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ON THE CLUSTER SET PROBLEM FOR THE GENERALIZED LAW OF THE ITERATED LOGARITHM IN EUCLIDEAN SPACE[1]

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In a recent paper by the author it has been shown that there exists a general law of the iterated logarithm (LIL) in Banach space, which contains the LIL of Ledoux and Talagrand and an LIL for infinite-dimensional random variables in the domain of attraction to a Gaussian law as special cases. We now investigate the corresponding cluster set problem, which we completely solve for random vectors in two-dimensional Euclidean space. Among other things, we show that all cluster sets arising from this generalized LIL must be sets of diameter 2, which are star-shaped and symmetric about the origin, and any closed set of this type occurs as a cluster set for a suitable random vector. Moreover, we show that if the random vectors under consideration have independent components, one only obtains cluster sets from the subclass of all sets, which can be represented as closures of countable unions of standard ellipses.

1. Introduction. Let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*. Denote for any sequence $\{x_n\}$ in B the set of its limit points by $C(\{x_n\})$ and call it the *cluster set* of $\{x_n\}$.

We assume throughout that $X, X_1, X_2, \ldots$ are iid B-valued random variables with $0 < E\|X\| < \infty$. As usual, set $S_n := X_1 + \cdots + X_n$, $n \geq 1$, and put $Lt := \log(t \vee e)$, $LLt := L(Lt)$, $t \geq 0$.

Using the separability of B and the 0–1 law of Hewitt and Savage, one can show that for any sequence $\alpha_n \uparrow \infty$ with probability 1,

$$C(\{S_n/\alpha_n\}) = A, \tag{1.1}$$

where A is nonrandom and depends only on $\{\alpha_n\}$ and the distribution of X. [Refer to Lemma 1, Kuelbs (1981).] If $\{\alpha_n\}$ is a sequence such that $S_n/\alpha_n \to 0$ a.s., then of course it follows that $A = \{0\}$, and determining the cluster set is a trivial task. This is no longer the case if one considers sequences $\{\alpha_n\}$ such that

$$0 < \limsup_{n\to\infty} \|S_n\|/\alpha_n < \infty \quad \text{a.s.} \tag{1.2}$$

The classical choice for $\{\alpha_n\}$ in (1.2) is $\{\sqrt{2nLLn}\}$, and the corresponding cluster set problem for that particular sequence has been studied by Goodman, Kuelbs and Zinn (1981), Kuelbs (1981), DeAcosta and Kuelbs (1983) and

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Alexander (1989a, b), among others. From the work of these authors it became clear that the cluster set must be a subset of a canonical set D, which can be described as the unit ball of a certain reproducing kernel Hilbert space determined by the covariance of the random variable X. Under suitable conditions on the underlying Banach space, the cluster set A is always equal to D. This is, in particular, the case if B is a Hilbert space or a Euclidean space. Moreover, in the latter case, the set D and consequently the cluster set A turns out to be an ellipsoid. Alexander (1989a, b) finally showed that if B is an arbitrary separable Banach space, then the cluster set is either the empty set or of the form ρD, $0 \le \rho \le 1$. He was also able to construct for each $0 \le \rho < 1$ a c_0-valued random variable so that the cluster set A in (1.1) is equal to ρD. A related example for a random variable, where the cluster set is empty, is due to Kuelbs (1981).

Much less, however, is known about the possible cluster sets if one uses different norming sequences in (1.2). Feller (1968) [see also Kesten (1972)] and Klass (1976, 1977) found extensions of the classical Hartman–Wintner LIL to certain (real-valued) random variables with possibly infinite variances. These results, in particular, can be applied to random variables in the domain of attraction to the normal distribution. Kuelbs (1985) and Einmahl (1989) generalized the latter results to B-valued random variables in the domain of attraction of infinite-dimensional Gaussian distributions. They also showed that the resulting cluster sets are equal to the canonical sets determined by the covariance of the limiting Gaussian distributions.

Recently, Einmahl (1993) has shown that there is a general LIL in Banach space, which not only contains the LIL results of Ledoux and Talagrand (1988) and Kuelbs (1985) as special cases, but also can be applied in many cases not covered by the above results. To formulate this generalized LIL, we need some further notation. Following Klass (1976) we associate with any *real-valued* random variable ξ satisfying $0 < E|\xi| < \infty$ a function $K(\cdot)$ which is defined as the inverse function of a further function $G(\cdot)$ given by

$$G(y) := y^2 \Big/ \int_0^y E|\xi| 1\{|\xi| > u\}\, du, \qquad y > 0.$$

For any functional $f \in B^*$ with $E|f(X)| > 0$, let K_f be the K-function corresponding to $f(X)$, and set

$$\tilde{K}(y) := \sup\{K_f(y) \colon \|f\| \le 1\}, \qquad y > 0, \tag{1.3}$$

$$\gamma_n := \sqrt{2}\, \tilde{K}(n/LLn)LLn, \qquad n \ge 1. \tag{1.4}$$

THEOREM A. *Let X be a mean zero r.v. with $0 < E\|X\| < \infty$. Then we have*

$$1 \le \limsup_{n\to\infty} \|S_n\|/\gamma_n < \infty \quad a.s. \tag{1.5}$$

if and only if

$$\{S_n/\gamma_n\} \text{ is bounded in probability} \tag{1.6}$$

and

$$\sum_n P\{\|X\| > \gamma_n\} < \infty. \tag{1.7}$$

It is natural now to ask what are the possible cluster sets in that situation. One of the fundamental difficulties is that we can no longer assume that the r.v. X has a covariance structure. This of course can also happen for random variables in the domain of attraction to a Gaussian law, but in this case we have at least a limiting distribution with a covariance structure, which determines the form of the cluster set. Having in general neither a finite covariance for X nor a suitable limiting distribution, there does not seem to be any "natural" candidate for a canonical cluster set in connection with Theorem A.

The purpose of the present paper is to find all possible cluster sets if $(B, \|\cdot\|)$ is a Euclidean space. For convenience we will formulate and prove our results only for two-dimensional random vectors. It is possible to extend our results to d-dimensional random vectors, but this would make the proofs much more technical. So let from now on $X = (X^{(1)}, X^{(2)})$ be a two-dimensional random vector and let $X_n = (X_n^{(1)}, X_n^{(2)})$ be independent copies of X. Set $S_n^{(1)} := X_1^{(1)} + \cdots + X_n^{(1)}$ and $S_n^{(2)} := X_1^{(2)} + \cdots + X_n^{(2)}$, $n \geq 1$. Using Corollary 2 in Einmahl (1993), Theorem A can be improved in this particular case as follows:

THEOREM B. *Let $X = (X^{(1)}, X^{(2)})$ be a mean zero random vector with $0 < E\|X\| < \infty$. Then we have*

$$\limsup_{n\to\infty} \|S_n\|/\gamma_n = 1 \quad a.s. \tag{1.8}$$

if and only if

$$\sum_n P\{|X^{(i)}| > \gamma_n\} < \infty, \qquad i = 1, 2. \tag{1.9}$$

Setting $A = C(\{S_n/\gamma_n\})$, it follows from (1.8) that A is a subset of the unit disk and, moreover,

$$\sup\{\|x\|: x \in A\} = 1. \tag{1.10}$$

We will show that A must also be symmetric and star-shaped with respect to the origin (see Theorem 3 below) and, somewhat surprisingly, it will turn out that any closed set of this type will actually occur as a cluster set for a suitable random vector X (see Theorem 4 below). This is a fairly large class of sets and one might ask whether, under additional assumptions, one can say more about the structure of the cluster sets. Indeed, assuming that

$$X^{(1)} \text{ and } X^{(2)} \text{ are independent}, \tag{1.11}$$

it will turn out that the cluster sets will be from the subclass of all sets which can be represented as closures of unions of (at most) countably many standard ellipses. Here we call any (possibly degenerate) ellipse, which is centered at the origin and whose axes are parallel to the coordinate axes, a "standard ellipse."

2. Statement of the main results. We first formulate the results for the independent case. Let for any $0 < a, b < \infty$,

$$\mathscr{E}(a,b) := \{(x_1,x_2)\colon\ (x_1/a)^2 + (x_2/b)^2 \le 1\}$$

and set

$$\mathscr{E}(a,0) := [-a,a] \times \{0\}, \qquad \mathscr{E}(0,b) = \{0\} \times [-b,b], \qquad a,b \ge 0,$$

which we consider as degenerate ellipses. Write $\mathrm{cl}(M)$ for the (topological) closure of a subset M of 2-space.

Further, put

$$\sigma_1 := \limsup_{n\to\infty} |S_n^{(1)}|/\gamma_n$$

and

$$\sigma_2 := \limsup_{n\to\infty} |S_n^{(2)}|/\gamma_n,$$

and observe that we have $\sigma_1 \vee \sigma_2 \le 1$.

THEOREM 1. *Let X be a mean zero random vector satisfying conditions* (1.9) *and* (1.11). *Then we have for suitable sequences $0 \le a_m \le \sigma_1$, $0 \le b_m \le \sigma_2$,*

$$A = \mathscr{E}(\sigma_1,0) \cup \mathscr{E}(0,\sigma_2) \cup \mathrm{cl}\left(\bigcup_{m=1}^{\infty} \mathscr{E}(a_m,b_m)\right). \tag{2.1}$$

Note that (2.1) in particular implies,

$$\mathscr{E}(\sigma_1,0) \cup \mathscr{E}(0,\sigma_2) \subset A \subset \mathscr{E}(\sigma_1,\sigma_2). \tag{2.2}$$

Recalling (1.8), we see that we must have $\sigma_1 \vee \sigma_2 = 1$. Our next result shows that any set as in (2.1) occurs as a cluster set.

THEOREM 2. *Given $\sigma_1, \sigma_2 \ge 0$ with $\sigma_1 \vee \sigma_2 = 1$ and sequences $0 \le a_m \le \sigma_1$, $0 \le b_m \le \sigma_2$, one can find a symmetric random vector X satisfying conditions* (1.9) *and* (1.11) *such that we have* (2.1) *for the cluster set A.*

Theorem 2 in particular implies that there exists a random vector X for which the cluster set A is equal to the cross $\mathscr{E}(\sigma_1,0) \cup \mathscr{E}(0,\sigma_2)$. (Set $a_m = b_m = 0$.) This is also the smallest possible cluster set in that situation. If X is in the domain of attraction to a Gaussian distribution, we obtain the largest possible cluster set, namely, the ellipse $\mathscr{E}(\sigma_1,\sigma_2)$. These are the two extreme cases, and there are many other possibilities, for instance, sets which can be represented as finite unions of standard ellipses. An interesting example of sets which can only be generated by infinitely many standard ellipses, is the class of generalized l_p-balls $\mathscr{E}_p(\sigma_1,\sigma_2)$, $0 < p < 2$, where we set

$$\mathscr{E}_p(\sigma_1,\sigma_2) := \{(x_1,x_2)\colon\ |x_1/\sigma_1|^p + |x_2/\sigma_2|^p \le 1\}, \qquad p > 0. \tag{2.3}$$

If $p > 2$, it is obvious that $\mathscr{E}_p(\sigma_1, \sigma_2)$ cannot be among the possible cluster sets, since then relation (2.2) would not be satisfied. One can even show that if one chooses a constant $0 < \rho < 1$ so that

$$\rho\mathscr{E}_p(\sigma_1, \sigma_2) \subset \mathscr{E}(\sigma_1, \sigma_2), \tag{2.4}$$

it is still impossible to find a representation (2.1) for the set

$$\tilde{A} := \mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2) \cup \rho\mathscr{E}_p(\sigma_1, \sigma_2).$$

In view of Theorem 1 this means that there is no random vector satisfying (1.9) and (1.11), for which the cluster set A coincides with $\tilde{A}$. From the subsequent Theorem 4, however, it will follow that $\tilde{A}$ is among the possible cluster sets for random vectors where $X^{(1)}$ and $X^{(2)}$ are *dependent*.

We now turn to the general case.

THEOREM 3. *Let X be a mean zero random vector satisfying* (1.9). *Then we have for the cluster set $A = C(\{S_n/\gamma_n\})$,*

$$A \text{ is symmetric about the origin,} \tag{2.5}$$

$$A \text{ is star-shaped with respect to the origin} \tag{2.6}$$

and

$$\sup\{\|x\|: x \in A\} = 1. \tag{2.7}$$

Our last theorem finally shows that any closed set of this type occurs as a cluster set.

THEOREM 4. *Let $\tilde{A}$ be a closed set satisfying conditions* (2.5)–(2.7). *There exists a symmetric random vector X satisfying* (1.9) *such that with probability 1, $C(\{S_n/\gamma_n\}) = \tilde{A}$.*

The proofs of Theorems 1, 2, 3 and 4 will be presented in Sections 3, 4, 5 and 6, respectively. The starting point is an infinite-dimensional version of a general criterion for clustering, which goes back to Kesten (1970) (see Lemma 1 below).

