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LARGE DEVIATIONS FOR INDEPENDENT RANDOM WALKS ON THE LINE[1]

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For a system of infinitely many independent symmetric random walks on $\mathbb{Z}$ let $K_n(x)$ be the number of visits to $x \in \mathbb{Z}$ from time 0 to $n-1$. The probabilities of some rare events involving $(K_n(0), K_n(1))$ are estimated as $n \to \infty$ and the corresponding large deviation rate functions are derived for both deterministic and invariant initial distributions. The dependence on the initial distributions is discussed. A simple method is used for guessing at the rate functions. This method is effective for independent random walks on the line and is worth exploring in more general settings.

1. Introduction. For a system of independent random walks on $\mathbb{Z}$ denote by $\zeta_n(x)$ the number of particles at $x \in \mathbb{Z}$ at time n. Each random walk is assumed to be symmetric and simple, that is, walking to the left or right neighboring site with equal probability $1/2$. We are interested in the occupation time at x before time n,

$$K_n(x) = \sum_{m=0}^{n-1} \zeta_m(x),$$

and the difference between two distinct sites x and y:

$$L_n(x,y) = K_n(x) - K_n(y).$$

The case $x = 0$, $y = 1$ will be adopted throughout. The reader is referred to Remillard (1990) and Lee and Remillard (1994) for ideas to generalize our results to $\sum_{x \in \mathbb{Z}} V(x) K_n(x)$ for a broader class of functions V.

A direct motivation comes from the work of Cox and Durrett (1990) and Remillard (1990). For a deterministic initial configuration with asymptotic density 1, that is,

$$\lim_{k\to\infty} (2k+1)^{-1} \sum_{x=-k}^{k} \zeta_0(x) = 1,$$

let Q be the associated probability measure. They showed that

$$\lim_{n\to\infty} n^{-1/2-3\theta/2} \log Q\{K_n > an^{1+\theta}\} = -4(a/3)^{3/2}, \qquad a > 0,\ \theta \in (0,1), \tag{1.1}$$

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$$\lim_{n\to\infty} n^{-1/2-6\theta/5} \log Q\{L_n > an^{3/4+\theta}\} = -(5/4)(2/3)^{6/5} b^{6/5}, \quad b > 0,\ \theta \in (0, 5/4). \tag{1.2}$$

It can be checked that $\zeta_n(x)$, $x \in \mathbb{Z}$, converges in distribution as $n \to \infty$ to independent mean 1 Poisson random variables.

When $\zeta_0(x)$, $x \in \mathbb{Z}$, are independent mean 1 Poisson random variables, let us write P for the probability measure. It was proved by Cox and Durrett [(1990), Theorem 4] that

$$\liminf_{n\to\infty} (\log n)^{-1} n^{-1/2-\theta} \log P\{L_n > bn^{3/4+\theta}\} \ge -(1+2\theta), \quad b > 0. \tag{1.3}$$

Thus the deviations from a Poisson initial configuration are much more likely than for a fixed configuration. They thought that $(\log n)n^{1/2+\theta}$ might be the right order of magnitude. One goal in this paper is to determine more precise behavior of the processes P and Q in problems like (1.1)–(1.3). In this respect our main result is the following theorem.

THEOREM 1. *Statements* (i) *and* (ii) *hold.*

(i) *Let* λ_n *be a sequence such that* $\lambda_n \to \infty$ *and* $(\log \lambda_n)/(\log n) \to \theta > 0$, *as* $n \to \infty$. *Then with respect to* P, *the system*

$$\left(\left(\lambda_n^{-1} n^{-1} K_n,\ \lambda_n^{-1} (\log n)^{-1/4} n^{-3/4} L_n\right),\ \lambda_n (\log n)^{1/2} n^{1/2}\right)$$

is a large deviation system with rate $I^{p,\theta}$:

$$I^{P,\theta}(a,b) = \begin{cases} (2\theta)^{1/2} a + \dfrac{b^2}{4a}, & \text{for } a > 0,\ b \in \mathbb{R}, \\ 0, & \text{for } a = 0,\ b = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

(ii) *With respect to* Q, *the system* $((n^{-\theta-1}K_n,\ n^{-5\theta/4-3/4}L_n),\ n^{3\theta/2+1/2})$ *is a large deviation system with rate* I^Q:

$$I^{Q}(a,b) = \begin{cases} 4 \cdot 3^{-3/2} a^{3/2} + \dfrac{b^2}{4a}, & \text{for } a > 0,\ b \in \mathbb{R}, \\ 0, & \text{for } a = 0,\ b = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

REMARK 1. As usual, statement (i) means (1.4), (1.5) and (1.6) as follows:

(1.4) The set $\{(a,b)\colon I^{P,\theta}(a,b) \le l\}$ is compact for each finite l. (Thus, the function $I^{p,\theta}$ is lower semicontinuous).

For each closed $F \subset \mathbb{R}^2$,

$$\limsup_{n\to\infty} \lambda_n^{-1}(\log n)^{-1/2} n^{-1/2} \times \log P\{(\lambda_n^{-1} n^{-1} K_n, \lambda_n^{-1}(\log n)^{-1/4} n^{-3/4} L_n) \in F\} \le -\inf_{(a,b)\in F} I^{P,\theta}(a,b). \tag{1.5}$$

For each open $G \subset \mathbb{R}^2$,

$$\liminf_{n\to\infty} \lambda_n^{-1}(\log n)^{-1/2} n^{-1/2} \times \log P\{(\lambda_n^{-1} n^{-1} K_n, \lambda_n^{-1}(\log n)^{-1/4} n^{-3/4} L_n) \in G\} \ge -\inf_{(a,b)\in G} I^{P,\theta}(a,b). \tag{1.6}$$

REMARK 2. An application of the contraction principle shows that if $(\log \lambda_n)/\log n \to \theta > 0$, then

$$\lim_{n\to\infty} \lambda_n^{-1}(\log n)^{-1/2} n^{-1/2} \log P\{K_n \ge a\lambda_n n\} = -\inf_{\substack{c\ge a\\ b\in\mathbb{R}}} I^{P,\theta}(c,b) = -(2\theta)^{1/2} a \quad \text{for } a > 0,$$

$$\lim_{n\to\infty} \lambda_n^{-1}(\log n)^{-1/2} n^{-1/2} \log P\{L_n \ge b\lambda_n(\log n)^{1/4} n^{3/4}\} = -\inf_{\substack{a\in\mathbb{R}\\ c\ge b}} I^{P,\theta}(a,c) = -(2\theta)^{1/4} b \quad \text{for } b > 0.$$

Thus problems like (1.1)–(1.3) can be resolved by using proper λ_n. For example, if $\lambda_n = n^\theta(\log n)^{-1/4}$, then the last equation becomes

$$\lim_{n\to\infty} (\log n)^{-1/4} n^{-1/2-\theta} \log P\{L_n \ge b n^{3/4+\theta}\} = -(2\theta)^{1/4} b, \qquad b > 0. \tag{1.7}$$

The right order is therefore $(\log n)^{1/4} n^{1/2+\theta}$, instead of $(\log n) n^{1/2+\theta}$. An interesting feature of Theorem 1(i) is that it is *not* obtained by proving an appropriate limit theorem for moment generating functions, and then inverting. Instead, a certain direct probability estimation is done, which explains what is "really" going on with the process.

