

LIMIT BEHAVIOR OF THE BAK–SNEPPEN EVOLUTION MODEL

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One of the key problems related to the Bak–Sneppen evolution model on the circle is computing the limit distribution of the fitness at a fixed observation vertex in the stationary regime as the size of the system tends to infinity. Some simulations have suggested that this limit distribution is uniform on $(f, 1)$ for some $f \sim 2/3$. In this article, we prove that the mean of the fitness in the stationary regime is bounded away from 1, uniformly in the size of the system, thereby establishing the nontriviality of the limit behavior. The Bak–Sneppen dynamics can easily be defined on any finite connected graph. We also present a generalization of the phase-transition result in the context of an increasing sequence of such graphs. This generalization covers the multidimensional Bak–Sneppen model as well as the Bak–Sneppen model on a tree. Our proofs are based on a “self-similar” graphical representation of the avalanches.

1. Introduction. The Bak–Sneppen model, introduced in [2], has received a lot of attention in the literature; see, for instance, [1], [4] and [7]. Bak [1] described how he and Sneppen were looking for a simple model which was supposed to exhibit evolutionary behavior and also supposed to fall into the class of processes that show self-organized critical behavior. For physicists, self-organized critical behavior refers to power law decay of temporal and spatial quantities. After a number of attempts, Bak and Sneppen arrived at the following simple process.

Consider a system with N species. These species are represented by N vertices on a circle, evenly spaced, say. Now each of these species is assigned a so-called *fitness*, a number between 0 and 1. The higher the fitness, the better chance the species has of surviving. The dynamics of evolution is modelled as follows. Every discrete time step, we choose the vertex with minimal fitness and we think of the corresponding species as disappearing completely. This species is then replaced by a new one, with a fresh and independent fitness, uniformly distributed on $[0, 1]$. So far, the dynamics does not have any interaction between the species and does not result in an interesting process. Indeed, if we only replace the species with the lowest fitness, then it is easy to see that the system converges to a situation with all fitnesses equal to 1. Interaction is introduced by also replacing the two neighbors of the vertex with lowest fitness by new species with independent fitnesses. This interaction represents co-evolution of related species. This neighbor interaction makes the model also very interesting from a mathematical point of view.

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It is simple to run this model on a computer. Simulations then suggest the following behavior, for large N (see [4] and [1] for simulation results). It appears that the one-dimensional marginals are uniform (in the limit for $N \rightarrow \infty$) on $(f, 1)$ for some f whose numerical value is supposed to be close to $2/3$. This threshold value f is the basis for self-organized critical behavior, according to [2], [1] and [4], as follows. Since in the limit there is no mass below f , one can look at so-called *avalanches* of fitnesses below this threshold: starting the counting at the moment all fitnesses are above f and finishing the counting at the first next moment all fitnesses are above f again. The random number of updates, for instance, counted this way, is supposed to follow a power law. For this to make sense, the conjecture had better be true. This we have not been able to prove. However, we have been able to prove the weaker result that the mean average fitness in the stationary regime is bounded away from 1, uniformly in the number of vertices. Hence, in the limit, there is probability mass of the fitnesses below 1. Since it is not difficult to show that in the limit there can be no probability mass of the fitnesses below $1/3$, our results establish that the one-dimensional marginals do not become trivial as $N \rightarrow \infty$. For a similar result in a discrete version of this process, see [6].

Let F_N be the distribution function of the one-dimensional marginal in the stationary regime in the system with N vertices. We prove the following result.

THEOREM 1.1. *If $q < 1$ is close enough to 1, then there exists $c_q > 0$, independent of N , such that*

$$(1.1) \quad F_N(q) > c_q.$$

In Sections 4 and 5, we define the Bak-Sneppen dynamics on arbitrary finite connected graphs and present a generalization of our results for an increasing sequences of such graphs.

We remark that corresponding results for a mean-field version of the model are quite simple to obtain (see, e.g., [3] and [4]). In the mean-field case, it is possible to prove, using essentially only combinatorics, that the one-dimensional marginals do indeed converge to a uniform distribution on $(c, 1)$ for some constant c which depends on the characteristics of the model.

In the next section we prepare for the proof of Theorem 1.1 by introducing the notion of an avalanche and establishing some monotonicity properties of the avalanches. The proof itself can be found in Section 3.

2. The self-similar graphical representation. Let $\Lambda(N) = \{-N + 1, \dots, -1, 0\}$ index the set of N vertices on the circle, so that 0 and $-N + 1$ are neighbors. We use negative indices to simplify notation in the future. We say that in the time interval $[n, n + d]$, an *avalanche from threshold $q \in [0, 1]$* (also referred to as a q -avalanche) with *origin* at $x \in \Lambda(N)$ and *duration* $d \geq 1$ occurs if at time n ,

x is the vertex with minimal fitness above threshold q and $n + d$ is the first moment after n with all fitnesses again above q . The *range set* of the q -avalanche is the collection of vertices updated during the avalanche and the *range* of the q -avalanche is the number of different vertices in the range set. Note that, according to this definition, if at times n and $n + 1$ all the fitnesses are above q , then in the time interval $[n, n + 1]$, an avalanche of range 3 and duration 1 occurs, even though there were no fitnesses below q .

The fitnesses of the vertices are random variables with values in $[0, 1]$ and we update them according to the uniform distribution on $[0, 1]$. For computational reasons however, it is convenient for the fitnesses to have values in $[0, \infty]$ and to update them according to the exponential distribution with parameter 1, say. In this new setup a threshold b corresponds to the threshold $q = 1 - e^{-b}$ in the old setup.

Suppose that a b -avalanche starts at time 0, with the origin at the vertex 0, so that the vertex 0 and its two neighbors are updated. We can now graphically illustrate the b -avalanche on $\Lambda(N) \times \mathbb{R}^+$ (space \times fitness) as follows. Look for the vertex with minimal fitness and call this vertex x . (Note that x must be the vertex 0 or one of its two neighbors.) Suppose that the fitness of x is equal to $s < b$. We then continue updating according to the appropriate rules and wait until all fitnesses are above the threshold s . This in itself constitutes an s -avalanche, starting at x . We denote by $\xi_N(x, s)$ the set of vertices involved in this s -avalanche. In the graphical representation, we draw an arrow from the space–fitness point (x, s) to the space–fitness points (i, s) for all $i \in \xi_N(x, s)$.

