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IMPROVED UPPER BOUNDS FOR THE CONTACT PROCESS CRITICAL VALUE[1]

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The best known upper bound for the critical value λ_c of the basic one dimensional contact process is 2. Most techniques for finding bounds on critical values have the property that they can be modified in order to obtain improved bounds. This seemed not to be the case for the approach which yielded $\lambda_c \leq 2$ for the basic contact process. In this paper, we propose a technique for generating better bounds in this context. To illustrate its use, we carry out the full program in one case, with the conclusion that $\lambda_c \leq 1.942$.

1. Introduction. The basic one dimensional contact process is the Markov process on $\{0, 1\}^Z$ in which a one flips to zero at rate 1 and a zero flips to one at rate

$$\lambda(\#\text{ neighbors with value one}).$$

As is well known, there is a critical value λ_c with the property that the process dies out for $\lambda \leq \lambda_c$ and survives for $\lambda > \lambda_c$. The best known bounds for the critical value are $1.539 < \lambda_c < 2$. For these and other facts, see Chapter 6 of Liggett (1985) and Bezuidenhout and Grimmett (1990).

We have nothing to say about the lower bound, except to mention that it is the fourteenth member of a sequence of lower bounds which is known to converge to λ_c. [See Grillenberger and Ziezold (1988).] The technique used in Holley and Liggett (1978) to obtain the upper bound, on the other hand, has not previously led to a similar sequence of upper bounds. The purpose of this paper is to propose a technique to generate improved bounds.

One might reasonably ask why it is important to obtain improved upper bounds, particularly since the bound 2 seems to be quite good, and the arguments needed to obtain the improvements are not simple. There are two answers one can give. The first is that it is intellectually unsatisfying to have a technique which produces an upper bound, however good, but does not yield to improvement. The second is perhaps more convincing: There are applica-

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tions in which a certain degree of precision in the bound is essential. We quote three examples:

1. In a recent paper, Liggett (1994) proved a complete result (in the sense of covering all cases) concerning the coexistence of threshold voter models. The proof is based on the fact that a nonnearest neighbor variant of the basic contact process survives for $\lambda = 1$. That fact is proved by the Holley–Liggett technique, which in that context, appears to work down to about $\lambda = 0.985$. It is simply a matter of luck that $0.985 < 1$. If that had not been the case, it would have been necessary to extend this technique along the lines of the present paper to obtain better upper bounds for the critical value.
2. Pemantle (1992) proved that there are three different types of limiting behavior for the contact process on large enough trees. Two of these are the familiar ones (extinction and survival in a strong sense) for the contact process on Z^d. The third, which is absent on Z^d, is that in which the finite process survives globally with positive probability, but dies out locally. His proof that this behavior actually occurs for some range of parameter values is based on finding upper and lower bounds on two natural critical values. These bounds must be good enough to separate the critical values. For the homogeneous tree in which each vertex has $n + 1$ neighbors, his first bounds are good enough to separate if $n \geq 5$. He then improved the bounds in such a way as to cover the cases $n = 3, 4$. The case $n = 2$ is still open.
3. A third example in which precise bounds are essential appeared recently in a somewhat different context. Hara and Slade (1992) proved that the self-avoiding walk has mean field behavior in five or more dimensions. Their proof requires good bounds on the connective constant for the self-avoiding walk. The previously known bounds sufficed for their application in high enough dimensions. For lower dimensions, however, they needed improved bounds. The best form of these appears in Hara, Slade and Sokal (1993).

We begin by giving a short summary of the Holley–Liggett technique. Let ν_t be the distribution of the one dimensional contact process at time t when the initial distribution is ν. The idea is to choose a nontrivial ν in such a way that ν_t increases in t. If this is possible, then the process survives. It turns out that the word "increases" in the next to last sentence should not be taken in the sense of stochastic monotonicity, but rather in the sense that

$$\nu_t\{\eta: \eta(k) = 0 \text{ for all } k \in A\} \downarrow \text{in } t$$

for all finite subsets A of Z. This is a weaker concept of monotonicity, and therefore one that is, in principle, easier to prove. Duality implies that in order to have this monotonicity for all t, it is enough to have

$$\frac{d}{dt}\nu_t\{\eta: \eta(k) = 0 \text{ for all } k \in A\}\bigg|_{t=0} \leq 0 \tag{1.1}$$

for all finite A. The problem is reduced to making a good choice of ν and verifying (1.1). Holley and Liggett took ν to be the renewal measure which satisfies (1.1) with equality for all intervals A. It turns out that such a renewal measure exists if and only if $\lambda \geq 2$. For the details of the proof, see Section 1 of Chapter 6, Liggett (1985). This approach has been used in various other contexts; see the paper by Katori and Kono (1993) and those by Liggett [(1991a, b), (1992), (1994)], for example. In each case, the initial measure ν was a (possibly inhomogeneous) renewal measure.

The first difficulty one encounters in trying to extend this technique to obtain better upper bounds is to find an appropriate generalization of a renewal measure. There are many one might try, but it is important to find the right one, so that all the necessary computations can be performed, and the correct inequalities hold. The choice we will make is obtained by using a variant of the Gibbs formalism to describe probability measures on $\{0,1\}^Z$. Write ν formally as

$$\nu\{\eta\} = \text{constant} \times \exp\left(\sum_{\eta = 0 \text{ on } A} J_A \right), \tag{1.2}$$

where the constant is chosen to make this a probability measure and $\{J_A\}$ is a reasonable collection of constants indexed by the finite subsets A of Z and satisfies $J_{A+k} = J_A$ for all A and k. This is only formal because the sum which appears in (1.2) is, in general, infinite. To see the connection with the usual definition of Gibbs states in statistical mechanics, one should compare (1.2) with (1.4) of Chapter 4 of Liggett (1985) and note that the main difference is that we are replacing the basis elements χ_A which are natural in the Gibbs context by

$$1_{\{\eta:\, \eta = 0 \text{ on } A\}},$$

which are more natural for the contact process. For example, this is the duality function which is used to express contact process duality; see Section 4 of Chapter 3, of Liggett (1985).

In making (1.2) precise, one could follow the example provided by the Gibbs states. Since it is not too easy to compute probabilities from (1.2), it is better to use the formal expression (1.2) to motivate a precise definition which is formulated in a manner similar to the one used to define renewal measures. This will be done in Section 2. We will see there that the ν defined formally by (1.2) is a renewal measure if and only if $J_A = 0$ for all A other than intervals. Therefore, it is natural to define the n-perturbations of renewal measures as those ν which are given formally by (1.2) with $J_A = 0$ for all A other than intervals and sets of diameter less than or equal to n. Let M_n be the class of such measures. We then define

$$\lambda_n = \inf\{\lambda: \text{there exists a } \nu \in M_n \text{ which satisfies (1.1) with equality for all } A \text{ which are intervals or have diameter less than or equal to } n\}.$$

M_1 is the class of renewal measures and $\lambda_1 = 2$ is the Holley–Liggett bound.

Section 3 is devoted to the problem of solving (1.1) with equality for the appropriate A's. In particular, we compute $\lambda_2 = 1.941227\dots$, the largest root of the polynomial

$$4\lambda^3 - 7\lambda^2 - 2\lambda + 1.$$

In order to compute λ_n, one must solve a large but finite number of nonlinear equations and then prove the existence (and monotonicity) of a solution of a one (discrete) parameter convolution equation. In order to avoid excessively complicated expressions, we will prove this existence and carry out the proof of the inequalities (1.1) only in the case $n = 2$. We do solve the finite set of equations (with the help of Mathematica) in case $n = 3$, however, to see what bound this technique would generate at that stage. They can be solved for $\lambda = 1.89349\dots$, but not for slightly smaller λ, so that one should expect that $\lambda_3 = 1.89349\dots$. Section 4 is devoted to proving (1.1) in case $n = 2$, $\lambda \geq 1.941227\dots$, thus obtaining the following theorem.

THEOREM 1.3. $\lambda_c \leq 1.941227\dots$.

In the final section, we prove some inequalities which are needed in Section 4.

There are two problems which we have left open. First, it is natural to expect that $\lambda_c \leq \lambda_n$ for every n. We have proved this for $n = 2$, but without a substantial simplification in our proof, it is not clear how to proceed in the general case. Second, it would not be surprising if $\lambda_n \downarrow \lambda_c$. We do not know how to prove this.

2. Perturbations of renewal measures. Our first task is to use the formal expression (1.2) to suggest a precise definition for the elements of M_n. We assume that ν is defined formally by (1.2), where $J_A = 0$ for all A other than intervals and sets of diameter less than or equal to n. (By the diameter of A, we mean the difference of the largest and smallest elements of A.) In the following, equalities involving $\nu\{\eta\}$'s are to be interpreted as those which are obtained by canceling the (infinite) sums of J_A's which are common to the $\nu\{\eta\}$'s which appear in these equalities. For any configuration η, let η_k be the configuration which satisfies $\eta_k(j) = \eta(j)$ for $j \neq k$ and $\eta_k(k) = 1 - \eta(k)$. Note that

$$\frac{\nu\{\eta\}}{\nu\{\zeta\}} = \frac{\nu\{\eta_k\}}{\nu\{\zeta_k\}}$$

whenever $\eta(l) = 1$, $\eta(j) = \zeta(j)$ for all $j \leq l$ and $l - k \geq n$. Therefore, under ν, the conditional distribution of $\{\eta(j), j > l\}$ given $\eta(l) = 1$ and $\{\eta(j), j < l\}$ depends only on $\{\eta(j), l - n < j < l\}$. Using renewal measures as a model, this suggests that we define the conditional densities of the spacings between successive ones by

$$f_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}}(k) = \nu\big\{\eta(l + k) = 1, \eta(j) = 0 \;\forall\; l < j < l + k \mid \eta(l) = 1, \\ \eta(l - j) = \varepsilon_j \;\forall\; 1 \leq j < n\big\}.$$

These conditional densities are not arbitrary. They satisfy some relations as a consequence of $\nu \in M_n$. Next we will find these relations.