REMARK 3. The following two statements can be deduced from Theorem 1:

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \lambda_n^{-1}(\log n)^{-1/2} n^{-1/2} \times \log P\left\{\frac{K_n}{\lambda_n n} \in (a_* - \varepsilon, a_* + \varepsilon) \,\middle|\, \frac{L_n}{\lambda_n(\log n)^{1/4} n^{3/4}} \in (b - \varepsilon, b + \varepsilon)\right\} = 0, \tag{i}$$

where $a_* = 2^{-5/4}\theta^{-1/4}b$ is the minimizer in $\inf_{a>0} I^{P,\theta}(a,b)$;

$$\text{(ii)} \qquad \lim_{\varepsilon\to 0}\lim_{n\to\infty} n^{-3\theta/2-1/2} \times \log Q\left\{ \frac{K_n}{n^{\theta+1}} \in (a_* - \varepsilon, a_* + \varepsilon) \,\middle|\, \frac{L_n}{n^{5\theta/4+3/4}} \in (b-\varepsilon, b+\varepsilon) \right\} = 0,$$

where $a^* = 2^{-6/5}3^{1/5}b^{4/5}$ is the minimizer in $\inf_{a>0} I^Q(a,b)$.

If the sequence $\lambda_n = n^{5\theta/4}(\log n)^{-1/4}$ is used, then the two conditioning events are identical, but the "almost sure" behavior of K_n is different [K_n is on the order of $n^{5\theta/4+1}(\log n)^{-1/4}$ under P and of the order of $n^{\theta+1}$ under Q].

This research was begun with the main goal of seeking an effective method to guess at the right order of decay and prove the corresponding large deviation result like (1.1)–(1.3). The guessing method that we used to foresee Theorem 1 is simple. In order to demonstrate this method, we present another large deviation principle for K_n, L_n as Theorem 2. The proof of Theorem 2 is not difficult since we have the results of Cox and Durrett (1990) and Kesten (1962) and references therein. We need some notation. Let $\tau_x(s)$ be the probability density function of the hitting time of the origin for a Brownian motion starting from $x \in \mathbb{R}$:

$$\tau_x(s) = |x| s^{-3/2} \exp(-x^2/2s), \qquad s > 0.$$

Let f be the moment generating function of a truncated normal distribution:

$$f(c) = \int_0^\infty \exp(cu)\sqrt{\frac{2}{\pi}}\exp(-u^2/2)\,du. \tag{1.8}$$

Let

$$\begin{aligned} J^P(\alpha,\beta) &= \int_{-\infty}^{\infty}\int_0^1 \Big[f\big((1-s)^{1/2}(\alpha+\beta^2)\big) - 1\Big]\tau_x(s)\,ds\,dx, \\ J^Q(\alpha,\beta) &= \int_{-\infty}^{\infty} \log\left\{1 + \int_0^1 \Big[f\big((1-s)^{1/2}(\alpha+\beta^2)\big) - 1\Big]\tau_x(s)\,ds\right\} dx \end{aligned} \tag{1.9}$$

and let $\hat{J}^P, \hat{J}^Q$ be the Legendre transforms:

$$\begin{aligned} \hat{J}^P(a,b) &= \sup_{\alpha,\beta}\big[a\alpha + b\beta - J^P(\alpha,\beta)\big], \\ \hat{J}^Q(a,b) &= \sup_{\alpha,\beta}\big[a\alpha + b\beta - J^Q(\alpha,\beta)\big]. \end{aligned} \tag{1.10}$$

Then we prove the following theorem.

THEOREM 2. *The system $((n^{-1}K_n, n^{-3/4}L_n), n^{1/2})$ is a large deviation system for both processes P and Q, with rates $\hat{J}^P$ and $\hat{J}^Q$, respectively.*

REMARK 1. Since $y \geq \log(1 + y)$ for $y > -1$, we see that $J^P(\alpha, \beta) \geq J^Q(\alpha, \beta)$, for all α, β, and thus $\hat{J}^P(a, b) \leq \hat{J}^Q(a, b)$, for all a, b. The following properties are easy to check:

(i) We have $\hat{J}^P(a, b) = \hat{J}^Q(a, b) = +\infty$ for $a < 0$, $b \in \mathbb{R}$, and both $\hat{J}^P(a, b)$ and $\hat{J}^Q(a, b)$ are finite for $a > 0$, $b \in \mathbb{R}$.

(ii) For $\hat{J} = \hat{J}^P$ or $\hat{J}^Q$,

$$\hat{J}(a, b) = 0 \quad \Leftrightarrow \quad (a, b) = (1, 0)$$

and

$$\hat{J}^P(a, b) < \hat{J}^Q(a, b) \quad \text{for} \quad a > 0,\ b \in \mathbb{R} \text{ and } (a, b) \neq (1, 0).$$

A simple-minded guess for Theorem 1(i) is obtained by wishfully thinking that Theorem 2(i) holds also for $a = a_n \to \infty$, $b = b_n \to \infty$; roughly speaking,

$$-\log P\{K_n \sim a_n n,\ L_n \sim b_n n^{3/4}\} \sim n^{1/2} \hat{J}^P(a_n, b_n), \qquad n \to \infty. \tag{1.11}$$

Then we estimate the asymptotic behavior of $\hat{J}^P(a_n, b_n)$, using the fact that $f(c) - \exp(c^2/2) \to 0$ as $c \to \infty$, as follows:

$$\begin{aligned} \hat{J}^P(a_n, b_n) &= \sup_{\alpha, \beta} \left[a_n \alpha + b_n \beta - J^P(\alpha, \beta) \right] \\ &\sim \sup_{\alpha, \beta} \left[a_n \alpha + b_n \beta - \exp\left(\frac{(\alpha + \beta^2)^2}{2} \right) \right] \\ &\sim a_n (2 \log a_n)^{1/2} + \frac{b_n^2}{4 a_n}, \qquad n \to \infty. \end{aligned}$$

The choice $a_n = a\lambda_n$, $b_n = b\lambda_n (\log n)^{1/4}$ makes the two terms of the same order $\lambda_n (\log n)^{1/2}$ and suggests that Theorem 1(i) might hold. Of course, this heuristic is far from a proof because a statement like (1.11) in general does not hold when one merely assumes a large deviation principle as given in Theorem 2. However, we think that the processes P and Q are "regular" enough to warrant this heuristic argument for many sequences (a_n, b_n), as $n \to \infty$.

