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RANDOM POLYTOPES AND THE EFRON–STEIN JACKKNIFE INEQUALITY

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Let K be a smooth convex body. The convex hull of independent random points in K is a random polytope. Estimates for the variance of the volume and the variance of the number of vertices of a random polytope are obtained. The essential step is the use of the Efron–Stein jackknife inequality for the variance of symmetric statistics. Consequences are strong laws of large numbers for the volume and the number of vertices of the random polytope. A conjecture of Bárány concerning random and best-approximation of convex bodies is confirmed. Analogous results for random polytopes with vertices on the boundary of the convex body are given.

1. Introduction and main results. Let $\mathcal{K}_+^k$ be the set of compact convex sets $K \in \mathbb{R}^d$, $d \geq 2$, with nonempty interior, with boundary of differentiability class $\mathcal{C}^k$, and with positive Gaussian curvature. Fix $K \in \mathcal{K}_+^2$, and choose points $X_1, \dots, X_n$ from K, randomly, independently and according to the uniform distribution. We call the convex hull $K_n = \operatorname{conv}[X_1, \dots, X_n]$ a random polytope. Denote by $N(K_n)$ the number of vertices of K_n, and by $V(K_n)$ the volume of K_n. It is of interest to determine the asymptotic behavior of $N(K_n)$ which tends to infinity with probability 1 as $n \to \infty$, and to specify how fast $V(K_n)$ tends to $V(K)$ as $n \to \infty$. We will always assume that $V(K) = 1$.

Most of the known results about $N(K_n)$ and $V(K_n)$ concern the expected volume $\mathbb{E}V(K_n)$, the case $d = 2$, and n fixed. The question to determine this value for $d = 2$ and $n = 3$ was raised by Sylvester in 1864 and so became known as Sylvester's problem. Out of a large number of contributions we only mention the work of Blaschke [8], Dalla and Larman [17], Giannopoulos [21], Buchta [9] and Buchta and Reitzner [12], all dealing with the case $d = 2$ and, for $d \geq 3$, Groemer [24, 25], Kingman [33], Affentranger [1] and Buchta and Reitzner [13].

Starting with two classical articles by Rényi and Sulanke [39, 40], the investigations focused on the asymptotic behavior of $\mathbb{E}V(K_n)$ as n tends to infinity: Rényi and Sulanke determined the asymptotic behavior of $V(K) - \mathbb{E}V(K_n)$ and $\mathbb{E}N(K_n)$ in the case $d = 2$ for smooth K (and also for polygons). This was generalized by Wieacker [51] ($d = 3$, $K \in \mathcal{K}_+^3$), Bárány [3] (arbitrary d,

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$K \in \mathcal{K}_+^3$) and Schütt [46] (arbitrary d, general convex bodies) to

$$\lim_{n\to\infty} \big(V(K) - \mathbb{E}V(K_n)\big) n^{2/(d+1)} = \Gamma_d \Omega(K), \tag{1}$$

where the constant Γ_d only depends on the dimension and is known explicitly (cf., e.g., [46]):

$$\Gamma_d = \frac{1}{2}\left(\frac{\kappa_{d-1}}{d+1}\right)^{-2/(d+1)} \frac{(d^2+d+2)(d^2+1)}{(d+3)(d+1)!} \Gamma\left(\frac{d^2+1}{d+1}\right), \tag{2}$$

with κ_d the volume of the d-dimensional unit ball, and $\Omega(K)$ denotes the affine surface area of K,

$$\Omega(K) = \int_{\partial K} \kappa(x)^{1/(d+1)}\, dx,$$

where $\kappa(x)$ is the Gaussian curvature of ∂K at x. (For the definition of $\Omega(K)$ for general, nonsmooth convex bodies using the floating body, see [47].) For an analogous result for polytopes we refer to [5].

In contrast to these results concerning the first moment of $V(K_n)$ it turned out to be extremely difficult to deduce more information about the random variable $V(K_n)$ itself. Already the question to deduce the asymptotic behavior of the variance of $V(K_n)$ and to establish limit laws seems to be highly nontrivial and out of reach in most cases. In the planar case, Schneider [42] proved a strong law of large numbers for $V(K_n)$ if K is smooth; and Cabo and Groeneboom [15] determined the asymptotic behavior of $\operatorname{Var} V(K_n)$ for convex polygons and proved a central limit theorem, but the stated asymptotic value for the variance appears to be incorrect (see [32], page 111) and for a corrected version Buchta [11]. Only in the special case that K is the unit ball further results are available: for all $d \geq 1$ Kuefer [34] gave the estimate

$$\operatorname{Var} V(K_n) \leq c(B^d) n^{-(d+3)/(d+1)}$$

with a positive constant $c(B^d)$, and Hsing [31] used an analogous estimate for proving a central limit theorem in the case $d = 2$.

In this paper we prove estimates for $\operatorname{Var} V(K_n)$ [and $\operatorname{Var} N(K_n)$] for all convex bodies $K \in \mathcal{K}_+^2$ in arbitrary dimensions which in turn imply strong laws of large numbers. The essential step in the proofs is the use of the Efron–Stein jackknife inequality [20] for the variance of symmetric statistics. This inequality is formulated in Section 3.2. It turns out to be a powerful tool for estimating the variance of functionals $f(\cdot)$ of random polytopes since in this context a version of the Efron–Stein jackknife inequality reads as follows:

$$\operatorname{Var} f(K_n) \leq (n+1)\mathbb{E}\big(f(K_n) - f(K_{n+1})\big)^2.$$

Hence by choosing an additional random point X_{n+1} in K and investigating the difference between K_n and $K_{n+1} = \operatorname{conv}[K_n, X_{n+1}]$, we obtain an upper bound for the variance of $f(K_n)$. As will be shown below, in many cases this upper bound is best possible up to a constant. Thus, using this version of the Efron–Stein jackknife inequality, rather elementary and short proofs yield rather sharp estimates for the variance.

THEOREM 1. *Let $K \in \mathcal{K}_+^2$ and choose n random points in K independently and according to the uniform distribution. Then there is a positive constant $c(K)$ depending on K such that*

$$\operatorname{Var} V(K_n) \le c(K) n^{-(d+3)/(d+1)}. \tag{3}$$

We mention a recent result of Buchta ([11], Corollary 2) who showed that for $d = 2$ and $K \in \mathcal{K}_+^6$ there is a constant $\underline{c}(K) > 0$ depending on K such that

$$\operatorname{Var} V(K_n) \ge \underline{c}(K) n^{-5/3}. \tag{4}$$

This indicates that, at least in dimension two, estimate (3) is best possible up to the constant. We *conjecture* that for convex bodies $K \in \mathcal{K}_+^2$ the normalized variance $\operatorname{Var} V(K_n) n^{(d+3)/(d+1)}$ tends to a positive constant (depending on K) as n tends to infinity, and thus, that the upper bound in Theorem 1 gives the right asymptotic order of $\operatorname{Var} V(K_n)$ in any dimension. Nevertheless, it seems likely that the constant $c(K)$ in Theorem 1 is far from the right asymptotic constant.

Theorem 1 can be used to deduce a strong law of large numbers for the random variable $V(K_n)$, thus generalizing Schneider's result [42] from $d = 2$ to arbitrary dimensions. Applying Chebyshev's inequality we almost immediately get:

THEOREM 2. *Let $K \in \mathcal{K}_+^2$, choose a sequence of random points X_i, $1 \le i < \infty$, in K independently and according to the uniform distribution, and let $K_n = \operatorname{conv}[X_1, \dots, X_n]$. Then*

$$\lim_{n\to\infty} \big(V(K) - V(K_n)\big) n^{2/(d+1)} = \Gamma_d \Omega(K) \tag{5}$$

with probability 1.

We turn our attention to the number of vertices of P_n. Using the identity

$$\mathbb{E} N(K_n) = n\big(V(K) - \mathbb{E} V(K_{n-1})\big) \tag{6}$$

due to Efron [18], the results concerning $\mathbb{E} V(K_n)$ can be used to determine the expected number of vertices of K_n: corresponding to (1),

$$\lim_{n\to\infty} \mathbb{E} N(K_n) n^{-(d-1)/(d+1)} = \Gamma_d \Omega(K). \tag{7}$$

By establishing a generalization of Efron's identity (6), Buchta ([11], Corollary 3) derived an estimate for the variance of $N(K_n)$ if $d \ge 4$ and $K \in \mathcal{K}_+^{d+6}$:

$$\operatorname{Var} N(K_n) \ge \underline{c}(K) n^{(d-1)/(d+1)}. \tag{8}$$

A straightforward modification of the proof of Theorem 1 (which will be sketched in Section 12) yields that $\mathbb{E}(V(K_{n+1}) - V(K_n))n^{(d+3)/(d+1)}$ tends to a positive constant as $n \to \infty$ for $K \in \mathcal{K}_+^2$. Together with Corollary 1 and Remark 4 in [11] this proves for all $K \in \mathcal{K}_+^2$ the existence of a constant $\underline{c}(K) > 0$ such that

$$\operatorname{Var} N(K_n) \geq \underline{c}(K) n^{(d-1)/(d+1)} \tag{9}$$

for $d \geq 4$. In the following theorem we establish an upper bound with the same order of magnitude.

THEOREM 3. *Let $K \in \mathcal{K}_+^2$ and choose n random points in K independently and according to the uniform distribution. Then there is a positive constant $c(K)$ depending on K such that*

$$\operatorname{Var} N(K_n) \leq c(K) n^{(d-1)/(d+1)}. \tag{10}$$

In the planar case, Groeneboom [26] determined the asymptotic behavior of $\operatorname{Var} N(K_n)$ if K is a convex polygon or the unit disc, and established a central limit theorem. By Groeneboom's result for the unit disc, by (9) and (10) we are led to *conjecture* that for convex bodies $K \in \mathcal{K}_+^2$ the normalized variance $\operatorname{Var} N(K_n) n^{-(d-1)/(d+1)}$ tends to a positive constant (depending on K) as n tends to infinity.

In the case $d \geq 4$, Theorem 3 implies a strong law of large numbers for $N(K_n)$. [The restriction to $d \geq 4$ comes from the fact that $N(K_n)$ is not necessarily a monotone function in n, see the discussion in Section 8.]

THEOREM 4. *Let $K \in \mathcal{K}_+^2$, $d \geq 4$, choose a sequence of random points X_i, $1 \leq i < \infty$, in K independently and according to the uniform distribution, and let $K_n = \operatorname{conv}[X_1, \ldots, X_n]$. Then*

$$\lim_{n \to \infty} N(K_n) n^{-(d-1)/(d+1)} = \Gamma_d \Omega(K) \tag{11}$$

with probability 1.

