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QUANTUM OPERATORS IN CLASSICAL PROBABILITY THEORY: II. THE CONCEPT OF DUALITY IN INTERACTING PARTICLE SYSTEMS

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Duality has proved to be a powerful tool in the theory of interacting particle systems. The approach in this paper is algebraic rather than via Harris diagrams. A form of duality is found which includes coalescing and annihilating duality as special cases. This enables new results for the branching annihilating random walk and the biased annihilating branching process to be derived.

1. Introduction. In a previous paper, Lloyd, Sudbury and Donnelly (1993), we showed how results for an exclusion process on the complete graph (the Bernoulli–Laplace model) could be derived using a fortuitous equality between the infinitesimal generator for the process, Q, and the negative of the Heisenberg Hamiltonian for a ferromagnetic insulator, $H = -Q$. In this paper we turn our attention to the concept of *duality*, which has proved a powerful tool in the stochastic theory of interacting particle systems [see Liggett (1985) for examples]. Often the dual of a given stochastic process is one that appears to be the original process, but "running backward in time," and with particles and holes (empty sites) interchanged. In quantum theory this is called a CT transform, where C stands for charge conjugation (it takes particles into antiparticles) and T stands for time reversal. Here, however, our emphasis will be algebraic.

In this paper the stochastic processes can be taken as being ones in which a set of particles occupy some or all of the N sites of a (crystal) lattice in which each site has a well-defined set of nearest neighbors. Each site can only be occupied by 0 or 1 particles at a time, and these occupations alter with time by means of probabilistic nearest neighbor transitions.

The common theme underlying this series of papers is that the Q-matrix can be expressed as a sum of matrices or operators, each representing interactions between a pair of sites. Further, each of these operators is a quadratic function of single-site operators. In terms of such representations for Q, a duality relation is essentially an algebraic transformation of the single-site operators.

In Section 2 we show how the Q-matrix for several nearest neighbor interacting particle systems can be expressed in terms of the single-site

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operators of quantum theory. In Section 3 we give an informal argument to suggest that the defining equation for duality should be of the form

$$E\left(a^{|\zeta_t^A \cap B|}\right) = E\left(a^{|\xi_t^B \cap A|}\right).$$

When ζ and ξ are spin systems, this is a special case of (2.16) in Holley and Stroock (1979).

Since $0^x = 0$ unless $x = 0$, the special case $a = 0$ becomes $P(|\zeta_t^A \cap B| = \varnothing) = P(|\xi_t^B \cap A| = \varnothing)$ and is the coalescing dual. The case $a = -1$ gives annihilating duals.

In Section 4 we give conditions for duality expressing the transition rates of one process in terms of the transition rates of the other. It is believed that this formula covers all well-known duals [see (20)]. In Section 5 we use (20) to derive these, as well as new duals for the biased annihilating branching process and for the branching annihilating random walk.

2. Interacting particle systems in terms of quantum operators. Although the methods we shall employ have been described by us elsewhere [Lloyd, Sudbury and Donnelly (1993)], they are sufficiently unfamiliar for a brief recapitulation here. The crystal lattice is represented by a graph G, in which neighbors are connected by edges. There are $2^{|G|}$ possible states on the graph, the members of $\{0,1\}^G = S$. These states may either be referred to as configurations or sets of occupied sites. We shall designate that the site $i \in G$ is occupied by saying that the process has value 1 at that site. "Unoccupied" will be represented by a value of 0. The members of S may be regarded as a basis of a vector space. If A is a configuration, then we write the corresponding vector $|A\rangle$, which in quantum mechanics is called a ket vector.

If $|G| = 2$, there are four possible basis vectors, which we write $|11\rangle$, $|10\rangle$, $|01\rangle$ and $|00\rangle$. A natural representation of these four vectors is $(1,0,0,0)^T$, $(0,1,0,0)^T$, $(0,0,1,0)^T$ and $(0,0,0,1)^T$, which we shall designate 1, 2, 3 and 4. A probability distribution on S is of the form $\sum_{A \in S} P_A |A\rangle$, where $0 \le P_A \le 1$ and $\sum P_A = 1$. Extending this to N sites, probability distributions would be vectors in $\mathbb{R}^{|S|}$ with basis states the 2^N possible states of S.

We now define a particularly simple set of $2^{|G|} \times 2^{|G|}$ matrices operating on the vector space. They are the single-site operators which only change the occupancy at one site of the graph. For example, consider the "raising" operator S_i^+. If site i is unoccupied, this changes the value at that site from 0 to 1 or, in other words, puts a particle there. If site i is occupied, it sends the whole vector to the zero vector (not the state "all unoccupied"). So,

$$S_1^+|1\cdots\rangle = 0, \qquad S_1^+|0\cdots\rangle = |1\cdots\rangle. \tag{1}$$

When there are only two sites, we have

$$S_1^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(It should be noted that in this paper we are reversing the usual convention in stochastic processes; here λ_{ij} is the transition rate from j to i.)

It is simple to check that this matrix gives the relations

$$S_1^+|11\rangle = 0, \qquad S_1^+|10\rangle = 0, \qquad S_1^+|01\rangle = |11\rangle, \qquad S_1^+|00\rangle = |10\rangle.$$

When $|G| = 1$, $|1\rangle = (1,0)^T$ and $|0\rangle = (0,1)^T$. Then all single-site operators are linear sums of the following four arrays:

$$(2) \quad n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \overline{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here S^- is the "lowering" operator and $S^-|1\rangle = |0\rangle$, $S^-|0\rangle = 0$; n is the "number" operator and $\overline{n}$ is the "hole" operator; $n|1\rangle = |1\rangle$, $n|0\rangle = 0$, $\overline{n}|1\rangle = 0$ and $\overline{n}|0\rangle = |0\rangle$.