It would be interesting to see some sufficient conditions worked out for the heuristic (1.11) to give correct results. Of course, it would be especially nice if these sufficient conditions hold and are easy to verify for independent random walks and more complex systems of infinite particles. We think that this kind of heuristics makes correct predictions for many infinite particle systems. It is also instructive to understand various limitations of the prediction. Some limitations are easy to detect and some are obscure; for example, the prediction must fail for $\theta > 1$ in Theorem 1(ii) because in this case Q-probability is 0 for sufficiently large n. However, it is not clear whether (1.11) would predict correctly for more general sequences of (a_n, b_n), for example, $a_n = a > 0$,

$b_n \to \infty$ as $n \to \infty$. We think the answer is in the negative and hope to clarify it in the future.

In Section 2 we shall establish some auxiliary results (seven lemmas). In Section 3 we then employ them to prove Theorems 1 and 2. Many ingredients for proving our result can be found in the references. When this is the case and only simple modification is required, we shall give the references and omit some details.

2. Auxiliary results. The first lemma is concerned with the behavior of the principal eigenvalue of perturbed discrete Laplacian operators.

LEMMA 1. *Let* $V_\varepsilon(x) = \alpha\varepsilon^2\chi_0(x) + \beta\varepsilon(\chi_0 - \chi_1)(x)$, *where* χ_y *is the indicator function of* y *and* $\alpha + \beta^2 > 0$. *For small* ε *there exist* $\theta = \theta_\varepsilon > 0$, $A = A_\varepsilon > 0$ *such that* $\theta = (\alpha + \beta^2)\varepsilon^2 + O(\varepsilon^3)$ *as* $\varepsilon \to 0$, *and the function*

$$f(x) = \begin{cases} e^{\theta x}, & \text{for } x \le 0, \\ Ae^{-\theta(x-1)}, & \text{for } x \ge 1, \end{cases} \tag{2.1}$$

satisfies the equation

$$\frac{f(x+1) + f(x-1)}{2} = (\cosh\theta)\exp(-V_\varepsilon(x))f(x), \qquad x \in \mathbb{Z}. \tag{2.2}$$

PROOF. Clearly any function f in the form of (2.1) satisfies (2.2) for all x except maybe $x = 0, 1$. For $x = 0$ or 1, equality (2.2) becomes

$$\frac{\exp(-\theta) + A}{2} = (\cosh\theta)\exp(-\alpha\varepsilon^2 - \beta\varepsilon),$$

$$\frac{1 + A\exp(-\theta)}{2} = (\cosh\theta)\exp(\beta\varepsilon)A,$$

respectively. That is,

$$\begin{aligned} A &= (2\cosh\theta)\exp(-\alpha\varepsilon^2 - \beta\varepsilon) - \exp(-\theta) \\ &= \exp(-\alpha\varepsilon^2 - \beta\varepsilon)\left[\exp(\theta) + \left(1 - \exp(\alpha\varepsilon^2 + \beta\varepsilon)\right)\exp(-\theta)\right], \\ A^{-1} &= (2\cosh\theta)\exp(\beta\varepsilon) - \exp(-\theta) \\ &= \exp(\beta\varepsilon)\left[\exp(\theta) + (1 - \exp(-\beta\varepsilon))\exp(-\theta)\right]. \end{aligned}$$

Let $\gamma = e^{2\theta} > 1$. The product of these two equations yields the quadratic equation in γ,

$$\gamma^2 - B\gamma + C = 0, \qquad \gamma > 1,$$

where

$$B = \exp(\alpha\varepsilon^2) + \exp(-\beta\varepsilon) + \exp(\alpha\varepsilon^2 + \beta\varepsilon) - 2 = 1 + (2\alpha + \beta^2)\varepsilon^2 + O(\varepsilon^3),$$

$$C = \left(1 - \exp(\alpha\varepsilon^2 + \beta\varepsilon)\right)(1 - \exp(-\beta\varepsilon)) = -\beta^2\varepsilon^2 + O(\varepsilon^3), \qquad \varepsilon \to 0.$$

The root γ_ε, $\gamma_\varepsilon > 1$, is

$$\begin{aligned}\gamma_\varepsilon &= \tfrac{1}{2}\Big[B + (B^2 - 4C)^{1/2}\Big]\\ &= \tfrac{1}{2}\Big[1 + (2\alpha + \beta^2)\varepsilon^2 + \big(1 + 2(2\alpha + \beta^2)\varepsilon^2 + 4\beta^2\varepsilon^2\big)^{1/2}\Big] + O(\varepsilon^3)\\ &= 1 + 2(\alpha + \beta^2)\varepsilon^2 + O(\varepsilon^3).\end{aligned}$$

This then yields easily the behavior of θ_ε as $\varepsilon \to 0$. □

Let S_n denote the position of a symmetric simple random walk on $\mathbb{Z}^1$ at time n. Lemma 1 is important to our large deviation study because

$$(\cosh \theta_\varepsilon)^{-n}\left[\exp \sum_{m=0}^{n-1} V_\varepsilon(S_m)\right] f(S_n) \text{ is a martingale.}$$

Using this family of martingales, the argument of Cox and Durrett can be slightly modified to prove the next lemma.

LEMMA 2. *Let $\gamma_n \to \infty$, $\gamma_n n^{-1/4} \to 0$ as $n \to \infty$ and $y \in \mathbb{Z}$. Then*:

(i) $\lim_{n\to\infty} \gamma_n^{-4} \log E_y\{\exp[\alpha(\gamma_n n^{-1/4})^2 \overline{K}_n + \beta\gamma_n n^{-1/4}\overline{L}_n]\} = ((\alpha + \beta^2)_+)^2/2$, *where $c_+ = c \vee 0$ and E_y means the expectation with respect to a symmetric random walk initially at y. Throughout this paper, $\overline{K}_n$ and $\overline{L}_n$ will be used to remind us that only one particle is being considered.*

(ii) $\lim\sup_{n\to\infty} \gamma_n^{-4} \log \max_{1\le m\le n} E_y\{\exp[\alpha(\gamma_n n^{-1/4})^2\overline{K}_m + \beta\gamma_n n^{-1/4}\overline{L}_m]\} \le (\alpha + \beta^2)^2/2$, *for α, β such that $\alpha + \beta^2 > 0$.*

(iii) *With respect to P_y, the system $((\gamma_n^{-2}n^{-1/2}\overline{K}_n, \gamma_n^{-3}n^{-1/4}\overline{L}_n), \gamma_n^4)$ is a large deviation system with rate function S*:

$$S(a,b) = \begin{cases} \dfrac{a^2}{2} + \dfrac{b^2}{4a}, & \text{for } a > 0,\ b \in \mathbb{R},\\ \infty, & \text{otherwise.}\end{cases}$$

REMARK 1. Via the Gartner–Ellis theorem [cf. Ellis (1985)], statement (iii) follows from (i) and the fact that

$$S(a,b) = \sup_{\alpha,\beta}\left[a\alpha + b\beta - \frac{\left[(\alpha + \beta^2)_+\right]^2}{2}\right].$$

REMARK 2. The proof of Theorem 1(i) (for the process P) will require only the case $\gamma_n = (\log n)^{1/4}$ of Lemma 2, whereas that of Theorem 1(ii) (for the process Q) will use $\gamma_n = n^{\theta/4}$, $0 < \theta < 1$.