If assumptions (1.9) and (1.11) are satisfied, we can use a truncation argument in combination with a nonuniform Berry–Esseen type inequality and an exponential inequality for the tail probabilities of the normal distribution to prove that a point (x_1, x_2) belongs to the cluster set if and only if a certain series condition is satisfied (see Proposition 1). We then can infer that the cluster set must be a closure of a (countable) union of standard ellipses, and we get Theorem 1.

The proof of Theorem 2 is probably the most delicate part of the present paper. We first show that the criterion for clustering can be further simplified if the random variables $X^{(1)}$ and $X^{(2)}$ are in the domain of attraction to the normal distribution. It will turn out that in this case the clustering behavior can essentially be described in terms of the truncated second moment functions corresponding to these random variables (see Proposition 2). Making appropriate use of this observation, we will be able to construct for any given closed set $\tilde{A}$ of the form (2.1) *discrete* random variables $X^{(1)}$ and $X^{(2)}$ with the above property so that the cluster set of X is equal to $\tilde{A}$.

The proof of Theorem 3 is similar to (but much easier than) that of Theorem 1. We now employ a truncation argument in combination with a multidimensional Berry–Esseen type result of Kuelbs and Kurtz (1974) to establish a criterion for clustering in terms of certain Gaussian random vectors. Using the symmetry and another property of Gaussian random vectors which goes back to Anderson (1955), we obtain Theorem 3.

The proof of Theorem 4 is based on the fact that any closed star-shaped set can be written as a closure of line segments. We start with a random vector where the cluster set would be a single line segment. We then change the coordinate system (infinitely often) so that we also obtain the limit points from the other line segments.

3. Proof of Theorem 1.

3.1. *A necessary and sufficient condition for clustering.* We first formulate a general criterion for clustering in Banach space, which can be proved by a modification of the proof of Theorem 3 in Kesten (1970) [see also relation (6.37) of Kuelbs (1985) for a slightly weaker version of Lemma 1].

LEMMA 1. *Let $X_1, X_2, \ldots$ be iid B-valued random variables and let $\alpha_n \uparrow \infty$ be a sequence such that*

$$\alpha_n/\sqrt{n} \text{ is eventually nondecreasing} \tag{3.1}$$

and

$$\alpha_n/n \text{ is eventually nonincreasing.} \tag{3.2}$$

Assume that

$$\{S_n/\alpha_n\} \text{ is bounded in probability.} \tag{3.3}$$

Then the following are equivalent:

$$x \in C(\{S_n/\alpha_n\}) \quad a.s. \tag{3.4}$$

and

$$\sum_n n^{-1} P\{\|S_n/\alpha_n - x\| < \varepsilon\} = \infty, \qquad \varepsilon > 0. \tag{3.5}$$

We next note that Lemma 1 can always be applied in the context of Theorem B. To see this, recall that by relations (2.3) and (2.4) in Einmahl (1993), the sequence $\{\gamma_n\}$ satisfies conditions (3.1) and (3.2). Moreover, by relation (2.4) and Corollary 3 of Klass (1977) we have

$$\liminf_{n\to\infty} \gamma_n/E|f(S_n)| > 0, \qquad f \in B^*, \tag{3.6}$$

from which we readily obtain (3.3) if $(B, \|\cdot\|)$ is finite dimensional. We thus have the following lemma.

LEMMA 2. *Let $X, X_1, X_2, \ldots$ be as in Theorem B. Then we have*

$$(x_1, x_2) \in C(\{S_n/\gamma_n\}) \quad a.s.$$

if and only if for any $\varepsilon > 0$,

$$\sum_n n^{-1} P\{\|S_n/\gamma_n - (x_1, x_2)\| < \varepsilon\} = \infty. \tag{3.7}$$

We now turn to random vectors satisfying conditions (1.9) and (1.11). It is obvious that if $X^{(1)}$ and $X^{(2)}$ are independent, then (3.7) is equivalent to

$$\sum_n n^{-1} P\{|S_n^{(1)}/\gamma_n - x_1| < \varepsilon\} P\{|S_n^{(2)}/\gamma_n - x_2| < \varepsilon\} = \infty, \qquad \varepsilon > 0. \tag{3.8}$$

The purpose of this subsection is to show that (3.8) can be significantly simplified for random vectors satisfying (1.9).

Let $K_i(\cdot)$ be the K-function corresponding to $X^{(i)}$ and set $\sigma_{n,i}^2 := \mathrm{Var}(X^{(i)} 1\{|X^{(i)}| \le K_i(n/LLn)\})$, $i = 1, 2$. As usual, let $x_+ := x \vee 0$. Then we have the following proposition.

PROPOSITION 1. *Let X be a mean zero random vector satisfying* (1.9) *and* (1.11). *The following are equivalent:*

$$(x_1, x_2) \in C(\{S_n/\gamma_n\}) \quad a.s., \tag{3.9}$$

$$\sum_n n^{-1} (Ln)^{-\beta_{n,1}^2 (|x_1| - \varepsilon)_+^2 - \beta_{n,2}^2 (|x_2| - \varepsilon)_+^2} = \infty, \qquad \varepsilon > 0, \tag{3.10}$$

where $\beta_{n,i} := \gamma_n / (2nLLn)^{1/2} \sigma_{n,i} \ge 1$, $i = 1, 2$.

PROOF. (i) We first show that (3.8) [and consequently (3.9)] is equivalent to

$$\sum_n n^{-1} P\{|S_{n,n}^{(1)}/\gamma_n - x_1| < \varepsilon\} P\{|S_{n,n}^{(2)}/\gamma_n - x_2| < \varepsilon\} = \infty, \qquad \varepsilon > 0, \tag{3.11}$$

where we set for $1 \le k \le n$, $i = 1, 2$, and $n \ge 1$:

$$S_{n,k}^{(i)} := \bar{S}_{n,k}^{(i)} - E\bar{S}_{n,k}^{(i)}$$

and

$$\bar{S}_{n,k}^{(i)} := \sum_{j=1}^{k} X_j^{(i)} 1\{|X_j^{(i)}| \le K_i(n/LLn)\}.$$

To that end we prove

$$\sum_n n^{-1} P\{|S_n^{(i)} - S_{n,n}^{(i)}| \geq \varepsilon\gamma_n\} < \infty, \qquad \varepsilon > 0,\ i = 1,2. \tag{3.12}$$

Using the trivial inequalities

$$P\{|S_{n,n}^{(i)}/\gamma_n - x_i| \leq \varepsilon\} \leq P\{|S_n^{(i)}/\gamma_n - x_i| < 2\varepsilon\} + P\{|S_n^{(i)} - S_{n,n}^{(i)}| \geq \varepsilon\gamma_n\}, \qquad i = 1,2,$$

it is easy to see that (3.11) implies

$$\sum_n n^{-1} P\{|S_n^{(1)}/\gamma_n - x_1| < 2\varepsilon\} P\{|S_n^{(2)}/\gamma_n - x_2| < 2\varepsilon\} = \infty, \qquad \varepsilon > 0,$$

which of course is equivalent to (3.8). A similar argument shows that (3.8) in combination with (3.12) implies (3.11).

In order to prove (3.12), we first note that by Lemma 5(b) of Einmahl (1993) and (1.9),

$$\sum_n P\{|X^{(i)}| > \delta\gamma_n\} < \infty, \qquad \delta > 0,\ i = 1,2, \tag{3.13}$$

which in turn via Lemma 7 in Einmahl (1993) implies

$$E|X^{(i)}| 1\{|X^{(i)}| > \delta\gamma_n\} = o(\gamma_n/n) \quad \text{as } n \to \infty,\ i = 1,2. \tag{3.14}$$

Setting

$$T_{n,k}^{(i)} := \overline{T}_{n,k}^{(i)} - E\overline{T}_{n,k}^{(i)},$$

$$\overline{T}_{n,k}^{(i)} := \sum_{j=1}^{k} X_j^{(i)} 1\{K_i(n/LLn) \leq |X_j^{(i)}| \leq \delta\gamma_n\}, \qquad 1 \leq k \leq n,\ i = 1,2,$$

where $\delta > 0$ will be specified later, we readily obtain, from (3.14),

$$P\{|S_n^{(i)} - S_{n,n}^{(i)}| \geq \varepsilon\gamma_n\} \leq P\left\{|T_{n,n}^{(i)}| \geq \frac{\varepsilon}{2}\gamma_n\right\} + nP\{|X^{(i)}| > \delta\gamma_n\}, \qquad i = 1,2, \tag{3.15}$$

Recalling (3.13), we see that (3.12) follows, once we have proved

$$\sum_n n^{-1} P\left\{|T_{n,n}^{(i)}| \geq \frac{\varepsilon}{2}\gamma_n\right\} < \infty, \qquad \varepsilon > 0,\ i = 1,2. \tag{3.16}$$

Letting $\delta = \varepsilon/16$ and using the Hoffmann–Jørgensen inequality [see, for instance, Proposition 6.7 of Ledoux and Talagrand (1991)], we can reduce the proof of (3.16) to proving

$$\sum_n n^{-1} P\left\{\max_{1\leq k\leq n} |T_{n,k}^{(i)}| \geq \frac{\varepsilon}{8}\gamma_n\right\}^2 < \infty,\ i = 1,2. \tag{3.17}$$

Next a straightforward application of Kolmogorov's maximal inequality yields for $i = 1, 2$,

$$P\left\{\max_{1\le k\le n} |T^{(i)}_{n,k}| \ge \frac{\varepsilon}{8}\gamma_n\right\} \le 64\varepsilon^{-2} n E(X^{(i)})^2 1\left\{K_i\left(\frac{n}{LLn}\right) < |X^{(i)}| \le \frac{\varepsilon}{16}\gamma_n\right\}\gamma_n^{-2},$$

which by the Hölder inequality is less than or equal to

$$64\varepsilon^{-2} n (E|X^{(i)}| 1\{|X^{(i)}| > K_i(n/LLn)\})^{1/2} (E|X^{(i)}|^3 1\{|X^{(i)}| \le \gamma_n\})^{1/2}\gamma_n^{-2}.$$

Since by the definition of the function $K_i(\cdot)$ [see relation (3.23) in Einmahl (1993)]

$$E|X^{(i)}| 1\{|X^{(i)}| > K_i(n/LLn)\} \le K_i(n/LLn) LLn/n,$$

we find that the last term is less than or equal to

$$64\varepsilon^{-2} n^{1/2} (E|X^{(i)}|^3 1\{|X^{(i)}| \le \gamma_n\})^{1/2}\gamma_n^{-3/2}, \qquad i = 1, 2.$$

The last bound in combination with Lemma 5(a) in Einmahl (1993) immediately implies (3.17).

(ii) Now let (x_1, x_2) be a point with $|x_1| \wedge |x_2| > 0$ and let Y be a standard normal r.v. If $\varepsilon < (|x_1| \wedge |x_2|)/2$, we can infer from a well known nonuniform bound on the rate of convergence in the central limit theorem [see, for instance, Theorem 13 on page 125 in Petrov (1975)] and the c_r-inequality,

$$\begin{aligned}|P\{|S^{(i)}_{n,n}/\gamma_n - x_i| < \varepsilon\} - P\{(|x_i| - \varepsilon)\gamma_n < \sqrt{n}\,\sigma_{n,i} Y < (|x_i| + \varepsilon)\gamma_n\}|/n \\ \le C_\varepsilon \gamma_n^{-3} E|\overline{X}^{(i)}_{n,n}|^3, \qquad i = 1, 2,\end{aligned}$$

where $C_\varepsilon > 0$ is a constant depending on ε.

Using once more Lemma 5(a) in Einmahl (1993), we find that (3.11) is equivalent to

$$\begin{aligned}\sum_n n^{-1} P\{(|x_1| - \varepsilon)\gamma_n < \sigma_{n,1}\sqrt{n}\, Y < (|x_1| + \varepsilon)\gamma_n\} \\ \times P\{(|x_2| - \varepsilon)\gamma_n < \sigma_{n,2}\sqrt{n}\, Y < (|x_2| + \varepsilon)\gamma_n\} = \infty, \qquad \varepsilon > 0.\end{aligned} \tag{3.18}$$

(iii) Relation (3.18) trivially implies

$$\begin{aligned}\sum_n n^{-1} P\{(|x_1| - \varepsilon)\gamma_n < \sigma_{n,1}\sqrt{n}\, Y\} \\ \times P\{(|x_2| - \varepsilon)\gamma_n < \sigma_{n,2}\sqrt{n}\, Y\} = \infty, \qquad \varepsilon > 0,\end{aligned} \tag{3.19}$$

which by a standard exponential inequality for the tail probabilities of normal random variables is the same as (3.10). Thus, in order to finish the proof for points (x_1, x_2) with $|x_1| \wedge |x_2| > 0$, it only remains to show that (3.19) also implies (3.18). We use the following inequality, the easy proof of which is omitted:

$$P\{Y \ge t\} \le 2P\{t \le Y \le t + \beta\}, \qquad \beta \ge t^{-1},\ t > 0. \tag{3.20}$$

Observing that by definition of the function K_i [see also relation (3.22) in Einmahl (1993)]

$$n\sigma_{n,i}^2 \le K_i^2(n/LLn)LLn, \qquad i = 1,2, \tag{3.21}$$

it is easy to see that if $|x_i| > 0$ and $\varepsilon < |x_i|/2$, we have for large enough n,

$$2\varepsilon\gamma_n/\sqrt{n}\,\sigma_{n,i} \ge \sqrt{n}\,\sigma_{n,i}/(|x_i| - \varepsilon)\gamma_n,$$

which implies via (3.20),

$$\begin{aligned} &P\{(|x_i| - \varepsilon)\gamma_n \le \sqrt{n}\,\sigma_{n,i}Y\} \\ &\quad \le 2P\{(|x_i| - \varepsilon)\gamma_n \le \sqrt{n}\,\sigma_{n,i}Y \le (|x_i| + \varepsilon)\gamma_n\}, \qquad i = 1,2. \end{aligned} \tag{3.22}$$

It is now evident that (3.19) implies (3.18), thereby completing the proof of Proposition 1 for points (x_1, x_2) with $|x_1| \wedge |x_2| > 0$.

