varepsilon})\end{aligned}$$

and

$$\begin{aligned}\omega_d G(z) &= \frac{2}{d-2}(n-\delta)^{2-d} + o(n^{-\varepsilon})\\ &= \frac{2}{d-2}n^{2-d} + 2n^{1-d}\delta + (d-1)n^{-d}\delta^2 + o(n^{-\varepsilon}).\end{aligned}$$

By subtracting, we get the result.

Similarly, if $d = 2$, we have by (7),

$$G_n(0,z) = G_n(z,0) = \mathbb{E}^z[a(S(\xi_n))] - a(z).$$

There exists a constant C ([4], Theorem 1.6.2) such that for $|x| \ge n/2$, $\varepsilon = 15/8$,

$$a(x) = \frac{2}{\pi}\ln\,|x| + \frac{2C}{\pi} + o(n^{-\varepsilon}).$$

Hence (because $\omega_2 = \pi$),

$$\begin{aligned}\omega_2 \mathbb{E}^z[a(S(\xi_n))] &= 2\mathbb{E}^z(\ln\,|S(\xi_n)| + C) + o(n^{-\varepsilon})\\ &= 2\mathbb{E}^z[\ln\, n + C + n^{-1}(|S(\xi_n)| - n) + O(n^{-2})] + o(n^{-\varepsilon})\\ &= 2\,\ln\, n + 2C + 2b(z)n^{-1} + o(n^{-\varepsilon}),\end{aligned}$$

and

$$\begin{aligned}\omega_2 a(z) &= 2\ln\,(n-\delta)+2C+o(n^{-\varepsilon})\\ &= 2\ \ln\ n+2C-2n^{-1}\delta-n^{-2}\delta^2+o(n^{-\varepsilon}).\end{aligned}$$

Again the result follows from subtracting the two terms. □

It follows immediately from Lemma 2 that for $z \in D(n)$,

$$G_n(z,z)\mathbb{E}(M) = 2n\delta + 2b(z)n + (d-1)\delta^2 + o(n^{1/8}),$$

where $\delta = n - |z|$ and $M = M(n,m,z)$. In order to estimate the expectation of $\tilde{L} = \tilde{L}(n,z)$, note that

$$G_n(z,z)\mathbb{E}(\tilde{L}) = \sum_{y\in B_n} G_n(y,z) = \sum_{y\in B_n} G_n(z,y) = \mathbb{E}^z(\xi_n).$$

LEMMA 3. *For $z \in D(n)$, $|z| = n - \delta$,*

$$G_n(z,z)\mathbb{E}(\tilde{L}) = 2n\delta + 2b(z)n - \delta^2 + O(1).$$

PROOF. By a simple martingale argument (see [4], (1.21)),

$$\mathbb{E}^z(\xi_n) = \mathbb{E}^z(|S(\xi_n)|^2) - |z|^2.$$

Note that

$$\begin{aligned}\mathbb{E}^z[|S(\xi_n)|^2] &= \mathbb{E}^z[n^2 + 2n(|S(\xi_n)| - n) + O(1)]\\ &= n^2 + 2b(z)n + O(1)\end{aligned}$$

and

$$|z|^2 = (n-\delta)^2 = n^2 - 2n\delta + \delta^2.$$ □

Summarizing the two lemmas we see that for $z \in D(n)$,

$$\begin{aligned}\mathbb{E}(M) &= [G_n(z,z)]^{-1}(2n\delta + O(n)),\\ \mathbb{E}(\tilde{L}) &= [G_n(z,z)]^{-1}(2n\delta + O(n))\end{aligned}$$

and for all n sufficiently large,

$$\mathbb{E}(M) - \mathbb{E}(\tilde{L}) \geq \delta^2[G_n(z,z)]^{-1}.$$

It is standard (see, e.g., [4], Theorem 1.6.6) that

$$G_n(z,z) \leq c\ \ln\ n,$$

the logarithmic term being needed only if $d = 2$. The basic idea of the remainder of the proof is that, with high probability, $M - \mathbb{E}(M)$ will be of order $\mathbb{E}(M)^{1/2}$ and $\tilde{L} - \mathbb{E}(\tilde{L})$ will be of order $\mathbb{E}(\tilde{L})^{1/2}$. If $\mathbb{E}(M) - \mathbb{E}(\tilde{L})$ is of a greater order than this, we will be able to conclude that $M - \tilde{L} > 0$. This will be true if

