The Annals of Probability
1995, Vol. 23, No. 3, 1439–1455

SYMMETRIC TWO-PARTICLE EXCLUSION-EATING PROCESSES[1]

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We consider infinite particle systems on a countable set S with two-particle exclusion-eating motion determined by a symmetric transition function $p(x, y)$. This is, in a certain sense, a mixture of the exclusion process and the voter model. We discuss the dual process of this process and use the dual process to give a description of the set of invariant measures and to prove an ergodic theorem.

1. Introduction. The exclusion process was introduced by Spitzer (1970). Let $p(x, y)$ be the transition function for a Markov chain on a countable set S. There is at most one particle per site. That is, the set of configurations is $X_1 = \{0, 1\}^S$. Each particle waits an exponential time with parameter 1 and then attempts a transition to another site in S chosen according to the probability $p(x, y)$. It makes the transition if that site is vacant, while if it is occupied, the particle remains where it was. A series of papers with rather complete results have been written on exclusion processes in the past two decades [see Chapter 8 of Liggett (1985); also see Saada (1987), Andjel, Bramson and Liggett (1988), Kipnis, Olla and Varadhan (1989)].

Motivated by the two-particle contact process of Durrett and Swindle (1991) and the cyclic particle systems of Bramson and Griffeath (1989), we consider two-particle motion with exclusion-eating interaction. Let $X = \{0, 1, 2\}^S$ be the set of configurations. Each site of S may be occupied by a particle, either type 1 or type 2, or be vacant. Each particle waits an exponential time with parameter 1 and then attempts a transition to another site in S chosen according to the probability $p(x, y)$. It makes the transition if that site is vacant. No transition is made if that site is occupied by a particle of the same type. When a type 2 particle attempts a move to a site occupied by a type 1 particle, the type 2 particle eats the type 1 particle, that is, both sites become occupied by type 2 particles. A type 1 particle eats a type 2 particle in a similar manner.

The generator of the process is the closure of the operator:

$$\Omega f(\eta) = \sum_{\substack{\eta(x)\neq 0 \\ \eta(x)\neq\eta(y)}} p(x, y)\left[f(\eta_{x\to y}) - f(\eta)\right], \tag{1.1}$$

f depends on finitely many sites,

Received June 1993; revised November 1994.
[1] Research supported in part by the U.S. Department of Energy Grant DE-FG05-86ER-75274.
AMS 1991 *subject classification*. Primary 60K36.
Key words and phrases. Exclusion process, voter model, coalescing random walk, duality, ergodic theorem.

where $\eta_{x \to y}$ is defined as

$$\eta_{x \to y}(z) = \begin{cases} \eta(z), & \text{when } z \neq x, y, \\ \eta(x), & \text{when } z = y, \\ [2\eta(y)] \wedge \eta(x), & \text{when } z = x. \end{cases}$$

Note that the last expression above means $\eta_{x \to y}(x) = \eta(y)$ if $\eta(y) = 0$ and $\eta_{x \to y}(x) = \eta(x)$ if $\eta(y) \neq 0$.

When there are no 1's or no 2's, this is the exclusion process, whereas when there are no 0's, it is equivalent to the voter model.

Throughout this paper, we assume that $p(x, y) = p(y, x)$ and that $p(x, y)$ is irreducible. The invariant measures of the symmetric two-particle exclusion-eating process are closely related to the bounded harmonic functions for $p(x, y)$. Let

$$\mathscr{H} = \left\{\alpha\colon S \to [0,1] \text{ such that } \sum_y p(x, y)\alpha(y) = \alpha(x) \text{ for all } x\right\}.$$

Let $X_i(t)$, $i = 1, \dots, n$, be independent copies of the continuous time Markov chain with transition probabilities

$$p_t(x, y) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} p^{(n)}(x, y), \tag{1.2}$$

where $p^{(n)}(x, y)$ are the n-step transition probabilities associated with $p(x, y)$. Let

$$\mathscr{E} = \{\text{there exists } t_n \uparrow \infty \text{ such that } X_1(t_n) = X_2(t_n)\}.$$

For $\alpha \in \mathscr{H}$, $\alpha(X_1(t))$ is a bounded martingale, so $\lim_{t \to \infty} \alpha(X_1(t))$ exists with probability 1. Now let

$$\mathscr{H}' = \{(\alpha_1, \alpha_2) \in \mathscr{H} \times \mathscr{H}\colon \alpha_1(x) \geq \alpha_2(x) \text{ for all } x\}$$

and

$$\mathscr{H}^* = \Big\{(\alpha_1, \alpha_2) \in \mathscr{H}'\colon \lim_{t \to \infty} \alpha_2(X_1(t))[\alpha_1(X_1(t)) - \alpha_2(X_1(t))] = 0 \text{ a.s. on } \mathscr{E}\Big\}.$$

Let

$$T_n = \left\{\vec{x} \in S^n\colon x_i \neq x_j \text{ for all } 1 \leq i \neq j \leq n\right\}.$$

Define a function g on $\bigcup_{n=2}^{\infty} S^n$ as follows: if $\vec{x} \in S^n$, then

$$g(\vec{x}) = P^{\vec{x}}[(X_1(t), \dots, X_n(t)) \notin T_n \text{ for some } t \geq 0].$$

For $(\alpha_1, \alpha_2) \in \mathscr{H}'$, define ν_{α_1, α_2} to be the product measure on X with marginals

$$\nu_{\alpha_1, \alpha_2}\{\eta: \eta(x) = i\} = \begin{cases} \alpha_2(x), & \text{when } i = 2, \\ \alpha_1(x) - \alpha_2(x), & \text{when } i = 1, \\ 1 - \alpha_1(x), & \text{when } i = 0. \end{cases}$$

In other words,

$$\nu_{\alpha_1, \alpha_2}\{\eta(x) \neq 0\} = \alpha_1(x), \qquad \nu_{\alpha_1, \alpha_2}\{\eta(x) = 2\} = \alpha_2(x).$$

Let $\mathscr{I}$ be the set of invariant measures of the process and let $\mathscr{I}_e$ be the set of its extreme points. We now state the main results of this paper.

THEOREM 1. (a) *For* $(\alpha_1, \alpha_2) \in \mathscr{H}'$, $\lim_{t \to \infty} \nu_{\alpha_1, \alpha_2} S(t) = \mu_{\alpha_1, \alpha_2}$ *exists and is in* $\mathscr{I}$.

(b) $\mathscr{I}_e = \{\mu_{\alpha_1, \alpha_2}: (\alpha_1, \alpha_2) \in \mathscr{H}^*\}$.