The interdependence between $N(K_n)$ and $V(K) - V(K_n)$ mentioned in (6) becomes even more interesting when one compares approximation of convex bodies by random polytopes to approximation of convex bodies by best-approximating inscribed polytopes. To this end one is interested in the minimum of the difference $V(K) - V(P)$ among all convex polytopes $P \subset K$ with at most N vertices. Let K_N^{best} be a polytope for which the minimum is attained. For convex bodies $K \in \mathcal{K}_+^2$ Gruber [27] proved

$$\lim_{N \to \infty} \bigl(V(K) - V(K_N^{\text{best}})\bigr) N^{2/(d-1)} = \tfrac{1}{2} \operatorname{del}_{d-1} \Omega(K)^{(d+1)/(d-1)}, \tag{12}$$

where del_{d-1} is a constant depending only on the dimension. It should be noted that in this context N is the number of vertices of the best-approximating polytope whereas in (5) n denotes the number of random points. Inspired by this difference Bárány [4] investigated the random variable $(V(K) - V(K_n))N(K_n)^{2/(d-1)}$ and proved that for $d = 2, 3$,

$$\lim_{n\to\infty} \mathbb{E}\big((V(K) - V(K_n))N(K_n)^{2/(d-1)}\big) = \Gamma_d{}^{(d+1)/(d-1)}\Omega(K)^{(d+1)/(d-1)}$$

as $n \to \infty$. Here we prove an even stronger result for the remaining cases $d \geq 4$. In fact, since convergence of random variables $X_n \to c_1$ and $Y_n \to c_2$ with probability 1 as $n \to \infty$ implies

$$X_n Y_n^{2/(d-1)} \to c_1 c_2^{2/(d-1)}$$

with probability 1 as $n \to \infty$, the following theorem is an immediate consequence of Theorems 2 and 4:

THEOREM 5. *Let $K \in \mathcal{K}_+^2$, $d \geq 4$, choose a sequence of random points X_i, $1 \leq i < \infty$, in K independently and according to the uniform distribution, and let $K_n = \operatorname{conv}[X_1, \dots, X_n]$. Then*

$$\lim_{n\to\infty} (V(K) - V(K_n))N(K_n)^{2/(d-1)} = \Gamma_d{}^{(d+1)/(d-1)}\Omega(K)^{(d+1)/(d-1)} \tag{13}$$

with probability 1.

Combining (12) and (13) proves that, for $K \in \mathcal{K}_+^2$,

$$\lim_{n\to\infty} \frac{V(K) - V(K_n)}{V(K) - V(K^{\mathrm{best}}_{N(K_n)})} = \frac{\Gamma_d{}^{(d+1)/(d-1)}}{\mathrm{del}_{d-1}/2}$$

with probability 1 independent of the convex body K. Thus it is of interest to compare the arising constants: whereas the constant Γ_d is given explicitly in (2) the constant del_{d-1} is explicitly known only for $d = 2$ and $d = 3$: $\mathrm{del}_1 = 1/6$ and $\mathrm{del}_2 = 1/(2\sqrt{3})$. This yields

$$\frac{\Gamma_2{}^3}{\mathrm{del}_1/2} \sim 5{,}885\ldots \qquad (d = 2), \qquad \frac{\Gamma_3{}^2}{\mathrm{del}_2/2} \sim 3{,}683\ldots \qquad (d = 3).$$

On the other hand, by Stirling's formula we have

$$\Gamma_d{}^{(d+1)/(d-1)} = \frac{1}{4\pi e} d + o(d)$$

and it was proved by Mankiewicz and Schütt [35] that

$$\frac{1}{2}\mathrm{del}_{d-1} = \frac{1}{4\pi e} d + o(d),$$

which proves

$$\frac{\Gamma_d{}^{(d+1)/(d-1)}}{\mathrm{del}_{d-1}/2} \to 1 \qquad \text{as } d \to \infty.$$

Thus, as the dimension tends to infinity, approximation of convex bodies by random polytopes in the interior of K is nearly as good as approximation of convex bodies by best-approximating inscribed polytopes.

A problem which is implicit in the proof of Theorem 1, and the solution of which is used explicitly in the proof of Theorem 3, is of independent interest. Consider the random polytope K_n and choose in addition a further independent random point X in K. The question one has to answer is the following: if $X \in K \setminus K_n$ how many facets of K_n can be seen from X? We show that the expected number of facets tends to a constant independent of n, and that also the second moment remains bounded as $n \to \infty$.

So denote by $F_n(X)$ the number of facets of K_n which can be seen from X, that is, which are [up to $(d-2)$-dimensional faces] contained in the interior of the convex hull of K_n and X. We set $F_n(X) = 0$ if X is contained in K_n.

THEOREM 6. *Let $K \in \mathcal{K}_+^2$ and choose random points $X_1, \dots, X_n, X$ in K independently and according to the uniform distribution. Then there is a positive constant* C_d *depending on the dimension such that*

$$\lim_{n\to\infty} \mathbb{E} F_n(X) n^{2/(d+1)} = \mathrm{C}_d \Omega(K). \tag{14}$$

Further there is a positive constant $c(K)$ depending on K such that

$$\mathbb{E} F_n(X)^2 \le c(K) n^{-2/(d+1)}. \tag{15}$$

The constant C_d can be given explicitly.

By (14) the mean number of facets which can be seen from X tends to 0 if X is chosen uniformly in K. But by (1) the random point X is contained in $K \setminus K_n$, and thus $F_n(X) = 0$, with probability $\Gamma_d \Omega(K) n^{-2/(d+1)} + o(n^{-2/(d+1)})$ as n tends to infinity. Thus the first statement of Theorem 6 can be equivalently formulated as follows:

$$\lim_{n\to\infty} \mathbb{E}\big(F_n(X) | X \in K \setminus K_n\big) = \frac{\mathrm{C}_d}{\Gamma_d}, \tag{16}$$

which is independent of the convex body K.

The bounds for the variance of $V(K) - V(K_n)$ and $N(K_n)$ in Theorem 1 and Theorem 3 are by definition bounds for the difference between the second moment and the square of the first moment of $V(K) - V(K_n)$ and $N(K_n)$. As results for the second moments seem to be unknown so far, we state the following immediate consequences of (1) and (7) explicitly:

COROLLARY 7. *Let $K \in \mathcal{K}_+^2$ and choose n random points in K independently and according to the uniform distribution. Then*

$$\lim_{n\to\infty} \mathbb{E}\big(V(K) - V(K_n)\big)^2 n^{4/(d+1)} = {\Gamma_d}^2 \Omega(K)^2$$

and

$$\lim_{n\to\infty} \mathbb{E}N(K_n)^2 n^{-2(d-1)/(d+1)} = \Gamma_d{}^2 \Omega(K)^2.$$

By a refinement of (1) in the sense of asymptotic expansions proved in [37] we also get an asymptotic expansion for the second moment for $K \in \mathcal{K}_+^{k+1}$, $k \le d+2$. There are constants $a_j(K)$ depending on K such that

$$\begin{aligned}\mathbb{E}\bigl(V(K) - V(K_n)\bigr)^2 \\ &= a_4(K)n^{-4/(d+1)} + a_6(K)n^{-6/(d+1)} \\ &\quad + a_8(K)n^{-8/(d+1)} + \cdots + O\bigl(n^{-(k+1)/(d+1)}\bigr)\end{aligned}$$

as $n \to \infty$. The coefficient $a_4(K)$ coincides with the coefficient given in Corollary 7. An analogous result also holds for the number of vertices of K_n.

Clearly there is a large number of functionals of the random polytope K_n which are of high interest. The maybe most interesting generalizations of $N(K_n)$ and $V(K_n)$ are the number of k-dimensional faces $N_k(K_n)$ of K_n, $k = 0, \dots, d-1$, and the intrinsic volumes $V_i(K_n)$ of K_n, $i = 1, \dots, d$, where, for instance, $V_d = V$ is the volume, $2V_{d-1}$ the surface area and V_1 is a multiple of the mean width.

As for $V_i(K_n)$ the asymptotic behavior of its expectations for $i = 1, \dots, d$ was determined by Bárány [3] and Reitzner [37] if K is smooth, whereas the computation of $\mathbb{E}N_k(K_n)$ seems to be an open problem for $k = 1, \dots, d-3$. For $k = d-2$ and $k = d-1$ see [51]. Concerning higher moments in the case $d \ge 3$ only the quantity $V_1(K_n)$ was investigated if K_n is a random polytope in a ball: Schreiber [44] determined the asymptotic behavior of all higher moments of $V_1(K_n)$ and then [45] even proved a central limit theorem for $V_1(K_n)$.

The maybe most natural way to measure the distance between K_n and K is to use the Hausdorff distance $\delta(K_n, K)$ between them. Yet for $\delta(K_n, K)$ not much is known in general: Bárány [2] stated estimates for $\delta(K_n, K)$ for smooth K and all $d \ge 2$. In the case $d = 2$, Bingham, Bräker and Hsing [7] succeeded in proving a central limit theorem both for smooth convex bodies and for polygons.

We also want to mention the connection between random approximation of convex bodies and the floating body. This has been used in the papers by Bárány and Larman [6] and Bárány [2] to deduce estimates for $\mathbb{E}N_k(K_n)$, $\mathbb{E}V_i(K_n)$ and $\mathbb{E}\delta(K_n, K)$ in terms of volumes of floating bodies. In the planar case it was shown by Buchta and Reitzner [12] that this connection can be made explicit.

The paper is organized in the following way. In the next section we investigate random polytopes with vertices chosen on the boundary of a smooth convex body and give an estimate for the variance of the volume also in that case. Section 3 contains the tools and Sections 4–12 contain the proofs of our results.

Surveys on random polytopes and related questions are due to Buchta [10], Schneider [42, 43] and Weil and Wieacker [50]. For further results concerning best-approximating polytopes we refer to a recent survey article by Gruber [29] where approximation of convex bodies by random polytopes is compared to approximation by best-approximating polytopes. We also want to mention a paper of Steele [49] which, to the best of our knowledge, is the only work where the Efron–Stein jackknife inequality is used in geometric probability.

2. Random points on the boundary of smooth convex bodies. From the point of view of approximation, it is more suitable to choose the random points on the boundary of K than from its interior: the rate of convergence of the volume of the random polytope to the volume of K increases from $n^{-2/(d+1)}$ [cf. (1)] to $n^{-2/(d-1)}$ [cf. (17)], which is the same rate of convergence as for best-approximating polytopes [cf. (12)]. Clearly this is due to the fact, that now by construction each random point is on ∂K and thus is a vertex of the random polytope.

