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COMPOSITION SEMIGROUPS AND RANDOM STABILITY

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A random variable X is *N-divisible* if it can be decomposed into a random sum of N i.i.d. components, where N is a random variable independent of the components; X is *N-stable* if the components are rescaled copies of X. These N-divisible and N-stable random variables arise in a variety of stochastic models, including thinned renewal processes and subordinated Lévy and stable processes. We consider a general theory of N-divisibility and N-stability in the case where $E(N) < \infty$, based on a representation of the probability generating function of N in terms of its limiting Laplace–Stieltjes transform ℓ. We analyze certain topological semigroups of such p.g.f.'s in detail, and on this basis we extend existing characterizations of N-divisible and N-stable laws in terms of ℓ. We apply the results to the aforementioned stochastic models.

1. Introduction. A random variable X is infinitely divisible in the classical sense if it can be decomposed (in distribution) into the sum of n i.i.d. components, for any natural number n; in particular, X is strictly stable in the classical sense if the components are rescaled copies of X itself. In this paper we study the problem of *random* divisibility and stability: we say that X is *N-divisible* if X can be decomposed into a random sum of N i.i.d. components, where N is a natural number-valued random variable independent of the components, and we say that X is *N-stable* if the components are rescaled copies of X. (Here we only consider strict N-stability, that is, no constant term.) The problem is: given the distribution of N, characterize the N-divisible and N-stable probability laws.

This problem first arose in the context of thinned renewal processes. Let $\{R(\tau),\ \tau \in [0,\infty)\}$ denote a renewal process; the *p-thinning* of R is the point process R_p formed by independently retaining each point of R with probability p or deleting it with probability $1 - p$. Rényi (1976) showed that, among renewal processes, the Poisson process is characterized by the fact that

$$\{R(\tau),\ \tau \in [0,\infty)\} =_D \{R_p(p^{-1}\tau),\ \tau \in [0,\infty)\}, \qquad p \in (0,1],$$

where $=_D$ denotes equality in distribution. Gnedenko (1970) considered arbitrary rescalings of the time axis, and he showed that

$$\{R(\tau),\ \tau \in [0,\infty)\} =_D \{R_p(c(p)\tau),\ \tau \in [0,\infty)\}, \qquad p \in (0,1],$$

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for some function $c(\cdot)$ if and only if the interval distribution of R has Laplace–Stieltjes transform (LST) $1/(1+\lambda s^{\alpha})$, $s \geq 0$, where $\alpha \in (0,1]$ and $\lambda > 0$. (The Poisson case corresponds to $\alpha = 1$.) Now an interval of R_p is the random sum of N_p intervals of R, where N_p is geometric with success probability p. So Gnedenko's result says that a distribution on the positive half-line is *geometric stable*—that is, N_p-stable for all $p \in (0,1]$—if and only if it has LST $1/(1+\lambda s^{\alpha})$. Geometric stable and geometric infinitely divisible laws (N_p-divisible for all $p \in (0,1]$) have subsequently been characterized on $\mathscr{R}$ and even in Banach space [see Gnedenko and Janjic (1983), Klebanov, Maniya and Melamed (1984), Pillai (1990), Melamed (1992), Fujita (1993) and Rachev and Samorodnitsky (1994)]. These distributions have various statistical applications, particularly as models for financial data [see Rachev and SenGupta (1992) and Rachev and Samorodnitsky (1994)].

In this paper we consider a theory of N-divisibility and N-stability for a general N with $E(N) < \infty$. Apart from its intrinsic interest, such a theory enables us to construct more flexible stochastic models based on random sums; in particular, it allows us to identify the renewal processes that are invariant up to time rescaling under a broad class of thinnings [see Rachev and Samorodnitsky (1994) for applications]. From another perspective, an N-divisible (respectively N-stable) random variable can be interpreted as the value of a subordinated Lévy (resp. stable) process, and its components as the increments of the process along a subordinated renewal process. We discuss these matters further in Section 3.

When $E(N) < \infty$, the N-divisible and N-stable laws can be characterized by a method based on the following representation of the probability generating function (p.g.f.) of N. The representation comes from the theory of branching processes; in that setting the LST ℓ is interpreted as the limiting LST of the normalized population size.

FACT 1 [Bingham, Goldie and Teugels (1987), page 404, and Asmussen and Hering (1983), page 84]. *Let g denote the p.g.f. of a probability measure on $\mathscr{N} := \{1, 2, \ldots\}$, with mean $\mu = g'(1-) \in (1,\infty)$. Then there is a unique LST ℓ (up to scale), with $\lim_{s\to\infty} \ell(s) = 0$, such that*

$$g(s) = \ell(\mu \ell^{\leftarrow}(s)), \qquad 0 < s \leq 1, \tag{1}$$

where $\ell^{\leftarrow}$ denotes the inverse function of ℓ. Furthermore, $1-\ell$ varies regularly at $0+$ with index 1.