After the s -avalanche has ended, the new fitnesses of all vertices involved in this avalanche are independent and identically distributed (i.i.d.) and exponentially distributed on $[s, \infty)$, due to the lack of memory property of the exponential distribution. We can now look for the minimal fitness among all vertices in $\xi_N(x, s)$. If this minimal fitness is above b , then the b -avalanche has stopped. If this minimal fitness is equal to t , where $s < t < b$, and is associated with the vertex y , say, then we start, as before, a t -avalanche with origin y . We continue updating until all fitnesses are above t . If $\xi_N(y, t)$ denotes the set of vertices involved in this t -avalanche, then we draw an arrow in the graphical representation from the space–fitness point (y, t) to all space–fitness points (i, t) for $i \in \xi_N(y, t)$. We continue in the obvious way. Under the assumption that all avalanches are finite, this process will stop a.s. as soon as all fitnesses are above b . The idea of avalanches which form a hierarchical structure of subavalanches is also mentioned in [5], in a slightly different context.

This graphical representation, denoted by GR_N , is a random graph on the space–fitness diagram $\Lambda(N) \times \mathbb{R}^+$. We can describe it more formally as follows.

Let $\{\Pi_k\}_{k \in \Lambda(N)}$ be a collection of independent homogeneous Poisson processes. For each process Π_k we perform the following procedure. At the j th arrival $\tau_{k,j}$ of Π_k , we draw a pair $(\hat{\xi}_N(k, \tau_{k,j}), \hat{\eta}_N(\tau_{k,j}))$, where $\hat{\xi}_N(k, \tau_{k,j})$ is distributed as the range set and $\hat{\eta}_N(\tau_{k,j})$ is distributed as the duration of a typical $\tau_{k,j}$ -avalanche,

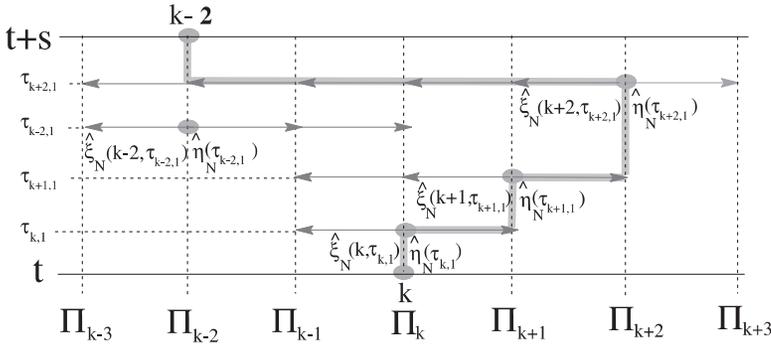


FIG. 1. The graphical representation GR_N , where, for instance, $(k, t) \rightsquigarrow (k - 2, t + s)$ in GR_N .

with origin at k . We draw arrows in $\Lambda(N) \times \mathbb{R}^+$ from $(k, \tau_{k,j})$ to $(y, \tau_{y,j})$ for all $y \in \hat{\xi}_N(k, \tau_{k,j})$. For any $t_1 < t_2$ we say that (x, t_1) is *connected* to (x, t_2) by a *time segment*. A *path* is a sequence $(x_0, s_0), \dots, (x_n, s_n)$ of points in $\Lambda(N) \times \mathbb{R}^+$ such that every pair $(x_j, s_j), (x_{j+1}, s_{j+1})$ is connected either by a time segment or an arrow. For any $x, y \in \mathbb{Z}$ and $t_1 \leq t_2 \in \mathbb{R}$, write $(x, t_1) \rightsquigarrow (y, t_2)$ in GR_N if there exists a path from (x, t_1) to (y, t_2) . See Figure 1 for an illustration.

For any $b > 0$, the range set $\xi_N(0, b)$ of a b -avalanche with origin at 0 consists of all vertices x such that $(0, 0) \rightsquigarrow (x, b)$ in GR_N , and the duration, denoted by $\eta_N(b)$, of this avalanche is the sum of $\hat{\eta}_N(\tau_{x,j})$ over all $\tau_{x,j} \leq b$ such that $(0, 0) \rightsquigarrow (x, \tau_{x,j})$ in GR_N .

The graphical representation provides us with the following monotonicity properties. For any $A \subset \Lambda(N)$ and $t, s \geq 0$, we denote by $\xi_N^{(A,t)}(s)$ the random set of vertices $x \in \Lambda(N)$ such that there exists $y = y(x) \in A$, and $(y, t) \rightsquigarrow (x, t + s)$ in GR_N . Similarly, for any $A \subseteq \Lambda(N)$ and $t, s \geq 0$, we denote by $\eta_N^{(A,t)}(s)$ the sum of $\hat{\eta}_N(\tau_{x,j})$ over all $\tau_{x,j} \leq t + s$ such that there exists $y = y(x) \in A$, and $(y, t) \rightsquigarrow (x, \tau_{x,j})$ in GR_N . Then for any $A \subseteq B \subseteq \Lambda(N)$, $0 \leq s_1 \leq s_2$ and $t \geq 0$,

$$(2.1) \quad \begin{aligned} \xi_N^{(A,t)}(s_1) &\subseteq \xi_N^{(B,t)}(s_2), \\ \eta_N^{(A,t)}(s_1) &\leq \eta_N^{(B,t)}(s_2). \end{aligned}$$

In particular, for $\xi_N(0, b) = \xi_N^{(\{-1,0,-N+1\},0)}(b)$ and $\eta_N(b) = \eta_N^{(\{-1,0,-N+1\},0)}(b)$, we have

$$(2.2) \quad \xi_N(0, b_1) \subseteq \xi_N(0, b_2), \quad \eta_N(b_1) \leq \eta_N(b_2), \quad \text{if } b_1 \leq b_2.$$

The last inequality allows us to couple two copies GR_N and GR'_N of the graphical representation in such a way that for any $k \in \Lambda(N)$, $\Pi_k(\cdot)$ restricted to $[0, b/2]$ is the same as $\Pi'_k(\cdot)$ restricted to $[b/2, b]$, and $\hat{\eta}(\tau_{k,j}) \leq \hat{\eta}'(\tau'_{k,j})$ for $\tau_{k,j} \in [0, b/2]$ and $\tau'_{k,j} \in [b/2, b]$. See Figure 2 for an illustration of the coupled GR_N and GR'_N . Note that we do *not* claim that the two copies together yield a realization of the