So denote by d_K a positive continuous density function on the boundary of K, and choose n random points $X_1, \dots, X_n$ on the boundary of K independently and according to d_K. Denote by K_n^{bd} the convex hull of $X_1, \dots, X_n$. Then for $K \in \mathcal{K}_+^2$

$$\lim_{n\to\infty} \big(V(K) - \mathbb{E}V(K_n^{\mathrm{bd}})\big) n^{2/(d-1)} = \Gamma_d^{\mathrm{bd}} \Omega_{d_K}(K), \tag{17}$$

where

$$\Omega_{d_K}(K) = \int_{\partial K} d_K(x)^{-2/(d-1)} \kappa(x)^{1/(d-1)}\, dx,$$

and where the constant Γ_d^{bd} only depends on the dimension. This result is due to Schneider [42] (for $d = 2$) and Schütt and Werner [48] (for arbitrary d and even more general convex bodies); compare also [38] for an alternative proof. Schneider proves an even stronger result: let $d = 2$ and $K \in \mathcal{K}_+^2$, choose a sequence of random points X_i, $1 \le i < \infty$, and let $K_n = \mathrm{conv}[X_1, \dots, X_n]$. Then

$$\lim_{n\to\infty} \big(V(K) - V(K_n^{\mathrm{bd}})\big) n^2 = \Gamma_2^{\mathrm{bd}} \Omega_{d_K}(K)$$

with probability 1. By giving an estimate for the variance of $V(K_n^{\mathrm{bd}})$ it is possible to generalize this result to arbitrary dimensions.

THEOREM 8. *Let $K \in \mathcal{K}_+^2$ and choose n random points on ∂K independently and according to a positive continuous density function d_K. Then there is a positive constant $c(K, d_K)$ depending on K and d_K such that*

$$\operatorname{Var} V(K_n^{\mathrm{bd}}) \le c(K, d_K) n^{-1-4/(d-1)}. \tag{18}$$

Theorem 8 and Chebyshev's inequality yield

$$\mathbb{P}\Big(\big|\big(V(K) - V(K_n^{\mathrm{bd}})\big) - \mathbb{E}\big(V(K) - V(K_n^{\mathrm{bd}})\big)\big| n^{2/(d-1)} \ge \varepsilon\Big) \le c(K, d_K)\varepsilon^{-2} n^{-1}$$

for $d \ge 2$. We thus see that the probabilities

$$\mathbb{P}\Big(\big|\big(V(K) - V\big(K_{n_k}^{\mathrm{bd}}\big)\big) - \mathbb{E}\big(V(K) - V\big(K_{n_k}^{\mathrm{bd}}\big)\big)\big| n_k^{2/(d+1)} \ge \varepsilon\Big)$$

are summable for $n_k = k^2$. By the Borel–Cantelli lemma this implies that the random variable $(V(K) - V(K_{n_k}^{\mathrm{bd}}))n_k{}^{2/(d-1)}$ tends to $\Gamma_d^{\mathrm{bd}}\Omega_{d_K}(K)$ with probability 1 as n tends to infinity. Since $V(K) - V(K_n^{\mathrm{bd}})$ is decreasing this proves:

COROLLARY 9. *Let $K \in \mathcal{K}_+^2$, choose a sequence of random points X_i, $1 \le i < \infty$, on ∂K independently and according to a positive continuous density function d_K, and let $K_n^{\mathrm{bd}} = \mathrm{conv}[X_1, \dots, X_n]$. Then*

$$\lim_{n\to\infty}\bigl(V(K) - V(K_n^{\mathrm{bd}})\bigr)n^{2/(d-1)} = \Gamma_d^{\mathrm{bd}}\Omega_{d_K}(K) \tag{19}$$

with probability 1.

Note that for $d_K(x) = \kappa(x)^{1/(d+1)}\Omega(K)^{-1}$ we have with probability 1

$$\lim_{n\to\infty}\bigl(V(K) - V(K_n^{\mathrm{bd}})\bigr)n^{2/(d-1)} = \Gamma_d^{\mathrm{bd}}\Omega(K)^{(d+1)/(d-1)}.$$

This result should be compared to the approximation of smooth convex bodies by best-approximating polytopes (12). In particular, by the work of Schütt and Werner [48] and Mankiewicz and Schütt [35] it follows that $\Gamma_d^{\mathrm{bd}}/(\frac{1}{2}\mathrm{del}_{d-1}) \to 1$ as the dimension tends to infinity.

The result of Theorem 6 can be stated much more clearly for random points on the boundary of K. This is due to the fact that an additional random point chosen on the boundary of K is contained in $K \setminus K_n^{\mathrm{bd}}$ with probability 1. So choose n random points and thus the random polytope K_n^{bd}, choose another random point X according to the density function d_K, and denote by $F_n(X)$ the number of facets of K_n^{bd} which can be seen from X. The following theorem states, without the additional remark leading to (16), that the expectation of $F_n(X)$ tends to a positive constant as n tends to infinity, and that the second moment remains bounded.

THEOREM 10. *Let $K \in \mathcal{K}_+^2$ and choose random points $X_1, \dots, X_n, X$ on ∂K independently and according to a positive continuous density function d_K. Then there is a positive constant $\mathrm{C}_d^{\mathrm{bd}}$ depending on the dimension such that*

$$\lim_{n\to\infty}\mathbb{E}F_n(X) = \mathrm{C}_d^{\mathrm{bd}} \tag{20}$$

and there is a positive constant $c(K, d_K)$ depending on K and on d_K such that

$$\mathbb{E}F_n(X)^2 \le c(K, d_K). \tag{21}$$

Note that $\mathrm{C}_d^{\mathrm{bd}}$ only depends on the dimension, and thus (20) is independent of K and, in particular, independent of the density function d_K.

For more information on convex hulls of random points chosen on the boundary of smooth convex bodies we refer to [14, 23, 28, 36, 38]. Their results should be compared to results on best-approximation due to Gruber [27] and Glasauer and Gruber [22].

3. Tools.

3.1. *The first tool.* The first tool is a precise description of the local behavior of the boundary of a smooth convex body. Fix $K \in \mathcal{K}_+^2$. At every boundary point x of K there is a paraboloid $Q_2^{(x)}$—given by a quadratic form $b_2^{(x)}$—osculating ∂K at x. $Q_2^{(x)}$ and $b_2^{(x)}$ can be defined in the following way: identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. Then there is a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y = (y^1, \dots, y^{d-1}) \in \mathbb{R}^{d-1}$ representing ∂K in a neighborhood of x, that is, $(y, f^{(x)}(y)) \in \partial K$. Denote by $f_{ij}^{(x)}(0)$ the second partial derivatives of $f^{(x)}$ at the origin. Then

$$b_2^{(x)}(y) := \tfrac{1}{2} \sum_{i,j} f_{ij}^{(x)}(0) y^i y^j$$

and

$$Q_2^{(x)} := \{(y, z) \mid z \geq b_2^{(x)}(y)\}.$$

The essential point in the following lemma is the fact that these paraboloids approximate the boundary of K uniformly for all $x \in \partial K$.

LEMMA 11. *Let $K \in \mathcal{K}_+^2$ be given. Choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$ only depending on δ and K, such that for each boundary point x of K the following holds: identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. The λ-neighborhood U^λ of x in ∂K defined by $\operatorname{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda B^{d-1}$ can be represented by a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y \in \lambda B^{d-1}$. Furthermore,*

$$(1+\delta)^{-1} b_2^{(x)}(y) \leq f^{(x)}(y) \leq (1+\delta) b_2^{(x)}(y), \tag{22}$$

$$\sqrt{1 + |\operatorname{grad} f^{(x)}(y)|^2} \leq (1+\delta) \tag{23}$$

and

$$(1+\delta)^{-1} 2 b_2^{(x)}(y) \leq (y, 0) \cdot n_K(y) \leq (1+\delta)\, 2 b_2^{(x)}(y) \tag{24}$$

for $y \in \lambda B^{d-1}$, where $n_K(y)$ is the outer unit normal vector of K at the boundary point $(y, f^{(x)}(y))$.

The proof of this lemma is contained in [38].

In particular, fix some δ_1 with $0 < \delta_1 \leq 1$ and denote by λ_1 the corresponding λ and define

$$h_1 := \tfrac{1}{2} \lambda_1^2 \min_{x \in \partial K, v \in S^{d-1}} b_2^{(x)}(v) > 0. \tag{25}$$

Let the cap $K \cap H_+$ be the intersection of K with a half space H_+ and define the height of the cap $K \cap H_+$ as the maximal distance of the points in $K \cap H_+$ to the hyperplane ∂H_+. Then the boundary of each cap of K with height less than h_1 can be represented by a convex function fulfilling (22) and (23) with $\delta = \delta_1$ and $\lambda = \lambda_1$.

3.2. *The second tool.* The second tool is the Efron–Stein jackknife inequality [20]; see also [19, 30].

If $S = S(Y_1, \dots, Y_n)$ is any real symmetric function of the independent identically distributed random vectors Y_j, $1 \le j < \infty$, we set $S_i = S(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_{n+1})$ and $S_{(\cdot)} = \frac{1}{n+1} \sum_{i=1}^{n+1} S_i$. The Efron–Stein jackknife inequality then says

$$\operatorname{Var} S \le \mathbb{E} \sum_{i=1}^{n+1} (S_i - S_{(\cdot)})^2 = (n+1)\mathbb{E}(S_{n+1} - S_{(\cdot)})^2. \tag{26}$$

Note that the right-hand side is not decreased if $S_{(\cdot)}$ is replaced by any other function of $Y_1, \dots, Y_{n+1}$.

We apply this inequality to the random variable $f(K_n)$ where $f(\cdot)$ is a function of the random polytope. Then $S_{n+1} = f(\operatorname{conv}[X_1, \dots, X_n]) = f(K_n)$, and we replace $S_{(\cdot)}$ by $f(K_{n+1})$ which is a function of the convex hull of K_n and a further random point X_{n+1}.

In the case that $f(\cdot)$ is the volume of the random polytope (26) implies

$$\operatorname{Var} V(K_n) \le (n+1)\mathbb{E}\big(V(K_{n+1}) - V(K_n)\big)^2, \tag{27}$$

and if $f(\cdot)$ denotes the number of vertices of K_n we obtain

$$\operatorname{Var} N(K_n) \le (n+1)\mathbb{E}\big(N(K_{n+1}) - N(K_n)\big)^2. \tag{28}$$

4. Proof of Theorem 1. Let $K \in \mathcal{K}_+^2$ be given, and assume that $V(K) = 1$. We have to prove that $\mathbb{E}(V(K_{n+1}) - V(K_n))^2$ is at least of order $n^{-1-(d+3)/(d+1)}$.