[The geometric p.g.f., for example, can be written as $\ell(p^{-1}\ell^{\leftarrow}(s))$ with $\ell(s) = 1/(1+s)$, the LST of the unit exponential distribution.] To apply the method, one embeds the p.g.f. $g(\cdot) = \ell(\mu\ell^{\leftarrow}(\cdot))$ in the parametric family $\{\ell(\mu^n \ell^{\leftarrow}(\cdot)),\ n \in \mathscr{N}\}$ generated by composing g with itself; by taking certain limits in n, one obtains a representation of the characteristic functions (ch.f.'s) of the N-divisible and N-stable laws in terms of ℓ (see Sections 3 and 4 for details). This was essentially the method used by Rényi (1976) and his successors in the exponential/geometric case; subsequently Klebanov, Maniya and Melamed (1985) used a more general version of the method for the case of a general ℓ with finite mean [see Melamed (1992) for details].

However, for our purposes Fact 1 is overly restrictive. Indeed, we will show below that, for *any* LST ℓ [with $\lim_{s\to\infty}\ell(s) = 0$], there is a nonempty parameter set H_ℓ such that $g_{\ell,t}(\cdot) := \ell(e^t\ell^{\leftarrow}(\cdot))$ is a p.g.f. for $t \in H_\ell$ and $g_{\ell,t}$ has finite mean, although the mean need not be e^t in general (cf. Section 2.2 and Proposition 3). For each ℓ, the set $G_\ell := \{g_{\ell,t}(\cdot),\ t \in H_\ell\}$ is a closed subsemigroup of the general topological semigroup of p.g.f.'s (under composition and weak convergence), and each (nontrivial) G_ℓ determines a nonempty class of "G_ℓ-infinitely divisible" ("G_ℓ-stable") laws, that is, distributions that are $N_{\ell,t}$-divisible ($N_{\ell,t}$-stable) for all $t \in H_\ell$, where $N_{\ell,t}$ has p.g.f. $g_{\ell,t}$.

We proceed as follows. In Section 2.1 we give several examples of p.g.f.'s generated by specific LSTs, including the exponential/geometric as a special case. In Section 2.2 we show that, given ℓ, H_ℓ and G_ℓ are isomorphic and homeomorphic closed "Hun" semigroups [cf. Ruzsa and Székely (1988)], and we organize the H_ℓ's (and hence the ℓ's) into a lattice. In Section 2.3 we compute the mean of $g_{\ell,t}$ as a function of $t \in H_\ell$, particularly when $1-\ell$ is regularly varying at 0+ (with index $\alpha \in [0,1]$), and we relate this to Fact 1. In Section 2.4 we find that the set of all p.g.f.'s of the form $g_{\ell,t}$, namely $\bigcup_\ell G_\ell$, is equal to the set of p.g.f.'s with finite mean. This set is not closed, but we give a closure property of the union of G_ℓ's over a tight set of ℓ's (i.e., the set of corresponding measures is tight). In Section 3 we show that any infinitely divisible (stable) law has a G_ℓ-infinitely divisible (G_ℓ-stable) counterpart for any ℓ. Conversely, returning to the case of Fact 1, we completely characterize the symmetric or one-sided G_ℓ-infinitely divisible and G_ℓ-stable laws when $1-\ell$ varies regularly at 0+ with index 1. In the course of this we discuss the interpretation in terms of subordinated Lévy (stable) processes mentioned above, and the application to renewal processes thinned by mechanisms more general than p-thinning. Finally, Section 4 contains the proofs.

2. Composition semigroups of probability generating functions. Let $\mathscr{G}$ denote the set of p.g.f.'s $g(s)$, $0 < s \le 1$, of probability measures on $\mathscr{N}$. With the operation of composition and the topology of weak convergence, $\mathscr{G}$ is a topological semigroup [Carruth, Hildebrant and Koch (1983)]. Here and throughout we use the term "semigroup" to mean a semigroup *with an identity element*; such a semigroup is also called a *monoid*. Note that $\mathscr{G}$ is not commutative.

Let $\mathscr{L}$ denote the set of LSTs of probability measures on $[0,\infty)$ without an atom at 0, so that $\lim_{s\to\infty}\ell(s) = 0$ for $\ell \in \mathscr{L}$. If $1-\ell$ varies regularly at 0+ with index $\alpha \in [0,1]$, we write $1-\ell \in \mathrm{RV}_\alpha^{0+}$ [see Bingham, Goldie and Teugels (1987)]. As before, given $\ell \in \mathscr{L}$, let $H_\ell = \{t \in \mathscr{R}\colon\ g_{\ell,t} \in \mathscr{G}\}$ and let $G_\ell = \{g_{\ell,t}\colon\ t \in H_\ell\}$. Finally, let $\mathscr{H}$ denote the topological semigroup $\{[0,\infty)\}$ with the operation of addition and the usual topology, and recall that a semigroup is called *monothetic* if it is generated by a single element.