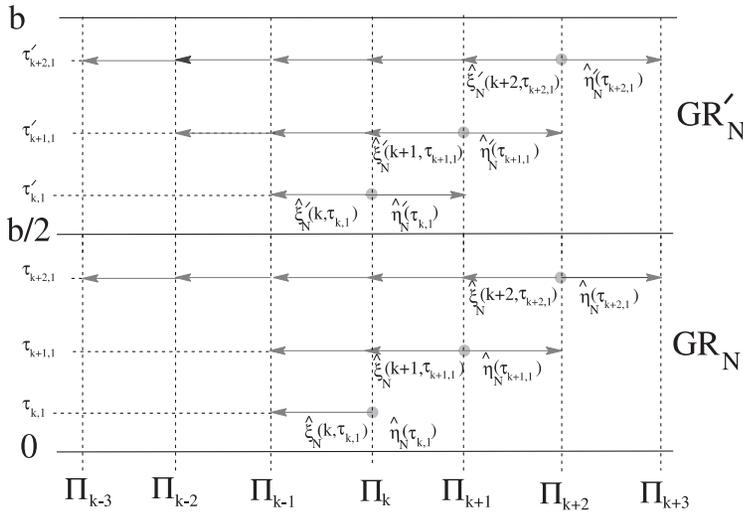


FIG. 2. The lower part of GR_N and the upper part of GR'_N coupled together.

evolution of the process from 0 to b . Nevertheless, this coupling gives us, for any $b > 0$,

$$\eta_N^{(\Lambda(N),0)}(b/2) \leq \eta_N^{(\Lambda(N),b/2)}(b/2)$$

and, hence,

$$(2.3) \quad 2E(\eta_N^{(\Lambda(N),b/2)}(b/2)) \geq E(\eta_N^{(\Lambda(N),0)}(b)).$$

3. Proving Theorem 1.1. An important step in the proof of Theorem 1.1 is the following lemma, which estimates the probability that an avalanche has range N , uniformly in N . For any $b \in \mathbb{R}_+$, define $P_N(b)$ as the probability that an avalanche has range N .

LEMMA 3.1. *If b is large enough, then $P_N(b) \geq 1/2$, uniformly in N .*

PROOF. For any $x \in A \subseteq \Lambda(N)$, denote by $\ell(x, A)$ the left corner of A with respect to x ,

$$\ell(x, A) = \min\{k \in (-\infty, x] \mid [k, x] \subseteq A \text{ mod } N\}$$

and write $l_N(s) := \ell(0, \xi_N(0, s))$ for the leftmost vertex involved in an s -avalanche with the origin at 0. We will have proved the lemma if we show that l_N is explosive in the sense that there exists a $0 < b_\infty < \infty$ such that for any $0 \leq i \leq i_{\max} = \lceil \log_{3/2}(N - 1) \rceil \in \mathbb{N}$,

$$(3.1) \quad P(l_N(b_\infty) \leq \max\{-(\frac{3}{2})^i, -N + 1\}) \geq \frac{1}{2} + (\frac{1}{2})^{i+1},$$

where the max actually works only at $i = i_{\max}$. Indeed, (3.1) implies that $P_N(b_\infty) \geq \frac{1}{2}$. To achieve this, choose a constant $b_0 \geq 17$. Define a converging sequence of thresholds b_1, b_2, b_3, \dots as follows:

$$b_i = b_{i-1} + \left(\frac{3}{4}\right)^i b_0, \quad i \geq 1,$$

$$b_\infty = \lim_{i \rightarrow \infty} b_i = 4b_0.$$

Observe that due to the monotonicity property (2.1), it suffices to prove that for all $i \in [0, i_{\max}]$,

$$(3.2) \quad P(l_N(b_i) \leq \max\{-(\frac{3}{2})^i, -N + 1\}) \geq \frac{1}{2} + (\frac{1}{2})^{i+1}.$$

We proceed by induction. First note that

$$(3.3) \quad P(l_N(b_0) \leq -1) = 1.$$

Next, suppose that (3.2) holds for some $i \in [0, i_{\max} - 1]$. Observe that

$$\left\{ \begin{aligned} &l_N(b_i) \leq -\left(\frac{3}{2}\right)^i, \\ &\exists x \in \left[-\left(\frac{3}{2}\right)^i, -\frac{1}{2}\left(\frac{3}{2}\right)^i\right] \cap \Lambda(N), \exists \tau_{x,j} \in \Pi_x \cap [b_i, b_{i+1}) \\ &\quad \text{such that } \ell(x, \hat{\xi}_N(x, \tau_{k,j})) \leq x - \left(\frac{3}{2}\right)^i \end{aligned} \right\}$$

implies

$$\{l_N(b_{i+1}) \leq \max\{-(\frac{3}{2})^{i+1}, -N + 1\}\}.$$

See Figure 3 for an illustration. Hence to finish the inductive step, it suffices to

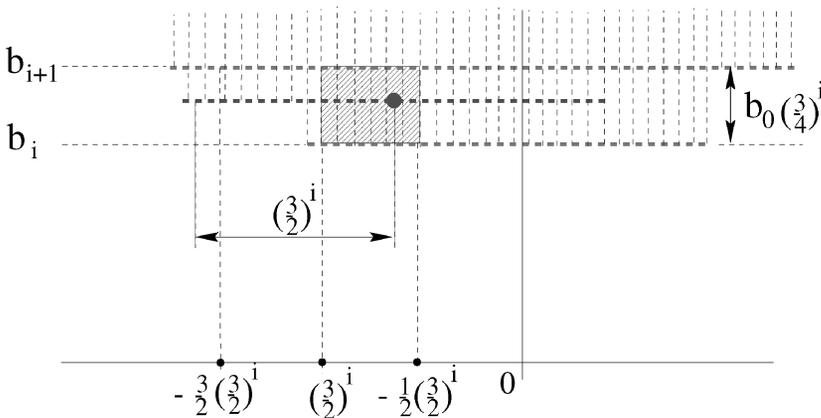


FIG. 3. Illustration of the induction step.