$$\delta^2[G_n(z,z)]^{-1} \gg [2n\delta/G_n(z,z)]^{1/2},$$

that is, if

$$\delta^{3/2} \gg n^{1/2}G_n(z,z)^{1/2}.$$

From this we see that $\delta = n^{1/3}(\ln\ n)^{\beta}$ for some $\beta > 0$ is the right order of magnitude to guarantee this. To make this rigorous, we call on a standard large deviation estimate for sums of indpendent indicator random variables.

LEMMA 4 ([5], Lemma 4). *Suppose X is a sum of independent indicator functions and $\mu = \mathbb{E}(X)$. Then for all sufficiently large n and all $\lambda \in (0, 1/4)$,*

$$P\{|X - \mu| \geq \mu^{1/2+\lambda}\} \leq 2\ \exp\{-\tfrac{1}{4}\mu^{2\lambda}\}.$$

By choosing λ so that $\mathbb{E}(M)^{\lambda} = \ln\ n$ and $\mathbb{E}(\tilde{L})^{\lambda} = \ln\ n$, we see that there exists a $\gamma_n \xrightarrow{q} 0$ such that, except on a set of probability at most γ_n,

$$|M - \mathbb{E}(M)| \leq \mathbb{E}(M)^{1/2}(\ln\ n), \qquad |\tilde{L} - \mathbb{E}(\tilde{L})| \leq \mathbb{E}(\tilde{L})^{1/2}(\ln\ n).$$

If $\delta \in [n^{1/3}(\ln\ n)^2, 2n^{1/3}(\ln\ n)^2]$,

$$(\ln\ n)^2[\mathbb{E}(M)^{1/2} + \mathbb{E}(\tilde{L})^{1/2}] \leq cn^{2/3}(\ln\ n)^3[G_n(z,z)]^{-1/2},$$

$$\mathbb{E}(M) - \mathbb{E}(\tilde{L}) \geq cn^{2/3}(\ln\ n)^4[G_n(z,z)]^{-1} \geq cn^{2/3}(\ln\ n)^{7/2}[G_n(z,z)]^{-1/2}.$$

Hence for all n sufficiently large,

$$\mathbb{E}(M) - \mathbb{E}(\tilde{L}) \geq 4(\ln\ n)^2[\mathbb{E}(M)^{1/2} + \mathbb{E}(L)^{1/2}]$$

and we get the theorem by setting $a = \mathbb{E}(M) - (\ln\ n)^2[\mathbb{E}(M)^{1/2} + \mathbb{E}(\tilde{L})^{1/2}]$ in (5).

3. Outer error. In this section we will prove (2). We will start by giving a slightly different construction of an internal DLA cluster. Fix n and let $m = m(n) = [\omega_d n^d]$. The construction we describe will depend on n. Essentially we let m random walks run until they either add to the cluster or reach distance n. After letting all m walks run this far, we then let those walks that have not added to the cluster run some more until they either add to the cluster or reach distance $n + 2n^{1/3}$; then those that have not added yet can go until distance $n + 4n^{1/3}$ and so forth, until all walkers eventually add to the cluster.

To be precise, let $S^1, \ldots, S^m$ be independent simple random walks starting at the origin. Define

$$\sigma_1^1 = 0, \qquad A_1(1) = \{0\}.$$

For $1 < j \le m$, we define

$$\sigma_1^j = \inf\{t\colon\ S^j(t) \notin A_1(j-1)\},$$

$$\rho_1^j = \sigma_1^j \wedge \xi_n^j,$$

$$A_1(j) = A_1(j-1) \cup \{S^j(\rho_1^j)\}.$$

Note that $A_1(j) \setminus A_1(j-1)$ is nonempty if and only if $\rho_1^j = \sigma_1^j$. Let $J_1(j)$ be the indicator function of the event $\{\sigma_1^j > \rho_1^j\}$. Note that