Choose random points $X_1, \dots, X_n$ in K independently and according to the uniform distribution. First we exclude those cases where the Hausdorff distance of K_n to K is greater than ε_K with

$$\varepsilon_K := \frac{1}{144} \min_{x \in \partial K, v \in S^{d-1}} b_2(v) \Big(\max_{x \in \partial K, v \in S^{d-1}} b_2(v) \Big)^{-1} h_1, \tag{29}$$

where h_1 is defined in (25). If $\delta(K_n, K) > \varepsilon_K$, then there exists a facet of K_n whose affine hull cuts off a cap of height at least ε_K from K and which contains no further random point. Since the volume of all caps of height at least ε_K is bounded from below by a positive constant $c_K > 0$, it is immediate that

$$\mathbb{P}\big(\delta(K_n, K) > \varepsilon_K\big) \le \binom{n}{d}(1 - c_K)^{n-d} = O\big(n^d (1 - c_K)^n\big).$$

Hence for computing $\operatorname{Var} V(K_n)$ we assume that $\delta(K_n, K) < \varepsilon_K$ and add an error term $O(n^d(1-c_K)^n)$. In particular, we assume that the origin is contained in K_n.

For $\mathrm{I} = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ denote by F_I the convex hull of $X_{i_1}, \dots, X_{i_d}$ which is a $(d-1)$-dimensional simplex. The affine hull of F_I is denoted by $H(F_\mathrm{I})$. This hyperplane dissects $\mathbb{R}^d$ into two (closed) half spaces, and we denote that half space which contains the origin by $H_0(F_\mathrm{I})$, and the other by $H_+(F_\mathrm{I})$. (The origin is contained in exactly one half space with probability 1.)

Observe that $K_{n+1} \backslash K_n$ is either empty (if $X_{n+1} \in K_n$) or consists of several simplices which are the convex hull of X_{n+1} and those facets of K_n which can be seen from X_{n+1}. Denote the set of these facets by $\mathcal{F}(X_{n+1})$, that is,

$$\begin{aligned} &\mathcal{F}(X_{n+1}) \\ &\quad = \mathcal{F}(X_1, \dots, X_n; X_{n+1}) \\ &\quad = \bigl\{F_\mathrm{I} : K_n \subset H_0(F_\mathrm{I}), X_{n+1} \in H_+(F_\mathrm{I}), \mathrm{I} = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}\bigr\}. \end{aligned} \tag{30}$$

Then

$$\begin{aligned} &\mathbb{E}\bigl(V(K_{n+1}) - V(K_n)\bigr)^2 \\ &\quad = \int_K \cdots \int_K \left(\sum_{F \in \mathcal{F}(X_{n+1})} V(\operatorname{conv}[F, X_{n+1}]) \right)^2 dX_1 \cdots dX_n \, dX_{n+1} \\ &\quad \le \int_K \cdots \int_K \left(\sum_\mathrm{I} I(F_\mathrm{I} \in \mathcal{F}(X_{n+1})) V_+(F_\mathrm{I}) \right)^2 dX_1 \cdots dX_n \, dX_{n+1} \\ &\qquad + O\bigl(n^d(1-c_K)^n\bigr), \end{aligned} \tag{31}$$

where $V_+(F_\mathrm{I})$ denotes the volume of $K \cap H_+(F_\mathrm{I})$, the summation extends over all subsets $\mathrm{I} = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$, and the integration over all n-tuples of points in K such that $\delta(K_n, K) < \varepsilon_K$.

In the first step we expand the integrand:

$$\begin{aligned} &\le \sum_\mathrm{I} \sum_\mathrm{J} \int_K \cdots \int_K I\bigl(F_\mathrm{I} \in \mathcal{F}(X_{n+1})\bigr) V_+(F_\mathrm{I}) \\ &\qquad\qquad \times I\bigl(F_\mathrm{J} \in \mathcal{F}(X_{n+1})\bigr) V_+(F_\mathrm{J}) \, dX_1 \cdots dX_n \, dX_{n+1} \\ &\quad + O\bigl(n^d(1-c_K)^n\bigr), \end{aligned} \tag{32}$$

where the summation extends over all subsets $\mathrm{I} = \{i_1, \dots, i_d\}$ and $\mathrm{J} = \{j_1, \dots, j_d\}$ of $\{1, \dots, n\}$.

The summation determines two subsets $\{i_1, \dots, i_d\}$ and $\{j_1, \dots, j_d\}$ of $\{1, \dots, n\}$ that may have nonempty intersection. If we fix the number of common integers, say k, then the corresponding term in (32) is independent of the choice of $i_1, \dots, i_d$ and $j_1, \dots, j_d$. For given $k \in \{0, \dots, d\}$ denote by F_1 the convex hull of $X_1, \dots, X_d$ and by F_2 the convex hull of $X_{d-k+1}, \dots, X_{2d-k}$. Then we obtain

$$\begin{aligned}
&\leq \sum_{k=0}^{d} \binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\qquad \times \int_K \cdots \int_K I\big(F_1 \in \mathcal{F}(X_{n+1})\big) V_+(F_1) \\
&\qquad\qquad \times I\big(F_2 \in \mathcal{F}(X_{n+1})\big) V_+(F_2)\, dX_1 \cdots dX_n\, dX_{n+1} \\
&\quad + O\big(n^d (1-c_K)^n\big).
\end{aligned}$$

(Note that for $k = d$ we have $F_1 = F_2$.) Since the integrand is a symmetric function of the two sets of random points $(X_1, \dots, X_{d-k})$ and $(X_{d+1}, \dots, X_{2d-k})$, we restrict our integration to those pairs of facets where the diameter of $K \cap H(F_2)$ is smaller than the diameter of $K \cap H(F_1)$:

$$\begin{aligned}
&\leq \sum_{k=0}^{d} 2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\qquad \times \int_K \cdots \int_K I\big(F_1 \in \mathcal{F}(X_{n+1})\big) V_+(F_1) I\big(F_2 \in \mathcal{F}(X_{n+1})\big) V_+(F_2) \\
&\qquad\qquad \times I\big(\mathrm{diam}(K \cap H(F_2)) < \mathrm{diam}(K \cap H(F_1))\big) \\
&\qquad\qquad \times\, dX_1 \cdots dX_n\, dX_{n+1} \\
&\quad + O\big(n^d (1-c_K)^n\big).
\end{aligned}$$

In the second step we estimate this expression by replacing $I(F_2 \in \mathcal{F}(X_{n+1}))$ by $I(H(F_2) \cap H(F_1) \neq \varnothing)$ since the sets $K \cap H_+(F_1)$ and $K \cap H_+(F_2)$ have at least the point X_{n+1} in common:

$$\begin{aligned}
&\leq \sum_{k=0}^{d} 2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\qquad \times \int_K \cdots \int_K I\big(F_1 \in \mathcal{F}(X_{n+1})\big) V_+(F_1) I\big(H(F_2) \cap H(F_1) \neq \varnothing\big) V_+(F_2) \\
&\qquad\qquad \times I\big(\mathrm{diam}(K \cap H(F_2)) < \mathrm{diam}(K \cap H(F_1))\big) \\
&\qquad\qquad \times\, dX_1 \cdots dX_n\, dX_{n+1} \\
&\quad + O\big(n^d (1-c_K)^n\big).
\end{aligned}$$

Because $F_1 \in \mathcal{F}(X_{n+1})$ if and only if the points $X_{2d-k+1}, \dots, X_n$ are contained in $H_0(F_1)$ and X_{n+1} is contained in $H_+(F_1)$, the integration with respect to

$X_{2d-k+1}, \dots, X_n, X_{n+1}$ yields

$$
\begin{aligned}
&\le \sum_{k=0}^{d} 2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\qquad \times \int_K \cdots \int_K (1 - V_+(F_1))^{n-2d+k} V_+(F_1)^2 I\big(H(F_2) \cap H(F_1) \ne \varnothing\big) V_+(F_2) \\
&\qquad\qquad \times I\big(\operatorname{diam}(K \cap H(F_2)) < \operatorname{diam}(K \cap H(F_1))\big)\, dX_1 \cdots dX_{2d-k} \\
&\quad + O\big(n^d (1 - c_K)^n\big).
\end{aligned}
$$

In the next step we investigate

$$
\begin{aligned}
&\int_K \cdots \int_K I\big(H(F_2) \cap H(F_1) \ne \varnothing\big) V_+(F_2) \\
&\qquad \times I\big(\operatorname{diam}(K \cap H(F_2)) < \operatorname{diam}(K \cap H(F_1))\big)\, dX_{d+1} \cdots dX_{2d-k}.
\end{aligned}
$$

Since the height $h(F_1)$ of the cap $K \cap H(F_1)$ is less than ε_K (and thus $< h_1$) we can apply Lemma 11 with $\delta = \delta_1$. To this end identify the hyperplane in $H_+(F_1)$ tangent to ∂K and parallel to $H(F_1)$ with $\mathbb{R}^{d-1}$. Then the boundary of K can be represented by a convex function $f(y)$ with $y \in \mathbb{R}^{d-1}$. We introduce polar coordinates: Set $y = rv$ with $r \in \mathbb{R}_+$ and $v \in S^{d-1}$. The representation (22) of the points $(rv, f(rv))$ in $\partial K \cap H(F_1)$ thus gives

$$
h(F_1) = f(rv) \ge \tfrac{1}{2} r^2 b_2(v)
$$

or, equivalently,

$$
r^2 \le 2h(F_1) b_2(v)^{-1},
$$

and thus the diameter of $K \cap H(F_1)$ is bounded from above:

$$
\operatorname{diam}\big(K \cap (H(F_1))\big) \le 2 \max_{v \in S^{d-1}} r \le 2^{3/2} h(F_1)^{1/2} \Big(\min_{v \in S^{d-1}} b_2(v)\Big)^{-1/2}.
$$

Since the diameter of $K \cap H(F_2)$ is bounded by the diameter of $K \cap H(F_1)$, it is clear that the projection of $K \cap H(F_2)$ onto $\mathbb{R}^{d-1}$ is contained in a ball with radius

$$
3 2^{3/2} h(F_1)^{1/2} \Big(\min_{v \in S^{d-1}} b_2(v)\Big)^{-1/2}.
$$

Using (22) we thus see that the maximal distance of $K \cap H(F_2)$ to $\mathbb{R}^{d-1}$ is bounded from above by

$$
\bar{c} h(F_1) = 144 \Big(\min_{v \in S^{d-1}} b_2(v)\Big)^{-1} \max_{v \in S^{d-1}} b_2(v) h(F_1) \qquad (< h_1),
$$

where $\bar{c}$ is independent of F_1 and F_2. This implies that the cap $K \cap H_+(F_2)$ and thus also the random points $X_{d+1}, \dots, X_{2d-k} \in K \cap H(F_2)$ are contained in a cap $K \cap H_+$ with height $\bar{c} h(F_1)$, where ∂H_+ is parallel to $H(F_1)$.