Take $0 \le l \le n-1$ and $k \ge n-l$. For $1 \le i < n$, take ε_i, $\delta_i \in \{0,1\}$ with $\delta_i = \varepsilon_i$ for $1 \le i \le l$. Consider two configurations η and ζ which satisfy

$$\begin{aligned} &\eta(0) = \zeta(0) = \eta(k) = \zeta(k) = 1, \\ &\eta(i) = \varepsilon_{-i}, \qquad \zeta(i) = \delta_{-i}, \qquad n < i < 0, \\ &\eta(i) = \zeta(i) = 0, \qquad 1 \le i < k, \end{aligned}$$

and $\eta(i) = \zeta(i)$ otherwise. Define η^* by

$$\begin{aligned} &\eta^*(i) = \eta(i), \qquad i < k, \\ &\eta^*(k) = 0, \\ &\eta^*(i) = \eta(i-1), \qquad i > k, \end{aligned}$$

and define ζ^* in terms of ζ in an analogous way. By canceling J_A's in (1.2), it is not hard to see that

$$\frac{\nu\{\eta^*\}}{\nu\{\zeta^*\}} = \frac{\nu\{\eta\}}{\nu\{\zeta\}},$$

so that

$$\frac{\nu\{\eta^*\}}{\nu\{\eta\}} = \frac{\nu\{\zeta^*\}}{\nu\{\zeta\}}.$$

Therefore,

$$f_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-1}}(k+1) f_{\delta_1,\delta_2,\dots,\delta_{n-1}}(k) = f_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-1}}(k) f_{\delta_1,\delta_2,\dots,\delta_{n-1}}(k+1)$$

whenever $k + l \ge n$ and $\varepsilon_i = \delta_i$ for $1 \le i \le l$. Equivalently, there is a function α on $\{0,1\}^{n-1}$ so that

$$\frac{f_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-1}}(k)}{f_{\delta_1,\delta_2,\dots,\delta_{n-1}}(k)} = \frac{\alpha(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-1})}{\alpha(\delta_1,\delta_2,\dots,\delta_{n-1})} \tag{2.1}$$

whenever $k + l \ge n$ and $\varepsilon_i = \delta_i$ for $1 \le i \le l$.

As a check, we now compute the number of free parameters, both in (1.2) when the parameters are the J_A's and in (2.1) when the parameters are taken to be the α's and f's. In the case of the J_A's, there is a one parameter family corresponding to the intervals and an additional $2^n - n - 1$ parameters corresponding to sets of diameter less than or equal to n which are not intervals. In the case of the α's and f's, there is the one parameter family $f_{1,1,\dots,1}(\cdot)$, $2^{n-1} - 1$ free α parameters (since one of the α's is arbitrary) and $2^{n-1} - n$ other f's, which can be taken to be the $f_{\varepsilon_1,\dots,\varepsilon_{n-1}}(k)$, where $2 \le k \le n-1$ and $\varepsilon_1,\dots,\varepsilon_{n-k}$ are not all 1 and $\varepsilon_{n-k+1} = \cdots = \varepsilon_{n-1} = 1$.

In order to compute probabilities for ν, we need more than just the conditional spacing densities $f_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}}(k)$—we need to have some (unconditional) cylinder probabilities. It will be convenient to use the notation

$$\nu(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) = \nu\{\eta: \eta(k+1) = \varepsilon_1, \dots, \eta(k+m) = \varepsilon_m\}$$

for such probabilities and use the tail probabilities

$$F_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}}(k) = \sum_{j=k}^{\infty} f_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}}(j).$$

Then (2.1) implies

$$\frac{F_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}}(k)}{F_{\delta_1, \delta_2, \dots, \delta_{n-1}}(k)} = \frac{\alpha(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})}{\alpha(\delta_1, \delta_2, \dots, \delta_{n-1})} \tag{2.2}$$

whenever $k + l \geq n$ and $\varepsilon_i = \delta_i$ for $1 \leq i \leq l$.

Now we can take $2 \leq k \leq n$ and $\varepsilon_i \in \{0, 1\}$ for $1 \leq i \leq n - k$ and compute the following relations among cylinder probabilities on n sites:

$$\begin{aligned}
&\nu(\varepsilon_1, \dots, \varepsilon_{n-k}, 1, 0, \dots, 0)\\
&\quad = \sum_{\delta_1, \dots, \delta_{k-1} \in \{0,1\}} \nu(\delta_1, \dots, \delta_{k-1}, \varepsilon_1, \dots, \varepsilon_{n-k}, 1, 0, \dots, 0)\\
&\quad = \sum_{\delta_1, \dots, \delta_{k-1} \in \{0,1\}} \nu(\delta_1, \dots, \delta_{k-1}, \varepsilon_1, \dots, \varepsilon_{n-k}, 1) F_{\varepsilon_{n-k}, \dots, \varepsilon_1, \delta_{k-1}, \dots, \delta_1}(k).
\end{aligned}$$

Using (2.2), it follows that

$$\begin{aligned}
&\alpha(\varepsilon_{n-k}, \dots, \varepsilon_1, 1, \dots, 1) \nu(\varepsilon_1, \dots, \varepsilon_{n-k}, 1, 0, \dots, 0)\\
&\quad = F_{\varepsilon_{n-k}, \dots, \varepsilon_1, 1, \dots, 1}(k) \sum_{\delta_1, \dots, \delta_{k-1} \in \{0,1\}} \alpha(\varepsilon_{n-k}, \dots, \varepsilon_1, \delta_{k-1}, \dots, \delta_1)\\
&\qquad \times \nu(\delta_1, \dots, \delta_{k-1}, \varepsilon_1, \dots, \varepsilon_{n-k}, 1).
\end{aligned} \tag{2.3}$$

To determine the cylinder probabilities for n consecutive sites of a shift invariant measure, one must determine 2^{n-1} quantities [e.g., the quantities $\nu(\varepsilon_1, \dots, \varepsilon_{n-1}, 1)$ for arbitrary choices of the ε's]. There are $2^{n-1} - 1$ equations of form (2.3). Therefore, one further equation is needed. It is provided by

$$\begin{aligned}
1 &= \nu(1,1) + 2\nu(1,0,1) + 3\nu(1,0,0,1) + \cdots\\
&= \sum_{\varepsilon_1, \dots, \varepsilon_{n-1} \in \{0,1\}} \nu(\varepsilon_1, \dots, \varepsilon_{n-1}, 1) \sum_{k=1}^{\infty} k f_{\varepsilon_1, \dots, \varepsilon_{n-1}}(k),
\end{aligned} \tag{2.4}$$

which requires that the conditional spacing densities have a finite mean.

Therefore, an n-perturbation of a renewal measure ν is parametrized by a decreasing summable sequence

$$F(k) = F_{1, \dots, 1}(k),$$

a collection

$$\alpha(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$$

(this is $2^{n-1} - 1$ parameters because only ratios of α's are relevant) and a collection of numbers

$$F_{\varepsilon_1, \dots, \varepsilon_{n-1}}(k),$$

where $2 \le k \le n-1$ and $\varepsilon_1, \dots, \varepsilon_{n-k}$ are not all 1, and $\varepsilon_{n-k+1} = \cdots = \varepsilon_{n-1} = 1$ which are decreasing in k for all ε's. The corresponding measure ν is defined by (2.3), (2.4) and the fact that the conditional distributions of the spacings are given by the $F_{\varepsilon_1, \dots, \varepsilon_{n-1}}(k)$'s defined by (2.2).

3. The equations. In this section, we consider the problem of finding an n-perturbation of a renewal measure which satisfies (1.1) with equality for all A which are intervals or have diameter less than or equal to n. These equations for all A which have diameter less than or equal to n are equivalent to

$$\frac{d}{dt}\nu_t(\varepsilon_1, \dots, \varepsilon_{n+1})\bigg|_{t=0} = 0 \tag{3.1}$$

for all choices of $\varepsilon_1, \dots, \varepsilon_{n+1} \in \{0,1\}$. The equation for A which is an interval of diameter $k-1$ is equivalent to

$$\sum_{j=1}^{k} \nu(0,\dots,0,1,0,\dots,0) = 2\lambda\nu(1,0,\dots,0), \tag{3.2}$$

where the cylinder probabilities on the left are on k consecutive sites, with the 1 appearing at the jth of these, and the cylinder probability on the right is on $k+1$ consecutive sites.