When $S = Z^d$ and $p(x, y) = p(0, y - x)$, it is known [see Corollary I.7.2 of Liggett (1985)] that $\mathscr{H}$ consists of constants. If $g \equiv 1$, then $P(\mathscr{E}) = 1$, $\mathscr{H}^* = \{(\alpha, 0), (\alpha, \alpha), 0 \leq \alpha \leq 1\}$ and $\mathscr{I}_e = \{\nu_{\alpha, 0}, \nu_{\alpha, \alpha}: 0 \leq \alpha \leq 1\}$. If $g \not\equiv 1$, then $P(\mathscr{E}) = 0$ and $\mathscr{H}^* = \mathscr{H}' = \{(\alpha_1, \alpha_2): 0 \leq \alpha_2 \leq \alpha_1 \leq 1\}$. Let $\mathscr{S}$ be the set of translation invariant measures on X and let $\mathscr{S}_e$ be the set of its extreme points.

THEOREM 2. *Suppose* $S = Z^d$, $p(x, y) = p(0, y - x)$ *and* $\mu \in \mathscr{S}_e$. *Let* $\alpha_1 = \mu\{\eta(x) \neq 0\}$ *and* $\alpha_2 = \mu\{\eta(x) = 2\}$.

(a) *If* $g \equiv 1$, *then*

$$\lim_{t \to \infty} \mu S(t) = \left(1 - \frac{\alpha_2}{\alpha_1}\right) \nu_{\alpha_1, 0} + \frac{\alpha_2}{\alpha_1} \nu_{\alpha_1, \alpha_1}.$$

(b) *If* $g \not\equiv 1$, *then*

$$\lim_{t \to \infty} \mu S(t) = \mu_{\alpha_1, \alpha_2}.$$

It is known that the symmetric exclusion process and the voter model have dual processes which are exclusion process and coalescing random walk, respectively [Liggett (1985)]. So it is natural to ask whether the two-particle exclusion-eating process has a dual process. Duality theory for this process is developed in Section 2. It turns out that its dual process can be represented by a coupling of an exclusion process and a coalescing random walk. This makes it possible to employ the techniques for voter models and for exclusion processes for the two-particle exclusion-eating process. However, unlike the voter model and the exclusion process, the two-particle exclusion-eating process is not a monotone process. It also loses certain monotonicity properties possessed by the voter model or the exclusion process [see Lemma V.1.32 and Proposition VIII.1.7 of Liggett (1985)]. The proofs of the main theorems will be given in Section 4.

2. Duality. Let

$$Y = \{A: A \text{ is a finite subset of } S\}.$$

For $\eta \in X$ and $i = 0, 1, 2$, let

$${}_i\eta(x) = \begin{cases} 1, & \text{when } \eta(x) = i, \\ 0, & \text{otherwise.} \end{cases}$$

For $A, B \in Y$, define a duality function

$$H(\eta, A, B) = \prod_{x \in A} {}_2\eta(x) \prod_{y \in B} [1 - {}_0\eta(y)].$$

For $B \in Y$, let $B_{xy} = B$ when $x, y \in B$ or $x, y \notin B$; $B_{xy} = B \setminus \{x\} \cup \{y\}$ when $x \in B$, $y \notin B$; $B_{xy} = B \setminus \{y\} \cup \{x\}$ when $y \in B$, $x \notin B$.

LEMMA 2.1. *If $A, B \in Y$ and $A \subseteq B$, then*

$$\begin{aligned} \Omega H(\cdot, A, B)(\eta) &= \sum_{\substack{x \in A \\ y \in S}} p(x, y)\big[H(\eta, A \setminus \{x\} \cup \{y\}, B_{xy}) - H(\eta, A, B)\big] \\ &\quad + \sum_{\substack{x \in B \setminus A \\ y \notin B}} p(x, y)\big[H(\eta, A, B_{xy}) - H(\eta, A, B)\big]. \end{aligned} \tag{2.2}$$

PROOF. It is convenient to use the notation η_{xy} of the exclusion process and η_x of the spin-flip process. Let $\eta_{xy}(z) = \eta(z)$, when $z \neq x, y$, $\eta_{xy}(x) = \eta(y)$ and $\eta_{xy}(y) = \eta(x)$; $\eta_x(z) = \eta(z)$, when $z \neq x$ and $\eta_x(x) = [3 - \eta(x)] \times [1 - {}_0\eta(x)]$. By (1.1),

$$\begin{aligned} \Omega H(\eta, A, B) &= \sum_{\substack{\eta(x) \neq 0 \\ \eta(y) = 0}} p(x, y)\big[H(\eta_{xy}, A, B) - H(\eta, A, B)\big] \\ &\quad + \sum_{\eta(x)\eta(y) = 2} p(x, y)[H(\eta_x, A, B) - H(\eta, A, B)] \\ &= \Omega_1 H(\eta, A, B) + \Omega_2 H(\eta, A, B). \end{aligned}$$

Then

$$\begin{aligned} &\Omega_1 H(\eta, A, B) \\ &\quad = \frac{1}{2} \sum_{\eta(x)\eta(y) = 0} p(x, y)\big[H(\eta_{xy}, A, B) - H(\eta, A, B)\big] \\ &\quad = \sum_{\substack{x \in A \\ y \notin B \\ \eta(x)\eta(y) = 0}} p(x, y)\big[H(\eta, A \setminus \{x\} \cup \{y\}, B_{xy}) - H(\eta, A, B)\big] \\ &\qquad + \sum_{\substack{x \in B \setminus A \\ y \notin B \\ \eta(x)\eta(y) = 0}} p(x, y)\big[H(\eta, A, B_{xy}) - H(\eta, A, B)\big], \end{aligned} \tag{2.3}$$

whereas

$$
\begin{aligned}
\Omega_2 H(\eta, A, B) &= \sum_{\substack{x\in A\\ \eta(x)\eta(y)=2}} p(x,y)\big[H(\eta_x, A, B) - H(\eta, A, B)\big]\\
&= \sum_{\substack{x\in A\\ \eta(x)=1\\ \eta(y)=2}} p(x,y)H(\eta_x, A, B) - \sum_{\substack{x\in A\\ \eta(x)=2\\ \eta(y)=1}} p(x,y)H(\eta, A, B)\\
&= \sum_{\substack{x\in A\\ y\in S}} p(x,y)H\big(\eta, A\setminus\{x\}\cup\{y\}, B_{xy}\big)_1\eta(x)\\
&\quad - \sum_{\substack{x\in A\\ y\in S}} p(x,y)H(\eta, A, B)_1\eta(y)\\
&= \sum_{\substack{x\in A\\ \eta(x)\eta(y)\neq 0}} p(x,y)\Big[H\big(\eta, A\setminus\{x\}\cup\{y\}, B_{xy}\big)[1 -{}_2\eta(x)]\\
&\qquad\qquad - H(\eta, A, B)[1 -{}_2\eta(y)]\Big]\\
&= \sum_{\substack{x\in A\\ \eta(x)\eta(y)\neq 0}} p(x,y)\Big[H\big(\eta, A\setminus\{x\}\cup\{y\}, B_{xy}\big) - H(\eta, A, B)\Big].
\end{aligned}
\tag{2.4}
$$