PROOF. For the detail of the proof of (i) for $\alpha + \beta^2 > 0$ and that of (ii), the reader is referred to the proof of Cox and Durrett [(1990), Lemma 1]. To prove (i) when $\alpha + \beta^2 \le 0$, note that the limit is increasing in α for fixed β. Thus

$$\limsup_{n\to\infty} \gamma_n^{-4} \log E_y\Big\{\exp\Big[\alpha\big(\gamma_n n^{-1/4}\big)^2\overline{K}_n + \beta\gamma_n n^{-1/4}\overline{L}_n\Big]\Big\} \le 0.$$

For the lower bound, a simple computation shows that [see Lemmas 2.1 and 2.3 of Lee and Remillard (1994)]

$$E_y\{\overline{K}_n\} < cn^{1/2}, \qquad E_y\{|\overline{L}_n|\} \le E_y\{\overline{L}_n^2\}^{1/2} < cn^{1/4} \quad \text{for some } c > 0.$$

An application of the Jensen inequality then yields

$$\begin{aligned}
&\liminf_{n\to\infty} \gamma_n^{-4} \log E_y\Big\{\exp\Big[\alpha\big(\gamma_n n^{-1/4}\big)^2 \overline{K}_n + \beta\gamma_n n^{-1/4}\overline{L}_n\Big]\Big\} \\
&\quad \ge \liminf_{n\to\infty} \gamma_n^{-4} E_y\big\{\alpha\gamma_n^2 n^{-1/2}\overline{K}_n + \beta\gamma_n n^{-1/4}\overline{L}_n\big\} \\
&\quad \ge \liminf_{n\to\infty} \gamma_n^{-4}\big(|\alpha|\gamma_n^2 + |\beta|\gamma_n\big)c = 0.
\end{aligned}$$

The proof is complete. □

LEMMA 3. *As in Lemma 2, let $\gamma_n \to \infty$ and $\gamma_n n^{-1/4} \to 0$ as $n \to \infty$. Then for each α, β such that $\alpha + \beta^2 > 0$ and for each positive ε there exists $n_* = n_*(\alpha, \beta, \varepsilon)$ such that*

$$\log E^P\Big\{\exp\Big[\alpha\big(\gamma_n n^{-1/4}\big)^2 K_n + \beta\gamma_n n^{-1/4}L_n\Big]\Big\} \le n^{1/2}\exp\Big\{\gamma_n^4(\alpha+\beta^2)^2(1+\varepsilon)/2\Big\}$$

for $n \ge n_*$.

PROOF. Let A be a finite subset of $\mathbb{Z}$. It follows from an elementary property of the Poisson distribution that the numbers of particles entering A first at site y and time $n \ge 0$ are independent Poisson variables with mean $\mu_{n,y}$, $n \ge 0$, $y \in A$. It is known [see Lee and Remillard (1994), Lemma 2.8, for a proof] that

$$\lim_{n\to\infty} n^{-1/2}\sum_{m=0}^{n-1}\sum_{y\in A}\mu_{m,y} = c, \quad \text{a positive constant.}$$

Let $A = \{0, 1\}$. Since

$$\begin{aligned}
&\log E^P\{\exp(\alpha\gamma_n^2 n^{-1/2}K_n + \beta\gamma_n n^{-1/4}L_n)\} \\
&\quad = \sum_{m=0}^{n-1}\sum_{y=0}^{1}\mu_{m,y}\Big(E_y\big\{\exp(\alpha\gamma_n^2 n^{-1/2}\overline{K}_{n-m} + \beta\gamma_n n^{-1/4}\overline{L}_{n-m})\big\} - 1\Big) \qquad (2.3)\\
&\quad \le \left(\sum_{m=0}^{n-1}\sum_{y=0}^{1}\mu_{m,y}\right)\max_{\substack{1\le k\le n\\ y=0,1}} E_y\big\{\exp(\alpha\gamma_n^2 n^{-1/2}\overline{K}_k + \beta\gamma_n n^{-1/4}\overline{L}_k)\big\},
\end{aligned}$$

the present lemma follows from Lemma 2(ii). □

Denote by $[b]$ the integer part of b. Let $\zeta_n'(x)$ be the number of particles at $x \in \mathbb{Z}$ at time n which are initially in $\mathbb{Z}\setminus\{1-[n^{1/2}],\dots,1,2,\dots,[n^{1/2}]\}$. Let K_n' and L_n' be defined as in the first paragraph of this paper, through ζ_n'

instead of ζ_n. Similarly, define $\zeta_n''(x)$, K_n'' and L_n'' for those particles which are initially in $\{1 - [n^{1/2}], \dots, [n^{1/2}]\}$. Note

$$\zeta_n(x) = \zeta_n'(x) + \zeta_n''(x),$$
$$K_n = K_n' + K_n'',$$
$$L_n = L_n' + L_n''.$$

LEMMA 4. *If $a_n n^{-1/2} \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} P\{|L_n'| \le a_n\} = 1.$$

PROOF. It was proved in Theorem 1 of Lee and Remillard (1994) that

$$\lim_{n\to\infty} E^P\{\exp(un^{-1/2}L_n)\} = \exp\left(\frac{cu^2}{2}\right),$$

where c is a positive constant. Among other things, this implies that, with respect to P, the sequence $n^{-1/2}L_n$ converges in distribution to a Gaussian distribution and that

$$E\{L_n^2\} \le (c+1)n \quad \text{for large } n.$$

Since $E\{L_n'\} = 0$, the Chebyshev inequality then yields that

$$P\{|L_n'| > a_n\} \le E\{(L_n')^2\}/a_n^2 \le E\{L_n^2\}/a_n^2 \le (c+1)n/a_n^2 \to 0 \quad \text{as } n \to \infty,$$

where the second to last inequality uses the independence of random walks. □

LEMMA 5. *For each $\alpha > 0$ and finite subset A of $\mathbb{Z}$,*

$$E_y\left\{\exp\left(\alpha n^{-1/2} \sum_{z\in A} \overline{K}_n(z)\right)\right\} \text{ is bounded in } y \in \mathbb{Z},\ n \in \mathbb{N}.$$