(iv) If (x_1, x_2) is a point with $|x_1| \wedge |x_2| = 0$, we have either $(x_1, x_2) = (0,0)$ (Case 1), $(x_1, x_2) = (x_1, 0)$, where $|x_1| > 0$ (Case 2) or $(x_1, x_2) = (0, x_2)$, where $|x_2| > 0$ (Case 3). Observing that $S_n/\gamma_n \to_P 0$ [use, for instance, Lemma 6 in Einmahl (1993)], it is clear that we always have $(0,0) \in C(\{S_n/\gamma_n\})$ a.s.

Since $\sum_n n^{-1} = \infty$, this is consistent with (3.10). To prove Proposition 1 in the second case, we can use the fact that $S_n^{(2)}/\gamma_n \to_P 0$ to show that (3.8) is then equivalent to

$$\sum_n n^{-1} P\{|S_n^{(1)}/\gamma_n - x_1| < \varepsilon\} = \infty, \qquad \varepsilon > 0. \tag{3.23}$$

Combining relations (3.12) and (3.22) with the nonuniform bound for the convergence speed in the central limit theorem used in part (ii), we see that this happens if and only if

$$\sum_n n^{-1}(Ln)^{-\beta_{n,1}^2(|x_1| - \varepsilon)_+^2} = \infty, \qquad \varepsilon > 0, \tag{3.24}$$

which again is consistent with (3.10).

Finally, one obtains Proposition 1 in Case 3 from Case 2 by symmetry. Recalling (3.21), we find that $\beta_{n,i} \ge 1$, $i = 1,2$. □

3.2. *Conclusion of the proof.* We set for $s_1, s_2 > 0$,

$$I_1(s_1) := \{n\colon\ \beta_{n,1} \le s_1\},$$

$$I_2(s_2) := \{n\colon\ \beta_{n,2} \le s_2\},$$

and

$$J(s_1, s_2) := I_1(s_1) \cap I_2(s_2).$$

Let $\{t_m\colon\ m \ge 1\}$ be an enumeration of the rationals in $[1, \infty)$ and set for $k, l \ge 1$,

$$\delta_{k,l} := \sup\left\{\delta > 0\colon\ \sum_{n \in J(t_k, t_l)} n^{-1}(Ln)^{-\delta^2} = \infty\right\},$$

where $\sup\varnothing := 0$.

Put $E_{k,l} := \mathscr{E}(\delta_{k,l}/t_k, \delta_{k,l}/t_l)$, $k,l \geq 1$. We claim that with probability 1,

$$A = C(\{S_n/\gamma_n\}) = \mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2) \cup \mathrm{cl}\left(\bigcup_{k,l=1}^{\infty} E_{k,l}\right), \tag{3.25}$$

which of course implies the assertion of Theorem 1.

Since A as a cluster set must be closed, (3.25) follows once it has been proven that

$$A' \subset \mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2) \cup \bigcup_{k,l=1}^{\infty} E_{k,l}, \tag{3.26}$$

$$\mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2) \subset A, \tag{3.27}$$

$$E_{k,l} \subset A, \qquad k,l \geq 1, \tag{3.28}$$

where $A' := \{(x_1, x_2) \in A\colon (\rho x_1, \rho x_2) \in A \text{ for some } \rho > 1\}$. Notice that by Proposition 1 we have $A = \mathrm{cl}(A')$.

We first prove (3.26). From the definition of the two quantities σ_1 and σ_2 it is plain that if $(x_1, x_2) \in A$ is a point with $|x_1| \wedge |x_2| = 0$, then we must have $(x_1, x_2) \in \mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2)$. Therefore, in order to prove (3.26) it is enough to show that if (x_1, x_2) is a point of A' with $|x_1| \wedge |x_2| > 0$, then one can find indices $k, l \geq 1$ such that

$$(x_1, x_2) \in E_{k,l}. \tag{3.29}$$

To that end, we first note that one can infer from Proposition 1 that for any point (x_1, x_2) with these properties, there exists some $0 < \eta < 1/2$ such that

$$\sum_n n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2)} = \infty. \tag{3.30}$$

Set $t' := \inf\{t > 1\colon \sum_{n \in I_1(t)} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2)} = \infty\}$ and observe that the above set is nonempty. To see this, simply let $\tilde{t} := 2/|x_1|\sqrt{1+\eta}$. Then we have

$$\sum_{n \notin I_1(\tilde{t})} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2)} \leq \sum_n n^{-1}(Ln)^{-4} < \infty.$$

This of course implies

$$\sum_{n \in I_1(\tilde{t})} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2)} = \infty. \tag{3.31}$$

Otherwise, the series in (3.30) would be convergent, which is impossible.

Since $t' \leq \tilde{t}$ is finite, we can find a $k \geq 1$ such that

$$\bar{t} := (1 - \eta/2)^{1/2} t_k < t' < t_k. \tag{3.32}$$

By the definition of t' we have

$$\sum_{n \in I_1(\bar{t})} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2)} < \infty \tag{3.33}$$

and

$$\sum_{n\in I_1(t_k)} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2+\beta_{n,2}^2 x_2^2)} = \infty, \tag{3.34}$$

which of course implies

$$\sum_{n\in I_0} n^{-1}(Ln)^{-(1+\eta)(\beta_{n,1}^2 x_1^2+\beta_{n,2}^2 x_2^2)} = \infty, \tag{3.35}$$

where $I_0 := I_1(t_k) - I_1(\bar{t})$.

Noting that for $n \in I_0$,

$$\begin{aligned}(1+\eta)(\beta_{n,1}^2 x_1^2 + \beta_{n,2}^2 x_2^2) &\geq (1+\eta)(1-\eta/2)t_k^2 x_1^2 + (1+\eta)\beta_{n,2}^2 x_2^2 \\ &\geq t_k^2 x_1^2 + (1+\eta)\beta_{n,2}^2 x_2^2,\end{aligned}$$

we readily obtain, from (3.35),

$$\sum_{n\in I_1(t_k)} n^{-1}(Ln)^{-t_k^2 x_1^2-(1+\eta)\beta_{n,2}^2 x_2^2} = \infty. \tag{3.36}$$

Next, set

$$t'' := \inf\left\{t > 1\colon \sum_{n\in J(t_k,t)} n^{-1}(Ln)^{-t_k^2 x_1^2-(1+\eta)\beta_{n,2}^2 x_2^2} = \infty\right\}.$$

Using essentially the same argument as above, we can infer from (3.36) that t'' is finite. Picking an $l \geq 1$ such that

$$(1-\eta/2)^{1/2} t_l < t'' < t_l, \tag{3.37}$$

we can show by an obvious modification of the proof of (3.36) that

$$\sum_{n\in J(t_k,t_l)} n^{-1}(Ln)^{-t_k^2 x_1^2 - t_l^2 x_2^2} = \infty. \tag{3.38}$$

However, by the definition of $\delta_{k,l}$ this implies $t_k^2 x_1^2 + t_l^2 x_2^2 \leq \delta_{k,l}^2$, which means that $(x_1, x_2) \in \mathscr{E}(\delta_{k,l}/t_k, \delta_{k,l}/t_l) = E_{k,l}$, thereby completing the proof of (3.26).

We now turn to the proof of (3.27). Observe that using the same arguments as in the proof of Proposition 1, one can infer from Lemma 1 that

$$x_i \in C(S_n^{(i)}/\gamma_n) \quad \text{a.s.} \tag{3.39}$$

if and only if

$$\sum_n n^{-1}(Ln)^{-\beta_{n,i}^2(|x_i|-\varepsilon)_+^2} = \infty, \qquad \varepsilon > 0. \tag{3.40}$$

By definition of σ_i we have

$$-\sigma_i \in C(\{S_n^{(i)}/\gamma_n\}) \text{ a.s.} \quad \text{or} \quad \sigma_i \in C(\{S_n^{(i)}/\gamma_n\}) \text{ a.s.}, \ i = 1,2, \tag{3.41}$$

which in combination with (3.40) implies

$$[-\sigma_i, \sigma_i] \subset C(\{S_n^{(i)}/\gamma_n\}), \qquad i = 1,2. \tag{3.42}$$

Arguing as in part (iv) of the proof of Proposition 1, we readily obtain (3.27) from (3.42).

It remains to prove (3.28). Since by (3.27), $(0,0) \in A$, we only need to prove (3.28) for indices $k, l \geq 1$ with $\delta_{k,l} > 0$. Also note that all points (x_1, x_2) in $E_{k,l}$ with $|x_1| \wedge |x_2| = 0$ are in $\mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2)$. We thus can focus on the points in $E_{k,l}$ with $|x_1| \wedge |x_2| > 0$. In view of Proposition 1 it is enough to show

$$(3.43) \qquad \sum_n n^{-1}(Ln)^{-\beta_{n,1}^2(|x_1|-\varepsilon)^2-\beta_{n,2}^2(|x_2|-\varepsilon)^2} = \infty, \qquad 0 < \varepsilon < |x_1| \wedge |x_2|.$$

Since $t_k^2(|x_1| - \varepsilon)^2 + t_l^2(|x_2| - \varepsilon)^2 < \delta_{k,l}^2$, $(x_1, x_2) \in E_{k,l}$, we can infer from the definition of $\delta_{k,l}$ that

$$(3.44) \qquad \sum_{n \in J(t_k, t_l)} n^{-1}(Ln)^{-t_k^2(|x_1|-\varepsilon)^2-t_l^2(|x_2|-\varepsilon)^2} = \infty, \qquad 0 < \varepsilon < |x_1| \wedge |x_2|.$$

Observing that $\beta_{n,1} \leq t_k$ and $\beta_{n,2} \leq t_l$, $n \in J(t_k, t_l)$, we readily obtain (3.43) from (3.44). This completes the proof of Theorem 1. □

4. Proof of Theorem 2.

4.1. *Preliminaries.* We will show that one can obtain all possible cluster sets for symmetric random vectors $X = (X^{(1)}, X^{(2)})$: $\Omega \to \mathbb{R}^2$ satisfying the additional assumption

(4.1) $X^{(i)}$ is in the domain of attraction of the normal distribution, $i = 1, 2$.

The purpose of this part of the proof is to further simplify the criterion for clustering in that case.

Set $H_i(t) := E(X^{(i)})^2 1\{|X^{(i)}| \leq t\}$, $t \geq 0$, $i = 1, 2$. Then the following is well known [see, for instance, the equivalence lemma on page 264 in Hahn and Klass (1980)].

FACT 1. (a) *$H_i(t)$ is slowly varying at infinity, $i = 1, 2$.*
(b) $M_i(t) := E|X^{(i)}|1\{|X^{(i)}| \geq t\} = o(H_i(t)/t)$ *as $t \to \infty$, $i = 1, 2$.*

Let $t_i := \inf\{t > 1\colon H_i(t) > 0\}$, $x_i := t_i^2/H_i(t_i)$ and set $a_i(x) := \inf\{t > t_i\colon H_i(t)/t^2 \leq 1/x\}$, $x \geq x_i$, $i = 1, 2$.

LEMMA 3. (a) *$a_i(x) \leq K_i(x)$, $x \geq x_i$, $i = 1, 2$.*
(b) *Under assumption* (4.1) *we have, as $x \to \infty$,*

$$(4.2) \qquad a_i(x) \sim K_i(x), \qquad i = 1, 2.$$

PROOF. Recall that $K_i(x)$ is the inverse function of

$$G_i(y) := y^2 \Big/ \int_0^y E|X^{(i)}|1\{|X^{(i)}| > u\}\, du, \qquad i = 1, 2.$$

Using integration by parts, we find that

$$(4.3) \qquad G_i(y) = y^2/(H_i(y) + yM_i(y)), \qquad i = 1, 2,$$

from which we readily obtain

$$H_i(K_i(x))/K_i(x)^2 \le 1/x, \qquad i = 1,2, \tag{4.4}$$

and consequently (a).