In order to write (3.2) in a more useful form, take $1 \le j < n$ and $k \ge n$ and write the following k-site cylinder probability in which the 1 is in the jth site:

$$\begin{aligned}
&\nu(0,\dots,0,1,0,\dots,0)\\
&\quad= \sum_{\delta_1,\dots,\delta_{n-j}\in\{0,1\}} \nu(\delta_1,\dots,\delta_{n-j},0,\dots,0,1)F_{0,\dots,0,\delta_{n-j},\dots,\delta_1}(k-j+1)\\
&\quad= \frac{F_{0,\dots,0,1,\dots,1}(k-j+1)}{\alpha(0,\dots,0,1,\dots,1)} \sum_{\delta_1,\dots,\delta_{n-j}\in\{0,1\}} \nu(\delta_1,\dots,\delta_{n-j},0,\dots,0,1)\\
&\qquad\times \alpha(0,\dots,0,\delta_{n-j},\dots,\delta_1)\\
&\quad= \frac{F_{0,\dots,0,1}(k-j+1)\nu(0,\dots,0,1,0,\dots,0)}{F_{0,\dots,0,1,\dots,1}(n-j+1)}.
\end{aligned} \tag{3.3}$$

In (3.3), all cylinder probabilities except the first are on n sites and the subscript on the $F(k-j+1)$ has $j-1$ initial 0's. The second equality is a consequence of (2.2), while the third comes from (2.3). In particular, if $j=1$, (3.3) becomes

$$\nu(1,0,\dots,0) = \frac{F(k)\nu(1,0,\dots,0)}{F(n)} \tag{3.4}$$

for $k \ge n$, where the cylinder probability on the left is on k sites and the one on the right is on n sites. Using (3.4) and (2.2), we have also that for $j \ge n$ and $k-j+1 \ge n$, the k-site cylinder probability in which the 1 appears at the jth site can be written as

$$\nu(0,\dots,0,1,0,\dots,0) = F(k-j+1)\frac{\alpha(0,\dots,0)}{\alpha(1,\dots,1)}\frac{F(j)\nu(1,0,\dots,0)}{F(n)}, \tag{3.5}$$

where the cylinder probability on the right is on n sites.

We turn now to obtaining a better form for (3.2). Take $k \geq n$ and use (3.3) and (3.5) to reexpress (3.2) as

$$\begin{aligned}\sum_{j=1}^{n-1} \nu(0,\dots,0,1,0,\dots,0)&\frac{F_{0,\dots,0,1,\dots,1}(k-j+1)}{F_{0,\dots,0,1,\dots,1}(n-j+1)}\\ &\times\left[1_{\{j\leq(k+1)/2\}}+1_{\{j\leq k/2\}}\right]\\ &+\sum_{n\leq j\leq k-n+1} F(k-j+1)F(j)\frac{\nu(1,0,\dots,0)}{F(n)}\frac{\alpha(0,\dots,0)}{\alpha(1,\dots,1)}\\ &=2\lambda\frac{F(k+1)\nu(1,0,\dots,0)}{F(n)}.\end{aligned} \tag{3.6}$$

All cylinder probabilities appearing in (3.6) are on n sites, and the first one has the 1 at the jth site. The subscripts on the F's have $j-1$ zeros and $n-j$ ones.

Next, we will find a necessary condition for (3.6) to have a bounded solution for $F(\cdot)$ if (2.2), (2.3), (2.4) and (3.1) are satisfied. Suppose that these equations all have solutions, multiply (3.6) by x^{k+1} and sum (3.6) for $k \geq n$. Let

$$M = \sum_{k=n}^{\infty} F(k)x^k,$$

which is finite for $|x| \leq 1$ by (2.4). The result is that M satisfies the following quadratic equation:

$$\begin{aligned}&\frac{\nu(1,0,\dots,0)\alpha(0,\dots,0)}{F(n)\alpha(1,\dots,1)}M^2\\ &\quad-2\left[\lambda\frac{\nu(1,0,\dots,0)}{F(n)}\right.\\ &\quad\left.-\sum_{j=1}^{n-1}\frac{\nu(0,\dots,0,1,0,\dots,0)\alpha(0,\dots,0,1,\dots,1)}{F_{0,\dots,0,1,\dots,1}(n-j+1)\alpha(1,\dots,1)}x^j\right]M\\ &\quad+2\lambda\nu(1,0,\dots,0)x^n\\ &\quad+\sum_{1<j<n\leq k<n+j-1}\frac{F_{0,\dots,0,1,\dots,1}(k-j+1)\nu(0,\dots,0,1,0,\dots,0)}{F_{0,\dots,0,1,\dots,1}(n-j+1)}\\ &\quad\times\left[1_{\{j\leq(k+1)/2\}}+1_{\{j\leq k/2\}}\right]=0.\end{aligned} \tag{3.7}$$

Therefore, the discriminant of this quadratic is nonnegative for every $|x| \le 1$. This is the necessary condition we were looking for. We will call it condition $C(n)$. If the discriminant is strictly positive for all these x's, we will say that strict $C(n)$ holds.

Next, we will carry out the computations for several values of n in order to see when $C(n)$ holds:

$n = 1$. After canceling a factor of $\nu(1)$, (3.7) becomes

$$M^2 - 2\lambda M + 2\lambda x = 0,$$

so condition $C(1)$ becomes $\lambda \ge 2$.

$n = 2$. After canceling a factor of $\nu(1,0)$ and multiplying by $F(2)$, (3.7) becomes

$$\frac{\alpha(0)}{\alpha(1)}M^2 - 2(\lambda - x)M + 2\lambda F(2)x^2 = 0.$$

Equation (3.2) with $k = 1$ gives

$$\nu(1,1) = (2\lambda - 1)\nu(1,0)$$

and with $k = 2$ gives

$$\alpha(1)\nu(1,0) = \lambda F(3)[\alpha(0)\nu(0,1) + \alpha(1)\nu(1,1)].$$

Equation (2.3) with $k = 2$ gives

$$\alpha(1)\nu(1,0) = F(2)[\alpha(0)\nu(0,1) + \alpha(1)\nu(1,1)].$$

Finally, (3.1) for the cylinder set $(0,1,0)$ gives

$$2\alpha(1) + 2\lambda\alpha(0)F(3) = (3 + 4\lambda)\alpha(0)F(2).$$

These four equations can be solved to give

$$\frac{\nu(1,1)}{\nu(1,0)} = 2\lambda - 1, \qquad \frac{\alpha(0)}{\alpha(1)} = \frac{4\lambda - 2}{4\lambda - 1},$$

$$F(2) = \frac{4\lambda - 1}{(4\lambda + 1)(2\lambda - 1)} \quad \text{and} \quad F(3) = \frac{4\lambda - 1}{\lambda(4\lambda + 1)(2\lambda - 1)}.$$

Therefore, condition $C(2)$ becomes

$$(\lambda - x)^2 \ge \frac{4\lambda x^2}{4\lambda + 1}$$

for all $|x| \le 1$ or, equivalently

$$\lambda \ge \left(\text{the largest zero of } 4\lambda^3 - 7\lambda^2 - 2\lambda + 1\right) = 1.941227\dots.$$

$n = 3$. After multiplying by $F(3)$ and dividing by $\nu(1,0,0)$, (3.7) becomes

$$\frac{\alpha(0,0)}{\alpha(1,1)}M^2 - 2\left[\lambda - x - \frac{\nu(0,1,0)}{\nu(1,0,0)}\frac{F(3)}{F_{0,1}(2)}\frac{\alpha(0,1)}{\alpha(1,1)}x^2\right]M + 2\lambda F(3)x^3 + \frac{\nu(0,1,0)}{\nu(1,0,0)}F(3)x^4 = 0. \tag{3.8}$$

To compute the values of the unknowns which appear in this expression, we (with the help of Mathematica) solve 10 equations in 10 unknowns. The equations are (3.2) with $k = 1, 2, 3$, (2.3) with $k = 2$ and $\varepsilon_1 = 0$, $k = 2$ and $\varepsilon_1 = 1$, $k = 3$, and (3.1) for the cylinder sets $(0, 1, 0)$, $(0, 0, 1, 0)$, $(0, 1, 0, 0)$ and $(0, 1, 1, 0)$. The solution can be expressed in terms of $h = \lambda F(4)/F(3)$, which is a zero of the polynomial

$$\begin{aligned}[]
[h(3+4\lambda) - (3+6\lambda)]\big[&2h^4(3+4\lambda) - h^3(27+42\lambda+12\lambda^2+16\lambda^3)\\
&+h^2(45+78\lambda+40\lambda^2+68\lambda^3+32\lambda^4)\\
&-h(33+62\lambda+43\lambda^2+88\lambda^3+64\lambda^4)\\
&+3(1+\lambda)(3+3\lambda+2\lambda^2+10\lambda^3)\big].
\end{aligned} \tag{3.9}$$

The solution is then given by

$$\frac{\nu(1,1,1)}{\nu(1,1,0)} = 2\frac{1+\lambda-\lambda^2-h}{-2-\lambda+2h},$$

$$\frac{\nu(1,1,0)}{\nu(0,1,0)} = \frac{2+\lambda-2h}{2(h-1)},$$

$$\frac{\nu(1,0,1)}{\nu(1,0,0)} = \lambda - 1,$$

$$F(2) = \frac{(2+\lambda-2h)(-1-2\lambda-\lambda^2+h+2\lambda h)}{1+\lambda+\lambda^2+2\lambda^3-2\lambda^4-h(2+\lambda+\lambda^2)+h^2},$$

$$F_{01}(2) = \frac{3+3\lambda-h(3+2\lambda)}{\lambda(\lambda-1)},$$

$$F(3) = \frac{-3(1+\lambda)^2(2\lambda+1)+h(6+22\lambda+23\lambda^2+4\lambda^3) - h^2(2\lambda+1)(4\lambda+3)}{1+\lambda^2+\lambda^3+2\lambda^4-2\lambda^5-\lambda h(2+\lambda+\lambda^2)+\lambda h^2}$$

and

$$\frac{\alpha(0,0)}{\alpha(0,1)} = \frac{(\lambda-1)(-3-5\lambda+h(3+4\lambda))}{3+3\lambda-h(3+2\lambda)},$$

$$\frac{\alpha(1,0)}{\alpha(1,1)} = \frac{(4\lambda+1)(h-1)(-1-\lambda+\lambda^2+h)}{(-2-\lambda+2h)(-1-2\lambda-\lambda^2)+h(1+2\lambda)},$$

$$\begin{aligned}
\frac{\alpha(0,1)}{\alpha(1,1)} = &\frac{(-3-3\lambda+h(3+2\lambda))}{2(\lambda-1)\lambda(h-1)}\\
&\times\frac{(-3-5\lambda+h(3+4\lambda))((1+\lambda-\lambda^2)(2\lambda^2+1) - h(2+\lambda+\lambda^2)+h^2)}{3(\lambda+1)^2(2\lambda+1)-h(6+22\lambda+23\lambda^2+4\lambda^3) + h^2(1+2\lambda)(3+4\lambda)}.
\end{aligned}$$