For instance, the flip operator, which changes the occupancy at a site, is

$$(3) \qquad C = S^+ + S^- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Although this representation as 2×2 matrices is useful, most of the time it is simpler to think of the single-site operators as operators rather than as $2^{|G|} \times 2^{|G|}$ matrices. Such considerations make it immediate that

single-site operators on different sites commute.

Most interacting particle systems develop via interactions between neighboring sites. If there are only two sites, the infinitesimal generator for the interaction between the pair of sites can be expressed in terms of a 4×4 matrix, which it will be seen can be written in terms of single-site operators. When 1 and 2 are neighboring sites on a graph G, that part of the infinitesimal generator that gives the interaction between sites 1 and 2 has the same form in terms of single-site operators, except that now they are $2^{|G|} \times 2^{|G|}$ matrices. The infinitesimal generator of the whole process on the graph G is the sum of the infinitesimal generators governing interacting pairs of sites.

To illustrate these principles, we shall give the infinitesimal generators on a graph with just two sites for several well-known processes. (The symbol above the arrow gives the rate of that particular transition.)

Biased voter model. Transitions

$$01 \xrightarrow{\lambda} 11, \qquad 01 \xrightarrow{1} 00, \qquad 10 \xrightarrow{\lambda} 11, \qquad 10 \xrightarrow{1} 00,$$

$$(4) \quad \begin{pmatrix} 0 & \lambda & \lambda & 0 \\ 0 & -(1+\lambda) & 0 & 0 \\ 0 & 0 & -(1+\lambda) & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \lambda(S_2^+ n_1 - \overline{n}_2 n_1) + \lambda(n_2 S_1^+ - n_2 \overline{n}_1) + (S_2^- \overline{n}_1 - n_2 \overline{n}_1) + (\overline{n}_2 S_1^- - \overline{n}_2 n_1).$$

From now on, in order to highlight transition rates in matrices we shall represent 0's by ·'s.

Annihilating random walk.

$$11 \overset{2}{\to} 00, \qquad 10 \overset{1}{\leftrightarrow} 01,$$

$$\begin{pmatrix} -2 & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot \\ \cdot & 1 & -1 & \cdot \\ 2 & \cdot & \cdot & \cdot \end{pmatrix} = 2(S_2^- S_1^- - n_2 n_1) + (S_2^+ S_1^- - \bar{n}_2 n_1) + (S_2^- S_1^+ - n_2 \bar{n}_1). \tag{5}$$

Coalescing random walk.

$$11 \overset{1}{\to} 01, \qquad 11 \overset{1}{\to} 10, \qquad 10 \overset{1}{\leftrightarrow} 01,$$

$$\begin{pmatrix} -2 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot \\ 1 & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (n_2 S_1^- - n_2 n_1) + (S_2^- n_1 - n_2 n_1) + (S_2^+ S_1^- - \bar{n}_2 n_1) + (S_2^- S_1^+ - n_2 \bar{n}_1). \tag{6}$$

Contact process.

$$10 \overset{\lambda}{\to} 11, \qquad 01 \overset{\lambda}{\to} 11, \qquad 11 \overset{1}{\to} 10, \qquad 11 \overset{1}{\to} 01,$$

$$10 \overset{1}{\to} 00, \qquad 01 \overset{1}{\to} 00,$$

$$\begin{pmatrix} -2 & \lambda & \lambda & \cdot \\ 1 & -(1+\lambda) & \cdot & \cdot \\ 1 & \cdot & -(1+\lambda) & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix} = \lambda(S_2^+ n_1 - \bar{n}_2 n_1) + \lambda(n_2 S_1^+ - n_2 \bar{n}_1) + (n_2 S_1^- - n_2 n_1) + (S_2^- n_1 - n_2 n_1) + (S_2^- \bar{n}_1 - n_2 \bar{n}_1) + (\bar{n}_2 S_1^- - \bar{n}_2 n_1). \tag{7}$$

3. Dual processes. Suppose we have two processes on G, ζ and ξ, which are governed by Q-matrices, Q_ζ and Q_ξ, respectively. We define

$$|\zeta_t^A\rangle = \exp(Q_\zeta t)|A\rangle, \qquad |\xi_t^B\rangle = \exp(Q_\xi t)|B\rangle, \tag{8}$$

as the probability distributions at time t of configurations starting from initial occupied sets A and B, respectively.