The second part of Lemma 3 is an immediate consequence of Fact 1(b). □

LEMMA 4. *Under assumption* (4.1) *we have, for* $i = 1,2$:
(a) $a_i(x)/\sqrt{x}$, $x \ge x_i$ *is nondecreasing.*
(b) *Given* $\varepsilon > 0$, *there exists a constant* $\alpha_\varepsilon > x_1 \vee x_2$ *such that*

$$a_i(z)/a_i(y) \le (1+\varepsilon)z/y, \qquad z \ge y \ge \alpha_\varepsilon. \tag{4.5}$$

PROOF. We only prove (4.5). On account of Lemma 3(b) we can find an α_ε so that

$$a_i(y) \ge (1+\varepsilon)^{-1} K_i(y), \qquad y \ge \alpha_\varepsilon,\ i = 1,2, \tag{4.6}$$

which enables us to conclude that

$$a_i(z)/a_i(y) \le (1+\varepsilon)K_i(z)/K_i(y), \qquad z \ge y \ge \alpha_\varepsilon,\ i = 1,2. \tag{4.7}$$

Recalling that $K_i(x)/x$ is nonincreasing [see relation (2.3) in Klass (1976)], we get (4.5) from (4.7). □

LEMMA 5. (a) *For any* $t > t_i$, *the inequality* $t^2/H_i(t) \ge x$ *implies* $t \ge a_i(x)$, $i = 1,2$.
(b) *If* α_ε *is defined as in Lemma* 4 *and* $x \ge \alpha_\varepsilon$, *the inequality* $t^2/H_i(t) \le x$ *implies* $t \le (1+\varepsilon)a_i(x)$, $i = 1,2$.

PROOF. We only show (b). Note that in this case we have $t^2/(H_i(t) + tM_i(t)) \le x$ and consequently $t \le K_i(x)$, $i = 1,2$. Recalling (4.6), we obtain (b). □

The next lemma will be crucial for the proof of Theorem 2. It shows that if $X^{(1)}$ and $X^{(2)}$ are independent, then there is an explicit formula for $\tilde{K}(x)$ in terms of $a_i(x)$, $i = 1,2$.

LEMMA 6. *Let* X *be a mean zero random vector satisfying conditions* (1.11) *and* (4.1). *Then we have, as* $x \to \infty$,

$$\tilde{K}(x) \sim a_1(x) \vee a_2(x). \tag{4.8}$$

PROOF. We first note that by Lemma 3(a),

$$\tilde{K}(x) \ge a_1(x) \vee a_2(x), \qquad x \ge x_1 \vee x_2. \tag{4.9}$$

Let now $0 < \varepsilon < 1/2$ be fixed. Using Fact 1(b) along with (1.11) it is easy to see that for $y = (y_1, y_2) \in \mathbb{R}^2$ with $y_1^2 + y_2^2 \le 1$ and large t,

$$\begin{aligned} H_y(t) &:= E(y_1 X^{(1)} + y_2 X^{(2)})^2 1\{|y_1 X^{(1)} + y_2 X^{(2)}| \le t\} \\ &\le t^2(P\{|X^{(1)}| \ge t\} + P\{|X^{(2)}| \ge t\}) \\ &\quad + E(y_1 X^{(1)} + y_2 X^{(2)})^2 1\{|X^{(1)}| \vee |X^{(2)}| \le t\} \\ &\le \tfrac{1}{3}\varepsilon(H_1(t) + H_2(t)) + y_1^2 H_1(t) + y_2^2 H_2(t) \\ &\quad + 2y_1 y_2 E X^{(1)} 1\{|X^{(1)}| \le t\} E X^{(2)} 1\{|X^{(2)}| \le t\} \\ &\le (1 + \tfrac{2}{3}\varepsilon)\overline{H}(t) + 2tE|X^{(2)}| 1\{|X^{(2)}| > t\} \\ &\le (1 + \tfrac{3}{4}\varepsilon)\overline{H}(t), \end{aligned}$$

where $\overline{H}(t) := H_1(t) \vee H_2(t)$, $t \ge 0$. Moreover, we have for $y \in \mathbb{R}^2$ with $y_1^2 + y_2^2 \le 1$ and large t,

$$\begin{aligned} M_y(t) &:= tE|y_1 X^{(1)} + y_2 X^{(2)}| 1\{|y_1 X^{(1)} + y_2 X^{(2)}| \ge t\} \\ &\le 2t\left(E|X^{(1)}| 1\left\{|X^{(1)}| \ge \frac{t}{2}\right\} + E|X^{(2)}| 1\left\{|X^{(2)}| \ge \frac{t}{2}\right\}\right) \\ &\le \frac{\varepsilon}{4}\overline{H}(t), \end{aligned}$$

where we use Fact 1(b).

Setting $G_y(t) := t^2/(H_y(t) + tM_y(t))$, $t \ge 0$, we see that for large enough t,

$$G_y(t) \ge t^2/(1+\varepsilon)\overline{H}(t) \tag{4.10}$$

provided that $y_1^2 + y_2^2 \le 1$.

Let K_y be the inverse function of G_y, which as in (4.3) is equal to the K-function corresponding to $y_1 X^{(1)} + y_2 X^{(2)}$. It then follows from (4.10) that for large x,

$$x(1+\varepsilon) \ge K_y^2(x)/\overline{H}(K_y(x)). \tag{4.11}$$

Recalling Lemma 5(b), we can infer that for large enough x,

$$K_y(x) \le (1+\varepsilon)a_1((1+\varepsilon)x) \quad \text{or} \quad \le (1+\varepsilon)a_2((1+\varepsilon)x)$$

according as $\overline{H}(K_y(x)) = H_1(K_y(x))$ or $= H_2(K_y(x))$. Noticing that

$$\tilde{K}(x) = \sup\{K_y(x)\colon\ y_1^2 + y_2^2 \le 1\}, \qquad x \ge 0,$$

it is now plain that

$$\tilde{K}(x) \le (1+\varepsilon)(a_1(x(1+\varepsilon)) \vee a_2(x(1+\varepsilon))), \tag{4.12}$$

which in view of (4.5) is bounded above by

$$(1+\varepsilon)^3(a_1(x) \vee a_2(x))$$

provided x is large enough.

This implies, in conjunction with (4.9), the assertion of Lemma 6. □

We are now ready to prove the following proposition.

PROPOSITION 2. *Let X be a mean zero random vector satisfying* (1.9), (1.11) *and* (4.1). *Then we have $(x_1, x_2) \in C(\{S_n/\gamma_n\})$ a.s. if and only if*

$$\sum_n n^{-1}(Ln)^{-\tilde{\beta}^2_{n,1}(|x_1|-\varepsilon)^2_+ - \tilde{\beta}^2_{n,2}(|x_2|-\varepsilon)^2_+} = \infty, \qquad \varepsilon > 0, \tag{4.13}$$

where $\tilde{\beta}_{n,i} := (a_1(n/LLn) \vee a_2(n/LLn))/a_i(n/LLn)$, $i = 1, 2$.

PROOF. In view of Proposition 1, Lemma 3(b) and Lemma 6, we only need to show that as $n \to \infty$,

$$\begin{aligned} \sigma^2_{n,i} &= \mathrm{Var}(X^{(i)} 1\{|X^{(i)}| \le K_i(n/LLn)\}) \\ &\sim K_i^2(n/LLn)LLn/n, \qquad i = 1, 2. \end{aligned} \tag{4.14}$$

To see this, note that we have by the definition of the K-function [see also (4.3)],

$$\begin{aligned} \sigma^2_{n,i} &= H_i(K_i(n/LLn)) - (EX^{(i)} 1\{|X^{(i)}| \le K_i(n/LLn)\})^2 \\ &= n^{-1}K_i^2(n/LLn)LLn - K_i(n/LLn)M_i(K_i(n/LLn)) \\ &\quad - (EX^{(i)} 1\{|X^{(i)}| \le K_i(n/LLn)\})^2 \\ &\ge n^{-1}K_i^2(n/LLn)LLn - 2K_i(n/LLn)M_i(K_i(n/LLn)), \qquad i = 1, 2. \end{aligned}$$

Recalling Fact 1(b), we see that for any given $\varepsilon > 0$ and large enough n,

$$\begin{aligned} \sigma^2_{n,i} &\ge n^{-1}K_i^2(n/LLn)LLn - \varepsilon H_i(K_i(n/LLn)) \\ &\ge (1-\varepsilon)n^{-1}K_i^2(n/LLn)LLn, \qquad i = 1, 2. \end{aligned}$$

Combining the last inequality with (3.21), we obtain (4.14) and Proposition 2 has been proved. □

4.2. *Construction of the random vectors.* We need two further auxiliary results.

LEMMA 7. *Let X be a mean zero random vector satisfying* (4.1) *and*

$$H_i(t) \le C\exp((LLt)^\alpha), \qquad t \ge 0, \ i = 1, 2, \tag{4.15}$$

where $C > 0$ and $0 \le \alpha < 1$.

Then we have

$$\sum_n P\{\|X\| > \gamma_n\} < \infty.$$

PROOF. We show that (4.15) in conjunction with (4.1) actually implies a slightly stronger statement, namely,

$$\sum_n P\{|X^{(i)}| > a_i(n/LLn)LLn\} < \infty, \qquad i = 1, 2, \tag{4.16}$$

which by relation (4.4) of Kuelbs (1985) is equivalent to

$$E(X^{(i)})^2/\tilde{H}_i(|X^{(i)}|/LL|X^{(i)}|)LL|X^{(i)}| < \infty, \tag{4.17}$$

where $\tilde{H}_i(t) := H_i(t) \vee 1$, $i = 1, 2$.

Using the Karamata representation [see, for instance, Theorem 1.2 in Seneta (1976)] of $\tilde{H}_i(t)$, which is slowly varying at infinity on account of Fact 1(b), we have for any $0 < \eta < 1$ and large enough t,

$$\tilde{H}_i(t) \le 2\tilde{H}_i(t/LLt)(LLt)^{\eta}, \qquad i = 1, 2. \tag{4.18}$$

Therefore, it is enough to show that for some $0 < \eta < 1$,

$$E(X^{(i)})^2/\tilde{H}_i(|X^{(i)}|)(LL|X^{(i)}|)^{1-\eta} < \infty, \qquad i = 1, 2. \tag{4.19}$$

From the definition of the functions $\tilde{H}_i$, $i = 1, 2$, it is easy to see that for any $\delta > 0$,

$$E(X^{(i)})^2/\tilde{H}_i(|X^{(i)}|)(L\tilde{H}_i(|X^{(i)}|))^{1+\delta} < \infty, \qquad i = 1, 2, \tag{4.20}$$

which in combination with (4.15) implies

$$E(X^{(i)})^2/\tilde{H}_i(|X^{(i)}|)(LL|X^{(i)}|)^{\alpha(1+\delta)} < \infty, \qquad i = 1, 2. \tag{4.21}$$

Choosing δ small enough, we obtain (4.19) and consequently (4.16). □

LEMMA 8. *Let* $\{c_n^{(i)}\}$, $i = 1, 2$, *be two nondecreasing sequences satisfying, for some* $0 < \alpha < 1$,

$$0 \le c_n^{(i)} \le \exp((Ln)^{\alpha}), \qquad n \ge 1, \tag{4.22}$$

and as $n \to \infty$,

$$c_{n+1}^{(i)}/c_n^{(i)} \to 1, \qquad i = 1, 2. \tag{4.23}$$

One can find a symmetric random vector $(X^{(1)}, X^{(2)})$ *satisfying* (1.9), (1.11), (4.1) *and*

$$H_i(t) = c_n^{(i)}, \qquad \exp(n) \le t < \exp(n+1), \qquad n \ge 1,\ i = 1, 2. \tag{4.24}$$

PROOF. Set $p_{n,i} := \frac{1}{2}(c_n^{(i)} - c_{n-1}^{(i)})\exp(-2n)$, $n \ge 1$, where $c_0^{(i)} = 0$, $i = 1, 2$, and let $X^{(i)}$, $i = 1, 2$, be *independent* random variables satisfying

$$P\{X^{(i)} = \exp(n)\} = P\{X^{(i)} = -\exp(n)\} = p_{n,i}, \qquad n \ge 1, \tag{4.25}$$

and

$$P\{X^{(i)} = 0\} = 1 - 2\sum_n p_{n,i}, \qquad i = 1, 2. \tag{4.26}$$

It is easily checked that (4.25) and (4.26) imply (4.24). Moreover, we get from (4.23),

$$\limsup_{t\to\infty} H_i(et)/H_i(t) = 1, \tag{4.27}$$

which means that H_i is slowly varying at infinity, $i = 1, 2$, or, equivalently, (4.1).

Finally, X satisfies condition (1.9) by Lemma 7. □

We are now ready to carry out the construction of the desired symmetric random vector for which the cluster set $A = C(\{S_n/\gamma_n\})$ is equal to a given set $\tilde{A}$ which has the form

$$\tilde{A} = \mathscr{E}(\sigma_1, 0) \cup \mathscr{E}(0, \sigma_2) \cup \mathrm{cl}\left(\bigcup_{m=1}^{\infty} \mathscr{E}(a_m, b_m)\right), \tag{4.28}$$

where $\sigma_1 \vee \sigma_2 = 1$ and $0 \le a_m \le \sigma_1$, $0 \le b_m \le \sigma_2$, $m \ge 1$. Without loss of generality we can and will assume that $\sigma_1 \wedge \sigma_2 > 0$ and $\sigma_1 = 1$. [If $\sigma_1 < 1$, construct a random vector X, for which the cluster set A is equal to $\mathscr{E}(\sigma_2, 0) \cup \mathscr{E}(0, \sigma_1) \cup \mathrm{cl}(\bigcup_{m=1}^{\infty} \mathscr{E}(b_m, a_m))$, and consider $\tilde{X} := (X^{(2)}, X^{(1)})$.]