show that for all $i \in [0, i_{\max} - 1]$,

$$(3.4) \quad P \left(\begin{array}{l} \forall x \in [-(\frac{3}{2})^i, -\frac{1}{2}(\frac{3}{2})^i] \cap \Lambda(N), \forall \tau_{x,j} \in \Pi_x \cap [b_i, b_{i+1}), \\ \ell(x, \hat{\xi}_N(x, \tau_{x,j})) > x - (\frac{3}{2})^i \end{array} \right) \leq (\frac{1}{2})^{i+2}.$$

Since the $\ell(x, \hat{\xi}_N(x, \tau_{x,j}))$'s are independent and since (due to the monotonicity property) for any $x \in \Lambda(N)$ and $\tau_{x,j} \geq b_i$,

$$P(\ell(x, \hat{\xi}_N(x, \tau_{x,j})) > x - (\frac{3}{2})^i) \leq P(l_N(b_i) > -(\frac{3}{2})^i) \leq \frac{1}{2},$$

the points

$$(3.5) \quad \bigcup_{x \in [-(\frac{3}{2})^i, -\frac{1}{2}(\frac{3}{2})^i] \cap \Lambda(N)} \{ \tau_{x,j} \in \Pi_x ; \ell(x, \hat{\xi}_N(x, \tau_{x,j})) \leq x - (\frac{3}{2})^i \}$$

constitute a thinning of the Poisson process $\bigcup_{x \in [-(\frac{3}{2})^i, -\frac{1}{2}(\frac{3}{2})^i] \cap \Lambda(N)} \Pi_x$ with deleting probability at most $1/2$. Thus the points in (3.5) contain a Poisson process of intensity at least

$$\begin{aligned} & \frac{1}{2} \cdot \left(\text{intensity of } \bigcup_{x \in [-(\frac{3}{2})^i, -\frac{1}{2}(\frac{3}{2})^i] \cap \Lambda(N)} \Pi_x \right) \\ & \geq \frac{1}{2} (\lfloor (\frac{3}{2})^i \rfloor - \lfloor \frac{1}{2}(\frac{3}{2})^i \rfloor) \geq \frac{1}{2} \cdot \frac{1}{3} (\frac{3}{2})^i. \end{aligned}$$

Observe that the event in (3.4) implies that the process (3.5) has no arrivals between time b_i and b_{i+1} , a time interval of length $b_0(\frac{3}{4})^i$, and hence it has probability at most

$$\exp \left\{ -\frac{1}{6} \left(\frac{3}{2}\right)^i b_0 \left(\frac{3}{4}\right)^i \right\} = \exp \left\{ -\frac{b_0}{6} \left(\frac{9}{8}\right)^i \right\} \leq \left(\frac{1}{2}\right)^{i+2}, \quad i \in \mathbb{N},$$

since $b_0 \geq 17$. So we have (3.4) and the proof is complete. \square

PROOF OF THEOREM 1.1. Suppose we start from an i.i.d. uniform distribution above the threshold $q < 1$, which we assume however is so close to 1 that in the model where we update fitnesses according to the exponential distribution, we would have $P_N(b/2) > 1/2$, uniformly in N . (Recall that b and q are related via $q = 1 - e^{-b}$.) We define the dynamics via the following independent sequences of i.i.d. random variables. Fix some $q' \in (q, 1)$. Let, for $i = 1, 2, 3$, $\mathcal{U}^i = (U_j^i)_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $[q', 1]$. We use \mathcal{U}^i to construct the dynamics above threshold q' . Let, for $i = 1, 2, 3$, $\mathcal{V}^i = (V_j^i)_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $[0, q']$. We use \mathcal{V}^i to construct the dynamics below threshold q' . Let, for $i = 1, 2, 3$, $\mathcal{S}^i = (S_j^i)_{j \in \mathbb{N}}$ be a sequence of i.i.d. Bernoulli distributed random variables taking the value 1 with

probability $1 - q'$ and the value 0 with probability q' . We use \mathcal{S}^i to choose between \mathcal{U}^i and \mathcal{V}^i , and store the result in the sequence $\mathcal{G}^i = (G_j^i)_{j \in \mathbb{N}}$, $i = 1, 2, 3$:

$$G_j^i = \begin{cases} U_j^i, & \text{if } S_j^i = 1, \\ V_j^i, & \text{if } S_j^i = 0, \end{cases} \quad j \in \mathbb{N}.$$

It is clear that for $i = 1, 2, 3$, the sequence \mathcal{G}^i consists of i.i.d. random variables uniformly distributed on $[0, 1]$. We use the beginning of the sequence \mathcal{G}^2 (this choice is arbitrary) to assign the initial fitnesses to the vertices. At time $n \in [0, N - 1]$, we use the random variable G_n^2 to assign an initial fitness to the vertex with number $-n$. Now we are ready to define the dynamics. At every time $n \geq N$ we choose the vertex k , say, with minimal fitness at time $n - 1$, and we assign to $(k - 1, k, k + 1) \pmod{N}$ the triple (G_n^1, G_n^2, G_n^3) .

Let $f_0(n)$ denote the fitness of a fixed observation vertex at time n . Let j_n be the moment that this vertex received its current fitness, that is,

$$j_n = \min\{j \leq n \mid f_0(j) = f_0(n)\}.$$

Let $i(n)$ be the number of the sequence providing this value, that is,

$$f_0(n) = G_{j_n}^{i(n)}.$$

Observe that (since $j_{j_n} = j_n$) $i(n) = i(j_n)$. We say that n is q' -good if during $[j_n, n]$ the minimal fitness of all vertices is always less than q' . Then, for any time $n \in \mathbb{N}$ and any $q'' \in (q', 1)$, we have

$$\begin{aligned} P(f_0(n) < q'') &= \sum_{j=0}^{n-1} P(f_0(n) < q'', j_n = j) \\ &= \sum_{i=1}^3 \sum_{j=0}^{n-1} P(G_j^i < q'', j_n = j, i(j) = i) \\ (3.6) \quad &\geq \sum_{i=1}^3 \sum_{j=0}^{n-1} P(G_j^i < q'', j_n = j, i(j) = i, n \text{ is } q'\text{-good}) \\ &\geq \sum_{i=1}^3 \sum_{j=0}^{n-1} P(U_j^i < q'', j_n = j, i(j) = i, n \text{ is } q'\text{-good}). \end{aligned}$$