Hence

$$
\begin{aligned}
&\int_K \cdots \int_K I\big(H(F_2) \cap H(F_1) \neq \varnothing\big) V_+(F_2) \\
&\qquad\qquad \times I\big(\operatorname{diam}(K \cap H(F_2)) < \operatorname{diam}(K \cap H(F_1))\big)\, dX_{d+1} \cdots dX_{2d-k} \\
&\qquad \le \int_K \cdots \int_K I\big(X_{d+1} \cdots dX_{2d-k} \in K \cap H_+\big) V_+(H_+)\, dX_{d+1} \cdots dX_{2d-k} \qquad (33)\\
&\qquad \le V_+(H_+)^{d-k+1} \\
&\qquad \le \bar{c}^{\,d(d-k+1)} V_+(F_1)^{d-k+1}
\end{aligned}
$$

since $\bar{c} > 1$ and thus the cap $K \cap H_+$ is contained in the cap $\bar{c}(K \cap H_+(F_1))$.

The last step is easy:

$$
\begin{aligned}
&\mathbb{E}\big(V(K_{n+1}) - V(K_n)\big)^2 \\
&\quad \le 2\bar{c}^{\,d^2+d} \sum_{k=0}^{d} \binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\qquad\qquad \times \int_K \cdots \int_K \big(1 - V_+(F_1)\big)^{n-2d+k} V_+(F_1)^{d-k+3}\, dX_1 \cdots dX_d \\
&\qquad + O\big(n^d (1 - c_K)^n\big),
\end{aligned}
$$

where now we skip the assumption that F_1 is sufficiently close to the boundary of K. By Lemma 12 in Section 11 we see that each of the summands is bounded from above by

$$
\begin{aligned}
&\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k}\big\{c_d \Omega(K) n^{-2d+k-2-2/(d+1)} + o\big(n^{-2d+k-2-2/(d+1)}\big)\big\} \\
&\qquad \le c n^{-2-2/(d+1)}
\end{aligned}
$$

with a suitable constant c since the binomial coefficients are of order n^{2d-k}. Combined with the Efron–Stein jackknife inequality (27) this proves Theorem 1.

5. Proof of Theorem 2. Choose a sequence of random points X_i, $1 \le i < \infty$, in K independently and according to the uniform distribution, and let $K_n = \operatorname{conv}[X_1, \ldots, X_n]$. Then Chebyshev's inequality together with Theorem 1 yields

$$
\begin{aligned}
&\mathbb{P}\Big(\big|\big(V(K) - V(K_n)\big) - \mathbb{E}\big(V(K) - V(K_n)\big)\big| n^{2/(d+1)} \ge \varepsilon\Big) \\
&\qquad \le \varepsilon^{-2} n^{4/(d+1)} \operatorname{Var} V(K_n) \\
&\qquad \le c(K) \varepsilon^{-2} n^{-(d-1)/(d+1)}.
\end{aligned}
$$

Since the sum $\sum n_k^{-(d-1)/(d+1)}$ is finite for $n_k = k^4$ we thus see that the probabilities

$$
\mathbb{P}\Big(\big|\big(V(K) - V(K_{n_k})\big) - \mathbb{E}\big(V(K) - V(K_{n_k})\big)\big| n_k^{2/(d+1)} \ge \varepsilon\Big)
$$

are summable. By the Borel–Cantelli lemma and (1), this implies that

$$\lim_{k\to\infty}\big(V(K)-V(K_{n_k})\big){n_k}^{2/(d+1)}=\Gamma_d\Omega(K) \tag{34}$$

with probability 1. Since $V(K)-V(K_n)$ is decreasing,

$$\begin{aligned}&\big(V(K)-V(K_{n_k})\big){n_{k-1}}^{2/(d+1)}\\&\qquad\le\big(V(K)-V(K_n)\big)n^{2/(d+1)}\le\big(V(K)-V(K_{n_{k-1}})\big){n_k}^{2/(d+1)}\end{aligned}$$

for $n_{k-1}\le n\le n_k$, where by definition $n_{k+1}/n_k\to 1$, and thus the subsequence limit (34) suffices to prove Theorem 2.

6. Proof of Theorem 6. Choose n random points $X_1,\dots,X_n$ in K independently and according to the uniform distribution, and further another random point X in K. Using the notation introduced in (30) we have

$$\mathbb{E}F_n(X)=\int_K\cdots\int_K\sum I\big(F_{\mathrm{I}}\in\mathcal{F}(X)\big)\,dX_1\cdots dX_n\,dX,$$

where the summation extends over all subsets $\mathrm{I}=\{i_1,\dots,i_d\}$ of $\{1,\dots,n\}$. This integral is independent of the choice of $i_1,\dots,i_d$, and hence is equal to

$$\binom{n}{d}\int_K\cdots\int_K I\big(F_1\in\mathcal{F}(X)\big)\,dX_1\cdots dX_n\,dX,$$

where F_1 denotes the convex hull of $X_1,\dots,X_d$. Integrating with respect to $X,X_{d+1},\dots,X_n$ gives

$$=\binom{n}{d}\int_K\cdots\int_K(1-V_+(F_1))^{n-d}V_+(F_1)\,dX_1\cdots dX_d+O\big(n^d(1-c_K)^{-n}\big),$$

where we assumed that $\delta(K_n,K)<\varepsilon_K$ and thus that the origin is contained in K_n. By Lemma 12 this implies

$$=\binom{n}{d}\big\{c_d\Omega(K)n^{-d-2/(d+1)}+o\big(n^{-d-2/(d+1)}\big)\big\}+O\big(n^d(1-c_K)^{-n}\big)$$

as $n\to\infty$. As $\binom{n}{d}$ is of order n^d, this proves the first part of Theorem 6.

To prove the second part of Theorem 6 one has to investigate the integral

$$\mathbb{E}F_n(X)^2=\int_K\cdots\int_K\Big(\sum I\big(F_{\mathrm{I}}\in\mathcal{F}(X)\big)\Big)^2\,dX_1\cdots dX_n\,dX,$$

which is the same as (31) but where the term $V_+(F_{\mathrm{I}})$ is removed. A trivial modification of the proof of Theorem 1 thus gives the desired result.

7. Proof of Theorem 3. We have to investigate

$$\mathbb{E}\big(N(K_{n+1}) - N(K_n)\big)^2.$$

Let K_n be fixed. Now, if the additional random point X_{n+1} is contained in K_n, the variable $N(K_{n+1}) - N(K_n)$ equals to 0. But, for $X_{n+1} \notin K_n$ some of the vertices of K_n are contained in the interior of K_{n+1}, say $N^-(X_{n+1})$, and we have

$$N(K_{n+1}) - N(K_n) = 1 - N^-(X_{n+1}).$$

Note that since K_n is simplicial with probability 1, it follows that

$$N^-(X_{n+1}) \le dF_n(X_{n+1}).$$

Summarizing, we obtain

$$|N(K_{n+1}) - N(K_n)| \le (d+1)F_n(X_{n+1}).$$

By the Efron–Stein jackknife inequality (28) and by Theorem 6 this implies

$$\begin{aligned}\operatorname{Var} N(K_n) &\le (n+1)\mathbb{E}\big(N(K_{n+1}) - N(K_n)\big)^2\\ &\le (d+1)^2(n+1)\mathbb{E}F_n(X_{n+1})^2\\ &\le (d+1)^2 cn^{(d-1)/(d+1)}.\end{aligned}$$

8. Proof of Theorem 4. Choose a sequence of random points X_i, $1 \le i < \infty$, in K independently and according to the uniform distribution, and let $K_n = \operatorname{conv}[X_1, \dots, X_n]$. Then using Chebyshev's inequality together with the upper bound (10) yields

$$\mathbb{P}\big(|N(K_n) - \mathbb{E}N(K_n)|n^{-(d-1)/(d+1)} \ge \varepsilon\big) \le \varepsilon^{-2} n^{-2(d-1)/(d+1)} \operatorname{Var} N(K_n) \tag{35}$$

$$\le c(K)\varepsilon^{-2} n^{-(d-1)/(d+1)}. \tag{36}$$

We thus see that for $d \ge 4$ the probabilities $\mathbb{P}(|N(K_{n_k}) - \mathbb{E}N(K_{n_k})| \times {n_k}^{-(d-1)/(d+1)} \ge \varepsilon)$ are summable for $n_k = k^2$. This implies that

$$\lim_{k\to\infty} N(K_{n_k}){n_k}^{-(d-1)/(d+1)} = \Gamma_d \Omega(K)$$

with probability 1.

Since $N(K_n)$ is not necessarily a monotone function [observe that it can happen that $N(K_n) > N(K_{n+1})$ if more then one vertex of K_n is contained in the interior of K_{n+1}] it is not trivial to deduce the convergence of the sequence $N(K_n)$ from the convergence of the subsequence $N(K_{n_k})$. What helps is the observation that $N(K_n)$ increases at most by one if an additional point is added to K_n. So

$$N(K_{n_k}) - (n_k - n) \le N(K_n) \le N\big(K_{n_{k-1}}\big) + (n - n_{k-1})$$

for $n_{k-1} = (k-1)^2 \le n \le n_k = k^2$, which proves that $N(K_n)n^{-(d-1)/(d+1)}$ is bounded from below by

$$N(K_{n_k})n_k^{-(d-1)/(d+1)}\left(\frac{n_{k-1}}{n_k}\right)^{-(d-1)/(d+1)} - (n_k - n_{k-1})n_k{}^{-(d-1)/(d+1)}$$

and form above by

$$N(K_{n_{k-1}})n_{k-1}^{-(d-1)/(d+1)}\left(\frac{n_k}{n_{k-1}}\right)^{-(d-1)/(d+1)} + (n_k - n_{k-1})n_k{}^{-(d-1)/(d+1)}.$$

Since $n_{k+1}/n_k \to 1$, and $(n_k - n_{k-1})n_k{}^{-(d-1)/(d+1)} \to 0$ for $d \ge 4$ this proves Theorem 4.