2.1. *Examples.* We begin with the following observation. Let $\ell \in \mathscr{L}$ and let $\ell_\alpha(s) = \ell(s^\alpha)$, $s \geq 0$, $\alpha \in (0, 1]$. Then $\ell_\alpha \in \mathscr{L}$ and

$$\ell_\alpha(e^{t/\alpha}\ell_\alpha^{\leftarrow}(s)) = \ell(e^t\ell^{\leftarrow}(s)), \qquad 0 < s \leq 1,$$

so $H_{\ell_\alpha} = \alpha^{-1}H_\ell$ and $G_{\ell_\alpha} = G_\ell$. Thus for our purposes the difference between ℓ and ℓ_α is inconsequential, although the corresponding distributions are quite different.

EXAMPLE 1. Suppose that $\ell(s) = (1+s)^{-\gamma}$, $s \geq 0$, for some $\gamma > 0$. This is the LST of the gamma distribution with shape parameter γ; when $\gamma = 1$, ℓ is the LST of the exponential distribution with mean 1. Then

$$g_{\ell,t}(s) = \ell(e^t\ell^{\leftarrow}(s)) = \left(\frac{e^{-t}s^{1/\gamma}}{1-(1-e^{-t})s^{1/\gamma}}\right)^\gamma, \qquad 0 < s \leq 1,$$

and

$$H_\ell = \begin{cases} \mathscr{H}, & \text{if } \gamma = 1/k \text{ for some } k \in \mathscr{N}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

If $\gamma = 1/k$, then

$$G_\ell = \left\{\left(\frac{e^{-t}s^k}{1-(1-e^{-t})s^k}\right)^{1/k},\ t \in \mathscr{H}\right\}$$

and

$$p_{\ell,t}\{j\} = \begin{cases} \left(\displaystyle\prod_{i=1}^{(j-1)/k} \frac{1+(i-1)k}{ik}\right)(e^{-t})^{1/k}(1-e^{-t})^{(j-1)/k}, & j = 1, k+1, 2k+1, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

$t \in \mathscr{H}$, where $p_{\ell,t}\{\cdot\}$ denotes the measure corresponding to $g_{\ell,t}$. The case $\gamma = 1/k$ was considered by Melamed (1992); when $k = 1$ this reduces to the exponential/geometric semigroup. We also note that, when $\gamma = 1$, $\ell_\alpha(s) = 1/(1+s^\alpha)$ is the LST of the *Mittag–Leffler distribution* of index α [Pillai (1990) and Fujita (1993)].

EXAMPLE 2. Let $\ell \in \mathscr{L}$ and let

$$\phi(s) = \frac{\ell(s)}{1-\ell(s)} = \sum_{n\geq 1}\ell^n(s) = \text{ LST of the renewal measure } \sum_{n\geq 1}F^{*n},$$

$s > 0$, where F^{*n} denotes the nth convolution of the d.f. F corresponding to ℓ. Suppose that

$$e^{\alpha\beta}\phi(e^\beta s) = \phi(s), \qquad s > 0, \tag{2}$$

for some $\alpha \in (0, 1]$ and $\beta > 0$. Such functions ϕ were studied by Dubuc (1990) and by Biggins and Bingham (1991); their results show that any such ϕ is very close to a multiple of $s^{-\alpha}$. Then $\ell(s) = \phi(s)/(1+\phi(s))$, where ϕ satisfies (2); Jayakumar and Pillai (1993) called such LSTs *semi-Mittag–Leffler*. In this case

$$g_{\ell,t}(s) = \ell(e^t \ell^{\leftarrow}(s)) = \frac{\phi(e^t \phi^{\leftarrow}(s/(1-s)))}{1+\phi(e^t \phi^{\leftarrow}(s/(1-s)))}, \qquad 0 < s \le 1,$$

$$\{n\beta, n \in \mathscr{N}_0\} \subset H_\ell, \qquad \left\{\frac{e^{-\alpha n\beta} s}{1-(1-e^{-\alpha n\beta})s},\ n \in \mathscr{N}_0\right\} \subset G_\ell$$

and

$$p_{\ell, n\beta}\{j\} = e^{-\alpha n\beta}(1-e^{-\alpha n\beta})^{j-1}, \qquad n \in \mathscr{N}_0,\ j \in \mathscr{N},$$

where $\mathscr{N}_0 := \mathscr{N} \cup \{0\}$. But $H_\ell \ne \mathscr{H}$ unless $\phi(s) = s^{-\alpha}$, which is the Mittag–Leffler case of Example 1.