Now observe that for any $j \in [0, n]$, the event $\{j_n = j, i(j) = i, n \text{ is } q'\text{-good}\}$ is measurable with respect to

$$\sigma\{U_{j'}^i, V_{j'}^i, S_{j'}^i, S_{j''}^i, V_{j''}^i, 0 \leq j' < j, j \leq j'' \leq n, i = 1, 2, 3\}.$$

Hence, for any $j \in [0, n]$, the events $\{U_j^i < q''\}$ and $\{j_n = j, i(j) = i, n \text{ is } q'\text{-good}\}$ are independent, and we can continue the estimate (3.6) as follows:

$$\begin{aligned}
 &= \sum_{i=1}^3 \sum_{j=0}^{n-1} P(U_j^i < q'') P(j_n = j, i(j) = i, n \text{ is } q'\text{-good}) \\
 (3.7) \quad &= \frac{q'' - q'}{1 - q'} \sum_{i=1}^3 \sum_{j=0}^{n-1} P(j_n = j, i(j) = i, n \text{ is } q'\text{-good}) \\
 &= \frac{q'' - q'}{1 - q'} P(n \text{ is } q'\text{-good}).
 \end{aligned}$$

It remains to estimate the probability that n is q' -good from below, uniformly in n and N .

Define a sequence $(\tau_j^N)_{j \in \mathbb{N}}$ of stopping times, with respect to the natural filtration, as follows: $\tau_0 = 0$ and for any $j \in \mathbb{N}$, τ_{j+1} is the end of the first q -avalanche of range N after τ_j . For any $j \in \mathbb{N}$, we call the time interval $I_j^N = [\tau_j, \tau_{j+1})$ the j th period. It is clear that at every τ_j^N the fitnesses are i.i.d. and uniformly distributed above the threshold q . Thus the period lengths are i.i.d. random variables. For any time n , we denote by $\tau_{j(n)}$ the maximal τ_j such that $\tau_j \leq n$, that is, $n \in [\tau_{j(n)}, \tau_{j(n)+1})$, and we say that n is q' -nice, if during $[\tau_{j(n)}, n]$ the minimal fitness is always less than q' . For any $n \in \mathbb{N}$, if n is q' -nice, then n is q' -good. Indeed, suppose that n is q' -nice. If $j_n \geq \tau_{j(n)}$, then n is clearly q' -good. Suppose $j_n < \tau_{j(n)}$. Since during the q -avalanche of range N of the previous period, every vertex has been updated, j_n belongs to this q -avalanche and hence the minimal fitness at the time interval $[j_n, \tau_{j(n)})$ is always less than $q < q'$ and n is q' -good. Thus it suffices to show that the probability that time n is q' -nice is bounded away from zero uniformly in N and n .

A period can be decomposed into two parts: the duration of the avalanche of range N and the waiting time until this avalanche. We denote by W_N a typical waiting time before the avalanche of range N and denote by A_N the duration of this avalanche. During a q -avalanche, the minimal fitness is always at most $q < q'$. Hence, if in the i th period, the waiting time W_N satisfies $W_N = 0$, and in addition there is at least one vertex at time τ_i^N with fitness between q and q' , then any time n within the i th period is q' -nice. The event that there is such a vertex with fitness between q and q' is independent of W_N and A_N associated to that period, and has probability

$$p_1 = 1 - \left(1 - \frac{q' - q}{1 - q}\right)^N \geq \frac{q' - q}{1 - q} > 0 \quad \text{uniformly in } N.$$

Hence, in the stationary regime with N vertices, we can write (using alternating

renewal process theory)

$$\begin{aligned}
 P(n \text{ is } q'\text{-nice}) &\geq p_1 P(n \text{ is in a period with } W_N = 0) \\
 &\rightarrow p_1 P(W_N = 0) \frac{E(A_N)}{E(W_N) + E(A_N)}
 \end{aligned}$$

for $n \rightarrow \infty$. At this point, we switch from q to b , since we use results from the previous section.

Since $P(W_N = 0) = P_N(b) \geq P_N(b/2) \geq 1/2$ uniformly in N , it suffices to prove that there exists a constant $0 < c(b) < \infty$, independent of N , such that

$$(3.8) \quad E(W_N) \leq c(b)E(A_N).$$

Denote by Y_N the number of b -avalanches preceding the b -avalanche of range N . Every avalanche has range N with probability $P_N(b)$, independently of all other avalanches. Hence $Y_N + 1$ has a geometrical distribution with parameter $P_N(b)$ and we have

$$E(Y_N) = \frac{1}{P_N(b)} - 1.$$

Let $(Z_i^N)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables distributed as the duration of a typical b -avalanche, conditioned on its range being smaller than N . Then we can use Y_N of those avalanches to obtain W_N , that is,

$$W_N = Z_1^N + \dots + Z_{Y_N}^N.$$

In words, at the beginning of a new avalanche, we first decide (with the correct probability) whether or not the avalanche has range N . If not, we choose one, conditioned on its range being smaller than N , and the resulting duration is the next Z_i^N . Since due to the construction, Y_N is independent of the sequence $(Z_i^N)_{i \in \mathbb{N}}$, we have that

$$\begin{aligned}
 E(W_N) &= E(Z_1^N + \dots + Z_{Y_N}^N) \\
 (3.9) \quad &= E(Y_N)E(Z_1^N) = \left(\frac{1}{P_N(b)} - 1 \right) E(Z_1^N).
 \end{aligned}$$