We say that ζ and ξ are *dual processes* if, for any graph,

$$Ef(\zeta_t^A \cap B) = Ef(\xi_t^B \cap A), \tag{9}$$

for some function $f: S \to R$ [see Liggett (1985), page 84]. Most usually f has been of two forms:

Coalescing dual: $$f(B) = \begin{cases} 1, & \text{if } B = \varphi, \\ 0, & \text{otherwise.} \end{cases}$$

Annihilating dual: $$f(B) = \begin{cases} 1, & \text{if } |B| \text{ is odd}, \\ 0, & \text{if } |B| \text{ is even.} \end{cases}$$

For example, the voter model ζ and the coalescing random walk ξ are coalescing duals since

$$P(\zeta_t^A \cap B = \varphi) = P(\xi_t^B \cap A = \varphi). \tag{10}$$

Now the relationship in (9) is to be true for all configurations A, B. These may, therefore, be of any shape. For instance, if G is a lattice, A might be a solid block of sites, whereas B might contain no sites that are neighbors. Because of this, it is hard to see how (9) could be satisfied by an $f(B)$ that depended on the structure of the set B. For this reason we confine ourselves to functions which only depend on the cardinality of the set. Then (9) would require

$$\frac{d}{dt} Ef(|\zeta_t^A \cap B|)\bigg|_{t=0} = \frac{d}{dt} Ef(|\xi_t^B \cap A|)\bigg|_{t=0}. \tag{11}$$

Suppose $|A \cap B| = 1$; then remembering $\zeta_0^A = A$, $\xi_0^B = B$, $|\zeta_0^A \cap B| = |\xi_0^B \cap A| = 1$. The changes in $|\zeta_t^A \cap B|$ and $|\xi_t^B \cap A|$ that can occur at $t = 0$ are

$$|\zeta_0^A \cap B| \overset{\lambda_1}{\to} 2, \ \overset{\lambda_2}{\to} 0,$$

$$|\xi_0^B \cap A| \overset{\lambda_3}{\to} 2, \ \overset{\lambda_4}{\to} 0,$$

for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Now consider the process on a "large" lattice. Let $A_1 = T(A)$ and $B_1 = T(B)$ be translations of A and B such that $A_1 \cap A = \varphi$, $A_1 \cap B = \varphi$, $B_1 \cap A = \varphi$ and $B_1 \cap B = \varphi$. Then, let $A_i = T^i(A)$ and $B_i = T^i(B)$, and suppose the lattice large enough so that the only members of the $2n$ sets A_i, B_i, $i = 1, \dots, n$, that intersect are the pairs (A_i, B_i), $i = 1, \dots, n$. Then, putting

$$A^n = \bigcup_{i=1}^n A_i, \qquad B^n = \bigcup_{i=1}^n B_i, \qquad |\zeta_0^{A^n} \cap B^n| = n \quad \text{and} \quad |\xi_0^{B^n} \cap A^n| = n.$$

The possible transitions at $t = 0$ are then

$$|\zeta_0^{A^n} \cap B^n| \overset{n\lambda_1}{\to} n + 1, \ \overset{n\lambda_2}{\to} n - 1,$$

$$|\xi_0^{B^n} \cap A^n| \overset{n\lambda_3}{\to} n + 1, \ \overset{n\lambda_4}{\to} n - 1.$$

Then (11) implies

$$\lambda_1[f(n+1)-f(n)]+\lambda_2[f(n-1)-f(n)]$$
$$=\lambda_3[f(n+1)-f(n)]+\lambda_4[f(n-1)-f(n)].$$

This is a linear difference equation with roots 1 and a, giving $f(n) = C_0 + C_1 a^n$. (In fact, with annihilating models, transitions $n \to n-2$ are possible.) When f is substituted into (9), the constants C_0 and C_1 are of no significance.

Thus we shall only consider duals of the form

$$E(a^{|\zeta_t^A \cap B|}) = E(a^{|\xi_t^B \cap A|}), \tag{12}$$

except when either ζ or ξ is an annihilating process (allowing $11 \to 00$), in which case it may be possible to have a mixture of two different a's.

4. Algebraic conditions for duality. Let $A, B \subset S$, so that $|A\rangle$ and $|B\rangle$ are basis states. We define C to be the universal flip operator which flips the occupancy at every site, that is, changes 0's to 1's and 1's to 0's. Then if $|B\rangle$ is the basis vector corresponding to the set of occupied sites B, $C|B\rangle = |B^c\rangle$, where complements are wrt G.

We define the operator

$$U = \prod_{i \in G}(1 + S_i^- + aS_i^+). \tag{13}$$

Now suppose A and B are configurations with corresponding basis vectors $|A\rangle$ and $|B\rangle$. We designate the transpose of $|A\rangle$ as the "bra" vector $\langle A|$ and the usual scalar product is defined by $\langle A|B\rangle = \delta_{AB}$. Suppose now that $A(j) = B(j)$, $j \neq i$, that is, that the configurations A and B agree everywhere except possibly at i. Then

$$\langle A|(1 + S_i^- + aS_i^+)C_i|B\rangle = \begin{cases} a, & \text{if } A(i) = B(i) = 1, \\ 1, & \text{otherwise.} \end{cases} \tag{14}$$

Thus, at every site occupied in both A and B, the operator scores a, whereas all other sites score 1.

From (14) we see that if A and B are basic vectors,

$$\langle A|UC|B\rangle = a^{|B \cap A|}$$

and thus

$$\begin{aligned} E(a^{|\zeta_t^A \cap B|}) &= E(\langle \zeta_t^A|UC|B\rangle) \\ &= \langle A|\exp(Q_\zeta^T t)UC|B\rangle, \end{aligned} \tag{15}$$

since $|\zeta_t^A\rangle = \exp(Q_\zeta t)|A\rangle$ by (8).