It is easy to see that

$$
\sum_{\substack{x\in B\setminus A\\ y\notin B\\ \eta(x)\eta(y)\neq 0}} p(x,y)\Big[H\big(\eta, A, B_{xy}\big) - H(\eta, A, B)\Big] = 0 \tag{2.5}
$$

and

$$
\sum_{\substack{x\in A\\ y\in B\\ \eta(x)\eta(y)=0}} p(x,y)\Big[H\big(\eta, A\setminus\{x\}\cup\{y\}, B_{xy}\big) - H(\eta, A, B)\Big] = 0. \tag{2.6}
$$

By (2.3), (2.4), (2.5) and (2.6), we have (2.2). □

Let (A_t, B_t) be the finite process with $A_0 \subseteq B_0 \in Y$ and the following transition matrix: $(A, B) \to (A, B_{xy})$ at rate $p(x, y)$ if $x \in B\setminus A$, $y \notin B$; $(A, B) \to (A\setminus\{x\}\cup\{y\}, B_{xy})$ at rate $p(x, y)$ if $x \in A$, $y \in S$. Note that its first and second marginals are coalescing random walks and the finite exclusion process, respectively, and $A_t \subseteq B_t$ for all $t \geq 0$.

THEOREM 2.7. *If $\eta \in X$, $A, B \in Y$ and $A \subseteq B$, then*

$$
P^{\eta}[\eta_t = 2 \text{ on } A,\ \eta_t \neq 0 \text{ on } B] = P^{A,B}[\eta = 2 \text{ on } A_t,\ \eta \neq 0 \text{ on } B_t]
$$

for all $t \geq 0$.

PROOF. Let

$$u_\eta(t, A, B) = P^\eta[\eta_t = 2 \text{ on } A, \eta_t \neq 0 \text{ on } B] = S(t)H(\cdot, A, B)(\eta).$$

By (2.2),

$$\begin{aligned}
\frac{d}{dt}u_\eta(t, A, B) &= S(t)\Omega H(\cdot, A, B)(\eta) \\
&= \sum_{\substack{x\in A\\ y\in S}} p(x, y)\big[S(t)H(\cdot, A\setminus\{x\} \cup \{y\}, B_{xy})(\eta) \\
&\qquad\qquad - S(t)H(\cdot, A, B)(\eta)\big] \\
&\quad + \sum_{\substack{x\in B\setminus A\\ y\notin B}} p(x, y)\big[S(t)H(\cdot, A, B_{xy})(\eta) \\
&\qquad\qquad - S(t)H(\cdot, A, B)(\eta)\big] \\
&= \sum_{\substack{x\in A\\ y\in S}} p(x, y)\big[u_\eta(t, A\setminus\{x\} \cup \{y\}, B_{xy}) - u_\eta(t, A, B)\big] \\
&\quad + \sum_{\substack{x\in B\setminus A\\ y\notin B}} p(x, y)\big[u_\eta(t, A, B_{xy}) - u_\eta(t, A, B)\big].
\end{aligned}$$

The unique solution to these differential equations with initial condition $H(\eta, A, B)$ is

$$E^{A,B}H(\eta, A_t, B_t) = P^{A,B}[\eta = 2 \text{ on } A_t, \eta \neq 0 \text{ on } B_t].$$

This proves the theorem. □

For any probability measure on X, define

$$\hat{\mu}(A, B) = \int H(\eta, A, B)\mu(d\eta) = \mu\{\eta: \eta = 2 \text{ on } A, \eta \neq 0 \text{ on } B\}$$

for $A, B \in Y$. Then we have the following corollary.

COROLLARY 2.8. *For any probability measure μ on X, let $\mu_t = \mu S(t)$. Then*

$$\hat{\mu}_t(A, B) = E^{A,B}\hat{\mu}(A_t, B_t)$$

for all $A, B \in Y$ and $A \subseteq B$.

3. Some results on finite exclusion processes. Let $B_1(t)$ and $B_2(t)$ be two finite simple exclusion processes with $B_1(0) \subseteq B_2(0)$. By basic coupling, they can be constructed such that $B_1(t) \subseteq B_2(t)$ for all $t \geq 0$. We denote such a pair of finite simple exclusion processes by $(B_1(t), B_2(t))$.

Assume $m \leq n$. Let

$$S_{m,n} = \big\{(\vec{x}, \vec{y}) \in S^m \times S^n: \{x_1, \dots, x_m\} \subseteq \{y_1, \dots, y_n\}\big\}$$

and

$$T_{m,n} = \{(\vec{x}, \vec{y}) \in T_m \times T_n : \{x_1, \dots, x_m\} \subseteq \{y_1, \dots, y_n\}\}.$$

For $(\vec{x}, \vec{y}) \in S_{m,n}$, define

$$f_{m,n}(\vec{x}, \vec{y}) = \hat{\mu}(\{x_1, \dots, x_m\}, \{y_1, \dots, y_n\})$$

and

$$h_{m,n}(\vec{x}, \vec{y}) = \prod_{i=1}^{m} [\alpha_2(x_i)/\alpha_1(x_i)] \prod_{j=1}^{n} \alpha_1(y_j),$$

where μ is a probability measure on X and $(\alpha_1, \alpha_2) \in \mathscr{H}'$. For $\vec{y} \in S^n$, define

$$f'_{m,n}(\vec{y}) = f_{m,n}((y_1, \dots, y_m), (y_1, \dots, y_n))$$

and

$$h'_{m,n}(\vec{y}) = h_{m,n}((y_1, \dots, y_m), (y_1, \dots, y_n)).$$

Let $V_{m,n}(t)$ be the semigroup of $(B_1(t), B_2(t))$ on $T_{m,n}$ and let $U_n(t)$ be the semigroup of independent walks $(X_1(t), \dots, X_n(t))$ on S^n.

LEMMA 3.1. *Assume* $g \not\equiv 1$.

(a) *The limit* $\lim_{t \to \infty} V_{m,n}(t) h_{m,n} = \tilde{h}_{m,n}$ *exists on* $T_{m,n}$ *for all* $m \le n$.

(b) $\lim_{t \to \infty} U_n(t) f'_{m,n} = h'_{m,n}$ *on* S^n *is equivalent to* $\lim_{t \to \infty} V_{m,n}(t) f_{m,n} = \tilde{h}_{m,n}$ *on* $T_{m,n}$.