Let $\{\delta_m\}$, $\{\eta_m\} \subset [0, 1]$ and $\{s_m\}$, $\{t_m\} \subset [1, \infty]$ be sequences such that

$$\bigcup_{m=1}^{\infty} \mathscr{E}(a_m, b_m) = \bigcup_{m=1}^{\infty} \delta_m \mathscr{E}(s_m^{-1}, 1) \cup \bigcup_{m=1}^{\infty} \eta_m \mathscr{E}(1, t_m^{-1}), \tag{4.29}$$

where $1/\infty := 0$.

Set, for $k \ge 6$,

$$L_k := \{1 \le j \le 2^k \colon s_j \le 2^{k/2},\ \delta_j \ge 1/k\}, \qquad l_k := \# L_k,$$

$$L'_k := \{1 \le j \le 2^k \colon t_j \le 2^{k/2},\ \eta_j \ge 1/k\}, \qquad l'_k := \# L'_k.$$

Further, if $l_k \ge 1$, let π_k be a permutation of $\{1, \dots, 2^k\}$ such that

$$s_{\pi_k(1)} < \cdots < s_{\pi_k(l_k)} \tag{4.30}$$

and

$$\{s_j \colon j \in L_k\} = \{s_{\pi_k(j)} \colon 1 \le j \le l_k\}. \tag{4.31}$$

Likewise, if $l'_k \ge 1$, let π'_k be another permutation of $\{1, \dots, 2^k\}$ such that

$$t_{\pi'_k(1)} < \cdots < t_{\pi'_k(l'_k)} \tag{4.32}$$

and

$$\{t_j \colon j \in L'_k\} = \{t_{\pi'_k(j)} \colon 1 \le j \le l'_k\}. \tag{4.33}$$

To simplify our notation, we set

$$s_{k,l} := s_{k,\pi_k(l)}, \qquad \delta_{k,l} := \delta_{k,\pi_k(l)}, \qquad 1 \le l \le l_k,$$

$$t_{k,l} := t_{k,\pi'_k(l)}, \qquad \eta_{k,l} := \eta_{k,\pi'_k(l)}, \qquad 1 \le l \le l'_k,$$

$$s_{k,l_k+1} := e^{k/2}, \qquad \delta_{k,l_k+1} := \sigma_2,$$

$$t_{k,l'_k+1} := e^{(k+1)/2}, \qquad s_{k,0} := t_{k,0} := e^{-k/2}.$$

Further define

$$m_{k,0} := m_k := 4^{k^4}, \qquad m_{k,l+1} := m_{k,l} + k^3 + 4^{\delta^2_{k,l+1}k^4},$$
$$n_{k,l} := m_{k,l} + k^3, \qquad 0 \le l \le l_k,$$
$$\bar{m}_{k,0} := m_{k,l_k+1}, \qquad \bar{m}_{k,l+1} := \bar{m}_{k,l} + k^3 + 4^{\eta^2_{k,l+1}k^4}, \qquad 0 \le l \le l'_k - 1,$$
$$\bar{n}_{k,l} := \bar{m}_{k,l} + k^3, \qquad 0 \le l \le l'_k, \qquad \bar{m}_{k,l'_k+1} := m_{k+1}.$$

Next introduce two sequences $\{c_n^{(1)}\}$ and $\{c_n^{(2)}\}$ as follows:

$$c_n^{(1)} := 0, \qquad 1 \le n \le m_6,$$

and set for $k \ge 6$,

$$c_n^{(1)} := \exp(k^2),\ m_k \le n \le \bar{m}_{k,0},$$
$$c^{(1)}_{\bar{m}_{k,l}+j} := t^2_{k,l}\exp(k^2 + k + \log(t^2_{k,l+1}/t^2_{k,l})jk^{-3}), \qquad 0 \le j \le k^3,\ 0 \le l \le l'_k,$$
$$c_n^{(1)} := t^2_{k,l+1}\exp(k^2 + k), \qquad \bar{n}_{k,l} \le n \le \bar{m}_{k,l+1},\ 0 \le l \le l'_k.$$

Likewise, define

$$c_n^{(2)} := 0, \qquad 1 \le n \le m_6,$$

and set, for $k \ge 6$,

$$c^{(2)}_{m_{k,l}+j} := s^2_{k,l}\exp(k^2 + \log(s^2_{k,l+1}/s^2_{k,l})jk^{-3}), \qquad 0 \le j \le k^3,\ 0 \le l \le l_k,$$
$$c_n^{(2)} := s^2_{k,l+1}\exp(k^2), \qquad n_{k,l} \le n \le m_{k,l+1},\ 0 \le l \le l_k,$$
$$c_n^{(2)} := \exp(k^2 + k), \qquad \bar{m}_{k,0} \le n \le m_{k+1}.$$

It is easy to see that we have for the two foregoing sequences

$$(4.34) \qquad 0 \le c_n^{(i)} \le \exp((Ln)^{1/2})$$

and

$$(4.35) \qquad c^{(i)}_{n+1}/c^{(i)}_n \to 1 \quad \text{as } n \to \infty,\ i = 1, 2.$$

We can apply Lemma 8 and we obtain a symmetric random vector X satisfying (4.1) and the assumptions of Theorem 1 such that

$$(4.36) \qquad H_i(t) = c_n^{(i)}, \qquad \exp(n) \le t < \exp(n+1), \qquad n \ge 1,\ i = 1, 2.$$

From (4.36) and the definition of the two sequences $\{c_n^{(1)}\}$ and $\{c_n^{(2)}\}$ we can infer

$$(4.37) \qquad H_2(t)/H_1(t) = s^2_{k,l}, \quad \exp(n_{k,l-1}) \le t \le \exp(m_{k,l}), \quad 1 \le l \le l_k + 1,$$

$$(4.38) \qquad H_1(t)/H_2(t) = t^2_{k,l}, \quad \exp(\bar{n}_{k,l-1}) \le t \le \exp(\bar{m}_{k,l}), \quad 1 \le l \le l'_k + 1.$$

4.3. *Conclusion of the proof.* We now use (4.37) and (4.38) to get some information about the sequences $\{\tilde{\beta}_{n,i}\}$, $i = 1, 2$. To that end, we need three further lemmas. The first one is an immediate consequence of Lemma 5.

LEMMA 9. *Suppose that $H_1(t) = c^2 H_2(t)$, $t \in M$, where $M \subset (t_1, \infty)$ and $c > 0$. Then we have*

$$a_2(xc^2) \le a_1(x) \le (1+\varepsilon) a_2(xc^2)$$

whenever $a_1(x) \in M$ and $xc^2 \ge \alpha_\varepsilon$, which is defined as in Lemma 4.

The next lemma gives us precise information about how close $a_2(xc^2)$ is to $ca_2(x)$.

LEMMA 10. *If we define $X^{(2)}$ as above, we have for large k, $e^{-k/2} \le c \le e^{(k+1)/2}$ and any x satisfying $a_2(xe^{-k}) \ge \exp(m_{k-1})$,*

$$\exp(-3/k) \le a_2(xc^2)/ca_2(x) \le \exp(3/k). \tag{4.39}$$

PROOF. We show that for large k, $1 \le d \le e^{(k+1)/2}$ and y satisfying $a_2(y) \ge \exp(m_{k-1})$,

$$a_2(yd^2)/da_2(y) \le \exp(3/k). \tag{4.40}$$

We then obtain the upper bound in Lemma 10 for $c \ge 1$ by setting $y = x$, $d = c$ and the lower bound for $c \le 1$ by setting $y = xc^2$, $d = 1/c$. If $c \ge 1$, the lower bound in (4.39) follows from Lemma 4(a); so does the upper bound in (4.39) if $c \le 1$.

In order to prove (4.40), observe that

$$a_2^2(yd^2)/a_2^2(y) = d^2 H_2(a_2(yd^2))/H_2(a_2(y)). \tag{4.41}$$

Moreover, by Lemma 4(b) we have, for large enough y,

$$H_2(a_2(yd^2)) \le H_2(2d^2 a_2(y)). \tag{4.42}$$

Next note that by the definition of $\{c_n^{(2)}\}$, we have, for large k,

$$H_2(et)/H_2(t) \le \exp(3k^{-2}), \qquad t \ge \exp(m_{k-1}). \tag{4.43}$$

Setting $f(d) := \min\{j\colon e^j \ge 2d^2\}$ and noting that $f(d) \le 2k$ since $d^2 \le e^{k+1}$, we get

$$H_2(a_2(yd^2)) \le H_2(e^{f(d)} a_2(y)) \le \exp(6/k), \tag{4.44}$$

which in combination with (4.41) implies (4.39). □

The subsequent lemma provides a sufficient condition for (4.39) in terms of the function a_1.

LEMMA 11. *For large enough k, the inequality $a_1(x) \ge \exp(m_k)$ implies $a_2(xe^{-k}) \ge \exp(m_{k-1})$.*

PROOF. Observing that $H_1(t)/H_2(t) \le e^k$, $t \le \exp(m_{k-1})$, we can conclude that if $a_2(xe^{-k}) < \exp(m_{k-1})$, we have

$$xe^{-k} = a_2^2(xe^{-k})/H_2(a_2(xe^{-k})) \le e^k a_2^2(xe^{-k})/H_1(a_2(xe^{-k})),$$

which implies, via Lemma 5(a),

$$a_1(xe^{-2k}) \le a_2(xe^{-k}) \le \exp(m_{k-1}). \tag{4.45}$$

Using Lemma 4(b), we obtain, for any x satisfying $x \ge \alpha_1 e^{2k}$,

$$a_1(x) \le 2e^{2k} a_1(xe^{-2k}), \tag{4.46}$$

which by (4.45) implies, for large k,

$$a_1(x) < \exp(m_k). \tag{4.47}$$

If $x < \alpha_1 e^{2k}$, (4.47) easily follows from the fact that $a_1(x)/x \downarrow 0$ as $x \uparrow \infty$. □

We need some further notation. Set for $n \ge 1$, $\tilde{a}(n) := a_1(n/LLn)$ and let, for $k \ge 6$,

$$I'_{k,l} := \{n\colon \exp(m_{k,l-1}) < \tilde{a}(n) \le \exp(n_{k,l-1})\}, \qquad 1 \le l \le l_k + 1,$$
$$I_{k,l} := \{n\colon \exp(n_{k,l-1}) < \tilde{a}(n) \le \exp(m_{k,l})\}, \qquad 1 \le l \le l_k + 1,$$
$$J'_{k,l} := \{n\colon \exp(\bar{m}_{k,l-1}) < \tilde{a}(n) \le \exp(\bar{n}_{k,l-1})\}, \qquad 1 \le l \le l'_k + 1,$$
$$J_{k,l} := \{n\colon \exp(\bar{n}_{k,l-1}) < \tilde{a}(n) \le \exp(\bar{m}_{k,l})\}, \qquad 1 \le l \le l'_k + 1.$$

Combining the three previous lemmas, we can conclude that for any $0 < \alpha < 1$ there exists a k_α such that for $k \ge k_\alpha$,

$$(1-\alpha)s_{k,l} \le \tilde{\beta}_{n,1} \le (1+\alpha)s_{k,l}, \qquad n \in I_{k,l},\ 1 \le l \le l_k + 1, \tag{4.48a}$$

$$(1-\alpha)t_{k,l} \le \tilde{\beta}_{n,2} \le (1+\alpha)t_{k,l}, \qquad n \in J_{k,l},\ 1 \le l \le l'_k + 1. \tag{4.48b}$$

After these preparations we are finally ready to show that $A = \tilde{A}$.

PART 1. ($A \supset \tilde{A}$). Since A as a cluster set is closed, it is enough to verify that

$$A \supset \mathscr{E}(1,0), \tag{4.49}$$

$$A \supset \mathscr{E}(0,\sigma_2), \tag{4.50}$$

$$A \supset \delta_m \mathscr{E}(s_m^{-1}, 1), \qquad m \ge 1, \tag{4.51}$$

and

$$A \supset \eta_m \mathscr{E}(1, t_m^{-1}), \qquad m \ge 1. \tag{4.52}$$

We will apply Proposition 2. Observe that, for large enough k,

$$\sum_{n\in I_{k,l}} n^{-1} \geq \tfrac{1}{2}(m_{k,l} - n_{k,l-1}), \qquad 1 \leq l \leq l_k + 1, \tag{4.53}$$

$$\sum_{n\in J_{k,l}} n^{-1} \geq \tfrac{1}{2}(\bar{m}_{k,l} - \bar{n}_{k,l-1}), \qquad 1 \leq l \leq l'_k + 1. \tag{4.54}$$

To see (4.53) we note that, by Lemma 4(b),

$$\tilde{a}(z)/\tilde{a}(y) \leq 2z/y, \qquad z \geq y \geq \alpha_1, \tag{4.55}$$

which enables us to conclude that, for large k,

$$\begin{aligned}\sum_{n\in I_{k,l}} n^{-1} &\geq \log((i'_{k,l}+1)/(i_{k,l}-1)) - 2 \\ &\geq \log(\tilde{a}(i'_{k,l}+1)/\tilde{a}(i_{k,l}-1)) - \log 2 - 2 \\ &\geq \tfrac{1}{2}(m_{k,l} - n_{k,l-1}), \qquad 1 \leq l \leq l_k,\end{aligned}$$

where $i_{k,l} := \min I_{k,l}$ and $i'_{k,l} := \max I_{k,l}$, $1 \leq l \leq l_k$, $k \geq 1$. The proof of (4.54) is similar.