The dual ξ has Q-matrix Q_ξ, so the equation (12) expressing duality becomes

$$\begin{aligned} \langle A|\exp(Q_\zeta^T t)UC|B\rangle &= \langle B|\exp(Q_\xi^T t)UC|A\rangle \\ &= \langle A|C^T U^T \exp(Q_\xi t)|B\rangle. \end{aligned}$$

Since we wish this equation to be true for arbitrary $|A\rangle$ and $|B\rangle$, we require

$$\exp\left(Q_\zeta^T t\right) UC = C^T U^T \exp\left(Q_\xi t\right). \tag{16}$$

At $t = 0$, this requires UC to be symmetric; C is symmetric and it is simple to check from (3) that $S^+ C = n$ and $S^- C = \bar{n}$; n and $\bar{n}$ are symmetric, so UC is also.

For (16) to be true for all t, we require

$$Q_\zeta^T UC = C^T U^T Q_\zeta = UC Q_\xi. \tag{17}$$

We note that since UC is symmetric, so is the dual relationship between ζ and ξ.

When interactions only occur between neighboring sites, we have seen that Q is a sum of Q-matrices that only act on two sites. A sufficient condition for (17) to hold is that, for every neighboring pair (i, j), the two-site Q-matrices satisfy

$$Q_\zeta^T(ij) U_{ij} C_{ij} = U_{ij} C_{ij} Q_\xi(ij),$$

or

$$Q_\zeta^T(ij) U_{ij} = U_{ij}\left(C_{ij} Q_\xi(ij) C_{ij}\right) = U_{ij} \overline{Q}_\xi(ij), \tag{18}$$

where for any 4×4 matrix M, $\overline{M}$ is M with subscript 1 swapped with 4 and 2 swapped with 3.

We shall drop the ij subscript for the next part of the development. U has matrix form

$$\begin{pmatrix} 1 & a & a & a^2 \\ 1 & 1 & a & a \\ 1 & a & 1 & a \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since Q_ζ and Q_ξ are stochastic, Q_ζ^T has rows adding to 0 and $\overline{Q}_\xi$ columns adding to 0. Thus, if $\mathit{1}$ is the 4×4 matrix which has every element 1, $Q_\zeta^T \mathit{1} = 0 = \mathit{1}\overline{Q}_\xi$. We thus find (18) equivalent to

$$Q_\zeta^T(\mathit{1} - U) = (\mathit{1} - U)\overline{Q}_\xi,$$

where

$$\mathit{1} - U = (1 - a)\begin{pmatrix} \cdot & 1 & 1 & 1+a \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{19}$$

Now, if the transition rate from l to k in $Q_\zeta = \lambda_{kl}$ and in $Q_\xi = \mu_{kl}$, and we put $\lambda_{\cdot l} = \sum_{k \neq l} \lambda_{kl}$ and $\mu_{\cdot l} = \sum_{k \neq l} \mu_{kl}$, (19) implies

$$Q_\zeta^T \frac{(\mathit{1} - U)}{(1 - a)} = \begin{pmatrix} -\lambda_{\cdot 1} & \lambda_{21} & \lambda_{31} & \lambda_{41} \\ \lambda_{12} & -\lambda_{\cdot 2} & \lambda_{32} & \lambda_{42} \\ \lambda_{13} & \lambda_{23} & -\lambda_{\cdot 3} & \lambda_{43} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & -\lambda_{\cdot 4} \end{pmatrix} \begin{pmatrix} \cdot & 1 & 1 & 1+a \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} \cdot & -\lambda_{21} - \lambda_{41} & -\lambda_{31} - \lambda_{41} & -\lambda_{41} - a\lambda_{\cdot 1} \\ \cdot & \lambda_{12} + \lambda_{32} & -\lambda_{32} - \lambda_{42} & -\lambda_{42} + a\lambda_{12} \\ \cdot & -\lambda_{23} - \lambda_{43} & \lambda_{13} + \lambda_{23} & -\lambda_{43} + a\lambda_{13} \\ \cdot & \lambda_{14} + \lambda_{34} & \lambda_{14} + \lambda_{24} & a\lambda_{14} + \lambda_{\cdot 4} \end{pmatrix}.$$

Remembering $\overline{Q}_\xi$ is Q_ξ with $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$,

$$\frac{(1-U)}{1-a}\overline{Q}_\xi = \begin{pmatrix} \cdot & 1 & 1 & 1+a \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} -\mu_{\cdot 4} & \mu_{43} & \mu_{42} & \mu_{41} \\ \mu_{34} & -\mu_{\cdot 3} & \mu_{32} & \mu_{31} \\ \mu_{24} & \mu_{23} & -\mu_{\cdot 2} & \mu_{21} \\ \mu_{14} & \mu_{13} & \mu_{12} & -\mu_{\cdot 1} \end{pmatrix}$$

$$= \begin{pmatrix} -\mu_{\cdot 4} + a\mu_{14} & -\mu_{43} + a\mu_{13} & -\mu_{42} + a\mu_{12} & -\mu_{41} - a\mu_{\cdot 1} \\ \mu_{24} + \mu_{14} & \mu_{23} + \mu_{13} & -\mu_{32} - \mu_{42} & -\mu_{41} - \mu_{31} \\ \mu_{34} + \mu_{14} & -\mu_{23} - \mu_{43} & \mu_{32} + \mu_{12} & -\mu_{41} - \mu_{21} \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Equating first columns shows $\mu_{14} = \mu_{24} = \mu_{34} = 0$. Equating last rows shows $\lambda_{14} = \lambda_{24} = \lambda_{34} = 0$. Thus there is no "spontaneous creation." The unique solution is below. It should be noted that $a = 0$ represents the coalescing dual and $a = -1$ represents the annihilating dual (for simplicity the diagonal terms have been left out):