$$m = |A_1(m)| + \sum_{j=1}^{m} J_1(j).$$

For general $i > 1$, we let $s_i = n + 2(i-1)n^{1/3}$,

$$\sigma_i^1 = 0, \qquad A_i(1) = A_{i-1}(m).$$

For $j > 1$, we set

$$\sigma_i^j = \sigma_{i-1}^j, \quad \text{if } J_{i-1}(j) = 0;$$

otherwise we set

$$\sigma_i^j = \inf\{t\colon\ S^j(t) \notin A_i(j-1)\},$$

$$\rho_i^j = \sigma_i^j \wedge \xi_{s_i}^j,$$

$$A_i(j) = A_i(j-1) \cup \{S^j(\rho_i^j)\}$$

and $J_i(j)$ equal to the indicator function of $\{\sigma_i^j > \rho_i^j\}$. Again

$$m = |A_i(m)| + \sum_{j=1}^{m} J_i(j).$$

For any j let $\bar{i} = \bar{i}(j)$ be the smallest i such that $\rho_i^j = \sigma_i^j$ and set

$$\rho^j = \sigma_{\bar{i}}^j.$$

Note that

$$A_1(m) \subset A_2(m) \subset A_3(m) \subset \cdots.$$

If we set

$$A(m) = \lim_{i\to\infty} A_i(m),$$

then $A(m)$ has the same statistics as an internal DLA cluster after m steps as defined in the Introduction. This fact is not immediately obvious, but it can be proved by labelling the m particles and using the rule that if more than one particle is at a site, we only move the particle with the highest index. For more details of this, see Section 6 of [5].

As before, we can do the preceding construction on a probability space on which are defined independent random walks $\{X^j\colon j = 1, 2, \cdots\}$ and $\{Y^x\colon x \in \mathbb{Z}^d\}$, where $X^j(0) = 0$ and $Y^x(0) = x$. We define σ_i^j, σ^j and so forth with X^j in place of S^j and then let

$$S^j(t) = \begin{cases} X^j(t), & t \leq \sigma^j, \\ Y^{X^j(\sigma^j)}(t - \sigma^j), & t \geq \sigma^j. \end{cases}$$

For each $s \geq n$, $z \in \partial B_s$, $m = [\omega_d n^d]$ and $r = n - n^{1/3}(\ln\ n)^2$, let

$$M(s,n,z) = \sum_{j=1}^{m} I\{S^j(\xi_s^j) = z\},$$

$$L(s,n,z) = \sum_{j=1}^{m} I\{S^j(\xi_s^j) = z;\ \xi_s^j > \sigma^j\},$$

$$W(s,n,z) = M(s,n,z) - L(s,n,z),$$

$$\tilde{L}(s,n,z) = \sum_{y \in B_r} I\{Y^y(\xi_s^y) = z\}.$$

For any $U \subset \partial B_s$, we write

$$W(s,n,U) = \sum_{z \in U} W(s,n,z), \qquad W(s,n) = W(s,n,\partial B_s).$$

Note that

$$\{\tilde{L}(s,n,z) > L(s,n,z)\} \subset \{B(r) \not\subset A_1(m)\}. \tag{8}$$

The proof in the previous section can be adapted verbatim to show that

$$P\{B(r) \not\subset A_1(m)\} \xrightarrow{q} 0. \tag{9}$$

Also,

$$|A(m) \cap B_s^c| \leq W(s,n).$$

What we are going to prove is that there is a $\gamma_n \xrightarrow{q} 0$ and a constant $c > \infty$ such that for all i with $s_i \leq 2n$,

$$\mathbb{E}(W(s_{i+1},n)) \leq (1 - c(\ln\ n)^{-2})\mathbb{E}(W(s_i,n)) + \gamma_n. \tag{10}$$

By iterating this inequality, we see that there is a $\gamma_n \xrightarrow{q} 0$ such that for all n sufficiently large,

$$\mathbb{E}(W(n + n^{1/3}(\ln\ n)^4, n)) \leq (1 - c(\ln\ n)^{-2})^{(\ln\ n)^4/2}\mathbb{E}(W(n,n)) + \gamma_n.$$

However, the right-hand side goes to zero faster than any power of n. With the aid of the Borel–Cantelli lemma, this gives (2). Hence it suffices to prove (10).