PROOF. Let $\pi_{yz}^{(k)}$ be the k-step transition probabilities of a symmetric simple random walk, $|A|$ be the number of elements of A and

$$b_{j,n} = \sum_{0\le k_1 \le \cdots \le k_j \le n-1}\ \sum_{y_1,\dots,y_j \in A} \pi_{yy_1}^{(k_1)}\pi_{y_1y_2}^{(k_2-k_1)} \cdots \pi_{y_{j-1}y_j}^{(k_j-k_{j-1})}.$$

The sequence $b_{j,n}$ is increasing in n for fixed j. Let

$$a_{j,n} = b_{j,n} - b_{j,n-1}, \qquad a_{j,0} = 0.$$

Clearly,

$$\begin{aligned} &E_y\left\{\exp\left(\alpha n^{-1/2} \sum_{z\in A} \overline{K}_n(z)\right)\right\} \\ (2.4)\qquad &= \sum_{j=0}^{\infty} \frac{E_y\left\{\left(\sum_{z\in A} \overline{K}_n(z)\right)^j\right\}(\alpha n^{-1/2})^j}{j!} \le \sum_{j=0}^{\infty} b_{j,n}(\alpha n^{-1/2})^j. \end{aligned}$$

It is easy to check that there exists $c > 0$ such that

$$\sum_{n=0}^{\infty} e^{-sn} b_{j,n} \le (1-e^{-s})^{-1} \left(\sup_{y \in \mathbb{Z}} \sum_{z \in A} \sum_{n=0}^{\infty} e^{-sn} \pi_{yz}^{(n)} \right)^j \tag{2.5}$$
$$\le (1-e^{-s})^{-1} s^{-j/2} (c|A|)^j \quad \text{for } 0 < s < 1,$$

and

$$b_{j,n} = \sum_{k=0}^{n} a_{j,k} \le e^{sn} \sum_{k=0}^{\infty} e^{-sk} a_{j,k} = e^{sn}(1-e^{-s}) \sum_{k=0}^{\infty} e^{-sk} b_{j,k}. \tag{2.6}$$

Combining (2.6) and (2.5), we get

$$b_{j,n} \le \left(\inf_{0<s<1} e^{sn} s^{-j/2} \right) (c|A|)^j = \left(\inf_{0<\gamma<n} e^{\gamma} \gamma^{-j/2} \right) (c|A| n^{1/2})^j.$$

This inequality, together with (2.4), implies

$$E_y \left\{ \exp\left(\alpha n^{-1/2} \sum_{z \in A} \overline{K}_n(z) \right) \right\} \le \sum_{j=0}^{\infty} \left(\inf_{0<\gamma<n} e^{\gamma} \gamma^{-j/2} \right) (\alpha c |A|)^j$$
$$\le \exp\left(4(\alpha c|A|)^2\right) \sum_{j=0}^{\infty} 2^{-j} = 2 \exp\left(4(\alpha c |A|)^2\right)$$
$$\text{for } n > 4(\alpha c|A|)^2,$$

where the last inequality uses $\gamma = 4(\alpha c|A|)^2$. The proof is completed. □

LEMMA 6. *For each $\beta \in \mathbb{R}$, $E_y\{\exp(\beta n^{-1/2} \overline{L}_n)\}$ is bounded in $y \in \mathbb{Z}$, $n \in \mathbb{N}$.*

PROOF. We shall reduce the present lemma to Lemma 5 via a martingale technique. Let

$$G(y) = -2\chi_+(y) = \begin{cases} -2, & \text{for } y \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$E_y\{G(S_1)\} - G(y) = -(\chi_0 - \chi_1)(y),$$

we have

$$M_n \equiv G(S_n) - G(S_0) + \overline{L}_n \text{ is a zero mean martingale.} \tag{2.7}$$

For $u \in \mathbb{R}$ a simple computation shows that

$$E_y\{\exp(uM_1)\} = \exp(g_u(y)),$$

where

$$g_u(y) = \begin{cases} \left(\log \dfrac{e^{2u}+1}{2}\right) - u, & \text{for } y = 0, \\ \left(\log \dfrac{e^{-2u}+1}{2}\right) + u, & \text{for } y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

(2.8) $\qquad \varphi_{u,n} \equiv \exp(uM_n - \sum_{z=0}^{1} g_u(z)\overline{K}_n(z))$ is a positive, mean 1 martingale.

The function g_u is nonnegative and

(2.9) $$g_u(y) \sim \frac{u^2}{2} + O(u^3), \quad \text{for } y = 0, 1 \text{ as } u \to 0.$$

Using $u = 2\beta n^{-1/4}$, (2.7) and (2.8), an application of the Schwarz inequality yields

$$\begin{aligned} &E_y\{\exp(\beta n^{-1/4}\overline{L}_n)\} \\ &\quad = E_y\left\{\exp\left(\frac{u}{2}\overline{L}_n\right)\right\} \\ &\quad = E_y\left\{\exp\left[\frac{u}{2}M_n - \frac{u}{2}(G(S_n) - G(S_0))\right]\right\} \\ &\quad = E_y\left\{\varphi_{u,n}^{1/2}\exp\left[\frac{1}{2}\sum_{z=0}^{1} g_u(z)\overline{K}_n(z) - \frac{u}{2}(G(S_n) - G(S_0))\right]\right\} \\ &\quad \le E_y\{\varphi_{u,n}\}^{1/2} E_y\left\{\exp\left[\sum_{z=0}^{1} g_u(z)\overline{K}_n(z) - u(G(S_n) - G(S_0))\right]\right\}^{1/2} \\ &\quad \le 1 \cdot E_y\left\{\exp\left(\sum_{z=0}^{1} g_u(z)\overline{K}_n(z)\right)\right\}^{1/2} \exp|u| \\ &\quad \le E_y\left\{\exp\left[(2\beta n^{-1/4})^2 \sum_{z=0}^{1} \overline{K}_n(z)\right]\right\}^{1/2} \exp(1), \quad \text{for large } n, \end{aligned}$$

where the last inequality uses (2.9). Lemma 5 then implies the present lemma. □

LEMMA 7. *Let f and J^P, J^Q be as given in* (1.8) *and* (1.9). *The following four properties hold*:

(i) *With respect to P_y, $y \in \mathbb{Z}$,*

$$\left(\overline{K}_n n^{-1/2}, \overline{L}_n n^{-1/4}\right) \Rightarrow_d (U, V) \quad \textit{as } n \to \infty,$$

where the joint distribution of (U, V) *is uniquely characterized by*

$$f_U(u) = \left(\frac{2}{\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right), \qquad u > 0,$$

$$f_{V|U}(v) = (4\pi u)^{-1/2} \exp\left(-\frac{v^2}{4u}\right), \qquad v \in \mathbb{R}.$$