9. Sketch of proof of Theorem 8. Analogously to the proof of Theorem 1 we restrict our investigations to random polytopes with $\delta(K_n^{\mathrm{bd}}, K) < \varepsilon_K$. Using the notation introduced in (30) we have

$$\begin{aligned}
&\mathbb{E}\big(V(K_{n+1}^{\mathrm{bd}}) - V(K_n^{\mathrm{bd}})\big)^2 \\
&\quad\le \int_{\partial K}\cdots\int_{\partial K}\left(\sum_{\mathrm{I}} I\big(F_{\mathrm{I}} \in \mathcal{F}(X_{n+1})\big)V_+(F_{\mathrm{I}})\right)^2 \\
&\qquad\qquad\times \prod_{m=1}^{n+1} d_K(X_m)\, dX_1 \cdots dX_n\, dX_{n+1} \\
&\qquad + O\big(n^d(1-c_K^{\mathrm{bd}})^n\big),
\end{aligned} \tag{37}$$

where the summation extends over all subsets $\mathrm{I} = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$ and the integration over all random points such that $\delta(K_n^{\mathrm{bd}}, K) < \varepsilon_K$. This expression can be estimated by

$$\begin{aligned}
&\sum_{k=0}^{d} 2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \\
&\quad\times \int_{\partial K}\cdots\int_{\partial K} I(F_1 \in \mathcal{F}(X))V_+(F_1)I(H(F_2)\cap H(F_1) \ne \varnothing)V_+(F_2) \\
&\qquad\qquad \times I\big(\mathrm{diam}(K\cap H(F_2)) < \mathrm{diam}(K \cap H(F_1))\big) \\
&\qquad\qquad \times \prod_{m=1}^{n+1} d_K(X_m)\, dX_1 \cdots dX_n\, dX_{n+1} \\
&\quad + O\big(n^d(1-c_K^{\mathrm{bd}})^n\big),
\end{aligned}$$

where F_1 denotes the convex hull of $X_1, \dots, X_d$ and F_2 the convex hull of $X_{d-k+1}, \dots, X_{2d-k}$. Integrating with respect to $X_{2d-k+1}, \dots, X_{n+1}$ yields

$$\begin{aligned}
\sum_{k=0}^{d} 2\binom{n}{d}\binom{n-d}{d-k} & \\
\times \int_{\partial K} \cdots \int_{\partial K} & \big(1 - S_+(F_1)\big)^{n-2d+k} S_+(F_1) V_+(F_1) \\
& \times I\big(H(F_2) \cap H(F_1) \neq \varnothing\big) V_+(F_2) \\
& \times I\big(\mathrm{diam}(K \cap H(F_2)) < \mathrm{diam}(K \cap H(F_1))\big) \\
& \times \prod_{m=1}^{2d-k} d_K(X_m)\, dX_1 \cdots dX_{2d-k} \\
+ O\big(n^d (1 - c_K^{\mathrm{bd}})^n\big). &
\end{aligned}$$

Here $S_+(F_1)$ is the weighted surface area $\int_{H_+(F_1)\cap\partial K} d_K(x)\, dx$.

The arguments used to deduce (33) show that the cap $K \cap H_+(F_2)$ and thus also the random points $X_{d+1}, \dots, X_{2d-k} \in K \cap H(F_2)$ are contained in a cap $K \cap H_+$ with height $\bar{c}h(F_1)$. Since the cap $K \cap H_+$ is contained in the cap $\bar{c}(K \cap H_+(F_1))$,

$$\begin{aligned}
\int_{\partial K} \cdots \int_{\partial K} & I\big(H(F_2) \cap H(F_1) \neq \varnothing\big) V_+(F_2) \\
& \times I\big(\mathrm{diam}(K \cap H(F_2)) < \mathrm{diam}(K \cap H(F_1))\big) \\
& \times \prod_{m=d+1}^{2d-k} d_K(X_m)\, dX_{d+1} \cdots dX_{2d-k} \\
\leq \int_{\partial K} \cdots \int_{\partial K} & I(X_{d+1} \cdots dX_{2d-k} \in K \cap H_+) V_+(H_+) \\
& \times \prod_{m=d+1}^{2d-k} d_K(X_m)\, dX_{d+1} \cdots dX_{2d-k} \\
\leq S_+(H_+)^{d-k} V_+(H_+) & \\
\leq \bar{\bar{c}}^{(d-1)(d-k)+d} S_+(F_1)^{d-k} V_+(F_1), &
\end{aligned}$$

where

$$\bar{\bar{c}} = \frac{\max_{\partial K} d_K(x)}{\min_{\partial K} d_K(x)} \bar{c}.$$

Now by Lemma 13 and since the binomial coefficients are of order n^{2d-k}, each of the summands

$$2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k}\overline{\overline{c}}^{(d-1)(d-k)+d}$$
$$\times \int_{\partial K}\cdots\int_{\partial K}(1-S_+(F_1))^{n-2d+k}S_+(F_1)^{d-k+1}V_+(F_1)^2$$
$$\times \prod_{m=1}^{d} d_K(X_m)\,dX_1\cdots dX_d$$

is bounded from above by

$$n^{2d-k}\overline{\overline{c}}^{d^2}\{c_d\Omega_{2,d_K}(K)n^{-2d+k-2(d+1)/(d-1)}+o(n^{-2d+k-2(d+1)/(d-1)})\}$$
$$\leq cn^{-2(d+1)/(d-1)}$$

with a suitable constant c.

Combined with the Efron–Stein jackknife inequality this shows

$$\operatorname{Var} V(K_n^{\mathrm{bd}}) \leq (n+1)\mathbb{E}\big(V(K_{n+1}^{\mathrm{bd}})-V(K_n^{\mathrm{bd}})\big)^2 \leq cn^{-1-4/(d-1)},$$

which proves Theorem 8.

10. Sketch of proof of Theorem 10. Analogously to the proof of Theorem 6 we have

$$\begin{aligned}\mathbb{E}F_n(X) &= \int_{\partial K}\cdots\int_{\partial K}\sum I(F_{\mathrm{I}}\in\mathcal{F}(X))\prod_{m=1}^{n} d_K(X_m)\,d_K(X)\,dX_1\cdots dX_n\,dX\\ &= \binom{n}{d}\int_{\partial K}\cdots\int_{\partial K}(1-S_+(F_1))^{n-d}S_+(F_1)\prod_{m=1}^{d} d_K(X_m)\,dX_1\cdots dX_d\\ &\quad + O\big(n^d(1-c_K^{\mathrm{bd}})^{-n}\big)\\ &= \binom{n}{d}\{c_d n^{-d}+o(n^{-d})\}\end{aligned}$$

by Lemma 13 as $n\to\infty$. As $\binom{n}{d}$ is of order n^d this proves the first part of Theorem 10.

To prove the second part of Theorem 10 one has to investigate the integral

$$\begin{aligned}&\mathbb{E}F_n(X)^2\\ &\quad = \int_{\partial K}\cdots\int_{\partial K}\Big(\sum I(F_{\mathrm{I}}\in\mathcal{F}(X))\Big)^2\prod_{m=1}^{n} d_K(X_m)\,d_K(X)\,dX_1\cdots dX_n\,dX,\end{aligned}$$

which is the same as (37) but where the term $V_+(F_{\mathrm{I}})$ is removed. A trivial modification of the proof of Theorem 8 thus gives the result.

11. The expected volume of random caps. This section contains the more technical part of the proofs of Theorems 1, 6, 8 and 10.

Let $K \in \mathcal{K}_+^2$ be given. Assume that $V(K) = 1$ and that the origin is contained in the interior of K. Choose random points $X_1, \dots, X_d$ in K independently and according to the uniform distribution. The convex hull of $X_1, \dots, X_d$ is a $(d-1)$-dimensional simplex whose affine hull is denoted by $H(X_1, \dots, X_d)$. This hyperplane dissects $\mathbb{R}^d$ into two (closed) halfspaces, and we denote that halfspace which contains the origin by $H_0(X_1, \dots, X_d)$, and the other by $H_+(X_1, \dots, X_d)$. Let $V_+ = V_+(X_1, \dots, X_d)$ be the volume of $H_+(X_1, \dots, X_d) \cap K$. In the following lemma we establish an asymptotic formula for the expectation of $(1 - V_+)^{n-i} V_+^k$. The method of proof of this lemma is well known. We include it here in detail for the sake of completeness.

LEMMA 12. *Let $K \in \mathcal{K}_+^2$ with $V(K) = 1$ and $0 \in K$, and choose random points $X_1, \dots, X_d$ in K independently and according to the uniform distribution. Then for fixed $i, k \in \mathbb{N}$,*

$$\mathbb{E}\big((1 - V_+)^{n-i} V_+^k\big) = c_d(k)\Omega(K) n^{-k-d+1-2/(d+1)} + o\big(n^{-k-d+1-2/(d+1)}\big) \tag{38}$$

as $n \to \infty$.

PROOF. In a first step we transform the integral

$$\mathbb{E}\big((1 - V_+)^{n-i} V_+^k\big) = \int_K \cdots \int_K (1 - V_+)^{n-i} V_+^k \, dX_1 \cdots dX_d$$

using the Blaschke–Petkantschin formula (cf., e.g., [41], II.12.3),

$$\begin{aligned} &\int_K \cdots \int_K \cdots dX_1 \cdots dX_d \\ &\qquad = (d-1)! \int_{H \in \mathcal{H}(d,d-1)} \int_{K \cap H} \cdots \int_{K \cap H} \cdots V_{d-1}(\mathrm{conv}[X_1, \dots, X_d]) \\ &\qquad\qquad \times dX_1 \cdots dX_d \, dH, \end{aligned} \tag{39}$$

where $V_{d-1}(A)$ denotes the $(d-1)$-dimensional volume of A. The Blaschke–Petkantschin formula relates the d-dimensional volume elements dX_j of the points $X_j \in K$ to the $(d-1)$-dimensional volume elements dX_j of points $X_j \in K \cap H$, where H is a random hyperplane in $\mathbb{R}^d$. (Throughout this section we denote by dX the j-dimensional volume element corresponding to the j-dimensional Hausdorff measure on a given space. The space itself, and thus its dimension, and thus the precise meaning of dX, is determined by the range of integration.) The differential dH corresponds to the suitably normalized rigid motion invariant Haar measure on the Grassmannian $\mathcal{H}(d, d-1)$ of hyperplanes in $\mathbb{R}^d$. A hyperplane can be parametrized by its unit normal vector $u \in S^{d-1}$ and its distance h to the origin. Denoting by du the element of surface area on S^{d-1} we have $dH = du \, dh$. Note that the integrand vanishes for $h > h_K(u)$ where $h_K(u)$ is the support function of K in direction u. Also note that $V_+ = V_+(h,u)$ only depends on $H(h,u)$ but not on the relative position of the points $X_j \in H(h,u)$. This yields

$$\int_K \cdots \int_K (1-V_+)^{n-i} V_+^k \, dX_1 \cdots dX_d$$
$$= (d-1)! \int_{S^{d-1}} \int_0^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathcal{I}_{K\cap H(h,u)} \, dh \, du$$

with

$$\mathcal{I}_{K\cap H(h,u)} = \int_{K\cap H(h,u)} \cdots \int_{K\cap H(h,u)} V_{d-1}(\operatorname{conv}[X_1,\dots,X_d]) \times dX_1 \cdots dX_d.$$

Now let $u \in S^{d-1}$ be fixed. The proof of Lemma 12 consists of showing that

$$\int_0^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathcal{I}_{K\cap H(h,u)} \, dh \sim c_4 \kappa(u)^{-1+1/(d+1)} n^{-k-d+1-2/(d+1)} \tag{40}$$

[equation (48)] uniformly in u.