$$Q_\xi = \frac{1}{1-a}$$

$$(20) \quad \times \begin{pmatrix} & \begin{matrix} \lambda_{13} + \lambda_{23} \\ -\lambda_{32} - \lambda_{42} \\ +\lambda_{31} + \lambda_{41} \end{matrix} & \begin{matrix} \lambda_{12} + \lambda_{32} \\ -\lambda_{23} - \lambda_{43} \\ +\lambda_{21} + \lambda_{41} \end{matrix} & \cdot \\ \begin{matrix} \lambda_{43} + a\lambda_{42} - a\lambda_{21} \\ -a\lambda_{31} - (1+a)\lambda_{41} \\ -a\lambda_{13} - a^2\lambda_{12} \end{matrix} & & \begin{matrix} -a(\lambda_{12} + \lambda_{32}) \\ +\lambda_{23} + \lambda_{43} \\ -\lambda_{21} - \lambda_{41} \end{matrix} & \cdot \\ \begin{matrix} \lambda_{42} + a\lambda_{43} - a\lambda_{31} \\ -a\lambda_{21} - (1+a)\lambda_{41} \\ -a\lambda_{12} - a^2\lambda_{13} \end{matrix} & \begin{matrix} -a(\lambda_{13} + \lambda_{23}) \\ +\lambda_{32} + \lambda_{42} \\ -\lambda_{31} - \lambda_{41} \end{matrix} & & \cdot \\ \begin{matrix} a^2(\lambda_{12} + \lambda_{13}) \\ -a(\lambda_{42} + \lambda_{43} - \lambda_{21} - \lambda_{31}) \\ +(1+a)\lambda_{41} \end{matrix} & \begin{matrix} \lambda_{31} + \lambda_{41} \\ +a(\lambda_{13} + \lambda_{23}) \\ -a(\lambda_{32} + \lambda_{42}) \end{matrix} & \begin{matrix} \lambda_{21} + \lambda_{41} \\ +a(\lambda_{12} + \lambda_{32}) \\ -a(\lambda_{23} + \lambda_{43}) \end{matrix} & \cdot \end{pmatrix}.$$

These results have been derived for processes on a finite graph. They do, however, naturally extend to an infinite lattice when either A or B is finite.

Suppose ξ and ζ are processes on $G = \{0,1\}^{\mathbb{Z}^d}$. Fix t. Let A be finite and let ζ_t^A be a process which is a.s. finite. Essentially this requires that unoccupied sites cannot flip without interaction with an occupied site. We say that a site x interacts with A by time t if there is a sequence of times $0 < t_1 < \cdots < t_n < t$ and of sites $x = x_1, x_2, \dots, x_n \in A$ with $|x_i - x_{i+1}| = 1$, $i = 1, \dots, n-1$, such that $\xi(x_i)$ flips at time t_i.

If ξ has a maximum rate of interaction at a site, then the set of sites of G that can interact with A by time t is a.s. finite.

Let $R(\varepsilon)$, $A \subset R(\varepsilon) \subset G$, be a finite set with the property that the probability that any member of $G/R(\varepsilon)$ interacts with A by time t is less than ε. Couple a process $\xi_t^B(\varepsilon)$ on $R(\varepsilon)$ to ξ_t^B in such a way that whenever ξ_s^B, $s \le t$, flips at site $x \in R(\varepsilon)$ due to interaction with a set of sites $T \subset R(\varepsilon)$, $\xi_s^B(\varepsilon)$ also flips at site x if its values at x, T are identical to those of ξ_s^B. We then have

$$P\big(\xi_t^B \cap A \ne \xi_t^B(\varepsilon) \cap A\big) < \varepsilon.$$

Thus,

$$\left|E\big(a^{|\xi_t^B \cap A|}\big) - E\big(a^{|\xi_t^B \cap A(\varepsilon)|}\big)\right| < \varepsilon |a|^{|A|}.$$

Define $\xi_t^A(\varepsilon)$ in a similar way. Then, since $R(\varepsilon)$ is finite,

$$E\big(a^{|\zeta_t^A(\varepsilon) \cap B|}\big) = E\big(a^{|\xi_t^B(\varepsilon) \cap A|}\big).$$

As $\varepsilon \to 0$ the r.h.s. $\to E(a^{|\xi_t^B \cap A|})$. The equation above shows that the l.h.s. is uniformly bounded by $\max(1, |a|^{|A|})$, and since ξ_t^A is a.s. finite as $\varepsilon \to 0$, $R(\varepsilon) \to G$ and we obtain

$$E\big(a^{|\zeta_t^A \cap B|}\big) = E\big(a^{|\xi_t^B \cap A|}\big).$$

THEOREM 1. *Suppose G is a finite graph or* $\{0,1\}^{\mathbb{Z}^d}$. *Let* ζ *be an interacting particle system on G with the following transitions between neighboring sites*:

$$11 \underset{\lambda_{21}}{\overset{\lambda_{12}}{\leftrightarrows}} 10, \qquad 11 \underset{\lambda_{31}}{\overset{\lambda_{13}}{\leftrightarrows}} 01, \qquad 11 \overset{\lambda_{41}}{\rightarrow} 00,$$

$$10 \underset{\lambda_{32}}{\overset{\lambda_{23}}{\leftrightarrows}} 01, \qquad 10 \overset{\lambda_{42}}{\rightarrow} 00, \qquad 10 \overset{\lambda_{43}}{\rightarrow} 00.$$

Then if the transition rates given by (20) *are nonnegative and* ξ *is a nearest-neighbor interacting particle system on G with those transition rates*,

$$E\big(a^{|\zeta_t^A \cap B|}\big) = E\big(a^{|\xi_t^B \cap A|}\big)$$

for any sets A, B with either A or B finite.