(ii) $$\lim_{n\to\infty} E_y\left\{\exp\left(\alpha \overline{K}_n n^{-1/4} + \beta \overline{L}_n n^{-1/4}\right)\right\} = f(\alpha + \beta^2).$$

(iii) $$\lim_{n\to\infty} n^{-1/2} \log E^P\left\{\exp\left(\alpha K_n n^{-1/2} + \beta L_n n^{-1/4}\right)\right\} = J^P(\alpha, \beta).$$

(iv) $$\lim_{n\to\infty} n^{-1/2} \log E^Q\left\{\exp\left(\alpha K_n n^{-1/2} + \beta L_n n^{-1/4}\right)\right\} = J^Q(\alpha, \beta).$$

PROOF. Statement (i) is a special case of Kesten's Theorem 2 [Kesten (1962)]. The convergence of the marginal distributions $\overline{K}_n n^{-1/2}$ and $\overline{L}_n n^{-1/4}$ can be found in Dobrushin (1955) and Darling and Kac (1957).

To prove (ii) first note that the expected value of $c^{\alpha U + \beta V}$ is $f(\alpha + \beta^2)$. Given (i), we only need to show that $E_y\{\exp(\alpha \overline{K}_n n^{-1/2} + \beta \overline{L}_n n^{-1/4})\}$ is bounded in n, which follows easily from Lemmas 5 and 6 and the Schwarz inequality.

Let $A = \{0, 1\}$ and $\mu_{n,y}$, $n \geq 0$, $y \in \{0, 1\}$, be as in the proof of Lemma 3. Then [cf. (2.3)],

$$\log E^P\left\{\exp\left(\alpha n^{-1/2} K_n + \beta n^{-1/4} L_n\right)\right\}$$
$$= \sum_{y=0}^{1} \sum_{m=0}^{n-1} \mu_{m,y}\left(E_y\left\{\exp\left(\alpha n^{-1/2} \overline{K}_{n-m} + \beta n^{-1/4} \overline{L}_{n-m}\right)\right\} - 1\right). \tag{2.10}$$

Let $[c]$ be the largest integer which is no greater than c. For $0 < t < 1$ it is not difficult to see that

$$\sum_{y=0}^{1} \sum_{m=0}^{[tn]} \mu_{m,y} \sim n^{1/2} \int_{-\infty}^{\infty} \int_0^t \tau_x(s)\, ds, \qquad n \to \infty. \tag{2.11}$$

Also, it follows from (ii) that, for $y = 0$ or 1, $0 < s < 1$,

$$\lim_{n\to\infty} E_y\left\{\exp\left(\alpha n^{-1/2} \overline{K}_{n-[sn]} + \beta n^{-1/4} \overline{L}_{n-[sn]}\right)\right\}$$
$$= \lim_{m\to\infty} E_y\left\{\exp\left(\alpha (1-s)^{1/2} m^{-1/2} \overline{K}_m + \beta (1-s)^{1/4} m^{-1/4} \overline{L}_m\right)\right\} \tag{2.12}$$
$$= f\left((1-s)^{-1/2}(\alpha + \beta^2)\right).$$

By (2.10), (2.11) and (2.12) one anticipates statement (iii) which is proved as in Lee and Remillard [(1994), Lemmas 2.8, 2.9 and 2.10].

The proof of (iv) begins with

$$\log E^Q\left\{\exp\left(\alpha n^{-1/2} K_n + \beta n^{-1/4} L_n\right)\right\}$$
$$= \sum_{z\in\mathbb{Z}} \zeta_0(z) \log E_z\left\{\exp\left(\alpha n^{-1/2} \overline{K}_n + \beta n^{-1/4} \overline{L}_n\right)\right\}.$$

Observe that, for $x \in \mathbb{R}$,

$$\begin{aligned} &E_{[n^{-1/2}x]}\Big\{\exp\Big(\alpha n^{-1/2}\overline{K}_n + \beta n^{-1/4}\overline{L}_n\Big)\Big\} \\ &\quad \to \int_1^\infty \tau_x(s)\,ds + \int_0^1 \tau_x(s) f\Big((1-s)^{1/2}(\alpha+\beta^2)\Big)\,ds \\ &\quad = 1 + \int_0^1 \Big[f\Big((1-s)^{1/2}(\alpha+\beta^2)\Big) - 1\Big]\tau_x(s)\,ds. \end{aligned}$$

Since, by the definition of Q,

$$\sum_{z=-[n^{1/2}y]}^{[n^{1/2}y]} \zeta_0(z) \sim n^{1/2}2y = n^{1/2}\int_{-y}^{y} dx, \qquad n \to \infty, \text{ for } y > 0,$$

statement (iv) is anticipated. We refer the reader to the proof of Theorem 2 of Cox and Durrett (1990) for more detail. □

3. Proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. First we prove (ii). Recall from Lemma 2(i), with $\gamma_n = n^{\theta/4}$, $0 < \theta < 1$, that

$$\begin{aligned} &\lim_{n\to\infty} n^{-\theta} \log E_0\Big\{\exp\Big(\alpha n^{\theta/2-1/2}\overline{K}_n + \beta n^{\theta/4-1/4}\overline{L}_n\Big)\Big\} \\ &\quad = \frac{\big[(\alpha+\beta^2)_+\big]^2}{2}. \end{aligned} \tag{2.13}$$

We refer the reader to Cox and Durrett [(1990), Lemma 2] or Remillard [(1990), Theorem 1.1] for a proof of the following extension of (2.13):

$$\begin{aligned} &\lim_{n\to\infty} n^{-\theta} \log E_{n^{1/2+\theta/2}y}\Big\{\exp\Big(\alpha n^{\theta/2-1/2}\overline{K}_n + \beta n^{\theta/4-1/4}\overline{L}_n\Big)\Big\} \\ &\quad = \left[\frac{\big((\alpha+\beta^2)_+\big)^2}{2} - (\alpha+\beta^2)_+|y|\right]_+, \end{aligned}$$

and for the exchangeability of $\lim_{n\to\infty}$ and integration described below:

$$\begin{aligned} &\lim_{n\to\infty} n^{-3\theta/2-1/2} \log E^Q\Big\{\exp\Big(\alpha n^{\theta/2-1/2}K_n + \beta n^{\theta/4-1/4}L_n\Big)\Big\} \\ &\quad = \lim_{n\to\infty} n^{-3\theta/2-1/2}\int_{-\infty}^{\infty} \log E_{[x]}\Big\{\exp\Big(\alpha n^{\theta/2-1/2}\overline{K}_n + \beta n^{\theta/4-1/4}\overline{L}_n\Big)\Big\}\,dx \\ &\quad = \int_{-\infty}^{\infty} \lim_{n\to\infty} n^{-\theta} \log E_{[n^{\theta/2-1/2}y]}\Big\{\exp\Big(\alpha n^{\theta/2-1/2}\overline{K}_n + \beta n^{\theta/4-1/4}\overline{L}_n\Big)\Big\}\,dy \\ &\quad = \int_{-\infty}^{\infty}\left[\frac{\big[(\alpha+\beta^2)_+\big]^2}{2} - (\alpha+\beta^2)_+|y|\right]dy = \frac{(\alpha+\beta^2)_+^3}{4}. \end{aligned} \tag{2.14}$$