We now estimate $E(Z_1^N)$ from above and $E(A_N)$ from below [recall that the pair $(\xi_N(b), \eta_N(b))$ represents the range set and duration of a b -avalanche],

$$\begin{aligned}
 E(Z_1^N) &= E(\eta_N(b) \mid |\xi_N(b)| \leq N - 1) \\
 &= \sum_{k=0}^{\infty} \frac{k P(\eta_N(b) = k, |\xi_N(b)| \leq N - 1)}{P(|\xi_N(b)| \leq N - 1)} \leq \sum_{k=0}^{\infty} \frac{k P(\eta_N(b) = k)}{P(|\xi_N(b)| \leq N - 1)} \\
 &\leq \frac{E(\eta_N(b))}{P(|\xi_N(b)| \leq N - 1)} = \frac{1}{1 - P_N(b)} E(\eta_N(b))
 \end{aligned}$$

and

$$\begin{aligned} E(A_N) &= E(\eta_N(b) \mid |\xi_N(b)| = N) \\ &\geq E(\eta_N(b) \mathbf{1}\{|\xi_N(b)| = N\}) \geq E(\eta_N(b) \mathbf{1}\{|\xi_N(b/2)| = N\}) \\ &\geq E\left(\eta_N^{(\wedge(N), b/2)}\left(\frac{b}{2}\right) \mathbf{1}\{|\xi_N(b/2)| = N\}\right), \end{aligned}$$

and since $\eta_N^{(\wedge(N), b/2)}(b/2)$ and $\mathbf{1}\{|\xi_N(b/2)| = N\}$ are independent, this is equal to

$$E\left(\eta_N^{(\wedge(N), b/2)}\left(\frac{b}{2}\right)\right) P\left(\left|\xi_N\left(\frac{b}{2}\right)\right| = N\right) \geq \frac{1}{2} E(\eta_N(b)) P_N\left(\frac{b}{2}\right),$$

where the last inequality follows from (2.3). Combining the estimates of $E(Z_1^N)$ and $E(A_N)$, we have

$$E(Z_1^N) \leq \frac{1}{1 - P_N(b)} \frac{2}{P_N(b/2)} E(A_N).$$

Since $P_N(b) \geq P_N(b/2) \geq 1/2$, the above inequality together with (3.9) gives us

$$\begin{aligned} E(W_N) &\leq \left(\frac{1}{P_N(b)} - 1\right) \frac{1}{1 - P_N(b)} \frac{2}{P_N(b/2)} E(A_N) \\ &\leq \frac{2}{P_N(b)P_N(b/2)} E(A_N) \leq 8E(A_N). \end{aligned}$$

Thus we have (3.8) and the theorem. \square

4. Extension to general graphs. Let G be a finite connected graph. One can define on G a Bak–Sneppen process in the following way. We call two vertices neighbors if they are connected by a bond in G . Every vertex of G accommodates a random variable (the fitness) with value in $[0, 1]$. At the initial moment, all the fitnesses are i.i.d. and uniformly distributed on $[0, 1]$. Every discrete time step we choose a vertex with minimal fitness and replace it, together with the fitnesses of all its neighbors, by new independent fitnesses, uniformly distributed in $[0, 1]$. We give bounds for the mean of the fitness in the stationary regime. These bounds depend on the local geometrical structure of G , but are independent of the number of vertices in G .

To state our main result, here follows some notation. Let V_G denote the set of vertices of G . For any two vertices $x, y \in V_G$, we denote by $\rho_G(x, y)$ the distance between them, that is, the number of bonds in the shortest path between x and y . Then for any vertex $x \in V_G$ and $k \in \mathbb{N}$, we can define the ball

$$B_G^x(k) = \{y \in V_G \mid \rho(y, x) \leq k\}$$

and the sphere

$$S_G^x(k) = \{y \in V_G \mid \rho_G(y, x) = k\}.$$

Observe that since G is finite, we have, for sufficiently large $k = k(G)$, that $B_G^x(k) = V_G$ and $S_G^x(k) = \emptyset$ for any $x \in V_G$.

For any $k \in \mathbb{N}$ denote by $m_G(k)$ the number of vertices in the smallest ball of radius k and denote by $M_G(k)$ the number of vertices in the largest sphere of radius k , that is,

$$m_G(k) = \min_{x \in V_G} |B_G^x(k)|,$$

$$M_G(k) = \max_{x \in V_G} |S_G^x(k)|.$$

Consider an increasing sequence of radii

$$(4.1) \quad r_i = \lfloor (\frac{4}{3})^{i+2} \rfloor, \quad i \in \mathbb{N}.$$

For any vertex x of G we denote by F_G^x the distribution function of the fitness at x in the stationary regime. In the following two theorems we establish an analogue of the phase-transition result for an infinite collection of finite connected graphs.

THEOREM 4.1. *Let \mathcal{G} be an infinite collection of finite connected graphs such that, for some $60 \leq b < \infty$,*

$$(4.2) \quad \frac{b}{4} \left(\frac{4}{5}\right)^i m_G\left(\left\lfloor \frac{r_i}{3} \right\rfloor\right) \geq \log\left(M_G\left(\left\lceil \frac{2r_i}{3} \right\rceil\right)\right) \quad \text{uniformly in } G \in \mathcal{G}, i \in \mathbb{N}.$$

Then there exists $c_b > 0$ and $q_b \in (0, 1)$ such that

$$F_G^x(q_b) > c_b \quad \text{uniformly in } G \in \mathcal{G}, x \in G.$$

THEOREM 4.2. *Let \mathcal{G} be an infinite collection of finite connected graphs of uniformly bounded degree, that is, there exists a constant $K \in \mathbb{N}$ such that*

$$(4.3) \quad \max_{x \in V_G} |B_G^x(1)| < K \quad \text{uniformly in } G \in \mathcal{G}.$$

Then for any sequence $(G_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ such that

$$|V_{G_n}| \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

we have

$$\lim_{n \rightarrow \infty} \left(\max_{x \in V_{G_n}} F_{G_n}^x(1/K) \right) = 0.$$

The proof uses a standard branching process argument and is omitted.

4.1. *Examples.*

1. *The original Bak–Sneppen model on the circle.* Here $\mathcal{G} = (P_n)_{n \in \mathbb{N}}$, where P_n is the regular polygon with n vertices. Observe that for any $k \geq n$, we have $M_{P_n}(\lceil 2k/3 \rceil) = 0$. For any $k < n$, we have

$$m_{P_n}(\lfloor k/3 \rfloor) \geq 2k/3, \quad M_{P_n}(\lceil 2k/3 \rceil) \leq 2.$$

Then condition (4.2) holds for $b \geq 60$ and condition (4.3) holds for $K = 3$.