Using these expressions, we see that the nonnegativity of the discriminant of (3.8) is equivalent to

$$0 \le \lambda^2 - 2\lambda x + (7 + 10\lambda - 6h - 8\lambda h)x^2 + \frac{(3 + 5\lambda - 3h - 4\lambda h)(1 + 5\lambda - h - 4\lambda h)}{\lambda - \lambda h}x^3. \tag{3.10}$$

To find the smallest value of λ for which this holds for all $|x| \le 1$, solve the equation obtained from (3.10) by replacing $\le$ with $=$ and x by 1 for h in terms of λ and put the result in (3.9). Except for factors which are nonzero for positive λ, the resulting polynomial is the product of

$$\begin{aligned} &9024\lambda^{11} + 158420\lambda^{10} - 4848\lambda^9 - 423864\lambda^8 - 380144\lambda^7 \\ &- 38056\lambda^6 + 86708\lambda^5 + 32667\lambda^4 - 2216\lambda^3 \\ &- 2142\lambda^2 - 180\lambda + 39 \end{aligned} \tag{3.11}$$

and

$$2\lambda^2 - 12\lambda + 3.$$

According to Mathematica, the largest root of (3.11) is $\lambda = 1.89349\ldots$. For this value of λ, the values of the other variables are (to the accuracy given) $h = 1.21798$,

$$\begin{aligned} F(2) &= 0.279018, \quad F_{01}(2) = 0.244753, \\ F(3) &= 0.149839, \quad F(4) = 0.0963829, \\ \frac{\nu(1,1,1)}{\nu(1,1,0)} &= 2.62059, \quad \frac{\nu(1,1,0)}{\nu(0,1,0)} = 3.34331, \\ \frac{\nu(1,0,1)}{\nu(1,0,0)} &= 0.893493, \quad \frac{\alpha(0,0)}{\alpha(0,1)} = 0.887718, \\ \frac{\alpha(0,1)}{\alpha(1,1)} &= 0.814073, \quad \frac{\alpha(1,0)}{\alpha(1,1)} = 0.963413. \end{aligned} \tag{3.12}$$

To the given degree of accuracy, the right side of (3.10) is then

$$3.58531 - 3.78699x + 0.177196x^2 + 0.0244748x^3,$$

which is nonnegative for $|x| \le 1$. Therefore, $C(3)$ is satisfied for $\lambda = 1.89349\ldots$. Setting the second factor in (3.9) equal to 0, we have an implicit definition of h as a function of λ near $(h, \lambda) = (1.21798\ldots, 1.89349\ldots)$. The derivative $h'(\lambda) = 0.0160094\ldots$ at this point and one can then compute the derivative of the right side of (3.10) with $x = 1$ at this point. This derivative turns out to be strictly positive. Therefore we conclude that strict $C(3)$ is satisfied for λ just to the right of $1.89349\ldots$, while $C(3)$ is not satisfied for λ just to the left of $1.89349\ldots$.

At this point and for the remainder of the paper, we restrict our attention to the case $n = 2$. First, we will show that $C(2)$ is sufficient for (3.6) to have a decreasing summable solution $F(\cdot)$. Let $\alpha = \alpha(0)/\alpha(1)$ and write (3.6) as

$$2F(k) + \alpha \sum_{j=2}^{k-1} F(j)F(k+1-j) = 2\lambda F(k+1), \qquad k \geq 2. \tag{3.13}$$

Let $f(k) = F(k) - F(k+1)$. In order to write (3.13) as equations for f, subtract (3.13) for $k+1$ from (3.13) for k, obtaining

$$\begin{aligned} &2f(k) + \alpha \sum_{j=2}^{k-1} F(j)f(k+1-j) \\ &\quad - \alpha F(2)F(k) = 2\lambda f(k+1), \qquad k \geq 2. \end{aligned} \tag{3.14}$$

Performing the same subtraction on (3.14) and then replacing k by $k-1$, we obtain

$$\begin{aligned} &2[1 - \alpha F(2)]f(k-1) + \alpha \sum_{j=2}^{k-2} f(j)f(k-j) + 2\lambda f(k+1) \\ &\quad = (2 + 2\lambda)f(k), \qquad k \geq 3. \end{aligned} \tag{3.15a}$$

The corresponding equation for $k = 2$ is

$$[1 - \alpha F(2)]f(1) + 2\lambda f(3) = (2 + 2\lambda)f(2). \tag{3.15b}$$

[This can be obtained from (3.14) with $k = 2$, using the known values of $\alpha, F(2), F(3)$.] Note that (3.15) can be solved recursively for $f(k)$, $k \geq 3$, and the solution is unique. The values of $f(1)$ and $f(2)$ were determined earlier. The problem is to show that this solution is nonnegative and sums (for $k \geq 1$) to 1.

We begin by defining the generating function

$$\phi(x) = \sum_{k=1}^{\infty} F(k)x^k. \tag{3.16}$$

Multiplying (3.13) by x^k, summing by $k \geq 2$ and using the value of $F(2)$ leads to a quadratic equation for ϕ whose solution is given by

$$\alpha[\phi(x) - x] = \lambda - x - \lambda\sqrt{1 - \frac{2x}{\lambda} + \frac{x^2}{\lambda^2(1+4\lambda)}}\,.$$

Note that the quantity in the square root is nonnegative for all $|x| \leq 1$ if and only if $\lambda \geq 1.941227\ldots$, which we assume from now on. Factor the quadratic in the square root as

$$1 - \frac{2x}{\lambda} + \frac{x^2}{\lambda^2(1+4\lambda)} = (1 - ax)(1 - bx),$$

with $b < a$. For the range of λ's we are considering, $a, b \in (0, 1]$ (e.g., if $\lambda = 1.941227\ldots$, $a = 1$, $b = 0.03027\ldots$). Now use the expansion for the square root

$$\sqrt{1-t} = 1 - 2\sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!}\left(\frac{t}{4}\right)^k \tag{3.17}$$

to write an expansion for ϕ. Equating coefficients with (3.16) leads to the following expression for $F(m)$ for $m \geq 2$:

$$\alpha F(m) = 2\lambda\frac{(2m-2)!}{m!(m-1)!}\left[\left(\frac{a}{4}\right)^m + \left(\frac{b}{4}\right)^m\right]$$
$$- 4\lambda \sum_{\substack{j+k=m\\ j,k\geq 1}} \frac{(2j-2)!}{j!(j-1)!}\frac{(2k-2)!}{k!(k-1)!}\left(\frac{a}{4}\right)^j\left(\frac{b}{4}\right)^k.$$

This can be written more simply in terms of the sequence

$$H(m) = \frac{(2m-2)!}{m!(m-1)!}\left(\frac{a}{4}\right)^m.$$

[Compare with (1.18) of Chapter 6, of Liggett (1985).] We get

$$F(m) = \frac{2\lambda}{\alpha}H(m) \times\left[1 + (b/a)^m - 2\sum_{\substack{j+k=m\\ j,k\geq 1}}\frac{H(j)H(k)}{H(m)}(b/a)^k\right]. \tag{3.18}$$

Put

$$c = \lim_{m\to\infty}\frac{F(m)}{H(m)} = \frac{2\lambda}{\alpha}\left[1 - 2\sum_{k=1}^{\infty}H(k)\left(\frac{b}{a^2}\right)^k\right] = \frac{2\lambda}{\alpha}\sqrt{1-\frac{b}{a}}. \tag{3.19}$$

We need an estimate on the rate of convergence in this limit. To obtain it, write

$$m\left[c - \frac{F(m)}{H(m)}\right] = \frac{2m\lambda}{\alpha}\left[2\sum_{k=1}^{m-1}H(k)\left(\frac{b}{a}\right)^k\left[\frac{H(m-k)}{H(m)} - a^{-k}\right]\right.$$
$$\left. -2\sum_{k=m}^{\infty}H(k)\left(\frac{b}{a^2}\right)^k - \left(\frac{b}{a}\right)^m\right].$$

Since $H(m)a^{-m}$ is decreasing, the terms in the first sum above are nonnegative. To get a lower bound, we may then neglect the first sum and replace $H(k)a^{-k}$ by 1 in the second sum, obtaining

$$m\left[c - \frac{F(m)}{H(m)}\right] \geq -\frac{2\lambda m}{\alpha}\frac{3a-b}{a-b}\left(\frac{b}{a}\right)^m.$$