LEMMA 5. *There exists a constant $c < \infty$ such that for every $x \in B_n$ and every $r < n$, if $\delta = n - r$,*

$$\sum_{y \in B_n \setminus B_r} G_n(x, y) \le c\delta^2.$$

PROOF. See, for example, the proof of (2.10) in [5].

LEMMA 6. *There exists a constant $c < \infty$ such that if $r = n - \delta$, $\delta \ge n^{1/3}$ and $x \in \partial B_r$,*

$$|\omega_d n^d G_n(x, 0) - \sum_{y \in B_r} G_n(x, y)| \le c\delta^2.$$

PROOF. Let $\tilde{\delta} = n - |x|$ (so that $|\delta - \tilde{\delta}| \le 1$). Then as in Lemmas 2 and 3, we can show that

$$\begin{aligned} \omega_d n^d G_n(x, 0) &= 2n\tilde{\delta} + 2b(x)n + O(\delta^2), \\ \textstyle\sum_{y \in B_n} G_n(x, y) &= 2n\tilde{\delta} + 2b(x)n + O(\delta^2). \end{aligned} \tag{11}$$

However, by Lemma 5,

$$\sum_{y \in B_n \setminus B_r} G_n(x, y) = O(\delta^2). \qquad \square$$

LEMMA 7. *Suppose $r = n - n^{1/3}(\ln\ n)^2$ and $x \in \partial B_r$. Then for every $s \ge n$,*

$$\omega_d n^d G_s(x, 0) = \sum_{y \in B_r} G_s(x, y)[1 + O(n^{-2/3}(\ln\ n)^2)],$$

where the $O(\cdot)$ is bounded uniformly in x and s.

PROOF. Let $\rho_1 = 0, \eta_1 = \xi_n$ and for $j > 1$,

$$\rho_j = \inf\{t > \eta_{j-1}\colon\ S(t) \in \partial B_r\},$$

$$\eta_j = \inf\{t > \rho_j\colon\ S(t) \in \partial B_n\}.$$

We can write

$$\begin{aligned} G_s(x, 0) &= \mathbb{E}^x\left[\sum_{t=0}^{\xi_s - 1} I\{S(t) = 0\}\right] \\ &= \mathbb{E}^x\left[\sum_{j=1}^{\infty} I\{\xi_s > \rho_j\} \sum_{t=\rho_j}^{\eta_j} I\{S(t) = 0\}\right] \\ &= \sum_{j=1}^{\infty} P^x\{\xi_s > \rho_j\} \mathbb{E}^x[G_n(S(\rho_j), 0) \mid \xi_s > \rho_j]. \end{aligned}$$

Similarly,

$$\sum_{y\in B_r} G_s(x,y) = \sum_{j=1}^{\infty} P^x\{\xi_s > \rho_j\}\mathbb{E}^x\left[\sum_{y\in B_r} G_n(S(\rho_j),y) \mid \xi_s > \rho_j\right].$$

By Lemma 6 and (11), we have for any $x \in \partial B_r$,

$$\sum_{y\in B_r} G_n(x,y) = \omega_d n^d G_n(x,0)[1 + O(n^{-2/3}(\ln\ n)^2)],$$

and hence

$$\mathbb{E}^x\left[\sum_{y\in B_r} G_n(S(\rho_j),y) \mid \xi_s > \rho_j\right]$$
$$= \omega_d n^d E^x[G_n(S(\rho_j),0) \mid \xi_s > \rho_j](1 + O(n^{-2/3}(\ln\ n)^2)). \qquad \square$$

LEMMA 8. *Suppose $r = r(n) = n - n^{1/3}(\ln\ n)^2$ and $m = m(n) = [\omega_d n^d]$. Then for any $s \geq n$ and $z \in \partial B_s$,*

$$\mathbb{E}(M(s,n,z)) = \mathbb{E}(\tilde{L}(s,n,z))(1 + O(n^{-2/3}(\ln\ n)^2)).$$