2. *The multidimensional Bak–Sneppen model.* Let $d \geq 2$ be the dimension. We consider $\mathcal{G} = (G_n)_{n \in \mathbb{N}}$, where for every $n \in \mathbb{N}$ the set of vertices $V_{G_n} = \{1, \dots, n\}^d$, and with the usual nearest neighbor structure with periodic boundary conditions. Observe that for any $k \geq n$ we have $M_{P_n}(\lceil 2k/3 \rceil) = 0$. For any $k < n$, we have

$$m_{P_n}(\lfloor k/3 \rfloor) \sim c_1 k^d, \quad M_{P_n}(\lceil 2k/3 \rceil) \sim c_2 k^{d-1}.$$

Property (4.2) holds for $b \geq 30d$ and condition (4.3) holds for $K = 2d + 1$.

3. *The Bak–Sneppen model on a tree.* Choose $d \geq 2$ and consider $\mathcal{G} = (T_d(n))_{n \in \mathbb{N}}$, where $T_d(n)$ is the regular d -ary tree with d offspring at each vertex and n generations. Observe that for any $k > 3n$ we have $M_{P_n}(\lceil 2k/3 \rceil) = 0$. For any $k \leq 3n$, we have

$$m_{P_n}(\lfloor k/3 \rfloor) \sim c_3 d^{k/6}, \quad M_{P_n}(\lceil 2k/3 \rceil) \sim c_4 d^{2k/3}.$$

One can check that property (4.2) holds for $b \geq 30d$ and that condition (4.3) holds for $K = d + 2$.

5. Proof of Theorem 4.1. The proof of Theorem 4.1 essentially follows the proof of Theorem 1.1. For any finite G , one can associate a graphical representation GR_G . The construction of this graphical representation is essentially the same as before. Let $\{\Pi_k\}_{k \in V_G}$ be a collection of independent homogeneous Poisson processes. For each process Π_k we perform the following procedure. At the j th arrival $\tau_{k,j}$ of Π_k , we draw a pair $(\hat{\xi}_G(k, \tau_{k,j}), \hat{\eta}_G(k, \tau_{k,j}))$, where $\hat{\xi}_G(k, \tau_{k,j})$ is distributed as the range set and $\hat{\eta}_G(k, \tau_{k,j})$ is distributed as the duration of a typical $\tau_{k,j}$ -avalanche with origin at k . We draw arrows in $V_G \times \mathbb{R}^+$ from $(k, \tau_{k,j})$ to $(y, \tau_{k,j})$ for all $y \in \hat{\xi}_G(k, \tau_{k,j})$. As before, we can define, for any $A \subseteq V_G$, $0 \leq t, s < \infty$, the processes $\xi_G^{(A,t)}(s)$ and $\eta_G^{(A,t)}(s)$ such that the monotonicity properties hold:

$$(5.1) \quad \begin{aligned} \xi_G^{(A,t)}(s_1) &\subseteq \xi_N^{(B,t)}(s_2), \\ \eta_G^{(A,t)}(s_1) &\leq \eta_N^{(B,t)}(s_2), \quad 0 \leq s_1 \leq s_2. \end{aligned}$$

The range set $\xi_G(x, b)$ and the duration $\eta_G(x, b)$ of a b -avalanche with origin at $x \in V_G$ can be written as $\xi_G(x, b) = \xi_G^{(B_G^x(1),0)}(b)$ and $\eta_G(x, b) = \eta_G^{(B_G^x(1),0)}(b)$.

The only place where we previously used the geometrical structure of G is Lemma 3.1 and the related definitions. We give a new lemma for a collection of finite connected graphs.

For any vertex x of G and any $q > 0$, define $P_G^x(q)$ as the probability that updating x and the neighbors in the configuration with all fitnesses above q results in a q -avalanche of range $|V_G|$.

LEMMA 5.1. *Let G be a finite connected graph such that for some $60 \leq b_0 < \infty$,*

$$\frac{b_0}{4} \left(\frac{4}{5}\right)^i m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor\right) \geq \log \left(M_G \left(\left\lceil \frac{2r_i}{3} \right\rceil \right) \right) \quad \text{uniformly in } i \in \mathbb{N}.$$

Then there exists $q_\infty(b_0) \in (0, 1)$, depending only on b_0 such that for any $q > q_\infty(b_0)$,

$$P_G^x(q) > 1/2 \quad \text{uniformly in } x \in G.$$

PROOF. As in the proof of Lemma 3.1, we work with fitnesses defined on $[0, \infty)$ and update them according to the exponential distribution with parameter 1, say. Recall that in the new setup a threshold b corresponds to the threshold $q = 1 - e^{-b}$ in the old setup.

We will have proved the lemma if we show that for any $x \in G$, the process $\xi_G(x, t)$ is *explosive* in the sense that there exists $b_\infty \in (0, \infty)$, depending only on b_0 , such that for any $i \in \mathbb{N}$ [essentially for $i \leq i_{\max} = \max_{x,y} \rho(y, z)$, because for $i > i_{\max}$ we have $B_G^x(r_i) \equiv V_G$],

$$(5.2) \quad P(B_G^x(r_i) \subseteq \xi_G(x, b_\infty)) \geq \frac{1}{2} + \left(\frac{1}{2}\right)^{i+1}.$$

Indeed, (5.2) implies that $P_G^x(b_\infty) \geq \frac{1}{2}$. To achieve this, choose a constant b_0 that satisfies the condition of the lemma. Define a converging sequence of thresholds b_1, b_2, b_3, \dots as

$$b_i = b_{i-1} + \left(\frac{4}{5}\right)^i b_0, \quad i \geq 1, \\ b_\infty = \lim_{i \rightarrow \infty} b_i = 5b_0.$$

Observe that due to the monotonicity property (5.1), it suffices to prove that for all $i \in \mathbb{N}$, $x \in V_G$,

$$(5.3) \quad P(B_G^x(r_i) \subseteq \xi_G(x, b_i)) \geq \frac{1}{2} + \left(\frac{1}{2}\right)^{i+1}.$$