PROOF. (a) By reordering if necessary, we need only to prove the limit exists on

$$R_{m,n} = \{(\vec{x}, \vec{y}) \in T_{m,n} : x_i = y_i, i = 1, \dots, m\}.$$

Assume $(\vec{x}, \vec{y}) \in R_{m,n}$. Take $B_1(0) = \{x_1, \dots, x_m\}$, $B_2(0) = \{x_1, \dots, x_n\}$ and $X_i(0) = x_i$, $i = 1, \dots, n$. Let

$$(\{X_1(t), \dots, X_m(t)\}, \{X_1(t), \dots, X_n(t)\}),$$

and $(B_1(t), B_2(t))$ move together before $(X_1(t), \dots, X_n(t))$ hits $S^n \setminus T_n$, and then let them move independently. We have

$$\left| V_{m,n}(t) h_{m,n}(\vec{x}, \vec{y}) - U_n(t) h'_{m,n}(\vec{y}) \right| \le g_n(\vec{y})$$

on $R_{m,n}$, where g_n is the restriction of g to S^n. Since $U_n(t) h'_{m,n} = h'_{m,n}$, we have

$$\begin{aligned} &\left| V_{m,n}(t') h_{m,n}(\vec{x}, \vec{y}) - V_{m,n}(t'') h_{m,n}(\vec{x}, \vec{y}) \right| \\ &\quad \le \left| V_{m,n}(t') h_{m,n}(\vec{x}, \vec{y}) - U_n(t') h'_{m,n}(\vec{y}) \right| \\ &\qquad + \left| U_n(t'') h'_{m,n}(\vec{y}) - V_{m,n}(t'') h_{m,n}(\vec{x}, \vec{y}) \right| \\ &\quad \le 2 g_n(\vec{y}) \end{aligned}$$

on $R_{m,n}$. Take $t'_k, t''_k \uparrow \infty$ such that

$$F'_{m,n} = \lim_{k\to\infty} V_{m,n}(t'_k) h_{m,n}$$

and

$$F''_{m,n} = \lim_{k\to\infty} V_{m,n}(t''_k) h_{m,n}$$

exist on $T_{m,n}$. By Lemma V.1.26 of Liggett (1985), $V_{m,n}(t)F'_{m,n} = F'_{m,n}$ and $V_{m,n}(t)F''_{m,n} = F''_{m,n}$. Thus

$$\left|F'_{m,n}(\vec{x},\vec{y}) - F''_{m,n}(\vec{x},\vec{y})\right| \le 2g_n(\vec{y})$$

on $R_{m,n}$. Notice that $V_{m,n}(t)g_n(\vec{y}) = V_n(t)g_n(\vec{y})$, when $\vec{y} \in T_n$, where $V_n(t)$ is the semigroup of $B_2(t)$ on T_n. By Lemma VIII.1.23 of Liggett (1985),

$$\lim_{t\to\infty} V_{m,n}(t) g_n(\vec{y}) = 0, \qquad \vec{y} \in T_n.$$

Hence $F'_{m,n} = F''_{m,n}$ on $R_{m,n}$. This proves (a).

Part (b) can be proved using the same technique. For detail see the proof of Theorem VIII.1.24 of Liggett (1985). □

LEMMA 3.2. *Suppose that $g \not\equiv 1$. Let $(\alpha_1, \alpha_2) \in \mathscr{H}'$ and let μ be a probability measure on X. Then*

$$\lim_{t\to\infty} E^{A,B}\left[\hat{\mu}(B_1(t), B_2(t)) - \prod_{x\in B_1(t)} \alpha_2(x) \prod_{y\in B_2(t)\setminus B_1(t)} \alpha_1(y)\right] = 0$$

for all $A \subseteq B$, $A, B \in Y$, if and only if

(3.3) $$\lim_{t\to\infty} \sum_u p_t(x,u)\hat{\mu}(\varnothing, \{u\}) = \alpha_1(x),$$

(3.4) $$\lim_{t\to\infty} \sum_u p_t(x,u)\hat{\mu}(\{u\}, \{u\}) = \alpha_2(x),$$

(3.5) $$\lim_{t\to\infty} \sum_{u,v} p_t(x,u)p_t(x,v)\hat{\mu}(\varnothing, \{u,v\}) = \alpha_1^2(x)$$

and

(3.6) $$\lim_{t\to\infty} \sum_{u,v} p_t(x,u)p_t(x,v)\hat{\mu}(\{u,v\}, \{u,v\}) = \alpha_2^2(x).$$

PROOF. The first statement of the lemma is equivalent to

$$\lim_{t\to\infty} V_{m,n}(t) f_{m,n} = \tilde{h}_{m,n}$$

on $T_{m,n}$ for each $m \le n$. On the other hand, conditions (3.3)–(3.6) together are equivalent to the assertion that

(3.7) $$\lim_{t\to\infty} \sum_y p_t(x,y)\phi\eta(y) = \alpha_1(x)$$

and

(3.8) $$\lim_{t\to\infty} \sum_y p_t(x,y)\psi\eta(y) = \alpha_2(x),$$

in probability relative to μ for each $x \in S$, where $\phi\eta = 1 - {}_0\eta$, and $\psi\eta = {}_2\eta$. It follows that they are in turn equivalent to

$$\lim_{t \to \infty} U_n(t) f'_{m,n} = h'_{m,n}$$

on S^n for each $n \geq 1$. Hence the lemma follows from Lemma 3.1(b). □

4. Ergodic theorem. In this section, we prove the main theorems. Theorem 1 is implied by Theorem 4.3, Lemma 4.8 and Theorem 4.19 [note that when $g \equiv 1$, Theorem 1(a) follows from Corollary 2.8, Lemma 4.1 and Theorem VIII.1.12 in Liggett (1985)]. Theorem 2 is given by Theorem 4.5 and Theorem 4.28. In this section, A_t, A_t^1, A_t^2 denote coalescing random walks, while B_t, $B_1(t)$, $B_2(t)$ denote simple exclusion processes. We use extensive coupling throughout the section. For two processes X_t and Y_t in Y, the pair (X_t, Y_t) means that they are coupled in such a way that $X_t \subseteq Y_t$.

Let ϕ and ψ be defined as those of (3.7) and (3.8). Then

$$\eta = \phi\eta + \psi\eta.$$

Let $\mu\phi^{-1}$ be the measure on X defined by

$$\mu\phi^{-1}\{\cdot\} = \mu\{\eta\colon \phi\eta \in \cdot\};$$

$\mu\psi^{-1}$ is defined similarly.

We first consider the case $g \equiv 1$.