PROOF. For any $x \in B_r$ and $z \in \partial B_s$, by a last-exit decomposition (see, e.g., [4], Lemma 2.1.1),

$$\begin{aligned} P^x\{S(\xi_s) = z\} &= \sum_{y\in\partial B_r} G_s(x,y)P^y\{S(\xi_s \wedge \xi_r) = z\} \\ &= \sum_{y\in\partial B_r} G_s(x,y)P^z\{S(\xi_s \wedge \xi_r) = y\} \\ &= P^z\{\xi_r < \xi_s\}\mathbb{E}^z(G_s(S(\xi_r),x) \mid \xi_r < \xi_s). \end{aligned}$$

Therefore,

$$\mathbb{E}(M(s,n,z)) = P^z\{\xi_r < \xi_s\}[\omega_d n^d]\mathbb{E}^z(G_s(S(\xi_r),0) \mid \xi_r < \xi_s),$$

$$\mathbb{E}(\tilde{L}(s,n,z)) = P^z\{\xi_r < \xi_s\}\mathbb{E}^z\left[\sum_{y\in B_r} G_s(S(\xi_r),y) \mid \xi_r < \xi_s\right].$$

The lemma now follows from Lemma 7. $\square$

There exist ([4], Lemma 1.7.4) $0 < c_1 < c_2 < \infty$ such that for all $n \leq s \leq 2n$ and all $z \in \partial B_s$,

$$c_1 n^{1-d} \leq P\{S(\xi_s) = z\} = c_2 n^{1-d}.$$

Hence we can deduce from Lemma 8 that if $m = [\omega_d n^d]$, $n \leq s \leq 2n$ and $r = n - n^{1/3}(\ln\ n)^2$,

$$c_1 n \leq \mathbb{E}(M(s,n,z)) \leq c_2 n, \tag{12}$$

$$\mathbb{E}(M(s,n,z) - \tilde{L}(s,n,z)) \leq c n^{1/3}(\ln\ n)^2. \tag{13}$$

For any V, we write

$$M(s,n,V) = M(s,n,V\cap \partial B_s) = \sum_{z\in V} M(s,n,z) = \sum_{j=1}^{m} I\{S^j(\xi_s)\in V\},$$

$$\tilde{L}(s,n,V) = \tilde{L}(s,n,V\cap \partial B_s) = \sum_{z\in V} \tilde{L}(s,n,z) = \sum_{x\in B_r} I\{Y^x(\xi_n^x)\in V\}.$$

Note that $M(s,n,V)$ and $\tilde{L}(s,n,V)$ are both sums of independent indicator random variables so that Lemma 4 applies to them.

LEMMA 9. *There exists a constant $c_3 < \infty$ and a sequence $\gamma_n \xrightarrow{q} 0$ such that if $n \le s \le 2n$, $m = [\omega_d n^d]$ and $V \subset \partial B_s$ with $|V| \le 10n^{(d-1)/3}$,*

$$P\{W(s,n,V) \ge c_3 n^{d/3}(\ln\ n)^2\} \le \gamma_n.$$

PROOF. Without loss of generality, we may assume $|V| = 10n^{(d-1)/3}$. We write W, M, L and $\tilde{L}$ for $W(s,n,V), M(s,n,V), L(s,n,V)$ and $\tilde{L}(s,n,V)$. For any $a_1, a_2 > 0$,

$$\begin{aligned} &P\{W \ge a_1 + a_2 + \mathbb{E}(M-\tilde{L})\} \\ &\quad\le P\{M \ge \mathbb{E}(M)+a_1\} + P\{L \le \mathbb{E}(\tilde{L}) - a_2\} \\ &\quad\le P\{M \ge \mathbb{E}(M)+a_1\} + P\{\tilde{L} \le \mathbb{E}(\tilde{L}) - a_2\} + P\{\tilde{L} > L\}. \end{aligned}$$