For the upper bound, begin with the simple inequality

$$\frac{2m}{2m-3} \le \left(\frac{2m-1}{2m-3}\right)^{3/2}, \qquad m > 1,$$

which implies that $H(m)a^{-m}(2m-1)^{3/2}$ is increasing. Using this and the fact that $(1+x)^{3/2} \le 1 + 2x$ for $0 \le x \le 1$, we see that

$$\frac{H(m-k)}{H(m)} - a^{-k} \le a^{-k}\frac{4k}{2m-2k-1}.$$

Therefore,

$$\begin{aligned} m\left[c - \frac{F(m)}{H(m)}\right] &\le \frac{4\lambda m}{\alpha}\sum_{1\le k<m/2} H(k)\left(\frac{b}{a^2}\right)^k \frac{4k}{2m-2k-1} \\ &\quad + \frac{4\lambda m}{\alpha}\sum_{m>k\ge m/2}\frac{H(k)H(m-k)}{H(m)}\left(\frac{b}{a}\right)^k \\ &\le \frac{16\lambda}{\alpha}\sum_{k=1}^{\infty} H(k)k\left(\frac{b}{a^2}\right)^k \\ &\quad + \frac{4\lambda m}{\alpha}\left(\frac{b}{a}\right)^{m/2}\sum_{k=1}^{m-1}\frac{H(k)H(m-k)}{H(m)} \\ &= \frac{4\lambda b}{\alpha\sqrt{a(a-b)}} + \frac{4\lambda m}{\alpha}\left(\frac{b}{a}\right)^{m/2}. \end{aligned}$$

The first part of the last equality comes from differentiating (3.17) and the second part can be proved by generating functions. By explicit computation of a, b, it is easy to check that

$$\frac{b}{a} \le \frac{1}{16\lambda}. \tag{3.20}$$

Using this, we see that

$$-\frac{1}{10} \le m\left[c - \frac{F(m)}{H(m)}\right] \le \frac{1}{2}$$

for $m \ge 4$.

Now put $h(m) = H(m) - H(m+1)$, which is easily seen to be nonnegative. Then

$$\frac{h(m)}{H(m)} = 1 - a\frac{2m-1}{2m+2} \ge \frac{3}{2m+2}$$

and hence

$$\begin{aligned} f(m) &\ge ch(m) - \frac{H(m)}{2m} - \frac{H(m+1)}{10(m+1)} \\ &\ge h(m)\left[c - \frac{2}{5}\frac{m+1}{m}\right] \ge h(m)\left[c - \frac{1}{2}\right] \end{aligned} \tag{3.21}$$

for $m \geq 4$. Since $h(m)$ is nonnegative, it follows that $f(m)$ is nonnegative for $m \geq 4$. The nonnegativity of the first few values of f can be checked directly:

$$f(1) = \frac{2\lambda(4\lambda - 3)}{(2\lambda - 1)(4\lambda + 1)}, \qquad f(2) = \frac{(\lambda - 1)(4\lambda - 1)}{\lambda(2\lambda - 1)(4\lambda + 1)},$$

$$f(3) = \frac{(4\lambda - 1)(4\lambda^2 - 4\lambda - 1)}{\lambda^2(2\lambda - 1)(4\lambda + 1)^2}, \qquad f(4) = \frac{(4\lambda - 1)(5\lambda^2 - 6\lambda - 1)}{\lambda^3(2\lambda - 1)(4\lambda + 1)^2}.$$

In Section 5, we will also need an upper bound similar to (3.21), which is derived in a similar way:

$$f(m) \leq h(m)\left[c + \frac{1}{4^4}\right], \qquad m \geq 4. \tag{3.22}$$

4. The inequalities. The purpose of this section is to show that the inequalities (1.1) are satisfied by the $\nu \in M_2$ corresponding to the $f(\cdot)$ determined by (3.15). Fix a finite subset A of Z and define functions L and R on Z by the conditional probabilities

$$L(k) = \nu\{\eta \equiv 0 \text{ on } A \cap (-\infty, k) \mid \eta(k) = \eta(k+1) = 1\},$$
$$R(k) = \nu\{\eta \equiv 0 \text{ on } A \cap (k, \infty) \mid \eta(k) = \eta(k-1) = 1\}.$$

We will also need the conditional probabilities in which the conditioning is on $\eta(k \pm 1) = 0$ instead of 1:

$$L_0(k) = \nu\{\eta \equiv 0 \text{ on } A \cap (-\infty, k) \mid \eta(k) = 1, \eta(k+1) = 0\},$$
$$R_0(k) = \nu\{\eta \equiv 0 \text{ on } A \cap (k, \infty) \mid \eta(k) = 1, \eta(k-1) = 0\}.$$

These can be written in terms of L and R by breaking up the relevant probability according to the value of the first l at which $\eta(l) = 1$. For example,

$$\begin{aligned} L_0(k) &= f_0(1)L(k-1)1_{\{k-1 \notin A\}} + \sum_{\substack{l<k-1 \\ l \notin A}} f_0(k-l)L_0(l), \\ L(k) &= f(1)L(k-1)1_{\{k-1 \notin A\}} + \sum_{\substack{l<k-1 \\ l \notin A}} f(k-l)L_0(l). \end{aligned} \tag{4.1}$$

Since $f_0(1) = 1 - \alpha F(2)$ and $f_0(k) = \alpha f(k)$ for $k \geq 2$, one can eliminate the sum from the two expressions in (4.1), obtaining

$$L_0(k) = \alpha L(k) + (1 - \alpha)L(k-1)1_{\{k-1 \notin A\}}. \tag{4.2}$$

Similarly,

$$R_0(k) = \alpha R(k) + (1 - \alpha)R(k+1)1_{\{k+1 \notin A\}}. \tag{4.3}$$

Next, we evaluate the expressions which appear when the derivative on the left (1.1) is computed. The positive terms in the derivative are

$$\sum_{k\in A} \nu\{\eta(k) = 1, \eta \equiv 0 \text{ on } A\setminus\{k\}\}$$

$$(4.4)\qquad = \sum_{\substack{j<k<l\\ k\in A\\ j,l\notin A}} \nu\{\eta(j) = \eta(k) = \eta(l) = 1, \eta \equiv 0$$

$$\text{on } (A\cap(-\infty,j))\cup(j,k)\cup(k,l)\cup(A\cap(l,\infty))\}.$$

Breaking up this sum according to whether $k-j=1$, $l-k=1$, neither or both, and using the "renewal-like" property of ν, we see that except for a factor of

$$\frac{\nu(1)[1-\alpha F(2)]}{1+(1-\alpha)F(2)},$$

the right side of (4.4) can be written as

$$(4.5)\qquad \begin{aligned} f(1) &\sum_{\substack{k\in A\\ k-1,k+1\notin A}} L(k-1)R(k+1)\\ &+\sum_{\substack{k<l-1\\ k\in A\\ k-1,l\notin A}} f(l-k)L(k-1)R_0(l)\\ &+\sum_{\substack{j+1<k\\ k\in A\\ j,k+1\notin A}} f(k-j)L_0(j)R(k+1)\\ &+\frac{\alpha}{1-\alpha F(2)}\sum_{\substack{j+1<k<l-1\\ k\in A\\ j,l\notin A}} f(k-j)f(l-k)L_0(j)R_0(l).\end{aligned}$$

Using (4.1), the third term above can be written as

$$\sum_{\substack{k\in A\\ k+1\notin A}} L(k)R(k+1) - f(1)\sum_{\substack{k\in A\\ k-1,k+1\notin A}} L(k-1)R(k+1).$$

Similarly, the second term in (4.5) can be written as

$$\sum_{\substack{k\in A\\ k-1\notin A}} L(k-1)R(k) - f(1)\sum_{\substack{k\in A\\ k-1,k+1\notin A}} L(k-1)R(k+1).$$

Using (3.15a), the last term in (4.5) can be written as

$$(4.6)\qquad \begin{aligned} &\frac{2}{1-\alpha F(2)}\sum_{\substack{l-j\ge 4\\ j,l\notin A}} L_0(j)R_0(l)[(1+\lambda)f(l-j)\\ &\qquad -\lambda f(l-j+1) - [1-\alpha F(2)]f(l-j-1)]\\ &-\frac{\alpha}{1-\alpha F(2)}\sum_{\substack{j+1<k<l-1\\ k,j,l\notin A}} f(k-j)f(l-k)L_0(j)R_0(l).\end{aligned}$$

Note that each of the sums above is trivially divergent; the meaning of the difference of the sums is the difference after identical summands are canceled. Using (4.1), half of the first term in (4.6) can be written as

$$\frac{1+\lambda}{1-\alpha F(2)}\Bigg[\sum_{l\notin A} R_0(l)L(l) - f(1)\sum_{l-1,\,l\notin A} R_0(l)L(l-1)$$
$$-f(2)\sum_{l-2,\,l\notin A} R_0(l)L_0(l-2) - f(3)\sum_{l-3,\,l\notin A} R_0(l)L_0(l-3)\Bigg]$$
$$-\frac{\lambda}{1-\alpha F(2)}\Bigg[\sum_{l\notin A} R_0(l)L(l+1) - f(1)\sum_{l\notin A} R_0(l)L(l)$$
$$-f(2)\sum_{l-1,\,l\notin A} R_0(l)L_0(l-1)$$
$$-f(3)\sum_{l-2,\,l\notin A} R_0(l)L_0(l-2) - f(4)\sum_{l-3,\,l\notin A} R_0(l)L_0(l-3)\Bigg]$$
$$-\sum_{l\notin A} R_0(l)L(l-1) + f(1)\sum_{l-2,\,l\notin A} R_0(l)L(l-2)$$
$$+f(2)\sum_{l-3,\,l\notin A} R_0(l)L_0(l-3).$$

Using (4.2) and (4.3), this can be written in terms of the L's and R's without the subscript 0. The other half of the first term in (4.6) is written in a similar fashion, with the roles of the L's and R's reversed. Using (4.1) and the corresponding relations for the R's, the last term in (4.6) becomes

$$-\frac{\alpha}{1-\alpha F(2)}\Bigg[\sum_{k\notin A} L(k)R(k) - 2f(1)\sum_{k-1,\,k\notin A} L(k-1)R(k)$$
$$+f^2(1)\sum_{k-1,\,k,\,k+1\notin A} L(k-1)R(k+1)\Bigg].$$

Putting all these computations together yields a rather long expression for (4.4). We will not record it here, but will use it shortly in giving the overall expression for (1.1).