LEMMA 4.1. *Suppose $g \equiv 1$. Then for any probability measure μ on X,*

$$\lim_{t \to \infty} \mu S(t)\{\eta(x) = 2, \eta(y) = 1\} = 0. \tag{4.2}$$

PROOF. Let $x \neq y$. Take $A_0^1 = \{x\}$ and $A_0^2 = B_0 = \{x, y\}$. Construct (A_t^1, B_t) and (A_t^2, B_t) such that $A_t^1 \subseteq A_t^2$ for all $t \geq 0$. It is easy to see that

$$E^{\{x\},\{x,y\}}\hat{\mu}(A_t^1, B_t) - E^{\{x,y\},\{x,y\}}\hat{\mu}(A_t^2, B_t) \leq P(\tau > t),$$

where τ is the first time that $A_t^1 = A_t^2$. Hence

$$\lim_{t \to \infty} \left[E^{\{x\},\{x,y\}}\hat{\mu}(A_t^1, B_t) - E^{\{x,y\},\{x,y\}}\hat{\mu}(A_t^2, B_t) \right] = 0$$

since $g = 1$. By Corollary 2.8, we have (4.2). □

Let

$$X' = \{\eta(x) \neq 2, \text{for all } x \text{ or } \eta(x) \neq 1 \text{ for all } x\}.$$

By Lemma 4.1, if $\mu \in \mathscr{I}$ and $g \equiv 1$, then $\mu(X') = 1$. On the other hand, when restricted to the set X', the two-particle exclusion-eating process is the simple exclusion process. By the result of the set of the invariant measures of the simple exclusion process [see Theorem VIII.1.12 of Liggett (1985)] and by the fact that $\mu\{\eta \neq 2\} = 1$ or $\mu\{\eta \neq 1\} = 1$ if $\mu \in \mathscr{I}_e$, we have the following theorem.

THEOREM 4.3. *Suppose* $g \equiv 1$. *Then*

$$\mathscr{I}_e = \{\nu_{\alpha,0}, \nu_{\alpha,\alpha}, 0 \le \alpha \le 1\}.$$

LEMMA 4.4. *Let* $S = Z^d$, $p(x, y) = p(0, y - x)$ *and* $\mu \in \mathscr{S}$. *Then*

$$\mu S(t)\{\eta(x) = i\} = \mu\{\eta(x) = i\}, \qquad i = 0, 1, 2.$$

PROOF. Follows from translation invariance and Corollary 2.8 by taking $A = \varnothing$, $B = \{x\}$ and by taking $A = B = \{x\}$. □

THEOREM 4.5. *Suppose that* $g \equiv 1$, $S = Z^d$ *and* $p(x, y) = p(0, y - x)$. *For* $\mu \in \mathscr{S}_e$, *let* $\alpha_1 = \mu\{\eta(0) \ne 0\} > 0$ *and* $\alpha_2 = \mu\{\eta(0) = 2\}$. *Then*

$$\lim_{t \to \infty} \mu S(t) = \frac{\alpha_2}{\alpha_1} \nu_{\alpha_1, \alpha_1} + \left(1 - \frac{\alpha_2}{\alpha_1}\right) \nu_{\alpha_1, 0}. \tag{4.6}$$

PROOF. Take a sequence t_n such that

$$\lim_{n \to \infty} \mu S(t_n) = \nu$$

exists. Then $\nu \in \mathscr{S}$. We need to prove ν equals the right side of (4.6). Notice that $\phi\eta_t$ is the simple exclusion process and that $\mu\phi^{-1}$ is ergodic since μ is ergodic. Applying the ergodic theorem of the exclusion process [see Theorem VIII.1.47 of Liggett (1985)] and using the fact that ϕ^{-1} commutes with $S(t)$, we see that

$$\lim_{n \to \infty} \mu\phi^{-1} S(t_n) = \nu\phi^{-1} = \nu_{\alpha_1, 0}.$$

On the other hand, by Lemma 4.1, we have $\nu\{X'\} = 1$. Let $\nu_1(\cdot) = \nu(\cdot|\eta \ne 2)$, $\nu_2(\cdot) = \nu(\cdot|\eta \ne 1)$ and $c = \nu(\eta \ne 2)$. Then we have $\nu_1, \nu_2 \in \mathscr{S}$,

$$\nu_1(\eta \ne 2) = 1, \qquad \nu_2(\eta \ne 1) = 1$$

and

$$\nu = c\nu_1 + (1 - c)\nu_2. \tag{4.7}$$

Hence

$$\nu_{\alpha_1, 0} = \nu\phi^{-1} = c\nu_1\phi^{-1} + (1 - c)\nu_2\phi^{-1}.$$

Assume $0 < c < 1$. Since $\nu_{\alpha_1, 0} \in \mathscr{S}_e$, and $\nu_1\phi^{-1}, \nu_2\psi^{-1} \in \mathscr{S}$, we have

$$\nu_1\phi^{-1} = \nu_2\phi^{-1} = \nu_{\alpha_1, 0}.$$

Thus,

$$\nu_1 = \nu_{\alpha_1, 0} \quad \text{and} \quad \nu_2 = \nu_{\alpha_1, \alpha_1}.$$

By Lemma 4.4 and (4.7), we have

$$c\nu_1\{\eta(x) = 1\} = \nu\{\eta(x) = 1\} = \alpha_1 - \alpha_2.$$

Hence,

$$c = \frac{(\alpha_1 - \alpha_2)}{\alpha_1} = 1 - \frac{\alpha_2}{\alpha_1}.$$

This proves ν equals the right side of (4.6). □

Now we turn to the case $g \not\equiv 1$.

LEMMA 4.8. *For any* $(\alpha_1, \alpha_2) \in \mathscr{H}'$,

$$\lim_{t\to\infty} \nu_{\alpha_1, \alpha_2} S(t) = \mu_{\alpha_1, \alpha_2}$$

exists and is in $\mathscr{I}$.