Via the Gartner–Ellis theorem, statement (ii) follows from the limiting formula (2.14) and the following Legendre transform formula:

$$\sup_{\alpha,\beta}\left[a\alpha + b\beta - \frac{(\alpha+\beta^2)_+^3}{4}\right] = \begin{cases} 4\cdot 3^{-3/2}a^{3/2} + b^2/4a, & \text{for } a > 0,\ b\in\mathbb{R},\\ 0, & \text{for } a = 0,\ b = 0,\\ \infty, & \text{otherwise.}\end{cases}$$

We now prove (i). Note that the Legendre transform of the rate function is

$$\sup_{a>0,\,b}\left\{\alpha a + \beta b - \left[(2\theta)^{1/2}a + \frac{b^2}{4a}\right]\right\} = \sup_{a>0}\left[\alpha + \beta^2 - (2\theta)^{1/2}\right]a$$
$$= \begin{cases} 0, & \text{for } \alpha+\beta^2 \le (2\theta)^{1/2},\\ \infty, & \text{for } \alpha+\beta^2 > (2\theta)^{1/2}.\end{cases}$$

So we should *not* use the Gartner–Ellis theorem for a proof. For the upper bound we use Lemma 3, with the special choice $\gamma_n = (\log n)^{1/4}$; that is,

$$\begin{aligned}\log E^P\left\{\exp\left(\alpha(\log n)^{1/2}n^{-1/2}K_n + \beta(\log n)^{1/4}n^{-1/4}L_n\right)\right\}\\ \le n^{\{1+[(\alpha+\beta^2)_+]^2(1+\varepsilon)\}/2}.\end{aligned} \tag{2.15}$$

The Chebyshev inequality, together with (2.15), implies for closed sets F in $\mathbb{R}^2$ that

$$\begin{aligned}&\log P\left\{\left(K_n\lambda_n^{-1}n^{-1}, L_n\lambda_n^{-1}(\log n)^{-1/4}n^{-3/4}\right)\in F\right\}\\ &\quad\le \log P\left\{\exp\left[\alpha(\log n)^{1/2}n^{-1/2}K_n + \beta(\log n)^{1/4}n^{-1/4}L_n\right]\right.\\ &\qquad\qquad \left.\ge \exp\left(\left[\inf_{(a,b)\in F}(a\alpha+b\beta)\right]\lambda_n(\log n)^{1/2}n^{1/2}\right)\right\}\\ &\quad\le n^{\{1+(\alpha+\beta^2)^2(1+\varepsilon)\}/2} - \left[\inf_{(a,b)\in F}(a\alpha+b\beta)\right]\lambda_n(\log n)^{1/2}n^{1/2}.\end{aligned}$$

Since $(\log\lambda_n)/\log n \to \theta$ as $n\to\infty$, the second term dominates the first term as $n\to\infty$ for α,β such that $\alpha+\beta^2 > 0$ and $(\alpha+\beta^2)^2(1+\varepsilon)/2 < \theta$; for example, for $(\alpha,\beta)\in H \equiv \{(\alpha+\beta^2)(1+\varepsilon) = (2\theta)^{1/2}\}$. We conclude that

$$\begin{aligned}&\limsup_{n\to\infty}\lambda_n^{-1}(\log n)^{-1/2}n^{-1/2}\log P\left\{\left(K_n\lambda_n^{-1}n^{-1}, L_n\lambda_n^{-1}(\log n)^{-1/4}n^{-3/4}\right)\in F\right\}\\ &\quad\le -\sup_{(\alpha,\beta)\in H}\ \inf_{(a,b)\in F'}(a\alpha+b\beta) \equiv -\Gamma.\end{aligned}$$

Since K_n is a nonnegative random variable, it suffices to consider $F\subset[0,\infty)\times\mathbb{R}$. Let $A, B > 0$ and $F' = F\cap\{a\le A\}\cap\{|b|\le B\}$, which is compact. Since $F\subset F'\cup\{a > A\}\cup\{|b| > B\}$,

$$\Gamma \ge \min\left\{\sup_H\inf_F(a\alpha+b\beta),\ \sup_H\inf_{a>A}(a\alpha+b\beta),\ \sup_H\inf_{|b|>B}(a\alpha+b\beta)\right\}.$$

Because F' is compact, a standard argument [see Donsker and Varadhan (1975), for example] shows that the order of sup and inf is exchangeable. Thus,

$$\begin{aligned}\sup_H \inf_{F'}(a\alpha + b\beta) &= \inf_{F'}\sup_H(a\alpha + b\beta) = \inf_{F'}\sup_\beta\left(a\left(\frac{(2\theta)^{1/2}}{(1+\varepsilon)} - \beta^2\right) + b\beta\right)\\ &= \inf_{F'}\left(\frac{(2\theta)^{1/2}a}{1+\varepsilon} + \frac{b^2}{4a}\right) \geq (1+\varepsilon)^{-1}\inf_{F'} I^{P,\theta}(a,b)\\ &\geq (1+\varepsilon)^{-1}\inf_F I^{P,\theta}(a,b).\end{aligned}$$

The following two estimates are easy:

$$\sup_H \inf_{a>A}(a\alpha + b\beta) \geq \inf_{a>A}\left(a\frac{(2\theta)^{1/2}}{1+\varepsilon} + b\cdot 0\right) = (1+\varepsilon)^{-1}(2\theta)^{1/2}A,$$

$$\begin{aligned}\sup_H \inf_{|b|>B}(a\alpha + b\beta) &\geq \inf_{|b|>B}\max\left\{a\cdot 0 + b\frac{(2\theta)^{1/4}}{(1+\varepsilon)^{1/2}}, a\cdot 0 + b\frac{-(2\theta)^{1/4}}{(1+\varepsilon)^{1/2}}\right\}\\ &= (1+\varepsilon)^{-1/2}(2\theta)^{1/4}B.\end{aligned}$$

Putting together these three estimates yields

$$\Gamma \geq \min\left\{(1+\varepsilon)^{-1}\inf_F I^{P,\theta}(a,b), (1+\varepsilon)^{-1}(2\theta)^{1/2}A, (1+\varepsilon)^{-1/2}(2\theta)^{1/4}B\right\}$$

for all $A, B, \varepsilon > 0$. The upper bound of probability (lower bound of Γ) follows from letting $A, B \to \infty$ and then $\varepsilon \to 0$.