The negative terms in the expression for the derivative in (1.1) are simpler. Except for a factor of $-\lambda$, they are

$$\sum_{\substack{k\in A\\ k+1\notin A}} \nu\{\eta(k+1)=1,\ \eta\equiv 0 \text{ on } A\}$$
$$+\sum_{\substack{k\in A\\ k-1\notin A}} \nu\{\eta(k-1)=1,\ \eta\equiv 0 \text{ on } A\}$$
$$(4.7)\qquad = \frac{\nu(1)}{1+(1-\alpha)F(2)}\Bigg[\sum_{\substack{k\in A\\ k+1\notin A}} L(k+1)R_0(k+1)$$
$$+\sum_{\substack{k\in A\\ k-1\notin A}} L_0(k-1)R(k-1)\Bigg].$$

Using (4.2) and (4.3), this can be written in terms of the L's and R's without the subscript 0.

Combining the results of these computations (the algebraic manipulations involved here are substantial; they are eased by the use of Mathematica) and using the values obtained in Section 3 for α and the f's, we find that, except for a factor of $\nu(1)$, the left side of (1.1) is given by

$$\begin{aligned}
&-\frac{4\lambda-3}{(4\lambda+1)(4\lambda-1)^2}\sum_{k,k+2\notin A} L(k)R(k+2)\\
&+\frac{2\lambda-1}{2\lambda(4\lambda-1)}\sum_{k\notin A} L(k)R(k+1)\\
&+\frac{2\lambda-1}{2\lambda(4\lambda-1)}\sum_{k+1\notin A} L(k)R(k+1)\\
&+\frac{(16\lambda^3-4\lambda^2-2\lambda-1)(2\lambda-1)}{\lambda(4\lambda-1)^2}\sum_{k\notin A} L(k)R(k)\\
&-\frac{2(16\lambda^4-16\lambda^3+5\lambda^2-4\lambda+2)}{\lambda(4\lambda-1)^2}\sum_{k,k+1\notin A} L(k)R(k+1)\\
&-\frac{(8\lambda^3-10\lambda^2+9\lambda-4)}{\lambda(4\lambda-1)^2}\sum_{k,k+1,k+2\notin A} L(k)R(k+2)\\
&+\frac{4\lambda-3}{(4\lambda+1)(4\lambda-1)^2}\Bigg[\sum_{k,k+2,k+3\notin A} L(k)R(k+3)\\
&\qquad\qquad+\sum_{k,k+1,k+3\notin A} L(k)R(k+3)\Bigg]\\
&-\frac{4\lambda-3}{(4\lambda+1)(4\lambda-1)^2}\sum_{k,k+1,k+3,k+4\notin A} L(k)R(k+4)\\
&-\frac{(4\lambda+1)(2\lambda-1)^2}{(4\lambda-1)^2}\\
&\quad\times\Bigg[\sum_{k\notin A} L(k)R(k-1)+\sum_{k-1\notin A} L(k)R(k-1)\Bigg]\\
&+\frac{(4\lambda-3)(2\lambda-1)(4\lambda+1)}{2(4\lambda-1)^2}\\
&\quad\times\Bigg[\sum_{k,k+1\notin A} L(k)R(k)+\sum_{k,k-1\notin A} L(k)R(k)\Bigg]
\end{aligned}\tag{4.8}$$

$$+\frac{\lambda-1}{\lambda(4\lambda-1)^2}\sum_{k,k+1,k+2,k+3\notin A} L(k)R(k+3) - \frac{2\lambda-1}{2\lambda(4\lambda-1)}$$
$$\times\left[\sum_{k,k+1\notin A} L(k)R(k+2) + \sum_{k+1,k+2\notin A} L(k)R(k+2)\right]$$
$$+\frac{(2\lambda-1)(4\lambda+1)}{2(4\lambda-1)^2}\left[\sum_{k-1,k,k+1\notin A} L(k)R(k+1)\right.$$
$$\left. + \sum_{k,k+1,k+2\notin A} L(k)R(k+1)\right].$$

Next, we rearrange (4.8) using the shorthand notation

$$l(k) = L(k) - L(k-1)1_{\{k-1\notin A\}}, \qquad r(k) = R(k) - R(k+1)1_{\{k+1\notin A\}}.$$

Expression (4.8) becomes

$$-\frac{4\lambda-3}{(4\lambda+1)(4\lambda-1)^2}\sum_{k-1,k+1\notin A} l(k-1)r(k+1)$$
$$+\frac{\lambda-1}{\lambda(4\lambda-1)^2}\sum_{k-1,k\notin A} l(k-1)r(k)$$
$$-\frac{(4\lambda^2+\lambda+1)(2\lambda-1)}{\lambda(4\lambda-1)^2}\sum_{k\notin A} l(k)r(k)$$
$$(4.9)\qquad +\frac{(2\lambda-1)(4\lambda+1)}{2(4\lambda-1)^2}\left[\sum_{\substack{k\notin A\\ k-1\in A}} l(k)r(k) + \sum_{\substack{k\notin A\\ k+1\in A}} l(k)r(k)\right]$$
$$-\frac{(4\lambda+1)(2\lambda-1)^2}{(4\lambda-1)^2}\left[\sum_{k\notin A} l(k+1)r(k) + \sum_{k+1\notin A} l(k+1)r(k)\right]$$
$$+\frac{2\lambda-1}{2\lambda(4\lambda-1)}\left[\sum_{\substack{k\notin A\\ k-1\in A}} L(k-1)r(k) + \sum_{\substack{k\notin A\\ k+1\in A}} l(k)R(k+1)\right].$$

In order to show that (4.9) is nonpositive, we need to show that the functions l and r satisfy some inequalities. The ones for l are stated below; the ones for r are analogous. The proposition will be proved in Section 5.

PROPOSITION 4.10. (a) $l(k) \geq 0$ *for all* k.
(b) $l(k+1) \leq l(k)$ *if* $k \notin A$.

We continue with the proof that (4.9) is nonpositive. Think of writing (4.9) as a sum over maximal intervals in the complement of A of the contributions corresponding to those intervals. Only in the case of the first term in (4.9) is there any ambiguity about which interval to associate it to, and that only occurs when the $k \in A$. In that case, we ignore the term, which is negative anyway. Suppose then that $[m, n]$ is a maximal interval in the complement of A, so that $m-1, n+1 \in A$ and $m, \dots, n \notin A$. We consider the case in which the interval is a singleton separately from the general case. In both cases, we will multiply (4.9) by $2\lambda(4\lambda-1)^2(4\lambda+1)$ to remove the denominators. It will sometimes be useful to write L and R in terms of the functions corresponding to sets obtained from A by removing a point. For example, suppose that $k \notin A$ and $k-1 \in A$. Let L^* be defined as L was, but with respect to the set $A \setminus \{k-1\}$. Then

$$L^*(k-1) = L(k-1), \qquad L^*(k) = L(k) + f(1)L(k-1), \tag{4.11}$$

$$\begin{aligned} L^*(k+1) &= L(k+1) + f^2(1)L(k-1) + f(2)L_0(k-1) \\ &= L(k+1) + [f^2(1) + \alpha f(2)]L(k-1) \\ &\quad + (1-\alpha)f(2)L(k-2)1_{\{k-2 \notin A\}} \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} L^*(k+2) &= L(k+2) + [f^3(1) + f(2)f_0(1)]L(k-1) \\ &\quad + [f(1)f(2) + f(3)]L_0(k-1) \\ &= L(k+2) + [f^3(1) + (1-\alpha)f(2) \\ &\qquad + 2\alpha f(1)f(2) + \alpha f(3)]L(k-1) \\ &\quad + (1-\alpha)[f(1)f(2) + f(3)]L(k-2)1_{\{k-2 \notin A\}}, \end{aligned} \tag{4.13}$$

where the last equality in (4.12) and (4.13) comes from (4.2). Equations (4.11)–(4.13) can be rewritten as

$$L(k-1) = L^*(k-1), \qquad L(k) = L^*(k) - f(1)L^*(k-1), \tag{4.14}$$

$$\begin{aligned} L(k+1) &= L^*(k+1) - [f^2(1) + \alpha f(2)]L^*(k-1) \\ &\quad - (1-\alpha)f(2)L^*(k-2)1_{\{k-2 \notin A\}} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} L(k+2) &= L^*(k+2) - [f^3(1) + (1-\alpha)f(2) \\ &\qquad + 2\alpha f(1)f(2) + \alpha f(3)]L^*(k-1) \\ &\quad - (1-\alpha)[f(1)f(2) + f(3)]L^*(k-2)1_{\{k-2 \notin A\}}. \end{aligned} \tag{4.16}$$

Case 1: $m = n$. We need to show that the following quantity is nonpositive:

$$\begin{aligned} &-(4\lambda-3)\lambda[l(m-2)r(m)1_{\{m-2 \notin A\}} + l(m)r(m+2)1_{\{m+2 \notin A\}}] \\ &\quad -2(2\lambda-1)(4\lambda+1)l(m)r(m) - 2\lambda(4\lambda+1)^2(2\lambda-1)^2 \\ &\qquad \times [l(m+1)r(m) + l(m)r(m-1)] \\ &\quad +(4\lambda-1)(4\lambda+1)(2\lambda-1)[L(m-1)r(m) + l(m)R(m+1)]. \end{aligned}$$

Ignoring the first term above, which is negative, it is easy to see that [using the nonnegativity of $l(m)$ and $r(m)$] we need to show that

$$(4.17)\quad (4\lambda - 1)L(m-1) \le 2\lambda(4\lambda+1)(2\lambda-1)l(m+1) + l(m)$$

and the analogous inequality for the R's and r's. Using (4.14) and (4.15) to write this in terms of the starred quantities, we see that we need to prove that

$$0 \le 2\lambda(4\lambda+1)(2\lambda-1)l^*(m+1) + l^*(m) + 2(\lambda-1)l^*(m-1).$$

However, this is true by part (a) of Proposition 4.10 (applied to the starred quantities).