When the transition rates are isotropic, that is, do not depend on the orientation of the interacting sites, (20) can be considerably simplified. Isotropy is equivalent to the transition rates being unaffected by swapping the labels 2 and 3, so that, for example, $\lambda_{12} = \lambda_{13}$, $\lambda_{23} = \lambda_{32}$ and so on.

Equation (20) then becomes

$$
Q_\xi = \frac{1}{1-a} \times \begin{pmatrix}
\cdot & \lambda_{12}+\lambda_{21}+\lambda_{41}-\lambda_{42} & \lambda_{12}+\lambda_{21}+\lambda_{41}-\lambda_{42} & \cdot \\
(1+a)(\lambda_{42}-\lambda_{41})-2a\lambda_{21}-a(1+a)\lambda_{12} & \cdot & (1-a)\lambda_{32}-a\lambda_{12}+\lambda_{42}-\lambda_{41}-\lambda_{21} & \cdot \\
(1+a)(\lambda_{42}-\lambda_{41})-2a\lambda_{21}-a(1+a)\lambda_{12} & (1-a)\lambda_{32}-a\lambda_{12}+\lambda_{42}-\lambda_{41}-\lambda_{21} & \cdot & \cdot \\
2a^2\lambda_{12}-2a(\lambda_{42}-\lambda_{21})+(1+a)\lambda_{41} & \lambda_{21}+\lambda_{41}+a\lambda_{12}-a\lambda_{42} & \lambda_{21}+\lambda_{41}+a\lambda_{12}-a\lambda_{42} & \cdot
\end{pmatrix}. \tag{21}
$$

5. Examples of dual processes. In these examples we shall give the Q-matrices of two-site interactions. We shall omit the diagonal terms. The columns must add to 0. As before, 0's have been replaced by ·'s.

The biased annihilating branching process (BABP) [introduced by Neuhauser and Sudbury (1993)].

$$
Q_\zeta = \begin{pmatrix} \cdot & \lambda & \lambda & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},
$$

$$
Q_\xi = \frac{1}{1-a}\begin{pmatrix} \cdot & 1+\lambda & 1+\lambda & \cdot \\ -2a-\lambda(a+a^2) & \cdot & -a\lambda-1 & \cdot \\ -2a-\lambda(a+a^2) & -a\lambda-1 & \cdot & \cdot \\ 2a^2\lambda+2a & 1+a\lambda & 1+a\lambda & \cdot \end{pmatrix}. \tag{22}
$$

The BABP is self-dual when $a = -1/\lambda$.

Annihilating/coalescing random walk (A/CRW). Particles perform independent random walks, but when they meet they coalesce with probability J_C or annihilate each other with probability J_A, $J_A + J_C = 1$:

$$
Q_\zeta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ J_C & \cdot & 1 & \cdot \\ J_C & 1 & \cdot & \cdot \\ 2J_A & \cdot & \cdot & \cdot \end{pmatrix},
$$

$$Q_\xi = \frac{1}{1-a}\begin{pmatrix} \cdot & J_C + 2J_A & J_C + 2J_A & \cdot \\ -2(a+J_A) & \cdot & -a-J_A & \cdot \\ -2(a+J_A) & -a-J_A & \cdot & \cdot \\ 2(a+J_A) & J_C+2J_A & J_C+2J_A & \cdot \end{pmatrix}.$$

This requires $a = -J_A$, giving the voter model (VM) with matrix

$$\begin{pmatrix} \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

It is well known that the VM and the CRW are coalescing duals ($a = -J_A = 0$) and that the VM and the ARW are annihilating duals ($a = -J_A = -1$).

Branching annihilating random walk (*BARW*) [*see Bramson and Gray* (1985)]

(23)

$$Q_\zeta = \begin{pmatrix} \cdot & \lambda & \lambda & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$Q_\xi = \frac{1}{1-a} \times \begin{pmatrix} \cdot & 2+\lambda & 2+\lambda & \cdot \\ -(a+a^2)\lambda - 2(a+1) & \cdot & -a(\lambda+1)-1 & \cdot \\ -(a+a^2)\lambda - 2(a+1) & -a(\lambda+1)-1 & \cdot & \cdot \\ 2a^2\lambda + 2(a+1) & 2+a\lambda & \cdot & \cdot \end{pmatrix}.$$

Thus the BARW is self-dual when $a = -2/\lambda$. This is believed to be a new dual.