PROOF. By Corollary 2.8, we need to prove

$$\lim_{t\to\infty} E^{A,B} \nu_{\alpha_1 \alpha_2}(A_t, B_t) \quad \text{exists when } A \subseteq B,\ A, B \in Y. \tag{4.9}$$

We use induction on $|A|$. When $|A| \le 1$, (A_t, B_t) is a pair of simple exclusion processes. By Corollary 2.8 and Lemma 3.1(a), the limit in (4.9) exists. Suppose that the limit in (4.9) exists for $|A| < m$. Take an A with $|A| = m$. Let τ be the first time that $|A| < m$. Then, by the strong Markov property,

$$\begin{aligned} &\lim_{t\to\infty} E^{A,B}\left[\hat{\nu}_{\alpha_1, \alpha_2}(A_t, B_t),\ \tau < \infty\right] \\ &\quad = \lim_{t\to\infty} E^{A,B}\left[E^{A_\tau, B_\tau}\hat{\nu}_{\alpha_1, \alpha_2}(A_t, B_t),\ \tau < \infty\right]. \end{aligned} \tag{4.10}$$

The limit in (4.10) exists by induction hypothesis, since $|A_\tau| < m$ on $\tau < \infty$. On the other hand, let $(B_1(t), B_2(t))$ be the simple exclusion process constructed on the same probability space as (A_t, B_t) in such a way that $B_1(0) = A_0$, $B_2(0) = B_0$, $A_t \subseteq B_1(t)$, $B_t = B_2(t)$ and $B_1(t) \subseteq B_2(t)$. We have $B_1(t) = A_t$ on $\tau = \infty$. Then

$$\begin{aligned} &\lim_{t\to\infty} E^{A,B}\left[\hat{\nu}_{\alpha_1, \alpha_2}(A_t, B_t),\ \tau = \infty\right] \\ &\quad = \lim_{t\to\infty} E^{A,B}\left[\hat{\nu}_{\alpha_1, \alpha_2}(B_1(t), B_2(t)),\ \tau = \infty\right] \\ &\quad = \lim_{t\to\infty} E^{A,B}\left[\hat{\nu}_{\alpha_1, \alpha_2}(B_1(t), B_2(t))\right] \\ &\qquad - \lim_{t\to\infty} E^{A,B}\left[E^{B_1(\tau), B_2(\tau)}\hat{\nu}_{\alpha_1, \alpha_2}(B_1(t), B_2(t)),\ \tau < \infty\right]. \end{aligned} \tag{4.11}$$

By Lemma 3.1(a), (4.10) and (4.11), the limit in (4.9) exists for $|A| = m$. By Lemma V.1.26 of Liggett (1985), $\lim_{t\to\infty} \nu_{\alpha_1, \alpha_2} S(t)$ converges. Hence $\mu_{\alpha_1, \alpha_2} \in \mathscr{I}$, because we are dealing with a Feller process. □

LEMMA 4.12. *Assume* $g \not\equiv 1$. *Let* μ *be a probability measure on* X. *Suppose that* $(\alpha_1, \alpha_2) \in \mathscr{H}'$ *satisfies* (3.3)–(3.6). *Then*

$$\lim_{t\to\infty} \mu S(t) = \mu_{\alpha_1, \alpha_2},$$

where μ_{α_1, α_2} *is as defined in Lemma* 4.8.

PROOF. By definition of μ_{α_1, α_2} and Corollary 2.8, it suffices to show that, for $A, B \in Y$, $A \subseteq B$,

$$\lim_{t\to\infty} E^{A,B}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y)\right] = 0. \tag{4.13}$$

Let $A = A_0 = B_1(0) = \{x_1, \dots, x_m\}$, $B = B_0 = B_2(0) = \{x_1, \dots, x_n\}$ with distinct $x_1, \dots, x_n$ and $m \le n$. Let τ be the first time that $|A_t| < m$. Let (A_t, B_t) and $(B_1(t), B_2(t))$ move together before τ. Then, by Lemma 3.2,

$$\begin{aligned}
&\lim_{t\to\infty} E^{A,B}[\hat{\mu}(A_t, B_t), \tau = \infty]\\
&\quad = \lim_{t\to\infty} E^{A,B}[\hat{\mu}(B_1(t), B_2(t)), \tau = \infty]\\
&\quad = \lim_{t\to\infty} E^{A,B}\hat{\mu}(B_1(t), B_2(t)) - \lim_{t\to\infty} E^{A,B}[\hat{\mu}(B_1(t), B_2(t)), \tau < \infty]\\
&\quad = \lim_{t\to\infty} E^{A,B}\hat{\mu}(B_1(t), B_2(t))\\
&\qquad - E^{A,B}\left[\lim_{t\to\infty} E^{B_1(\tau), B_2(\tau)}\hat{\mu}(B_1(t), B_2(t)), \tau < \infty\right]\\
&\quad = \lim_{t\to\infty} E^{A,B}\left[\prod_{x\in B_1(t)} \alpha_2(x) \prod_{y\in B_2(t)\setminus B_1(t)} \alpha_1(y)\right]\\
&\qquad - E^{A,B}\left\{\lim_{t\to\infty} E^{B_1(\tau), B_2(\tau)}\left[\prod_{x\in B_1(t)} \alpha_2(x) \prod_{y\in B_2(t)\setminus B_1(t)} \alpha_1(y)\right], \tau < \infty\right\}\\
&\quad = \lim_{t\to\infty} E^{A,B}\left[\prod_{x\in B_1(t)} \alpha_2(x) \prod_{y\in B_2(t)\setminus B_1(t)} \alpha_1(y), \tau = \infty\right]\\
&\quad = \lim_{t\to\infty} E^{A,B}\left[\prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y), \tau = \infty\right].
\end{aligned}$$

Hence

$$\lim_{t\to\infty} E^{A,B}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y), \tau = \infty\right] = 0. \tag{4.14}$$

Now we prove (4.13) by induction on $|A|$. When $|A| \le 1$, (4.13) follows from Lemma 3.2. Assume that (4.13) is true for $|A| < m$. Take $|A| = m$. Then

$$\begin{aligned}
&\lim_{t\to\infty} E^{A,B}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y)\right]\\
&\quad = \lim_{t\to\infty} E^{A,B}\left\{E^{A_\tau, B_\tau}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y)\right], \tau < \infty\right\}\\
&\qquad + \lim_{t\to\infty} E^{A,B}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t} \alpha_2(x) \prod_{y\in B_t\setminus A_t} \alpha_1(y), \tau = \infty\right].
\end{aligned}$$

The first term is zero by the induction hypothesis since $|A_\tau| < m$. The second term is zero by (4.14). Hence (4.13) is true. □

THEOREM 4.15. *Suppose that* $(\alpha_1, \alpha_2) \in \mathscr{H}^*$ *and that* μ *is a probability measure on* X. *Then* $\lim_{t\to\infty} \mu S(t) = \mu_{\alpha_1,\alpha_2}$ *if and only if* (3.3)–(3.6) *are true.*