Case 2: $m < n$. We need to show that the following quantity is nonpositive:

$$\begin{aligned}
-2(4\lambda-3)\lambda &\sum_{m<k<n} l(k-1)r(k+1) \\
&+ 2(\lambda-1)(4\lambda+1)\sum_{m\le k<n} l(k)r(k+1) \\
&-(4\lambda^2+\lambda+2)(2\lambda-1)(4\lambda+1)[l(m)r(m)+l(n)r(n)] \\
&-2(4\lambda^2+\lambda+1)(2\lambda-1)(4\lambda+1)\sum_{m<k<n} l(k)r(k) \\
&-2\lambda(4\lambda+1)^2(2\lambda-1)^2 \\
&\quad\times\Big[l(m)r(m-1) + 2\sum_{m\le k<n} l(k+1)r(k) + l(n+1)r(n)\Big] \\
&+(4\lambda-1)(4\lambda+1)(2\lambda-1)[L(m-1)r(m)+l(n)R(n+1)].
\end{aligned}$$

Taking our cue from Case 1, we use (4.17) to remove $L(m-1)$ from the above expression and the analogue for the R's and r's to remove $R(n+1)$. Then replace $\lambda(4\lambda-3)$ in the coefficient of the first sum by the smaller quantity $(\lambda-1)(4\lambda+1)$ and then cancel the common factor of $(4\lambda+1)$ from all the terms. We then see that we must show that the following expression is nonpositive:

$$\begin{aligned}
-2(\lambda-1)&\sum_{m<k<n} l(k-1)r(k+1) + 2(\lambda-1)\sum_{m\le k<n} l(k)r(k+1) \\
&-(4\lambda^2+\lambda+1)(2\lambda-1)\Big[l(m)r(m) + 2\sum_{m<k<n} l(k)r(k) + l(n)r(n)\Big] \\
&-2\lambda(2\lambda-1)^2(4\lambda+1)\Big[l(m)r(m-1) + l(m+1)r(m)1_{\{n>m+1\}} \\
&\qquad + 2\sum_{m<k<n-1} l(k+1)r(k) \\
&\qquad + l(n)r(n-1)1_{\{n>m+1\}} + l(n+1)r(n)\Big].
\end{aligned}$$

This can be rewritten as

$$
\begin{aligned}
-(\lambda-1) & \sum_{m<k<n} \big(l(k-1)[r(k+1)-r(k)]+[l(k-1)-l(k)]r(k+1)\big) \\
& -(4\lambda^2+\lambda+1)(2\lambda-1) \\
& \quad \times\Big[l(m)r(m)1_{\{n>m+1\}}+l(m+1)r(m+1)1_{\{n>m+2\}} \\
& \qquad +2\sum_{m+1<k<n-1} l(k)r(k)+l(n-1)r(n-1)1_{\{n>m+2\}} \\
& \qquad\qquad +l(n)r(n)1_{\{n>m+1\}}\Big] \\
& -2\lambda(2\lambda-1)^2(4\lambda+1)\Big[l(m)r(m-1)1_{\{n>m+1\}}+l(m+1)r(m)1_{\{n>m+2\}} \\
& \qquad +l(m+2)r(m+1)1_{\{n>m+3\}} \\
& \qquad +2\sum_{m+1<k<n-2} l(k+1)r(k) \\
& \qquad +l(n-1)r(n-2)1_{\{n>m+3\}} \\
& \qquad +l(n)r(n-1)1_{\{n>m+2\}} \\
& \qquad +l(n+1)r(n)1_{\{n>m+1\}}\Big] \\
& +l(n-1)\big[(\lambda-1)r(n)-(4\lambda^2+\lambda+1)(2\lambda-1)r(n-1) \\
& \qquad -2\lambda(4\lambda+1)(2\lambda-1)^2 r(n-2)\big] \\
& +r(m+1)\big[(\lambda-1)l(m)-(4\lambda^2+\lambda+1)(2\lambda-1)l(m+1) \\
& \qquad -2\lambda(4\lambda+1)(2\lambda-1)^2 l(m+2)\big].
\end{aligned}
$$

By Proposition 4.10, it suffices to show that

$$
\begin{aligned}
(\lambda-1)l(m)-(4\lambda^2+\lambda+1)(2\lambda-1)l(m+1)& \\
-2\lambda(4\lambda+1)(2\lambda-1)^2 l(m+2) &\le 0
\end{aligned}
$$

(and the corresponding statement for the r's). This is equivalent to

$$
\begin{aligned}
(8\lambda^3-2\lambda^2+2\lambda-2)L(m)&+(32\lambda^4-32\lambda^3+2\lambda^2+1)L(m+1) \\
&\le(32\lambda^4-24\lambda^3+2\lambda)L(m+2).
\end{aligned}
$$

Rewriting this in terms of the starred quantities using (4.14)–(4.16), this is equivalent to

$$0 \le (32\lambda^4 - 24\lambda^3 + 2\lambda)l^*(m+2) + (8\lambda^3 - 2\lambda^2 + \lambda - 1)l^*(m+1) + (\lambda - 1)[l^*(m-1) - l^*(m)] + \frac{2(20\lambda^3 - 13\lambda^2 + 1)}{\lambda(4\lambda+1)} l^*(m-1) + \frac{(4\lambda - 1)(8\lambda^2 - \lambda - 1)}{\lambda(4\lambda+1)} L^*(m-2) 1_{\{m-2 \notin A\}}.$$

However, this is a consequence of Proposition 4.10.

5. Proof of Proposition 4.10. We begin by computing

$$\begin{aligned} 1 - L(k) &= \nu\{\eta(j) = 1 \text{ for some } j \in A \cap (-\infty, k) \mid \eta(k) = \eta(k+1) = 1\} \\ &= \sum_{\substack{j<k \\ j \in A}} \nu\{\eta(j) = 1, \eta = 0 \text{ on } A \cap (-\infty, j) \mid \eta(k) = \eta(k+1) = 1\} \\ &= \sum_{\substack{j<k \\ j \in A}} \nu\{\eta(j) = 1, \eta(j+1) = 0 \mid \eta(k) = \eta(k+1) = 1\} L_0(j) \\ &\quad + \sum_{\substack{j<k \\ j \in A}} \nu\{\eta(j) = 1, \eta(j+1) = 1 \mid \eta(k) = \eta(k+1) = 1\} L(j) \\ &= \sum_{\substack{j<k \\ j \in A}} [v(k-j)L_0(j) + w(k-j)L(j)], \end{aligned} \tag{5.1}$$

where the sequences $v(k)$ and $w(k)$ $(k \ge 0)$ are defined by the last equality. These satisfy the initial conditions $v(0) = v(1) = 0$ and $w(0) = 1$, $w(1) = f(1)$. Also, the argument which led to (4.1) and (4.2) implies that they satisfy the recursions

$$v(k) = f(1)v(k-1) + \sum_{j=2}^{k-1} f(j)[\alpha v(k-j) + (1-\alpha)v(k-j-1)] + f(k)$$

and

$$w(k) = f(1)w(k-1) + \sum_{j=2}^{k-1} f(j)[\alpha w(k-j) + (1-\alpha)w(k-j-1)]$$

for $k \ge 2$. Later, we will prove that if $x(k) = v(k) + w(k)$ or $x(k) = \alpha v(k) + w(k)$, then $x(k)$ is decreasing and concave:

$$x(k+1) \le x(k) \quad \text{and} \quad 2x(k+1) \le x(k+2) + x(k) \tag{5.2}$$

for $k \ge 0$. Now we will use these facts to complete the proof of Proposition 4.10.

The first statement is immediate if $k-1 \in A$, so assume that $k-1 \notin A$. Then by (5.1) and (4.2),

$$\begin{aligned}
l(k) &= \sum_{\substack{j<k-1\\ j\in A}} \big[v(k-1-j)L_0(j) + w(k-1-j)L(j) \\
&\qquad\qquad -v(k-j)L_0(j) - w(k-j)L(j)\big] \\
&= \sum_{\substack{j<k-1\\ j\in A}} \big[L(j)[\alpha v(k-1-j) - \alpha v(k-j) \\
&\qquad\qquad +w(k-1-j) - w(k-j)] \\
&\qquad\quad +L(j-1)1_{\{j-1\notin A\}}(1-\alpha)[v(k-1-j) - v(k-j)]\big] \\
&= \sum_{\substack{j<k-1\\ j\in A}} l(j)[(\alpha v + w)(k-1-j) - (\alpha v + w)(k-j)] \\
&\quad + \sum_{\substack{j<k-1\\ j-1\notin A\\ j\in A}} L(j-1)[(v+w)(k-1-j) - (v+w)(k-j)].
\end{aligned}$$

The expressions in brackets are nonnegative by (5.2), so the nonnegativity of $l(k)$ follows by induction on k.