Branching coalescing random walk (*BCRW*) [*see, e.g., Durrett* (1988) *and Sudbury* (1993)]

$$Q_\zeta = \begin{pmatrix} \cdot & \lambda & \lambda & \cdot \\ 1 & \cdot & \mu & \cdot \\ 1 & \mu & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

(24) $$Q_\xi = \frac{1}{1-a} \times \begin{pmatrix} & 1+\lambda & 1+\lambda & \cdot \\ -2a - \lambda a(1+a) & \cdot & -a\lambda - 1 + (1-a)\mu & \cdot \\ -2a - \lambda a(1+a) & -a\lambda - 1 + (1-a)\mu & \cdot & \cdot \\ 2a(a\lambda+1) & 1+a\lambda & 1+a\lambda & \cdot \end{pmatrix}.$$

When $a = -1/\lambda$, the BCRW is self-dual. When $a = 0$, we have

$$(25)\qquad Q_\xi = \begin{pmatrix} \cdot & 1+\lambda & 1+\lambda & \cdot \\ \cdot & \cdot & \mu-1 & \cdot \\ \cdot & \mu-1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

This is the biased voter model when $\mu = 1$. With $\mu > 1$, we have the biased voter model with neighbors allowed to swap positions at rate $\mu - 1$.

The exclusion process

$$Q_\zeta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad Q_\xi = \frac{1}{1-a}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & (1-a) & \cdot \\ \cdot & (1-a) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Thus the exclusion process is self-dual for all a.

Biased voter model (BVM)

$$Q_\zeta = \begin{pmatrix} \cdot & \lambda & \lambda & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix},$$

$$Q_\xi = \begin{pmatrix} \cdot & \lambda-1 & \lambda-1 & \cdot \\ (1+a)(1-a\lambda) & \cdot & 1-a\lambda & \cdot \\ (1+a)(1-a\lambda) & 1-a\lambda & \cdot & \cdot \\ 2a(a\lambda-1) & a(\lambda-1) & a(\lambda-1) & \cdot \end{pmatrix} \times \frac{1}{1-a}.$$

$a = 1/\lambda$ makes the BVM self-dual. This is the only dual so far discovered with $a > 0$.

6. Some results for the branching annihilating random walk (BARW). The BARW was described via its Q-matrix in Section 4. Particles perform independent random walks on a finite graph or $\mathbb{Z}^d$ and when they meet they annihilate each other. However, they are also able to branch onto or create a new particle at neighboring sites. The rate at which a particle moves to a particular neighboring site is 1 and the rate of branching is λ. It has been shown that the BARW survives when λ is large, but not when it is small in low dimensions. Survival always occurs in high dimensions.

If we designate the configuration for the BARW as η, we have

$$(26)\qquad E\left(\left(-\frac{2}{\lambda}\right)^{|\eta_t^B \cap A|}\right) = E\left(\left(-\frac{2}{\lambda}\right)^{|\eta_t^A \cap B|}\right).$$

As is usual in the theory of interacting particles, (26) is true for random sets. In this section we shall take B to be the product measure $\nu_{\lambda/(2+\lambda)}$ on the graph, but we shall abbreviate this to ν. Our aim is to show that η_t^ν has a limiting measure.

$$\text{If} \quad |\eta_t^A| = r \geq 0, \quad \text{then} \quad |\eta_t^A \cap \nu| \sim \text{Bin}(r, \lambda/(2+\lambda)).$$

Then

$$E\left(s^{|\eta_t^A \cap \nu|}\right) = \left(\frac{2}{2+\lambda} + \frac{\lambda}{2+\lambda}s\right)^r = 0 \quad \text{when } s = -\frac{2}{\lambda}.$$

This implies

$$E\left(\left(-\frac{2}{\lambda}\right)^{|\eta_t^A \cap \nu|}\right) = P\left(|\eta_t^A| = 0\right) = 1 - s_A(t), \tag{27}$$

where $s_A(t)$ means the probability that the system survives to time t with initial set A. Since $s_A(t)$ is nonincreasing in t, $s_A(t) \to s_A$, the probability of nonextinction. Equation (27) then implies that, for all finite sets A,

$$E\left(\left(-\frac{2}{\lambda}\right)^{|\eta_t^A \cap \nu|}\right) \to 1 - s_A \quad \text{as } t \to \infty. \tag{28}$$

We consider (27) for the special cases $A = \{1\}$ and $A = \{1,2\}$. We define $P_t(m) = P(\eta_t^\eta(1) = m)$ and $P_t(m,n) = P(\eta_t^\nu(1) = m,\ \eta_t^\nu(2) = n)$, $m, n = 0, 1$. Equation (27) then gives

$$-\frac{2}{\lambda}P_t(1) + \left(1 - P_t(1)\right) = 1 - s_{\{1\}}(t), \tag{29}$$

$$\frac{4}{\lambda^2}P_t(1,1) - \frac{2}{\lambda}\left[P_t(1,0) + P_t(0,1)\right] + P_t(0,0) = s_{\{1,2\}}(t). \tag{30}$$

Equation (29) gives

$$P_t(1) = \frac{\lambda}{2+\lambda}s_{\{1\}}(t) \to \frac{\lambda}{2+\lambda}s_{\{1\}}, \tag{31}$$

and since ν is translation and reflection invariant, expressing $P_t(1,1)$ and $P_t(0,0)$ in terms of $P_t(1)$ and $P_t(1,0)$, (29) and (30) give

$$\frac{2}{2+\lambda}P_t(1) - P_t(1,0) = \left(\frac{\lambda}{2+\lambda}\right)^2 \left[s_{\{1\}}(t) - s_{\{1,2\}}(t)\right].$$