We return to the proof of (4.49). In view of Proposition 2 it is obviously enough to show that

$$\sum_n n^{-1}(Ln)^{-\tilde{\beta}^2_{n,1}} = \infty. \tag{4.56}$$

Using the trivial fact that $\tilde{\beta}_{n,1} \wedge \tilde{\beta}_{n,2} = 1$, it is clear from (4.48) that for large k,

$$\tilde{\beta}_{n,1} = 1, \qquad n \in J_{k,l'_k+1}, \tag{4.57}$$

which in combination with (4.54) implies

$$\sum_{n\in J_{k,l'_k+1}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,1}} \geq \tfrac{1}{2}(m_{k+1} - \bar{n}_{k,l'_k})4^{-(k+1)^4} \geq \tfrac{1}{4}. \tag{4.58}$$

Since the sets $J_{k,l}$ are disjoint, we readily obtain (4.56) from (4.58).

In order to show (4.50) it is by Proposition 2 enough to prove,

$$\sum_n n^{-1}(Ln)^{-a^2\tilde{\beta}^2_{n,2}} = \infty, \qquad 0 < a < \sigma_2. \tag{4.59}$$

Using the same argument as above, we find that for $n \in I_{k,l_k+1}$ and large k,

$$\tilde{\beta}_{n,2} = 1, \tag{4.60}$$

from which we can infer via (4.53) that

$$\sum_{n\in I_{k,l_k+1}} n^{-1}(Ln)^{-a^2\tilde{\beta}^2_{n,2}} \geq 4^{\sigma_2^2 k^4 - a^2(k+1)^4}/2, \tag{4.61}$$

where the last term goes to infinity if $a < \sigma_2$. This establishes (4.50).

We next prove (4.51). We only need to consider points (x_1, x_2) in $\delta_m \mathscr{E}(s_m^{-1}, 1)$ with $|x_1| \wedge |x_2| > 0$. [If $|x_1| \wedge |x_2| = 0$, it already follows from (4.49) and (4.50) that (x_1, x_2) belongs to A.] If such a point exists, we trivially must have $\delta_m > 0$ and $s_m < \infty$, and we can find for large enough k a unique index $1 \le r(k,m) \le l_k$ such that

$$\delta_{k,r(k,m)} = \delta_m, \qquad s_{k,r(k,m)} = s_m. \tag{4.62}$$

Set $I_k(m) := I_{k,r(k,m)}$ and let $0 < \varepsilon < |x_1| \wedge |x_2|$ be fixed. Recalling (4.47), it is easy to see that we have, if $s_m > 1$,

$$\tilde{\beta}_{n,1} \le (1+\varepsilon)s_m, \qquad \tilde{\beta}_{n,2} = 1, \qquad n \in I_k(m). \tag{4.63}$$

Moreover, if $s_m = 1$, we obtain from Lemma 9 for large enough k,

$$\tilde{\beta}_{n,1} \vee \tilde{\beta}_{n,2} \le 1 + \varepsilon, \; n \in I_k(m). \tag{4.64}$$

Combining (4.63) and (4.64) and using the obvious fact that $|x_1| \vee |x_2| \le 1$, it follows that

$$\begin{aligned}\tilde{\beta}_{n,1}^2(|x_1| - \varepsilon)^2 + \tilde{\beta}_{n,2}^2(|x_2| - \varepsilon)^2 &\le (1-\varepsilon^2)^2(s_m^2 x_1^2 + x_2^2) \\ &\le (1-\varepsilon^2)^2 \delta_m^2,\end{aligned}$$

which in turn implies that as $k \to \infty$,

$$\sum_{n \in I_k(m)} n^{-1}(Ln)^{-\tilde{\beta}_{n,1}^2(|x_1|-\varepsilon)^2 - \tilde{\beta}_{n,2}^2(|x_2|-\varepsilon)^2} \ge 4^{\delta_m^2(k^4-(k+1)^4(1-\varepsilon^2)^2)}/2 \to \infty.$$

Using Proposition 2, we get (4.51). The proof of (4.52) is similar.

PART 2. $(A \subset \tilde{A})$. Employing a similar argument as in the proof of (4.53) and noticing that $\tilde{\alpha}(x)/x^{1/3}$ is eventually nondecreasing, we get for sufficiently large k,

$$\sum_{n \in I'_{k,l}} n^{-1} \le 4(n_{k,l-1} - m_{k,l-1}), \qquad 1 \le l \le l_k + 1, \tag{4.65}$$

$$\sum_{n \in I_{k,l}} n^{-1} \le 4(m_{k,l} - n_{k,l-1}), \qquad 1 \le l \le l_k + 1, \tag{4.66}$$

$$\sum_{n \in J'_{k,l}} n^{-1} \le 4(\bar{n}_{k,l-1} - \bar{m}_{k,l-1}), \qquad 1 \le l \le l'_k + 1, \tag{4.67}$$

$$\sum_{n \in J_{k,l}} n^{-1} \le 4(\bar{m}_{k,l} - \bar{n}_{k,l-1}), \qquad 1 \le l \le l'_k + 1. \tag{4.68}$$

In particular, we have for any $\delta > 0$,

$$\sum_k \sum_{l=1}^{(l_k \vee l'_k)+1} \sum_{n \in I'_{k,l} \cup J'_{k,l}} n^{-1}(Ln)^{-\delta} < \infty, \tag{4.69}$$

where we set $I'_{k,l} = \varnothing$, $l > l_k + 1$ and $J'_{k,l} = \varnothing$, $l > l'_k + 1$.

We show that if (x_1, x_2) is a point outside $\tilde{A}$, it cannot belong to A. There are three possible cases: $x_1 = 0$ (Case 1), $x_2 = 0$ (Case 2) and $|x_1| \wedge |x_2| > 0$ (Case 3).

CASE 1. If $(0, x_2) \notin A$, we must have $|x_2| > \sigma_2$. We show

$$\sum_n n^{-1}(Ln)^{-\tilde{\beta}^2_{n,1}\rho^2} < \infty, \qquad \rho > \sigma_2, \tag{4.70}$$

which via Proposition 2 implies $(0, x_2) \notin \tilde{A}$.

Since $\tilde{\beta}_{n,2} \geq 1$ and $\delta_m \leq \sigma_2$, $m \geq 1$, we can infer from (4.66) that for large k,

$$\sum_{n \in I_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,2}\rho^2} \leq 4 \cdot 4^{-(\rho^2-\sigma_2^2)k^4}, \qquad 1 \leq l \leq l_k + 1.$$

Since $l_k \leq 2^k$, we readily obtain

$$\sum_k \sum_{l=1}^{l_k+1} \sum_{n \in I_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,2}\rho^2} < \infty, \qquad \rho > \sigma_2. \tag{4.71}$$

Next choose $\alpha > 0$ small enough so that

$$(1-\alpha)^2\rho^2 \geq \sigma_2^2(1 + \alpha/2).$$

Then we get from (4.48) for $n \in J_{k,l}$ and $k \geq k_\alpha$,

$$\tilde{\beta}^2_{n,2}\rho^2 \geq \sigma_2^2 t^2_{k,l}(1 + \alpha/2),$$

which in turn is

$$\geq \eta^2_{k,l}(1 + \alpha/2).$$

Here we have used the fact that $\sigma_2 \geq \eta_{k,l} t^{-1}_{k,l}$ which follows from the definition of σ_2.

Recalling that $\eta_{k,l} \geq 1/k$, $1 \leq l \leq l'_k$, we can infer that for $1 \leq l \leq l'_k$ and large k,

$$\sum_{n \in J_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,2}\rho^2} \leq 4 \times 2^{-\alpha k^2}.$$

Moreover, by (4.48) we have for $n \in J_{k,l'_k+1}$ and large enough k,

$$\tilde{\beta}^2_{n,2}\rho^2 \geq 2,$$

whence

$$\sum_k \sum_{n \in J_{k,l'_k+1}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,2}\rho^2} < \infty.$$

It is now plain that

$$\sum_k \sum_{l=1}^{l'_k+1} \sum_{n \in J_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,2}\rho^2} < \infty, \qquad \rho > \sigma_2. \tag{4.72}$$

Combining (4.69), (4.71) and (4.72), we get (4.70).

CASE 2. If $(x_1, 0) \notin \tilde{A}$, we must have $|x_1| > 1$. However, in this case $(x_1, 0)$ cannot be among the limit points of $\{S_n/\gamma_n\}$, since this would imply

$$\limsup_{n\to\infty} \|S_n\|/\gamma_n \geq |x_1| > 1 \quad \text{a.s.,} \tag{4.73}$$

which would be in contradiction to Theorem B.

CASE 3. We need a further lemma, the easy proof of which is omitted.

LEMMA 12. *Let* (y_1, y_2) *be a point in* $\mathbb{R}^2$.
(a) *If* $E = \delta\mathscr{E}(s^{-1}, 1)$, *where* $1 \leq s < \infty$ *and* $\operatorname{dist}((y_1, y_2), E) \geq \varepsilon > 0$, *then* $s^2 y_1^2 + y_2^2 \geq (\delta + \varepsilon)^2$.
(b) *If* $F = \eta\mathscr{E}(1, t^{-1})$, *where* $1 \leq t < \infty$ *and* $\operatorname{dist}((y_1, y_2), F) \geq \varepsilon > 0$, *then* $y_1^2 + t^2 y_2^2 \geq (\delta + \varepsilon)^2$.

Let now (x_1, x_2) be a point outside $\tilde{A}$ with $|x_1| \wedge |x_2| > 0$. Set $\rho := \operatorname{dist}((x_1, x_2), \tilde{A}))$ and choose $0 < \varepsilon < (|x_1| \wedge |x_2|)/2$ so small that

$$(1-\varepsilon)^2(|x_i| - \varepsilon)^2 \geq x_i^2/(1+\rho), \qquad i = 1, 2. \tag{4.74}$$

We show for any ε satisfying (4.74),

$$\sum_n n^{-1}(Ln)^{-\tilde{\beta}_{n,1}^2(|x_1|-\varepsilon)^2 - \tilde{\beta}_{n,2}^2(|x_2|-\varepsilon)^2} < \infty, \tag{4.75}$$

which by Proposition 2 implies $(x_1, x_2) \notin A$.

Observe that we have by Lemma 12, if $s_m < \infty$,

$$s_m^2 x_1^2 + x_2^2 \geq (\delta_m + \rho)^2. \tag{4.76}$$

Applying (4.47) with $\alpha = \varepsilon$, we obtain from (4.74), for large k,

$$\tilde{\beta}_{n,1}^2(|x_1| - \varepsilon)^2 + \tilde{\beta}_{n,2}^2(|x_2| - \varepsilon)^2 \geq (1+\rho)\delta_{k,l}^2, \qquad n \in I_{k,l},\ 1 \leq l \leq l_k. \tag{4.77}$$

Recalling (4.66) and the fact that $\delta_{k,l} \geq 1/k$, we get for large k,

$$\sum_{n\in I_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}_{n,1}^2(|x_1|-\varepsilon)^2 - \tilde{\beta}_{n,2}^2(|x_2|-\varepsilon)^2} \leq 4 \times 4^{-\rho k^2}, \qquad 1 \leq l \leq l_k. \tag{4.78}$$

Further, note that since $\varepsilon < |x_1|/2$, we also have for large enough k,

$$\tilde{\beta}_{n,1}^2(|x_1| - \varepsilon)^2 \geq \tilde{\beta}_{n,1}^2 x_1^2/4 \geq 2, \qquad n \in I_{k,l_k+1}. \tag{4.79}$$

Combining (4.78) and (4.79) we find that

$$\sum_k \sum_{l=1}^{l_k+1} \sum_{n\in I_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}_{n,1}^2(|x_1|-\varepsilon)^2 - \tilde{\beta}_{n,2}^2(|x_2|-\varepsilon)^2} < \infty. \tag{4.80}$$

A straightforward modification of the proof of (4.80) yields

$$\sum_k \sum_{l=1}^{l'_k+1} \sum_{n \in J_{k,l}} n^{-1}(Ln)^{-\tilde{\beta}^2_{n,1}(|x_1|-\varepsilon)^2-\tilde{\beta}^2_{n,2}(|x_2|-\varepsilon)^2} < \infty \tag{4.81}$$

provided $\varepsilon > 0$ has been chosen as in (4.74). Recalling (4.69), we see that (4.75) holds true, thereby completing the proof of Theorem 2. □

5. Proof of Theorem 3. We first note that by an obvious modification of part(i) of the proof of Proposition 1 one can show that if X is a random vector satisfying (1.9), one has $(x_1, x_2) \in C(\{S_n/\gamma_n\})$ a.s. if and only if

$$\sum_n n^{-1} P\{\|S_{n,n}/\gamma_n - (x_1, x_2)\| < \varepsilon\} = \infty, \qquad \varepsilon > 0, \tag{5.1}$$

where $S_{n,n} := \bar{S}_{n,n} - E\bar{S}_{n,n}$ and $\bar{S}_{n,n} := \sum_{j=1}^n X_j 1\{\|X_j\| \le \tilde{K}(n/LLn)\}$. To further simplify criterion (5.1), we need a lemma which is implicitly contained in Theorem 2 in Kuelbs and Kurtz (1975). [See also Lemma 5 in Einmahl (1991), for the special case $x = 0$.] Though we need this result only for finite-dimensional random vectors, we give a general version for Hilbert space-valued random variables.