The proof of the second part of Proposition 4.10 is similar. Take $k \notin A$ and write

$$\begin{aligned}
&l(k) - l(k+1) \\
&\quad = 2L(k) - L(k+1) - L(k-1)1_{\{k-1\notin A\}} \\
&\quad = 2L(k) - L(k+1) - L(k-1) + L(k-1)1_{\{k-1\in A\}} \\
&\quad = \big[-2w(1)L(k-1) \\
&\qquad\quad + [v(2)L_0(k-1) + w(2)L(k-1)] + L(k-1)\big]1_{\{k-1\in A\}} \\
&\qquad\quad + \sum_{\substack{j<k-1\\ j\in A}} \big[[v(k-1-j) + v(k+1-j) - 2v(k-j)]L_0(j) \\
&\qquad\qquad\qquad + [w(k-1-j) + w(k+1-j) - 2w(k-j)]L(j)\big] \\
&\quad = [F^2(2) + \alpha f(2)]L(k-1)1_{\{k-1\in A\}} \\
&\qquad\quad + (1-\alpha)f(2)L(k-2)1_{\{k-2\notin A,\, k-1\in A\}} \\
&\qquad\quad + \sum_{\substack{j<k-1\\ j\in A}} l(j)[(\alpha v + w)(k+1-j) \\
&\qquad\qquad\qquad + (\alpha v + w)(k-1-j) - 2(\alpha v + w)(k-j)] \\
&\qquad\quad + \sum_{\substack{j<k-1\\ j-1\notin A\\ j\in A}} L(j-1)[(v+w)(k-1-j) \\
&\qquad\qquad\qquad + (v+w)(k+1-j) - 2(v+w)(k-j)],
\end{aligned}$$

which is nonnegative by part (a) of Proposition 4.10 and (5.2).

We next turn to the proof of (5.2). Define a sequence $x(k)$ by $x(0) = 1$, $x(1) = f(1)$ and

$$x(k) = f(1)x(k-1) + \sum_{j=2}^{k-1} f(j)[\alpha x(k-j) + (1-\alpha)x(k-j-1)] + df(k) \tag{5.3}$$

for $k \geq 2$. We need to show that this sequence is decreasing and concave for $d = 1$ and for $d = \alpha$. The proof is based on the following identity, which holds for $k \geq 4$:

$$\begin{aligned} df(k) + (1-\alpha)f(k-1) = 2\lambda x(k+1) &- \frac{32\lambda^3 - 12\lambda^2 - 5\lambda - 1}{(4\lambda+1)(2\lambda-1)} x(k) \\ &+ \frac{32\lambda^4 - 40\lambda^3 + 4\lambda^2 - \lambda + 3}{(4\lambda+1)(2\lambda-1)^2} x(k-1) \\ &+ \frac{32\lambda^4 + 14\lambda^2 - 12\lambda - 3}{(4\lambda+1)^2(2\lambda-1)^2} x(k-2) \\ &- \frac{12\lambda^2 - 9\lambda - 1}{(4\lambda+1)^2(2\lambda-1)^2} x(k-3) \\ &+ \frac{4\lambda^2 - 3\lambda}{(4\lambda+1)^2(2\lambda-1)^2} x(k-4). \end{aligned} \tag{5.4}$$

The proof involves a rather lengthy and unenlightening computation which begins by using (5.3) twice to express $x(k)$ in terms of a convolution of the f's and then uses (3.15) to write this convolution in terms of f itself. After a substantial amount of cancellation, one is led to (5.4). Rather than carrying out this computation, the reader is encouraged to check (5.4) for a few values of k.

Letting $\Delta(k) = x(k+1) - 2x(k) + x(k-1)$, (5.4) can be rewritten as

$$\begin{aligned} \Delta(k) = zf(k) + yf(k-1) - u\Delta(k-1) \\ + v\Delta(k-2) - w\Delta(k-3) \end{aligned} \tag{5.5}$$

for $k \geq 4$, where the coefficients are given by

$$z = \frac{d}{2\lambda},$$

$$y = \frac{(4\lambda-1)(2\lambda-1) - 2\lambda d(4\lambda-3)}{2\lambda(2\lambda-1)(4\lambda+1)},$$

$$u = \frac{4\lambda^2 + \lambda + 1}{2\lambda(4\lambda+1)(2\lambda-1)},$$

$$v = \frac{\lambda - 1}{2\lambda(4\lambda + 1)(2\lambda - 1)^2},$$
$$w = \frac{4\lambda - 3}{2(4\lambda + 1)^2(2\lambda - 1)^2}.$$

We need some information on the behavior of ratios of f's. Using (3.19), (3.20) and $\lambda \geq 1.941227\ldots$, it follows that $c \geq 4.48\ldots$, and hence by (3.21) and (3.22) that

$$\frac{f(m)}{f(m+1)} \leq 1.127\frac{h(m)}{h(m+1)}, \qquad m \geq 4. \tag{5.6}$$

Next, let $\sigma(k) = \Delta(k)/f(k+1)$. Then (5.5) becomes

$$\sigma(k) = [z - u\sigma(k-1)]\frac{f(k)}{f(k+1)} + [y + v\sigma(k-2)]\frac{f(k-1)}{f(k+1)}$$
$$- w\sigma(k-3)\frac{f(k-2)}{f(k+1)}.$$

We wish to show by induction that

$$0 \leq \sigma(i) \leq 1 \tag{5.7}$$

for all $i \geq 3$. This will prove (5.2). In fact, it is easier to show it only for λ slightly larger than $1.941227\ldots$, and that is all that is needed to prove the theorem. We will perform the following computations with $\lambda = 1.941227\ldots$ (which implies $a = 1$). Since all the inequalities will be strict and only finitely many are needed, these will be valid for slightly larger λ as well. Here are the values (to the accuracy given) of the relevant quantities:

$$z = 0.2195 \quad (\text{if } d = \alpha), \qquad z = 0.2576 \quad (\text{if } d = 1),$$
$$y = 0.0381 \quad (\text{if } d = \alpha), \qquad z = 0.0102 \quad (\text{if } d = 1),$$
$$u = 0.1837, \qquad v = 0.0033, \qquad w = 0.0037.$$

Suppose (5.7) holds for $i = k-3, k-2, k-1$. We then have the recursion for σ that

$$\sigma(k) \geq 0.036\frac{f(k)}{f(k+1)} - 0.004\frac{f(k-2)}{f(k+1)}$$

and

$$\sigma(k) \leq 0.258\frac{f(k)}{f(k+1)} + 0.041\frac{f(k-1)}{f(k+1)}.$$

For any $m \geq 4$, (5.6) and the logconvexity of h imply that

$$\frac{f(m)}{f(m+1)} \leq 1.127\frac{h(m)}{h(m+1)} \leq 1.127\frac{h(4)}{h(5)} = 1.127\frac{12}{7} \leq 2.$$

Therefore it follows that (5.7) holds for $i = k$ provided that $k \geq 6$. It remains to show that $\sigma(k) \geq 0$ for $1 \leq k \leq 5$ and $\sigma(k) \leq 1$ for $3 \leq k \leq 5$. This is done

TABLE 1

n	$f(n)$	$x(n)$	$\sigma(n)$	$x(n)$	$\sigma(n)$
1	0.732236	0.7322	1.5522	0.7322	1.4044
2	0.129829	0.6660	0.7675	0.6468	1.0955
3	0.051142	0.6390	0.3460	0.6174	0.4337
4	0.025869	0.6210	0.3244	0.5992	0.3345
5	0.015161	0.6079	0.3223	0.5861	0.3285

by direct computation. In Table 1 the middle two columns correspond to $d = 1$ and the last two columns correspond to $d = \alpha$.

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REFERENCES

BEZUIDENHOUT, C. and GRIMMETT, G. (1990). The critical contact process dies out. *Ann. Probab.* **18** 1462–1482.

GRILLENBERGER, C. and ZIEZOLD, H. (1988). On the critical infection rate of the one dimensional basic contact process: numerical results. *J. Appl. Probab.* **25** 1–8.

HARA, T. and SLADE, G. (1992). The lace expansion for self-avoiding walk in five or more dimensions. *Rev. Math. Phys.* **4** 235–327.

HARA, T., SLADE, G. and SOKAL, A. (1993). New lower bounds on the self-avoiding walk connective constant. *J. Statist. Phys.* **72** 479–517.

HOLLEY, R. A. and LIGGETT, T. M. (1978). The survival of contact processes. *Ann. Probab.* **6** 198–206.

KATORI, M. and KONNO, N. (1993). Bounds for the critical line of the θ-contact processes with $1 \leq \theta \leq 2$. *J. Phys. A* **26** 6597–6614.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, Berlin.

LIGGETT, T. M. (1991a). Spatially inhomogeneous contact processes. In *Spatial Stochastic Processes. A Festschrift in Honor of Ted Harris on his Seventieth Birthday* (K. S. Alexander and J. C. Watkins, eds.) 105–140. Birkhäuser, Boston.

LIGGETT, T. M. (1991b). The periodic threshold contact process. In *Random Walks, Brownian Motion and Interacting Particle Systems. A Festschrift in Honor of Frank Spitzer* (R. Durrett and H. Kesten, eds.) 339–358. Birkhäuser, Boston.

LIGGETT, T. M. (1992). The survival of one dimensional contact processes in random environments. *Ann. Probab.* **20** 696–723.

LIGGETT, T. M. (1994). Coexistence in threshold voter models. *Ann. Probab.* **22** 764–802.

PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116.

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