Since a particle at 1 cannot be annihilated until it has produced a birth at a neighboring site, $s_{\{1\}}(t) - s_{\{1,2\}}(t) \to 0$ as $t \to \infty$, and so

$$P_t(1,0) \to \frac{2\lambda}{(2+\lambda)^2}s_{\{1\}}. \tag{32}$$

We now proceed by induction to show that $P(\eta_t^\nu(x_l) = i_l,\ l = 1, \dots, r)$, $i_l = 0$ or 1, also converges. Suppose this has been demonstrated for all sets of sites $x_1, \dots, x_m$, $m \leq r$. Now consider $P(\eta_t^\nu(x_l) = i_l,\ l = 1, \dots, r+1)$. Put $A = \{x_1, \dots, x_{r+1}\}$. Then from (27),

$$1 - s_A(t) = \sum_{\{i_1, \dots, i_{r+1})} P(\eta_t^\nu(x_l) = i_l, l = 1, \dots, r+1)\left(-\frac{2}{\lambda}\right)^{\Sigma i_l}. \tag{33}$$

Now the probabilities on the right-hand side of (33) can all be expressed in terms of $P(\eta_t^{\nu}(x_l) = 1,\ l = 1, \dots, r+1)$ and probabilities on subsets of A of size r. Since we have assumed that the latter probabilities converge as $t \to \infty$, it follows from (33) that $P(\eta_t^{\nu}(x_l) = 1,\ l = 1, \dots, r+1)$ converges, and thus that $P(\eta_t^{\nu}(x_l) = i_l,\ l = 1, \dots, r+1)$ converges for all possible values of i_l.

We thus have the following theorem.

THEOREM 2. *If η is a BARW on a finite set of $\mathbb{Z}^d$, $\eta_t^{\nu_{\lambda/(2+\lambda)}}$ converges to a limiting measure. Further,*

$$P\big(\eta_t^{\nu_{\lambda/(2+\lambda)}}(0) = 1\big) = \frac{\lambda}{2+\lambda} P\big(\eta_t^{\{0\}} \neq \varphi\big)$$
$$\to \frac{\lambda}{2+\lambda} P\big(\eta_\infty^{\{0\}} \neq \varphi\big),$$

where $P(\eta_\infty^{\{0\}} \neq \varphi)$ is the probability that the BARW starting with one particle does not die out. When this probability is 0, δ_φ is the limiting measure. Also,

$$P\big(\eta^{\nu_{\lambda/(2+\lambda)}}(1) = 1 | \eta_t^{\nu_{\lambda/(2+\lambda)}}(0) = 1\big) \to \frac{\lambda}{2+\lambda}.$$

It should be noted that $\nu_{\lambda/(2+\lambda)}$ is not an equilibrium measure.

7. Results for the biased annihilating branching process (BABP). The process, designated by β, allows breeding onto vacant neighboring sites at rate λ and murdering of neighbors at rate 1. The Q-matrix is given in (21); $\nu_{\lambda/(1+\lambda)}$ is an invariant measure for this process on any graph. Neuhauser and Sudbury (1993) showed that on $\mathbb{Z}$ the only two invariant measures were $\nu_{\lambda/(1+\lambda)}$ and δ_φ. Mountford (1993) used their results to show that if the initial occupied set $A \subset \mathbb{Z}$ is finite and $\lambda > \frac{1}{3}$, then β_t^A tends to $\nu_{\lambda/(1+\lambda)}$. When $\lambda = 1$ on $\mathbb{Z}^d$, Sudbury (1990) and Bramson, Ding and Durrett (1991) showed that $\nu_{1/2}$ was the limiting measure. We explore now some consequences when the initial measure is ν_p, $0 < p < 1$.

Theorem 1 implies

$$E\big((-1/\lambda)^{|\beta_t^{\nu_p} \cap \{1\}|}\big) = E\big((-1/\lambda)^{|\beta_t^{\{1\}} \cap \nu_p|}\big), \tag{34}$$

When $|\beta_t^{\{1\}}| = r$, the r.h.s. of (34) equals $(1 - (1+\lambda)p/\lambda)^r$. Using the notation $P_t(1) = P(\beta_t^{\nu_p}(1) = 1)$ in the same manner as before, (34) gives

$$1 - \frac{1+\lambda}{\lambda} P_t(1) = \sum_{r=1}^{\infty} P\big(|\beta_t^{\{1\}}| = r\big)\left(1 - \frac{(1+\lambda)p}{\lambda}\right)^r, \tag{35}$$

$r = 0$ being impossible as the BABP cannot die out. When $1 - (1+\lambda)p/\lambda > 0$, the r.h.s. of (35) is less than $1 - (1+\lambda)p/\lambda$, giving $P_t(1) > p$.

When $0 > 1 - (1 + \lambda)p/\lambda > -1$, the first term on the r.h.s. is negative and thus the r.h.s. is less than $(1 - (1 + \lambda)p/\lambda)^2$, giving

$$P_t(1) > p\left[2 - \frac{(1 + \lambda)p}{\lambda}\right] > 0.$$

THEOREM 3.

$$P(\beta_t^{\nu_p}(1) = 1) > \begin{cases} p, & \text{for } 0 < p < \dfrac{\lambda}{1 + \lambda}, \\ p\left[2 - \dfrac{(1 + \lambda)p}{\lambda}\right], & \text{for } \dfrac{\lambda}{1 + \lambda} < p < \dfrac{2\lambda}{1 + \lambda}. \end{cases}$$

Theorem 3 gives results for arbitrarily small λ. It shows that for all λ the particle density remains bounded away from 0, even with arbitrarily small initial densities.

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