As K is of class $\mathcal{K}_+^2$ there is an unique point $x \in \partial K$ with outer normal vector u. Choose $\delta > 0$ sufficiently small. By Lemma 11 there exists a $\lambda = \lambda(\delta) > 0$ such that the λ-neighborhood of x in ∂K can be represented by a convex function $f^{(x)}(y)$ fulfilling (22) and (23). Choose $\varepsilon > 0$ such that for each u the intersection $H_+(h_K(u) - \varepsilon, u) \cap \partial K$ is contained in the λ-neighborhood of the boundary point x. Furthermore we assume that $h_K(u) \geq \varepsilon$ for any u. This implies that for all u the halfspace $H_+(h_K(u) - \varepsilon)$ cuts off from ∂K the point x.

We split the integral (40) into two parts: $h \in [0, h_K(u) - \varepsilon]$ and $h \in [h_K(u) - \varepsilon, h_K(u)]$.

Estimating the integral

$$\int_0^{h_K(u)-\varepsilon} (1-V_+)^{n-i} V_+^k \mathcal{I}_{K\cap H(h,u)} \, dh$$

is easy. As $\mathcal{I}_{K\cap H(h,u)}$ is always bounded by a constant γ_1 independent of h and u, as $V_+ \leq 1$, and since there exists a constant $\gamma_2 = \gamma_2(\delta) > 0$ independent of u with $V_+ = V_+(h,u) \geq \gamma_2$ for $h \leq h_K(u) - \varepsilon$, we have

$$0 \leq \int_0^{h_K(u)-\varepsilon} (1-V_+)^{n-i} V_+^k \mathcal{I}_{K\cap H(h,u)} \, dh \leq \gamma_1 (1-\gamma_2)^{n-i} \big(h_K(u) - \varepsilon\big). \tag{41}$$

Estimating the second part of the integral is more difficult. Let $\mathbb{R}^d = \{(y,z) \mid y \in \mathbb{R}^{d-1}, z \in \mathbb{R}\}$. For the moment identify the tangent hyperplane to ∂K at x with the plane $z = 0$ and x with the origin such that K is contained in the halfspace $z \geq 0$ and u coincides with $(0, -1)$. Define $H(h)$ to be the hyperplane parallel to $z = 0$

with distance $z = h_K(u) - h$ to the origin, and let $H_+(h)$ be the corresponding halfspace cutting off from ∂K the point x, that is, the new origin.

We introduce polar coordinates: let $\mathbb{R}^d = (\mathbb{R}^+ \times S^{d-2}) \times \mathbb{R}$ and thus denote by (rv, z) a point in $\mathbb{R}^d$, $r \in \mathbb{R}^+$, $v \in S^{d-2}$, $z \in \mathbb{R}$. Since $K \in \mathcal{K}_+^2$ the λ-neighborhood of x in ∂K can be represented by a convex function $f^{(x)}(rv)$. For abbreviation write $b_2(\cdot)$ and $f(\cdot)$ instead of $b_2^{(x)}(\cdot)$ and $f^{(x)}(\cdot)$:

$$(1+\delta)^{-1} b_2(v) r^2 \le z = f(rv) \le (1+\delta) b_2(v) r^2. \tag{42}$$

By choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ the coefficient $b_2(v)$ can be written as

$$b_2(v) = \tfrac{1}{2}\big(k_1 \langle v, e_1\rangle^2 + \cdots + k_{d-1}\langle v, e_{d-1}\rangle^2\big)$$

and since for all boundary points of K the principal curvatures k_i are bounded from below and above by positive constants the same holds for $b_2(v)$. Inequality (42) implies

$$(1+\delta)^{-1/2} b_2(v)^{-1/2} z^{1/2} \le r = r(v,z) \le (1+\delta)^{1/2} b_2(v)^{-1/2} z^{1/2}, \tag{43}$$

where r is the radial function of $K \cap H(h)$. From this we obtain estimates for the $(d-1)$-dimensional volume of $K \cap H(h)$:

$$\begin{aligned} &(1+\delta)^{-(d-1)/2} c_1 \kappa(u)^{-1/2} z^{(d-1)/2} \\ &\qquad \le V_{d-1}(K \cap H(h)) \le (1+\delta)^{(d-1)/2} c_1 \kappa(u)^{-1/2} z^{(d-1)/2} \end{aligned} \tag{44}$$

with a suitable constant $c_1 > 0$. [Recall that $z = h_K(u) - h$.]

For given z (43) means that $K \cap H(h)$ contains an ellipsoid $\mathcal{E}_-$ defined by $(1+\delta)^{-1} b_2(v) r^2 = z$, respectively, is contained in an ellipsoid $\mathcal{E}_+$ defined by $(1+\delta) b_2(v) r^2 = z$. We are interested in the value of

$$\mathcal{I}_{K\cap H(h)} = \int_{K\cap H(h)} \cdots \int_{K\cap H(h)} V_{d-1}(\mathrm{conv}[X_1, \dots, X_d])\, dX_1 \cdots dX_d.$$

Clearly if the range of integration is increased, respectively, decreased, $\mathcal{I}$ will increase, respectively, decrease:

$$\mathcal{I}_{\mathcal{E}_-} \le \mathcal{I}_{K\cap H} \le \mathcal{I}_{\mathcal{E}_+}.$$

Note that these integrals are invariant under volume-preserving affinities, that is, they do not depend on the shape of the ellipsoids but only on their volumes. Hence $\mathcal{I}_{\mathcal{E}_-}$, respectively, $\mathcal{I}_{\mathcal{E}_+}$, is proportional to $V_{d-1}(\mathcal{E}_-)^{d+1}$, respectively, $V_{d-1}(\mathcal{E}_+)^{d+1}$. Thus there exists a suitable constant c_2 for which

$$\begin{aligned} &(1+\delta)^{-(d^2-1)/2} c_2 \kappa(u)^{-(d+1)/2} z^{(d^2-1)/2} \\ &\qquad \le \mathcal{I}_{K\cap H(h)} \le (1+\delta)^{(d^2-1)/2} c_2 \kappa(u)^{-(d+1)/2} z^{(d^2-1)/2}. \end{aligned} \tag{45}$$

By definition,

$$V_+ = \int_h^{h_K(u)} V_{d-1}(K \cap H(p))\, dp \tag{46}$$

which by (44) implies

$$\begin{aligned}(1+\delta)^{-(d-1)/2}&\frac{2}{d+1}c_1\kappa(u)^{-1/2}z^{(d+1)/2}\\ &\le V_+ \le (1+\delta)^{(d-1)/2}\frac{2}{d+1}c_1\kappa(u)^{-1/2}z^{(d+1)/2}.\end{aligned} \tag{47}$$

Now we are ready to estimate the integral

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathbb{1}_{K\cap H(h)}\, dh.$$

Note that (46) is equivalent to

$$\frac{dV_+}{dh} = -V_{d-1}(K \cap H(h)),$$

which implies

$$\begin{aligned}&\int_{h_K(u)-\varepsilon}^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathbb{1}_{K\cap H(h)}\, dh\\ &\quad = \int_0^{V_+(h=h_K(u)-\varepsilon)} (1-V_+)^{n-i} V_+^k \mathbb{1}_{K\cap H(V_+)} V_{d-1}(K\cap H(V_+))^{-1}\, dV_+,\end{aligned}$$

where $H(V_+)$ denotes the hyperplane parallel to $z=0$ cutting off from K a cap of volume V_+.

Combining this with (45) and (44) yields

$$\begin{aligned}(1+\delta)^{(d-1)d(d+3)/(d+1)}&c_3\kappa(u)^{-1+1/(d+1)}\\ &\times \int_0^{V_+(h=h_K(u)-\varepsilon)} (1-V_+)^{n-i} V_+^{k+d-2+2/(d+1)}\, dV_+\\ &\le \int_{h_K(u)-\varepsilon}^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathbb{1}_{K\cap H(h)}\, dh\\ &\le (1+\delta)^{-(d-1)d(d+3)/(d+1)} c_3\kappa(u)^{-1+1/(d+1)}\\ &\qquad \times \int_0^{V_+(h=h_K(u)-\varepsilon)} (1-V_+)^{n-i} V_+^{k+d-2+2/(d+1)}\, dV_+\end{aligned}$$

with a suitable constant c_3.

Thus we are interested in the asymptotic behavior of the integral

$$\begin{aligned}\int_0^{V_+(h=h_K(u)-\varepsilon)} &(1-V_+)^{n-i} V_+^{k+d-2+2/(d+1)}\, dV_+ \\ &= \mathbf{B}\left(n-i+1, k+d-1+\frac{2}{d+1}\right) \\ &\quad + O\big((1-\gamma_2)^{n-i}\big)\end{aligned}$$

as $n \to \infty$. [Recall that $V_+ \geq \gamma_2$ for $h \leq h_K(u)-\varepsilon$.] Since the asymptotic behavior of the Beta-function $\mathbf{B}(\cdot,\cdot)$ is well known (cf., e.g., [16], page 60),

$$\mathbf{B}(n-i+1, t) = \Gamma(t) n^{-t} + O(n^{-t-1})$$

as $n \to \infty$, this yields the following bounds:

$$\begin{aligned}&(1+\delta)^{(d-1)d(d+3)/(d+1)} \\ &\quad \times c_4 \kappa(x)^{-1+1/(d+1)} \Big(n^{-k-d+1-2/(d+1)} + O\big(n^{-k-d-2/(d+1)}\big)\Big) \\ &\quad \leq \int_{h_K(u)-\varepsilon}^{h_K(u)} (1-V_+)^{n-i} V_+^k \mathbb{1}_{K\cap H}\, dp \qquad (48)\\ &\quad \leq (1+\delta)^{-(d-1)d(d+3)/(d+1)} \\ &\qquad \times c_4 \kappa(x)^{-1+1/(d+1)} \Big(n^{-k-d+1-2/(d+1)} + O\big(n^{-k-d-2/(d+1)}\big)\Big)\end{aligned}$$

as $n \to \infty$, where the constant in $O(\cdot)$ and the constant c_4 are independent of x.