PROOF. Sufficiency has been proved in Lemma 4.12. Now let $\lim_{t\to\infty} \mu S(t) = \mu_{\alpha_1,\alpha_2}$. Then (3.3) and (3.4) follow from Corollary 2.8 applied to $A = \varnothing$, $B = \{x\}$ and to $A = B = \{x\}$. (3.5) follows from the fact that $\phi\eta_t$ is the simple exclusion process, that $\lim_{t\to\infty} \mu\phi^{-1}S(t) = \nu_{\alpha_1,0}$ and Theorem VIII.1.24 of Liggett (1985). To prove (3.6), we use the same technique as that in the proof of Theorem V.1.9 of Liggett (1985). Notice that $\lim_{t\to\infty} \mu S(t) = \mu_{\alpha_1,\alpha_2}$ implies (4.13) is true. Thus,

$$\lim_{t\to\infty} E^{\{x\}\{x,y\}}\big[\hat{\mu}(A_t, B_t) - \hat{\nu}_{\alpha_1,\alpha_2}(A_t, B_t)\big] = 0.$$

Notice that when $|A_0| = 1$, (A_t, B_t) have the same distribution as the simple exclusion processes. By Lemma 3.1(b),

$$\lim_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t))\big] = 0, \qquad x, y \in S, \tag{4.16}$$

where $X(t)$ and $Y(t)$ are independent walks with transition probability (1.2). Let $\tau_1' = \inf\{t: X(t) \neq Y(t)\}$ and $\tau_1 = \inf\{t > \tau_1': X(t) = Y(t)\}$. By (4.16) and strong Markov property,

$$\lim_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t)),\, \tau_1 < \infty\big] = 0, \qquad x, y \in S.$$

Repeating this idea, we see that

$$\lim_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t)),\, \mathscr{E}^c\big] = 0.$$

So that, by (4.16),

$$\lim_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t)),\, \mathscr{E}\big] = 0.$$

Since $\hat{\mu}(\{u,v\},\{u,v\}) \le \hat{\mu}(\{u\},\{u,v\})$, it follows that

$$\limsup_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t)),\, \mathscr{E}\big] \le 0.$$

Since $(\alpha_1, \alpha_2) \in \mathscr{H}^*$, taking $X_1(t) = X(t)$ and $X_2(t) = Y(t)$ and noting that

$$\alpha_2(X(t))\big[\alpha_1(Y(t)) - \alpha_2(Y(t))\big]$$

must converge since it is the product of two bounded martingales, we deduce that

$$\lim_{t\to\infty} \alpha_2(X(t))\big[\alpha_1(Y(t)) - \alpha_2(Y(t))\big] = 0 \quad \text{a.s. on } \mathscr{E},$$

hence

$$\limsup_{t\to\infty} E^{x,y}\big[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_2(Y(t)),\, \mathscr{E}\big] \le 0. \tag{4.17}$$

By (4.13) and the strong Markov property, we have

$$\lim_{t\to\infty} E^{\{x,y\},\{x,y\}}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t}\alpha_2(x)\prod_{y\in B_t\setminus A_t}\alpha_1(y),\ \tau_1<\infty\right] = 0, \qquad x\neq y.$$

Hence

$$\lim_{t\to\infty} E^{\{x,y\},\{x,y\}}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t}\alpha_2(x)\prod_{y\in B_t\setminus A_t}\alpha_1(y),\ \tau_1=\infty\right] = 0, \qquad x\neq y.$$

Let (A_t, B_t) and $(X(t), Y(t))$ move together before $X(t) = Y(t)$. We have

$$\begin{aligned}&\lim_{t\to\infty} E^{x,y}[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\alpha_2(Y(t)),\ \tau_1=\infty]\\ &\quad= \lim_{t\to\infty} E^{\{x,y\},\{x,y\}}\left[\hat{\mu}(A_t, B_t) - \prod_{x\in A_t}\alpha_2(x)\prod_{y\in B_t\setminus A_t}\alpha_1(y),\ \tau_1=\infty\right] = 0,\\ &\qquad\qquad x\neq y.\end{aligned}$$

By the strong Markov property using τ_1', we have

$$\begin{aligned}&\lim_{t\to\infty} E^{x,y}[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\})\\ &\qquad -\alpha_2(X(t))\alpha_2(Y(t)),\ \tau_1=\infty] = 0, \qquad x, y\in S.\end{aligned}$$

Define $\tau_{i+1}' = \inf\{t\geq \tau_i\colon X(t)\neq Y(t)\}$, $\tau_{i+1} = \inf\{t\geq \tau_{i+1}'\colon X(t) = Y(t)\}$, $i = 1, 2, 3, \ldots,$ iteratively. Applying the strong Markov property with τ_i, we see that

$$\begin{aligned}&\lim_{t\to\infty} E^{x,y}[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\})\\ &\qquad -\alpha_2(X(t))\alpha_2(Y(t)),\ \mathscr{E}^c] = 0\end{aligned} \tag{4.18}$$

for $x, y \in S$. Combining (4.17) and (4.18), we have

$$\limsup_{t\to\infty} E^{x,y}[\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\alpha_2(Y(t))] \leq 0.$$

On the other hand, we have

$$\begin{aligned}0 &\leq \int\left[\sum_y p_t(x,y)\psi\eta(y) - \alpha_2(x)\right]^2 d\mu\\ &= \sum_{u,v} p_t(x,u)p_t(x,v)\hat{\mu}(\{u,v\},\{u,v\}) + \alpha_2^2(x)\\ &\quad - 2\alpha_2(x)\sum_y p_t(x,y)\hat{\mu}(\{y\},\{y\}).\end{aligned}$$

Thus, by (3.4), we have

$$\liminf_{t\to\infty}\sum_{u,v} p_t(x,u)p_t(x,v)\hat{\mu}(\{u,v\},\{u,v\}) \geq \alpha_2^2(x).$$

This proves (3.6). □

THEOREM 4.19. *Assume* $g \not\equiv 1$. *Then*

$$\mathscr{I}_e = \left\{ \mu_{\alpha_1, \alpha_2} : (\alpha_1, \alpha_2) \in \mathscr{H}^* \right\}.$$