By (8) and (9), $P\{\tilde{L} > L\} \le \gamma_n$ for some $\gamma_n \xrightarrow{q} 0$. By Lemma 4, if we set $a_1 = E(M)^{1/2}(\ln\ n)$ and $a_2 = \mathbb{E}(\tilde{L})^{1/2}(\ln\ n)$, we can see that the right-hand side is bounded by some $\gamma_n \xrightarrow{q} 0$. By (12),

$$\mathbb{E}(M) \le cn^{(d+2)/3}$$

and similarly

$$\mathbb{E}(\tilde{L}) \le cn^{(d+2)/3}.$$

Hence,

$$(\ln\ n)[\mathbb{E}(M)^{1/2} + \mathbb{E}(\tilde{L})^{1/2}] \le cn^{(d+2)/6}(\ln\ n) \le cn^{d/3}(\ln\ n),$$

the last inequality holding because $d \ge 2$. Finally, by (13),

$$\mathbb{E}(M-\tilde{L}) \le cn^{d/3}(\ln\ n)^2,$$

which completes the lemma. □

We write $B(x,R) = \{y \in \mathbb{Z}^d\colon\ |x-y| < R\}$. Because there are only $O(n^d)$ points in B_{2n} the following is an immediate corollary of Lemma 9.

COROLLARY 10. *Let $F_n = F_n(c_3)$ be the event*

$$\{W(s, n, B(x, n^{1/3})) \leq c_3 n^{d/3} (\ln\ n)^2 \text{ for all } n \leq s \leq 2n,\ x \in B_n\}.$$

There exists a constant $c_3 < \infty$ and a sequence $\gamma_n \xrightarrow{q} 0$ such that $P(F_n) \geq 1 - \gamma_n$.

We will need one simple estimate for the probability of hitting a set before leaving the ball of radius n.

LEMMA 11. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $A \subset B_n$ with $|A| \geq \varepsilon n^d$, then*

$$P\{\tau_A < \xi_n\} \geq \delta.$$

PROOF. Let $\varepsilon > 0$. It suffices to prove the result for n sufficiently large. Find $u = u(\varepsilon) < 1$ such that (for all n sufficiently large)

$$|B_n \setminus B_{un}| \leq \frac{\varepsilon}{2} n^d.$$

Then any $A \subset B_n$ with $|A| \geq \varepsilon n^d$ satisfies $|A \cap B_{un}| \geq (\varepsilon/2) n^d$.

Let $V = \sum_{t=0}^{\xi_n - 1} I\{S(t) \in A\}$ denote the number of visits to A. Then

$$\mathbb{E}(V) = \sum_{y \in A} G_n(y) \geq \sum_{y \in B_{un} \cap A} G_n(y) \geq \frac{\varepsilon}{2} n^d \inf_{y \in B_{un}} G_n(y).$$

There exists a constant $c(\varepsilon) > 0$ (see [4], Propositions 1.5.9 and 1.6.7) such that

$$\inf_{y \in B_{un}} G_n(y) \geq c(\varepsilon) n^{2-d}.$$

Hence $\mathbb{E}(V) \geq c(\varepsilon) n^2$. However, it is standard that for all $x \in B_n$, $\mathbb{E}^x(V) \leq \mathbb{E}^x(\xi_n) \leq cn^2$. Hence,

$$P\{V > 1\} = \mathbb{E}(V)[\mathbb{E}(V \mid V \geq 1)]^{-1} \geq c(\varepsilon) n^2 [cn^2]^{-1} = \delta(\varepsilon) > 0. \qquad \square$$

We will also need one simple geometric fact about "locally finite coverings." We will state it as a lemma, but we omit the proof.

LEMMA 12. *There exists a constant $K = K_d$ such that for every $R > 0$ and $V \subset \mathbb{Z}^d$, there exists a subset $\tilde{V} \subset V$ satisfying*

$$V \subset \bigcup_{x \in \tilde{V}} B(x, R)$$

and such that for every $y \in \mathbb{Z}^d$,

$$\big|\{x \in \tilde{V}\colon |x - y| \leq 2R\}\big| \leq K,$$

where $|\cdot|$ denotes cardinality.