To prove the lower bound, note that the event

$$\left\{K_n/(\lambda_n n) \in (a-\varepsilon, a+\varepsilon), L_n/\left(\lambda_n(\log n)^{1/4}n^{3/4}\right) \in (b-\varepsilon, b+\varepsilon)\right\}$$

contains the event described in (i)–(iv) as follows. Let u, δ be positive numbers and $[c]$ denote the integer part of c.

(i) $\zeta_0(x) = \alpha_n = [u\lambda_n(\log n)^{1/2}/2]$ for $1 - [n^{1/2}] \leq x \leq [n^{1/2}]$.

(ii) All particles which are initially located in $\{1-[n^{1/2}], \dots, [n^{1/2}]\}$ ($2[n^{1/2}]\alpha_n$ of them) pay their first visit to site 1 before time $A(\delta n)$.

(iii) $K''_n/(u^{-1}n^{1/2}(\log\ n)^{1/2}) \in (a-\varepsilon/2, a+\varepsilon/2)$ and $L''_n/(u^{-1}n^{1/4}(\log n)^{3/4}) \in (b-\varepsilon/2, b+\varepsilon/2)$.

(iv) $K'_n/(\lambda_n n) < \varepsilon/4$ and $|L'_n|/(\lambda_n(\log n)^{1/4}n^{3/4}) < \varepsilon/4$.

For the definition of K'_n, L'_n, K''_n and L''_n see the paragraph preceding Lemma 4. Due to independence, we have

$$P\{(\mathrm{i}), (\mathrm{ii}), (\mathrm{iii}) \text{ and } (\mathrm{iv})\} = P\{(\mathrm{i})\}P\{(\mathrm{ii})\}P\{(\mathrm{iii})\}P\{(\mathrm{iv})\}.$$

Since $K'_n \le K_n$ and $E^P\{K_n\} = n$, the Chebyshev inequality implies that

$$\lim_{n\to\infty} P\left\{\frac{K'_n}{\lambda_n n} < \frac{\varepsilon}{4}\right\} = 1 \quad \text{for all } \varepsilon > 0.$$

This and Lemma 4 show that

$$\lim_{n\to\infty} P\{(\text{iv})\} = 1. \tag{2.16}$$

Next we determine the logarithmic lower bounds of events (i)–(iii). Since $\zeta_0(x)$ are independent Poisson mean 1 random variables and $\log \lambda_n/(\log n) \to \theta$ as $n \to \infty$,

$$\begin{aligned} P\{(\text{i})\} &= P\{\zeta_0(x) = \alpha_n \text{ for } 1 - [n^{1/2}] \le x \le [n^{1/2}]\} \\ &\asymp \exp(-2n^{1/2}\alpha_n \log \alpha_n) \\ &\asymp \exp\left(-u\theta\lambda_n n^{1/2}(\log n)^{1/2}\right), \end{aligned}$$

where "$\asymp$" means logarithmic equivalence; that is,

$$\lim_{n\to\infty} \lambda_n^{-1} n^{-1/2}(\log n)^{-1/2} P\{(\text{i})\} = -u\theta. \tag{2.17}$$

Because $P_{1-[n^{1/2}]}\{\text{visiting 1 before time } \delta n\} \to c_\delta \equiv \int_0^\delta \tau_1(s)\,ds > 0$ as $n \to \infty$,

$$P\{(\text{ii})\} \ge \exp(2[n^{1/2}]\alpha_n \log c_\delta) \asymp \exp\left(n^{1/2}u\lambda_n(\log n)^{-1/2}\log c_\delta\right);$$

hence,

$$\begin{aligned} &\lim_{n\to\infty} \lambda_n^{-1} n^{-1/2}(\log n)^{-1/2} P\{(\text{ii})\} \\ &\quad \ge \lim_{n\to\infty} \frac{(\log c_\delta) u\lambda_n n^{1/2}(\log n)^{-1/2}}{\lambda_n n^{1/2}(\log n)^{1/2}} = 0. \end{aligned} \tag{2.18}$$

Notice that

$$P\{(\text{iii})\} \ge (q_{\delta,n})^{2[n^{1/2}]\alpha_n},$$

where

$$\begin{aligned} q_{\delta,n} \equiv \inf_{(1-\delta)n \le m \le n} P_1\Big\{&\overline{K}_m/\left(u^{-1}n^{1/2}(\log n)^{1/2}\right) \in (a - \varepsilon/2, a + \varepsilon/2), \\ &\overline{L}_n/\left(u^{-1}n^{1/4}(\log n)^{3/4}\right) \in (b - \varepsilon/2, b + \varepsilon/2)\Big\}. \end{aligned}$$

If δ is sufficiently small, then

$$\begin{aligned} q_{\delta,n} \ge \inf_{(1-2\delta)n \le m \le n} P_1\Big\{&K_m/\left(u^{-1}m^{1/2}(\log m)^{1/4}\right) \in (a - \varepsilon/4, a + \varepsilon/4) \text{ and} \\ &L_m/\left(u^{-1}m^{1/4}(\log m)^{3/4}\right) \in (b - \varepsilon/4, b + \varepsilon/4)\Big\}. \end{aligned}$$

It now follows from Lemma 2(iii), with $\gamma_n = (\log n)^{1/4}$, that

$$\begin{aligned} &\liminf_{n\to\infty} \lambda_n^{-1} n^{-1/2}(\log n)^{-1/2} \log P\{(\text{iii})\} \\ &\quad \ge \liminf_{n\to\infty} u(\log n)^{-1} \log q_{\delta,n} \\ &\quad \ge -uS(au^{-1}, bu^{-1}). \end{aligned} \tag{2.19}$$

Combining the four bounds (2.16), (2.17), (2.18) and (2.19) and noting that $u > 0$ is arbitrary, we conclude that

$$\begin{aligned}\liminf_{n \to \infty} \lambda_n^{-1} n^{-1/2} (\log n)^{-1/2} \log P\Big\{ & K_n/(\lambda_n n) \in (a - \varepsilon, a + \varepsilon), \\ & L_n/\big(\lambda_n (\log n)^{1/4} n^{3/4}\big) \in (b - \varepsilon, b + \varepsilon)\Big\} \\ \geq \sup_{u>0} & \left(-u\theta - uS(au^{-1}, bu^{-1})\right) \\ = - \inf_{u>0} & \left(u\theta + \frac{a^2 u^{-1}}{2} + \frac{b^2}{4a}\right) \\ = - & \left((2\theta)^{1/2} a + \frac{b^2}{4a}\right) = -I^{P,\theta}(a,b). \qquad \square\end{aligned}$$

PROOF OF THEOREM 2. In view of Lemma 7(iii) and (iv), an application of the Gartner–Ellis theorem yields the result. □

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