PROOF. Take $(\alpha_1, \alpha_2) \in \mathscr{H}^*$. By Lemma 4.8, $\mu_{\alpha_1, \alpha_2} \in \mathscr{I}$. Let

$$\mu_{\alpha_1, \alpha_2} = \lambda \mu_1 + (1 - \lambda) \mu_2, \tag{4.20}$$

where $0 < \lambda < 1$ and $\mu_1, \mu_2 \in \mathscr{I}$. Then

$$\lim_{t \to \infty} \sum_y p_t(x, y) \phi \eta(y) = \alpha_1(x)$$

and

$$\lim_{t \to \infty} \sum_y p_t(x, y) \psi \eta(y) = \alpha_2(x)$$

in probability relative to μ_{α_1, α_2} by Theorem 4.15 and by the equivalence of conditions (3.3)–(3.6) and statements (3.7) and (3.8). By (4.20), the same is true relative to μ_1 and μ_2. By Theorem 4.15 again, we have $\mu_1 = \mu_2 = \mu_{\alpha_1, \alpha_2}$. Hence $\mu_{\alpha_1, \alpha_2} \in \mathscr{I}_e$. For the converse, take $\mu \in \mathscr{I}_e$ and define $\alpha_1(x) = \mu\{\eta(x) \neq 0\}$ and $\alpha_2(x) = \mu\{\eta(x) = 2\}$. Then, $(\alpha_1, \alpha_2) \in \mathscr{H}'$. By Corollary 2.8, we have (3.3) and (3.4). Conditions (3.5) and (3.6) can be verified using the same procedure as that of proving Lemma VIII.1.36 of Liggett (1985). Here we give an outline of the proof of (3.6). Similar to (VIII.1.38) of Liggett (1985), we have

$$\lim_{t \to \infty} \sum_z p_t(y, z) \hat{\mu}(\{x, z\}, \{x, z\}) = \alpha_2(x) \alpha_2(y). \tag{4.21}$$

Let $V(t)$ be the semigroup of (A_t, B_t) on $T = \{(\vec{x}, \vec{y}) \in S^2 \times S^2 : \{x_1, x_2\} \subseteq \{y_1, y_2\},\ y_1 \neq y_2\}$. Let $f_{2,2}$ and $f'_{2,2}$ be defined as those in Section 3. By a natural coupling, we have

$$\left| V(t) f_{2,2}(\vec{x}, \vec{x}) - U_2(t) f'_{2,2}(\vec{x}) \right| \leq g_2(\vec{x}), \qquad \vec{x} \in T_2.$$

Let h be the limit of $U_2(t) f'_{2,2}$ along any sequence of times which tend to ∞. Since $\mu \in \mathscr{I}$, we have

$$\left| f_{2,2}(\vec{x}, \vec{x}) - h(\vec{x}) \right| \leq g_2(\vec{x}), \qquad \vec{x} \in T_2, \tag{4.22}$$

and $U_2(t) h = h$ for all t by Lemma V.1.26 of Liggett (1985). Therefore,

$$\sum_y p_t(x, y) h(y, z) = h(x, z) \tag{4.23}$$

by Corollary II.7.3 of Liggett (1985). Combining (4.21), (4.22), (4.23) and (VIII.1.40) of Liggett (1985), we have (3.6). By Theorem 4.15,

$$\mu = \mu_{\alpha_1, \alpha_2}.$$

Let $X(t)$ and $Y(t)$ be independent Markov chains with transition probability (1.2). We wish to prove

$$\lim_{t \to \infty} \alpha_2(X(t)) \left[\alpha_1(X(t)) - \alpha_2(X(t)) \right] = 0 \quad \text{a.s. on } \mathscr{E}. \tag{4.24}$$

By (3.3)–(3.6) (which are equivalent to $\lim_{t\to\infty} U_n f'_{m,n} = h'_{m,n}$; see Section 3), we have

$$\lim_{t\to\infty} E^{x,y}[\,\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_1(Y(t))] = 0$$

and

$$\lim_{t\to\infty} E^{x,y}[\,\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\,\alpha_2(Y(t))] = 0.$$

By the strong Markov property using $\{\tau_i\}$, we have

$$\lim_{t\to\infty} E^{x,y}[\,\hat{\mu}(\{X(t)\},\{X(t),Y(t)\}) - \alpha_2(X(t))\alpha_1(Y(t)),\mathscr{E}] = 0 \tag{4.25}$$

and

$$\begin{aligned}\lim_{t\to\infty} E^{x,y}[\,&\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\})\\ &-\alpha_2(X(t))\,\alpha_2(Y(t)),\mathscr{E}] = 0.\end{aligned} \tag{4.26}$$

On the other hand, by the invariance of μ and coupling, we have

$$\begin{aligned}0 &\le \hat{\mu}(\{x\},\{x,y\}) - \hat{\mu}(\{x,y\},\{x,y\})\\ &= E^{\{x\},\{x,y\}}\hat{\mu}(A_t,B_t) - E^{\{x,y\},\{x,y\}}\hat{\mu}(A_t,B_t)\\ &\le P^{\{x,y\}}(|A_t| = 2).\end{aligned}$$

Passing to the limit as $t \to \infty$, this gives

$$0 \le \hat{\mu}(\{x\},\{x,y\}) - \hat{\mu}(\{x,y\},\{x,y\}) \le 1 - g(x,y).$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty} [\,&\hat{\mu}(\{X(t)\},\{X(t),Y(t)\})\\ &-\hat{\mu}(\{X(t),Y(t)\},\{X(t),Y(t)\})] = 0 \quad \text{a.s. on } \mathscr{E}.\end{aligned} \tag{4.27}$$

Combining (4.25), (4.26) and (4.27), we have

$$\lim_{t\to\infty} E^{x,y}\{\alpha_2(X(t))[\,\alpha_1(Y(t)) - \alpha_2(Y(t))],\mathscr{E}\} = 0$$

Since $\alpha_2(X(t))$ and $\alpha_i(Y(t))$, $i = 1,2$, are bounded martingales, by convergence theorem of martingales, with probability 1, the limit

$$\lim_{t\to\infty} \alpha_2(X(t))[\,\alpha_1(Y(t)) - \alpha_2(Y(t))]$$

exists. This limit equals the left side of (4.24) almost surely on $\mathscr{E}$. Hence (4.24) is true. □

When $S = Z^d$, $p(x,y) = p(0, y-x)$ and $g \not\equiv 1$, it is known that [see Liggett (1985)] $\mathscr{H}$ consists of constants and $P(\mathscr{E}) = 0$. Hence,

$$\mathscr{H}' = \{(\alpha_1,\alpha_2)\colon 0 \le \alpha_2 \le \alpha_1 \le 1\}.$$

THEOREM 4.28. *Suppose that* $S = Z^d$, $p(x, y) = p(0, y - x)$, $g \not\equiv 1$ *and* $\mu \in \mathscr{S}_e$. *Let* $\alpha_1 = \mu\{\eta(x) \neq 0\}$ *and* $\alpha_2 = \mu\{\eta(x) = 2\}$. *Then*

$$\lim_{t \to \infty} \mu S(t) = \mu_{\alpha_1, \alpha_2}.$$

PROOF. Follows from Theorem 4.15, Corollary II.8.20 of Liggett (1985) and the note following that corollary. □

Acknowledgments. I would like to thank David Griffeath for pointing out the connection between the two-particle exclusion-eating process and the voter model and an error in an earlier version of this paper. I would like to thank Thomas Liggett for his help, suggestion and encouragement. I would also like to thank a referee for a careful reading and suggestions to eliminate errors in this paper.

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