Concerning the remaining integration note that the term

$$\int_{S^{d-1}} \kappa(u)^{-1+1/(d+1)}\, du = \int_{\partial K} \kappa(x)^{1/(d+1)}\, dx$$

is the affine surface area $\Omega(K)$. Since the terms in (41) are of smaller order, (48) implies

$$\begin{aligned}&(1+\delta)^{(d-1)d(d+3)/(d+1)} c_5 \Omega(K) \Big(n^{-k-d+1-2/(d+1)} + O\big(n^{-k-d-2/(d+1)}\big)\Big) \\ &\quad \leq \mathbb{E}\big((1-V_+)^{n-i} V_+^k\big) \\ &\quad \leq (1+\delta)^{-(d-1)d(d+3)/(d+1)} c_5 \Omega(K) \Big(n^{-k-d+1-2/(d+1)} + O\big(n^{-k-d-2/(d+1)}\big)\Big)\end{aligned}$$

as $n \to \infty$ with a suitable constant c_5. This holding for each $\delta, \delta > 0$ proves Lemma 12. $\square$

For the proofs of Theorems 8 and 10 we need an analogous result for random points chosen on the boundary of the convex body K. Let K be given, assume that the origin is contained in the interior of K, and let d_K be a positive continuous density function on the boundary of K. Choose random points $X_1, \dots, X_d$ independently and according to the density function d_K. The affine hull of $X_1, \dots, X_d$ dissects the space into two halfspaces, into the halfspace $H_0(X_1, \dots, X_d)$ which contains the origin, and into $H_+(X_1, \dots, X_d)$. Denote by V_+ the volume of $H_+(X_1, \dots, X_d) \cap K$, and by S_+ the weighted surface area of $H_+(X_1, \dots, X_d) \cap \partial K$:

$$S_+ = \int_{H_+(X_1,\dots,X_d)\cap\partial K} d_K(x)\,dx.$$

LEMMA 13. *Let $K \in \mathcal{K}_+^2$, and choose random points $X_1, \dots, X_d$ in K independently and according to a positive continuous density function d_K on the boundary of K. Then for fixed $i, j, k \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{E}\big((1 - S_+)^{n-i} S_+^j V_+^k\big) &= c_d(j,k)\Omega_{k,d_K}(K) n^{-j-d+1-k(d+1)/(d-1)} \\ &\quad + o\big(n^{-j-d+1-k(d+1)/(d-1)}\big)\end{aligned} \tag{49}$$

as $n \to \infty$, where

$$\Omega_{k,d_K}(K) = \int_{\partial K} d_K(x)^{1-k(d+1)/(d-1)} \kappa(x)^{k/(d-1)}\,dx.$$

The proof of this lemma is similar to the proof of Lemma 12. For details we refer the reader to [38] where an analogous proof is worked out in detail in Sections 5.2–5.5. Here we only give a sketch of the proof.

PROOF OF LEMMA 13. We transform the integral

$$\begin{aligned}&\mathbb{E}\big((1 - S_+)^{n-i} S_+^j V_+^k\big) \\ &\qquad = \int_{\partial K} \cdots \int_{\partial K} (1 - S_+)^{n-i} S_+^j V_+^k \prod_{m=1}^{d} d_K(X_m)\,dX_1 \cdots dX_d\end{aligned}$$

using an analogue of the Blaschke–Petkantschin formula due to Zähle [52]:

$$\begin{aligned}&dX_1 \cdots dX_d \\ &\quad = (d-1)! V_{d-1}(\mathrm{conv}[X_1, \dots, X_d]) \prod_{m=1}^{d} p_H(X_m)^{-1}\,dX_1|_H \cdots dX_d|_H\,dH,\end{aligned}$$

where $dX_j|_H$ denotes the $(d-1)$-dimensional volume element of X_j in $H \cap \partial K$. The additional term $p_H(X_m)$ is the length of the projection of the outer unit normal vector of K at X_m onto the hyperplane H. This yields

$$\begin{aligned}&\mathbb{E}\big((1-S_+)^{n-i} S_+^j V_+^k\big)\\ &\qquad = (d-1)! \int_{S^{d-1}} \int_0^{h_K(u)} (1-S_+)^{n-i} S_+^j V_+^k \mathcal{I}_{\partial K \cap H(h,u)}\, dh\, du,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{I}_{\partial K \cap H(h,u)}\\ &\quad = \int_{\partial K \cap H(h,u)} \cdots \int_{\partial K \cap H(h,u)} V_{d-1}(\mathrm{conv}[X_1, \dots, X_d])\\ &\qquad\qquad \times \prod_{m=1}^{d} d_K(X_m) p_{H(h,u)}(X_m)^{-1}\\ &\qquad\qquad\qquad \times dX_1|_{H(h,u)} \cdots dX_d|_{H(h,u)}.\end{aligned}$$

Now let $u \in S^{d-1}$ be fixed. As K is of class $\mathcal{K}_+^2$ there is an unique point $x \in \partial K$ with outer normal vector u. Using Lemma 11 we obtain

$$\begin{aligned}S_+ &\sim c_1 d_K(x) \kappa(x)^{-1/2} z^{(d-1)/2},\\ V_+ &\sim c_2 \kappa(x)^{-1/2} z^{(d+1)/2}\end{aligned}$$

and

$$\mathcal{I}_{\partial K \cap H} \sim c_3 d_K(x)^d \kappa^{-(d+1)/2} z^{(d^2-2d-1)/2},$$

where $\sim$ indicates formulae analogously to (44), (47) and (45).

Combining this yields

$$\begin{aligned}&\int_0^{h_K(u)} (1-S_+)^{n-i} S_+^j V_+^k \mathcal{I}_{\partial K \cap H(h,u)}\, dh\, du\\ &\quad \sim c_4 d_K(x)^{1-k(d+1)/(d-1)} \kappa(x)^{k/(d-1)-1}\\ &\qquad \times \int_0^{S_+(h=0)} (1-S_+)^{n-i} S_+^{j+d-2+k(d+1)/(d-1)}\, dS_+\\ &\quad \sim c_4 d_K(x)^{1-k(d+1)/(d-1)} \kappa(x)^{k/(d-1)-1} n^{-j-d+1-k(d+1)/(d-1)}.\end{aligned}$$

The integration with respect to u now yields Lemma 13. $\square$

12. Sketch of proof of $\lim \mathbb{E}(V(K_{n+1}) - V(K_n))n^{(d+3)/(d+1)} = c\Omega(K)$. Analogously to (31) and using the notation introduced in Section 5, we have

$$\begin{aligned}
&\mathbb{E}\big(V(K_{n+1}) - V(K_n)\big)\\
&\quad= \int_K \cdots \int_K \sum_{F\in\mathcal{F}(X_{n+1})} V(\mathrm{conv}[F, X_{n+1}])\, dX_1 \cdots dX_n\, dX_{n+1}\\
&\quad= \binom{n}{d} \int_K \cdots \int_K I\big(F_1 \in \mathcal{F}(X_{n+1})\big) V(\mathrm{conv}[F_1, X_{n+1}])\, dX_1 \cdots dX_n\, dX_{n+1}\\
&\quad= \binom{n}{d} \int_K \cdots \int_K \big(1 - V_+(F_1)\big)^{n-d} I\big(X_{n+1} \in H_+(F_1)\big)\\
&\qquad\qquad\qquad \times V(\mathrm{conv}[F_1, X_{n+1}])\, dX_1 \cdots dX_d\, dX_{n+1}\\
&\qquad + O\big(n^d (1 - c_K)^n\big).
\end{aligned}$$

Since the volume of the simplex $\mathrm{conv}[F_1, X_{n+1}]$ is determined by $V_{d-1}(F_1)$ and the height $h_{F_1}(X_{n+1})$ of the simplex, the use of the Blaschke–Petkantschin formula (39) yields

$$\begin{aligned}
&\mathbb{E}\big(V(K_{n+1}) - V(K_n)\big)\\
&\quad= \binom{n}{d} \frac{(d-1)!}{d} \int_{S^{d-1}} \int_0^{h_K(u)} (1 - V_+)^{n-d} \mathcal{I}^{(2)}_{K\cap H(h,u)} \mathcal{J}_{K\cap H(h,u)}\, dh\, du\\
&\qquad + O\big(n^d (1 - c_K)^n\big)
\end{aligned}$$

with

$$\mathcal{I}^{(2)}_{K\cap H(h,u)} = \int_{K\cap H(h,u)} \cdots \int_{K\cap H(h,u)} V_{d-1}(\mathrm{conv}[X_1, \ldots, X_d])^2 dX_1 \cdots dX_d$$

and

$$\begin{aligned}
\mathcal{J}_{K\cap H(h,u)} &= \int_{K\cap H_+(h,u)} h_{F_1}(X_{n+1})\, dX_{n+1}\\
&= \int_0^{h_K(u)-h} t V_{d-1}\big(K \cap H\big(h_K(u) - h + t, u\big)\big)\, dt.
\end{aligned}$$

Fix u and recall that (as in Section 10) $z = h_K(u) - h$. Then by (44),

$$\begin{aligned}
\mathcal{J}_{K\cap H(h,u)} &\sim c_1 \kappa(u)^{-1/2} \int_0^z (z - t) t^{(d-1)/2}\, dt\\
&\sim c_2 \kappa(u)^{-1/2} z^{(d+3)/2},
\end{aligned} \tag{50}$$

where $\sim$ indicates formulae involving upper and lower bounds analogously to (44). On the other hand, analogously to (45),

$$\mathcal{I}^{(2)}_{K\cap H(h,u)} \sim c_3 \kappa(u)^{-(d+2)/2} z^{(d^2+d+2)/2},$$

which combined with (50) implies

$$\mathcal{I}^{(2)}_{K\cap H(h,u)}\mathcal{J}_{K\cap H(h,u)} \sim c_4\kappa(u)^{-(d+3)/2}z^{(d+1)^2/2}.$$

Now the proof proceeds as in the proof of Lemma 12: by (47) we obtain

$$\mathcal{I}^{(2)}_{K\cap H(h,u)}\mathcal{J}_{K\cap H(h,u)}\,dh \sim c_5\kappa(u)^{-1+1/(d+1)}V_+^{d+2/(d+1)}\,dV_+,$$

and since

$$\begin{aligned}\int_0^1 (1-V_+)^{n-d}V_+^{d+2/(d+1)}\,dV_+ \\ = \Gamma\biggl(d+1+\frac{2}{d+1}\biggr)n^{-d-1-2/(d+1)} + O\bigl(n^{-d-2-2/(d+1)}\bigr),\end{aligned}$$

we obtain

$$\mathbb{E}\bigl(V(K_{n+1})-V(K_n)\bigr) \sim c_6\Omega(K)n^{-(d+3)/(d+1)}$$

with a suitable constant c_6.

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