LEMMA 13. *Let $X_1, \dots, X_n$ be independent mean zero random variables taking values in a separable Hilbert space H satisfying $E\|X_j\|^3 < \infty$, $1 \le j \le n$. Let x be a point in H and let $\alpha_n \uparrow \infty$. If $Y_1, \dots, Y_n$ are independent Gaussian mean zero r.v.'s with $\operatorname{cov}(Y_j) = \operatorname{cov}(X_j)$, $1 \le j \le n$, we have for any $\varepsilon > 0$,*

$$\text{(a)} \qquad P\left\{\left\|\sum_{j=1}^n X_j/\alpha_n - x\right\| < \varepsilon\right\} \le P\left\{\left\|\sum_{j=1}^n Y_j/\alpha_n - x\right\| < 2\varepsilon\right\} + A\varepsilon^{-3}\alpha_n^{-3}\sum_{j=1}^n E\|X_j\|^3,$$

$$\text{(b)} \qquad P\left\{\left\|\sum_{j=1}^n Y_j/\alpha_n - x\right\| < \varepsilon\right\} \le P\left\{\left\|\sum_{j=1}^n X_j/\alpha_n - x\right\| < 2\varepsilon\right\} + A\varepsilon^{-3}\alpha_n^{-3}\sum_{j=1}^n E\|X_j\|^3,$$

where $A > 0$ is an absolute constant.

Recalling Lemma 5(a) in Einmahl (1993), we now can infer that (5.1) holds true if and only if

$$\sum_n P\{\|\sqrt{n}\, Y_n - \gamma_n(x_1, x_2)\| < \varepsilon\gamma_n\} = \infty, \qquad \varepsilon > 0, \tag{5.2}$$

where Y_n is a sequence of two-dimensional mean zero random vectors with $\operatorname{cov}(Y_n) = \operatorname{cov}(X 1\{\|X\| \le \tilde{K}(n/LLn)\})$, $n \ge 1$.

Since Y_n has a symmetric distribution, we immediately get property (2.5) from (5.2).

To prove (2.6), we use another well-known property of Gaussian random vectors which follows from a classical result of Anderson (1955).

FACT 2. *Let Y be a two-dimensional normal mean zero random vector and let $z = (z_1, z_2)$ be a point in $\mathbb{R}^2$. Then we have for any $0 \le t \le 1$, $\delta > 0$,*

$$P\{\|Y - tz\| < \delta\} \ge P\{\|Y - z\| < \delta\}.$$

Fact 2 in conjunction with (5.2) implies that whenever $(x_1, x_2) \in C(\{S_n/\gamma_n\})$ a.s., we must have $(tx_1, tx_2) \in C(\{S_n/\gamma_n\})$ a.s., $0 \le t \le 1$, which means that A has to be star-shaped. □

REMARK. The reader might have noticed that the above proof can be extended to Hilbert space-valued random variables when using a slightly more complicated argument for proving (5.1) such as that employed in the proof of Lemma 8 in Einmahl (1993). This means that the cluster set is also star-shaped and symmetric about zero in this case. Property (2.7), however, will not be satisfied, in general [see Theorem 4 in Einmahl (1993)].

6. Proof of Theorem 4.

6.1. *The construction.* We first observe that any closed set $\tilde{A}$ satisfying (2.5)–(2.7) can be written as a closure of (at most) countably many line segments; that is, we have

$$\tilde{A} = \mathrm{cl}\left(\bigcup_{j=1}^{\infty} \mathscr{L}_j\right), \tag{6.1}$$

where $\mathscr{L}_j := \{(t\cos\theta_j,\ t\sin\theta_j)\colon |t| < \sigma_j\}$, $j \ge 1$, for suitable sequences $0 \le \theta_j < 2\pi$ and $0 \le \sigma_j \le 1$. Without loss of generality, we assume that $\sigma_1 = 1$. Then the sets $L_k := \{1 \le j \le k\colon \sigma_j \ge 1/k\}$, $k \ge 1$, are nonempty and we can write

$$L_k = \{j_1(k), \dots, j_{l_k}(k)\}, \tag{6.2}$$

where $j_1(k) < \cdots < j_{l_k}(k)$ and $l_k := \#L_k \ge 1$, $k \ge 1$. Set

$$\sigma_{k,l} := \sigma_{j_{l+1}(k)}, \qquad 1 \le l \le l_k - 1, \qquad \sigma_{k,l_k} := \sigma_1.$$

Likewise,

$$\theta_{k,l} := \theta_{j_{l+1}(k)}, \qquad 1 \le l \le l_k - 1, \qquad \theta_{k,l_k} := \theta_1.$$

Further, define for $k \ge 1$,

$$m_k := 4^{k^4}, \qquad m_{k,0} := m_k,$$

$$m_{k,l+1} := m_{k,l} + k^3 + 4^{\sigma_{k,l+1}^2 k^4}, \qquad 0 \le l \le l_k - 2,$$

$$n_{k,l} := m_{k,l} + k^3, \qquad 1 \le l \le l_k - 1, \qquad m_{k,l_k} := m_{k+1}.$$

Let Z be a symmetric random variable such that the truncated second moment function

$$H(t) := EZ^2 1\{|Z| \le t\}, \qquad 0 \le t < \infty,$$

satisfies

$$H(t) = c_n, \qquad \exp(n) \le t < \exp(n+1), \qquad n \ge 1, \tag{6.3}$$

where

$$c_n := 0,\ 0 \le n < m_4,$$

and for $k \ge 4$,

$$\begin{aligned}
c_{m_k} &:= \exp(k^3),\\
c_{m_{k,l}+j} &:= \exp(k^3 + lk + j/k^2), \qquad 0 \le j \le k^3,\\
c_m &:= \exp(k^3 + (l+1)k), \qquad n_{k,l} \le m \le m_{k,l+1}, \qquad 0 \le l \le l_k - 2,\\
c_{m_{k,l_k-1}+j} &:= \exp((3k^2 + 4k + 1 - kl_k)jk^{-3} + k(l_k - 1) + k^3), \qquad 0 \le j \le k^3,\\
c_m &:= \exp((k+1)^3), \qquad n_{k,l_k-1} \le m \le m_{k,l_k} = m_{k+1}.
\end{aligned}$$

The existence of such a random variable follows by the same argument as in Lemma 8.

Define the two-dimensional random vector $X = (X^{(1)}, X^{(2)})$ by

$$X^{(1)} := \sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \cos(\theta_{k,l}) Z 1\{\exp(m_{k,l-1}) < |Z| \le \exp(m_{k,l})\},$$

$$X^{(2)} := \sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \sin(\theta_{k,l}) Z 1\{\exp(m_{k,l-1}) < |Z| \le \exp(m_{k,l})\}.$$

For a similar construction in a different context, refer to Example 4 in Hahn and Klass (1980).

We now claim that if we define X as above, then we have almost surely

$$A := C(\{S_n/\gamma_n\}) = \tilde{A}. \tag{6.4}$$

To prove (6.4), we need some auxiliary results.

6.2. *Preliminaries.* Since the above function $H(t)$ is slowly varying at infinity, the random variable Z is in the domain of attraction to the standard normal distribution and we can infer the following lemma from Fact 1.

LEMMA 14. *We have as $t \to \infty$:*

(a) $E|Z|1\{|Z| > t\} =: M(t) = o(H(t)/t)$.

(b) $P\{|Z| > t\} = o(H(t)/t^2)$.

Let $\overline{K}$ be the K-function corresponding to Z and set

$$\bar{\gamma}_n := \sqrt{2}\, \overline{K}(n/LLn) LLn, n \ge 1.$$

Then arguing as in the proof of Lemma 7 and using Lemma 5(b) in Einmahl (1993), we find that

$$\sum_{n=1}^{\infty} P\{|Z| > \delta\bar{\gamma}_n\} < \infty, \qquad \delta > 0. \tag{6.5}$$

Next set for any $-\infty < \theta < \infty$,

$$X(\theta) := (\cos\theta)X^{(1)} + (\sin\theta)X^{(2)}$$

and observe that

$$X(\theta) = \sum_{k=1}^{\infty}\sum_{l=1}^{l_k} \cos(\theta - \theta_{k,l})Z1\{\exp(m_{k,l-1}) < |Z| \le \exp(m_{k,l})\}, \tag{6.6}$$

from which we readily obtain,

$$|X(\theta)| \le |Z|. \tag{6.7}$$

Letting K_θ be the K-function corresponding to $X(\theta)$, we obviously have

$$\tilde{K}(x) := \sup_\theta K_\theta(x), \qquad x \ge 0, \tag{6.8}$$

and we find that

$$\gamma_n \le \bar{\gamma}_n, \qquad n \ge 1. \tag{6.9}$$

We next need upper bounds for $\bar{\gamma}_n/\gamma_n$, $n \ge 1$. Set for $-\infty < \theta < \infty$ and $t > 0$,

$$H_\theta(t) := EX^2(\theta)1\{|X(\theta)| \le t\},$$
$$\tilde{H}_\theta(t) := EX^2(\theta)1\{|Z| \le t\},$$
$$M_\theta(t) := E|X(\theta)|1\{|X(\theta)| > t\}$$

and observe that as in (4.3), K_θ is the inverse function of $t^2/(H_\theta(t)+tM_\theta(t)) =: G_\theta(t)$.

Using the above definitions and (6.7), and recalling Lemma 14, one can easily prove the next lemma.

LEMMA 15. *We have:*
(a) $\tilde{H}_\theta(t) \le H_\theta(t)$, $t \ge 0$, $-\infty < \theta < \infty$.
(b) $H_\theta(t) \le \tilde{H}_\theta(t) + t^2P\{|Z| > t\}$, $t \ge 0$, $-\infty < \theta < \infty$.
(c) $\sup_\theta tM_\theta(t)/H(t) \to 0$ *as* $t \to \infty$.

From formula (6.6) and the definition of the function H, we get the following lemma.

LEMMA 16. *We have for* $1 \le l \le l_k$ *and* $k \ge 1$:
(a) $\tilde{H}_\theta(t) \ge \cos^2(\theta - \theta_{k,l})H(t)(1 - e^{-k})$ *and*
(b) $\tilde{H}_\theta(t) \le \cos^2(\theta - \theta_{k,l})(H(t) - H(\exp(m_{k,l-1}))) + H(\exp(m_{k,l-1}))$
provided that $\exp(n_{k,l-1}) \le t \le \exp(m_{k,l})$.

We are now ready to prove a further lemma which will be crucial for the proof of Theorem 4.

LEMMA 17. (a) *There exists a sequence* $\delta_k \downarrow 0$ *such that*

$$\bar{\gamma}_n \le (1+\delta_k)\gamma_n, \qquad n \in J_{k,l},\ 1 \le l \le l_k, \tag{6.10}$$

where $J_{k,l} := \{n\colon \exp(n_{k,l-1}) \le \bar{K}(n/LLn) \le \exp(m_{k,l})\}$.
(b) $\limsup_{n\to\infty} \bar{\gamma}_n/\gamma_n \le 2$.