We proceed by induction. First note that for any $x \in V_G$,

$$(5.4) \quad P(B_G^x(1) \subseteq \xi_G(x, b_0)) = 1.$$

Next, suppose that (5.3) holds for some $i \in \mathbb{N}$ and all $x \in V_G$. Observe that

$$(5.5) \quad \left\{ B_G^x(r_i) \subseteq \xi_G(x, b_i), \right. \\ \left. \forall z \in S_G^x(\lceil 2r_i/3 \rceil), \exists y \in B_G^z(\lfloor r_i/3 \rfloor), \exists \tau_{y,j} \in \Pi_y \cap [b_i, b_{i+1}) \right. \\ \left. \text{such that } B_G^y(r_i) \subseteq \hat{\xi}_G(y, \tau_{y,j}) \right\}$$

implies

$$B_G^x(r_{i+1}) \subseteq \xi_G(x, b_{i+1}).$$

Indeed, if $a \in B_G^x(r_{i+1}) \setminus B_G^x(r_i)$, then there exists $z = z(a) \in S^x(\lceil 2r_i/3 \rceil)$ such that $\rho(a, z) + \rho(z, x) \leq r_{i+1}$, and hence $\rho(a, z) \leq \lceil 2r_i/3 \rceil$. Then if there exists $\tau \in \bigcup_{y \in B_G^z(\lfloor r_i/3 \rfloor)} \Pi_y \cap [b_i, b_{i+1})$ such that $B_G^y(r_i) \subseteq \hat{\xi}_G(y, \tau)$, then $\rho(a, y) \leq \rho(a, z) + \rho(z, y) \leq r_i$, and hence $a \in \hat{\xi}_G(y, \tau)$. See Figure 4 for an illustration of (5.5). Hence to finish the inductive step, it suffices to show that, uniformly in $i \in \mathbb{N}$,

$$(5.6) \quad P \left(\begin{array}{l} \exists z \in S_G^x(\lceil 2r_i/3 \rceil) \text{ such that} \\ \forall y \in B_G^z(\lfloor r_i/3 \rfloor), \forall \tau_{y,j} \in \Pi_y \cap [b_i, b_{i+1}), \\ B_G^y(r_i) \not\subseteq \hat{\xi}_G(y, \tau_{y,j}) \end{array} \right) \leq \left(\frac{1}{2}\right)^{i+2}.$$

Since the events $B_G^y(r_i) \subseteq \hat{\xi}_G(y, \tau_{y,j})$ are independent and since (due to the monotonicity property) for any $y \in V_G$ and $\tau_{y,j} \geq b_i$,

$$P(B_G^y(r_i) \subseteq \hat{\xi}_G(y, \tau_{y,j})) \leq P(B_G^y(r_i) \subseteq \hat{\xi}_G(y, b_i)) \leq \frac{1}{2},$$

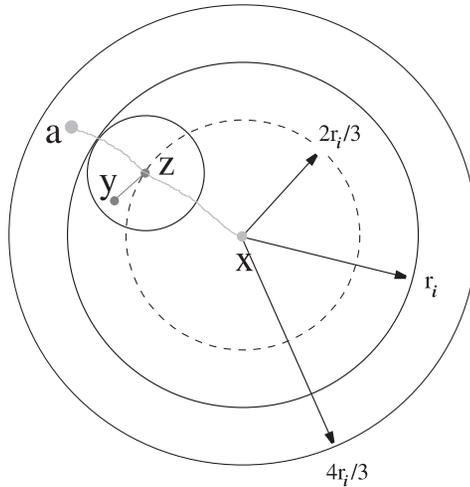


FIG. 4. Illustration of (5.5).

for any $z \in S_G^x(\lceil 2r_i/3 \rceil)$, the points

$$(5.7) \quad \bigcup_{y \in B_G^z(\lfloor r_i/3 \rfloor)} \{ \tau_{y,j} \in \Pi_y; B_G^y(r_i) \subseteq \hat{\xi}_G(y, \tau_{y,j}) \}$$

constitute a thinning of the Poisson process $\bigcup_{y \in B_G^z(\lfloor r_i/3 \rfloor)} \Pi_y$ with deleting probability at most $1/2$. Thus the points in (5.7) contain a Poisson process of intensity at least

$$\frac{1}{2} \cdot \left(\text{intensity of } \bigcup_{y \in B_G^z(\lfloor r_i/3 \rfloor)} \Pi_y \right) \geq \frac{1}{2} \cdot m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right).$$

Observe that the event in (5.6) implies that for some $z \in S_G^x(\lceil 2r_i/3 \rceil)$, the process (5.7) has no arrivals between time b_i and b_{i+1} , a time interval of length $b_0(\frac{4}{5})^i$. The last event has probability at most

$$\exp \left\{ -\frac{1}{2} m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right) b_0 \left(\frac{4}{5} \right)^i \right\}$$

uniformly in $z \in S_G^x(\lceil 2r_i/3 \rceil)$. Thus we can estimate the probability in (5.6) by

$$M_G \left(\left\lceil \frac{2r_i}{3} \right\rceil \right) \exp \left\{ -\frac{b_0}{2} m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right) \left(\frac{4}{5} \right)^i \right\}.$$

Split the above expression into two terms:

$$\left(M_G \left(\left\lceil \frac{2r_i}{3} \right\rceil \right) \exp \left\{ -\frac{b_0}{4} m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right) \left(\frac{4}{5} \right)^i \right\} \right) \exp \left\{ -\frac{b_0}{4} m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right) \left(\frac{4}{5} \right)^i \right\}.$$

The first term is less than or equal to 1 under the conditions of the lemma. For the second term, we write

$$\begin{aligned} & \exp \left\{ -\frac{b_0}{4} m_G \left(\left\lfloor \frac{r_i}{3} \right\rfloor \right) \left(\frac{4}{5} \right)^i \right\} \\ & \leq \exp \left\{ -\frac{b_0}{4} \left(1 + \left\lfloor \frac{r_i}{3} \right\rfloor \right) \left(\frac{4}{5} \right)^i \right\} \\ & \leq \exp \left\{ -\frac{b_0}{4} \left(\frac{1}{3} \left(\frac{4}{3} \right)^i \right) \left(\frac{4}{5} \right)^i \right\} \\ & \leq \exp \left\{ -\frac{b_0}{12} \left(\frac{16}{15} \right)^i \right\} \leq \left(\frac{1}{2} \right)^{i+2}, \quad i \in \mathbb{N}, \end{aligned}$$

since $b_0 \geq 60$. So we have (5.6) and the proof is complete. \square

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