We are now ready to prove (10). Let $F_n = F_n(c_3)$ be the event described in Corollary 10. By Lemma 12, for each $n \le s \le 2n$ we can find $x_1, \dots, x_u$ in ∂B_s with

$$\partial B_s \subset \bigcup_{k=1}^{u} B(x_k, n^{1/3}) \tag{14}$$

and such that each $y \in \mathbb{Z}^d$ is contained in at most K of the sets $\{B(x_k, 2n^{1/3})\}$. Fix i and $x \in \partial B_{s_i}$. Write $U = B(x, n^{1/3})$ and $2U = B(x, 2n^{1/3})$. For any set A, either

$$|A \cap (B_{s_{i+1}} \setminus B_{s_i}) \cap U| \ge 0.01 n^{d/3}$$

or, for every $y \in \partial B_{s_i} \cap U$,

$$|A^c \cap (B_{s_{i+1}} \setminus B_{s_i}) \cap B(y, n^{1/3})| \ge 0.01 n^{d/3}.$$

Recall the definitions of ρ^j and $J_i(j)$ given at the beginning of this section. Let $Y = Y(i, j, U)$ be the indicator function of the event $\{S^j(\xi_i^j) \in U,\ J_i(j) = 1\}$. By Lemma 11, for some $\delta > 0$,

$$\begin{aligned} P\{\rho^j < \xi_{i+1}^j, S^j(\rho^j) \in 2U, J_i(j) = 1, S^j(\xi_i^j) \in U\} \\ &\ge \delta \mathbb{E}(YI\{|A_{i+1}(j)^c \cap B(S^j(\xi_i^j), n^{1/3}) \cap B_{s_i}^c| \ge 0.01 n^{d/3}\}) \\ &\ge \delta \mathbb{E}(YI\{|A_{i+1}(m)^c \cap B(S^j(\xi_i^j), n^{1/3}) \cap B_{s_i}^c| \ge 0.01 n^{d/3}\}) \\ &\ge \delta \mathbb{E}(YI\{|A_{i+1}(m) \cap U \cap B_{s_i}^c| \le 0.01 n^{d/3}\}). \end{aligned}$$

If we sum this over all j, we see that

$$\mathbb{E}[|(A_{i+1}(m) \setminus A_i(m)) \cap 2U|] \ge \delta \mathbb{E}(W(s_i, n, U) I_Z),$$

where Z denotes the event

$$\{|A_{i+1}(m) \cap U \cap B_{s_i}^c| \le 0.01 n^{d/3}\},$$

but on Z^c,

$$|(A_{i+1}(m) \setminus A_i(m)) \cap 2U| \ge 0.01 n^{d/3}$$

and, hence,

$$\mathbb{E}[|(A_{i+1}(m) \setminus A_i(m) \cap 2U)|] \ge 0.01 n^{d/3} \mathbb{E}(I_{Z^c}).$$

Therefore, if $s = s_i$,

$$\begin{aligned} \mathbb{E}[|(A_{i+1}(m) \setminus A_i(m)) \cap 2U|] &\ge \tfrac{1}{2} \mathbb{E}[\delta W(s, n, U) I_Z + 0.01 n^{d/3} I_{Z^c}] \\ &\ge c \mathbb{E}[\min\{W(s, n, U), n^{d/3}\}] \\ &\ge c (\ln n)^{-2} \mathbb{E}[W(s, n, U) I_{F_n}] \\ &\ge c (\ln n)^{-2} \mathbb{E}[W(s, n, U)] - \gamma_n, \end{aligned}$$

for some $\gamma_n \xrightarrow{q} 0$. If we cover ∂B_s by balls $U_k = B(x_k, n^{1/3})$ as in (14), we get

$$\begin{aligned}\mathbb{E}[|(A_{i+1}(m) \setminus A_i(m)) \cap B_{s_{i+1}}|] &\geq K^{-1} \sum_k \mathbb{E}[|(A_{i+1}(m) \setminus A_i(m)) \cap 2U_k|] \\ &\geq c(\ln\ n)^{-2}\mathbb{E}\left[\sum_k W(s,n,U_k)\right] - \gamma_n \\ &\geq c(\ln\ n)^{-2}\mathbb{E}[W(s,n)] - \gamma_n,\end{aligned}$$

for some $\gamma_n \xrightarrow{q} 0$. However,

$$W(s_i,n) - W(s_{i+1},n) \geq |(A_{i+1}(m) \setminus A_i(m)) \cap B_{s_{i+1}}|,$$

which gives (10).

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DEPARTMENT OF MATHEMATICS
BOX 90320
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27708-0320