PROOF. By Lemma 16(a) we have for a suitable sequence $\delta_{k,1} \downarrow 0$ and $1 \le l \le l_k$, $k \ge 1$,

$$\tilde{H}_{\theta_{k,l}}(t) \ge (1-\delta_{k,1})H(t), \qquad \exp(n_{k,l-1}) \le t \le \exp(m_{k,l}), \tag{6.11}$$

which of course implies via Lemma 14 and Lemma 15(a),

$$H_{\theta_{k,l}}(t) \ge (1-\delta_{k,2})(H(t)+tM(t)), \qquad \exp(n_{k,l-1}) \le t \le \exp(m_{k,l}), \tag{6.12}$$

where $\delta_{k,2} \downarrow 0$.
We now can infer from (6.12),

$$\begin{aligned} G_{\theta_{k,l}}(\bar{K}(n/LLn)) &\le (1-\delta_{k,2})^{-1}G(\bar{K}(n/LLn)) \\ &= (1-\delta_{k,2})^{-1}n/LLn, \qquad n \in J_{k,l}, \end{aligned} \tag{6.13}$$

where $G(t) := t^2/(H(t)+tM(t))$ is the inverse function of $\bar{K}(x)$ [refer to (4.3)].
Using the fact that $K_{\theta_{k,l}}(x)/x$ is nonincreasing, we get from (6.13),

$$\bar{K}(n/LLn) \le (1-\delta_{k,2})^{-1}K_{\theta_{k,l}}(n/LLn), \qquad n \in J_{k,l}, \tag{6.14}$$

which proves (a).
To see (b), let $k \ge 5$ and assume that $t \in (\exp(m_{k,l-1}), \exp(m_{k,l})]$, where $1 \le l \le l_k$. Then we have either $H(t)/H(\exp(m_{k,l-1})) \ge 2$ or $H(t)/H(\exp(m_{k,l-1})) < 2$. In the first case, we obtain from formula (6.6),

$$\tilde{H}_{\theta_{k,l}}(t) \ge H(t)/2. \tag{6.15}$$

As for the second case, let $\theta^*_{k,l} := \theta_{k,l-1}$, $2 \le l \le l_k$, and $\theta^*_{k,1} := \theta_{k-1,l_{k-1}}$. Using the monotonicity of $H_{\theta^*_{k,l}}$, we can infer from Lemma 16(a),

$$\tilde{H}_{\theta^*_{k,l}}(t) \ge H(\exp(m_{k,l-1}))(1-e^{-k+1}). \tag{6.16}$$

Recalling that in the second case, $H(\exp(m_{k,l-1}))/H(t) \ge 1/2$, we can infer from (6.15) and (6.16) that

$$\liminf_{t\to\infty} \sup_{\theta} \tilde{H}_\theta(t)/H(t) \ge 1/2. \tag{6.17}$$

Arguing as in the proof of (a), we readily obtain assertion (b) from (6.17). □

Using Lemma 17(b) in conjunction with relations (6.5) and (6.7), we get

$$\sum_{n=1}^{\infty} P\{\|X\| > \delta\gamma_n\} < \infty, \qquad \delta > 0. \tag{6.18}$$

We now can argue as in part (i) of the proof of Proposition 1 to conclude that we have $(x_1, x_2) \in C(\{S_n/\gamma_n\}$ a.s. if and only if

$$\sum_{n=1}^{\infty} n^{-1} P\{\|S_{n,n}/\gamma_n - (x_1, x_2)\| < \varepsilon\} = \infty, \qquad \varepsilon > 0, \tag{6.19}$$

where $S_{n,n} := \sum_{j=1}^{n} X_j 1\{\|X_j\| \le \bar{K}(n/LLn)\}$.

Using Lemma 13, it follows that (6.19) in turn is equivalent to

$$\sum_{n=1}^{\infty} n^{-1} P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} = \infty, \qquad \varepsilon > 0, \tag{6.20}$$

where Y_n are two-dimensional Gaussian random vectors with mean zero and $\operatorname{cov}(Y_n) = \operatorname{cov}(X1\{\|X\| \le \bar{K}(n/LLn)\})$, $x_{n,i} := x_i\gamma_n/\sqrt{n}$, $i = 1, 2$.

6.3. *Conclusion of the proof.* We first show $A \supset \tilde{A}$. Since A is a closed set, it is enough to prove

$$\mathscr{L}_j \subset A, \qquad j \ge 1. \tag{6.21}$$

Recalling (6.20), this can be accomplished by showing for $\varepsilon > 0$, $|t| < \sigma_j$, $j \ge 1$,

$$\sum_n n^{-1} P\{\|Y_n - t\gamma_n/\sqrt{n}(\cos\theta_j, \sin\theta_j)\| < \varepsilon\gamma_n/\sqrt{n}\} = \infty, \tag{6.22}$$

which after an orthonormal transformation becomes

$$\sum_n n^{-1} P\{\|(Y_n(\theta_j), Y_n(\theta_j')) - (t\gamma_n/\sqrt{n}, 0)\| < \varepsilon\gamma_n/\sqrt{n}\} = \infty, \tag{6.23}$$

where $Y_n(\theta) := (\cos\theta)Y_n^{(1)} + (\sin\theta)Y_n^{(2)}$ if $Y_n = (Y_n^{(1)}, Y_n^{(2)})$ and $\theta' = \theta + \pi/2$.

Next observe that

$$\begin{aligned} P\left\{\left\|(Y_n(\theta_j),\ Y_n(\theta_j')) - \left(\frac{t\gamma_n}{\sqrt{n}}, 0\right)\right\| < \frac{\varepsilon\gamma_n}{\sqrt{n}}\right\} &\ge P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}},\ |Y_n(\theta_j')| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \\ &\ge P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} - P\left\{|Y_n(\theta_j')| \ge \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\}. \end{aligned}$$

Further note that if k is large enough and $\sigma_j > 0$, we can find an index $1 \le r_k(j) \le l_k$ such that $\theta_{k,r_k(j)} = \theta_j$. Setting $J_k(j) := J_{k,r_k(j)}$, where $J_{k,l}$ is defined as in Lemma 17, it is easy now to see that (6.23) follows once it has been proven that as $k \to \infty$,

$$\sum_{n\in J_k(j)} n^{-1}P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \to \infty, \tag{6.24}$$

and for large k,

$$P\left\{|Y_n(\theta'_j)| \geq \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \leq \frac{1}{2}P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\}, \qquad n \in J_k(j). \tag{6.25}$$

We first prove (6.24). Observing that $\|X\| = |Z|$, we get for any θ,

$$\mathrm{Var}(Y_n(\theta)) = \mathrm{Var}(X(\theta)1\{|Z| \leq \overline{K}(n/LLn)\}) = \tilde{H}_\theta(\overline{K}(n/LLn)), \tag{6.26}$$

where by Lemma 16(b) and Lemma 17(b) the last term is less than or equal to

$$H(\overline{K}(n/LLn)) = O(\gamma_n^2/(nLLn)).$$

Assuming that $|t| > \sigma_j/2 > \varepsilon$, we can use the same argument as in part (iii) of the proof of Proposition 1, and we find that for large enough n,

$$\begin{aligned} &P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \\ &\quad\geq \frac{1}{2}P\left\{Y_n(\theta_j) > \left(|t| - \frac{\varepsilon}{2}\right)\frac{\gamma_n}{\sqrt{n}}\right\}. \end{aligned} \tag{6.27}$$

Using (6.26) and Lemma 16(a), we get for $n \in J_k(j)$ and large k,

$$\mathrm{Var}(Y_n(\theta_j)) \geq H(\overline{K}(n/LLn))(1 - \delta'_{k,1}),$$

which by the definition of the function $\overline{K}$, Lemma 14 and (6.9) is greater than or equal to

$$(1 - \delta'_{k,2})\gamma_n^2/(2nLLn),$$

where $\delta'_{k,i} \downarrow 0$, $i = 1, 2$.

Combining the last bound with (6.27), we get for $\sigma_j/2 < |t| < \sigma_j$, $n \in J_k(j)$ and large enough k,

$$P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \geq (Ln)^{-t^2}. \tag{6.28}$$

Using a 1-dimensional version of Fact 2, we can infer that for *any* t with $|t| < \sigma_j$, $n \in J_k(j)$ and large k,

$$P\left\{\left|Y_n(\theta_j) - \frac{t\gamma_n}{\sqrt{n}}\right| < \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \geq (Ln)^{-\sigma_j^2+\beta}, \tag{6.29}$$

where $\beta > 0$ depends on t.

A similar calculation as in (4.54) yields for large k,

$$\sum_{n\in J_k(j)} n^{-1} \geq 4^{k^4\sigma_j^2}/2. \tag{6.30}$$

Combining (6.29) and (6.30), we readily obtain (6.24). We now turn to the proof of (6.25). In view of (6.29) it is enough to show that for large k,

$$P\left\{|Y_n(\theta_j')| \geq \frac{\varepsilon}{2}\frac{\gamma_n}{\sqrt{n}}\right\} \leq (Ln)^{-2}, \qquad n \in J_k(j). \tag{6.31}$$

To see (6.31), simply observe that by (6.26) and Lemma 16(b),

$$\operatorname{Var}(Y_n(\theta_j')) \leq H(\exp(m_{k,l-1})), \qquad n \in J_k(j). \tag{6.32}$$

By definition of the function H the last term is less than or equal to

$$e^{-k}H(\exp(m_{k,l})) = e^{-k}H(\overline{K}(n/LLn)),$$

which in turn is less than or equal to

$$e^{-k}\bar{\gamma}_n^2/(2nLLn).$$

Using the last bound and Lemma 17(b), we immediately get (6.31). This completes the proof of the inclusion $A \supset \tilde{A}$.

To prove the other inclusion, we show that if (x_1, x_2) is a fixed point outside $\tilde{A}$, it cannot belong to A.

Set $\beta := \operatorname{dist}((x_1, x_2), \tilde{A}) > 0$ and $\varepsilon := \beta/2$. In view of (6.20) it is enough to prove that

$$\sum_n n^{-1}P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} < \infty. \tag{6.33}$$

To establish (6.33), we first note that

$$P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} \leq P\{\operatorname{dist}(\sqrt{n}\,Y_n/\gamma_n, \tilde{A}) \geq \beta/2\}. \tag{6.34}$$

Moreover, it is easy to see that if $n \in J_{k,l}$, we have

$$\begin{aligned} &P\{\operatorname{dist}(\sqrt{n}\,Y_n/\gamma_n, \tilde{A}) \geq \beta/2\} \\ &\quad \leq P\{\operatorname{dist}(\sqrt{n}\,Y_n/\gamma_n, \mathscr{L}_{k,l}) \geq \beta/2\} \\ &\quad \leq P\{|Y_n(\theta_{k,l})| \geq (\sigma_{k,l} + \beta/4)\gamma_n/\sqrt{n}\} + P\{|Y_n(\theta_{k,l}')| \geq \beta\gamma_n/4\sqrt{n}\}, \end{aligned}$$

where $\mathscr{L}_{k,l} := \{t(\cos\theta_{k,l}, \sin\theta_{k,l})\colon |t| < \sigma_{k,l}\}$, $1 \leq l \leq l_k$, $k \geq 1$. As in (6.31) we have for $n \in J_{k,l}$, $1 \leq l \leq l_k$ and large k,

$$P\{|Y_n(\theta_{k,l}')| \geq \beta\gamma_n/4\sqrt{n}\} \leq (Ln)^{-2}, \tag{6.35}$$

and we can infer that

$$\sum_{k=1}^{\infty}\sum_{l=1}^{l_k}\sum_{n\in J_{k,l}} n^{-1}P\{|Y_n(\theta_{k,l}')| \geq \beta\gamma_n/4\sqrt{n}\} < \infty. \tag{6.36}$$

Recalling (6.26), Lemma 16(b) and Lemma 17(a), it is easy to see that for $n \in J_{k,l}$, $1 \leq l \leq l_k$ and large k,

$$P\{|Y_n(\theta_{k,l})| \geq (\sigma_{k,l} + \beta/4)\gamma_n/\sqrt{n}\} \leq (Ln)^{-\sigma_{k,l}^2 - \beta^2/17}. \tag{6.37}$$

As in (4.69) we have for $1 \le l \le l_k$ and large k,

$$\sum_{n \in J_{k,l}} n^{-1} \le 4(m_{k,l} - n_{k,l-1}). \tag{6.38}$$

Combining (6.37) and (6.38), we find that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \sum_{n \in J_{k,l}} n^{-1} P\{|Y_n(\theta_{k,l})| \ge (\sigma_{k,l} + \beta/4)\gamma_n/\sqrt{n}\} < \infty. \tag{6.39}$$

In view of (6.36) this means that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \sum_{n \in J_{k,l}} n^{-1} P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} < \infty. \tag{6.40}$$

It remains to show that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \sum_{n \in J'_{k,l}} n^{-1} P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} < \infty, \tag{6.41}$$

where $J'_{k,l} := \{n\colon \exp(m_{k,l-1}) < \overline{K}(n/LLn) \le \exp(n_{k,l-1})\}$, $1 \le l \le l_k$, $k \ge 1$.

By the definition of $\{m_{k,l}\}$ and $\{n_{k,l}\}$ we obviously have

$$\sum_{k=1}^{\infty} \sum_{l=1}^{l_k} \sum_{n \in J'_{k,l}} n^{-1} (Ln)^{-\delta} < \infty, \qquad \delta > 0, \tag{6.42}$$

and we can complete the proof of (6.42) by proving that for large n and a suitable $\delta > 0$,

$$P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} \le (Ln)^{-\delta}. \tag{6.43}$$

Observing that $\|(x_1, x_2)\| \ge \beta$, which follows from the facts that $0 \in \tilde{A}$ and $\operatorname{dist}((x_1, x_2), \tilde{A}) \ge \beta$, we get

$$P\{\|Y_n - (x_{n,1}, x_{n,2})\| < \varepsilon\gamma_n/\sqrt{n}\} \le P\left\{\|Y_n\| \ge \beta\gamma_n/2\sqrt{n}\right\}$$

and we obtain (6.43) by using a standard exponential inequality for normal random vectors in conjunction with (6.26). This completes the proof of Theorem 4. □

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