EXAMPLE 3. Suppose that ℓ is the LST of the unit mass at 1, $\ell(s) = e^{-s}$. Then

$$g_{\ell,t}(s) = \ell(e^t \ell^{\leftarrow}(s)) = s^{e^t}, \qquad 0 < s \le 1,$$

$$H_\ell = \{\log n,\ n \in \mathscr{N}\}, \qquad G_\ell = \{s^n,\ n \in \mathscr{N}\}$$

and

$$p_{\ell, \log n}\{j\} = \delta_{j,n}, \qquad n \in \mathscr{N},\ j \in \mathscr{N},$$

where $\delta_{j,n}$ is the Kronecker delta. Note that in this case H_ℓ has Lebesgue measure 0 but is not monothetic. Note also that $\ell_\alpha(s) = e^{-s^\alpha}$ is the LST of a positive α-stable distribution.

EXAMPLE 4. Let $\ell \in \mathscr{L}$ and suppose that $\ell(s) = \exp(-\rho(s))$, where ρ is a positive function with completely monotone derivative. Suppose further that

$$e^{-\log k}\rho(e^{(\log k)/\alpha} s) = \rho(s), \qquad s \ge 0,$$

for some $\alpha \in (0,1]$ and $k \in \mathscr{N}$; in this case the results of Dubuc (1990) and Biggins and Bingham (1991) apply to ρ'. Then

$$g_{\ell,t}(s) = \ell(e^t \ell^{\leftarrow}(s)) = \exp(-\rho(e^t \rho^{\leftarrow}(-\log s))), \qquad 0 < s \le 1.$$

In this case

$$\left\{\frac{n \log k}{\alpha},\ n \in \mathscr{N}_0\right\} \subset H_\ell \quad \text{and} \quad \left\{s^{k^n},\ n \in \mathscr{N}_0\right\} \subset G_\ell$$

and

$$p_{\ell, (n\log k)/\alpha}\{j\} = \delta_{j,k^n}, \qquad n \in \mathscr{N}_0,\ j \in \mathscr{N}.$$

2.2. *Basic structure.* We note first that $0 \in H_\ell$ for any $\ell \in \mathscr{L}$, since $g_{\ell,0}(s) = \ell(e^0 \ell^{\leftarrow}(s)) = s$, which is the p.g.f. of the unit mass at 1. Thus H_ℓ is always nonempty. In general, however, it seems to be difficult to derive H_ℓ and G_ℓ directly from ℓ, even when ℓ can be inverted in closed form. However, we have the following structural description of H_ℓ, G_ℓ and their relationship.

PROPOSITION 1. *For each $\ell \in \mathscr{L}$,*

(a) *H_ℓ is a closed subsemigroup of $\mathscr{H}$;*
(b) *G_ℓ is a closed commutative subsemigroup of $\mathscr{G}$;*
(c) *the map Φ_ℓ: $t \mapsto g_{\ell,t}$ from H_ℓ to G_ℓ is an isomorphism and a homeomorphism.*

Now let $\langle \mathscr{H} \rangle$ denote the family of closed subsemigroups of $\mathscr{H}$, partially ordered by inclusion and with binary operations $H \wedge J := H \cap J$ and $H \vee J :=$ the closed semigroup generated by $H \cup J$, where $H, J \in \langle \mathscr{H} \rangle$. Then $\langle \mathscr{H} \rangle$ is a lattice, with unit $\mathscr{H}$ and zero $\{0\}$. However, while $\langle \mathscr{H} \rangle$ partially orders the H_ℓ's and partitions $\mathscr{L}$ into $\bigcup_{H \in \langle \mathscr{H} \rangle} \{\ell \in \mathscr{L}\colon H_\ell = H\}$, we cannot at present guarantee that for *every* $H \in \langle \mathscr{H} \rangle$ there is some $\ell \in \mathscr{L}$ such that $H_\ell = H$.

We can shed some light on the internal structure of the H_ℓ's by noting that they are "Hun" semigroups, as defined and analyzed by Ruzsa and Székely (1988). For this we need the following definitions, which we adapt to our additive notation. First, let S denote a commutative Hausdorff topological semigroup, and let $t, u \in S$. If there is some $v \in S$ such that $t = u + v$, we say that *u divides t* and write $u|t$.

DEFINITION 1 [Ruzsa and Székely (1988), page 15]. *The semigroup S is Hun if ($u|t$ and $t|u$) implies $t = u$, and $\{u\colon u|t\}$ is compact $\forall\, t \in S$.*

PROPOSITION 2. *A closed subsemigroup of $\mathscr{H}$ is Hun.*

Ruzsa and Székely (1988), Section 2, gave a detailed account of *decomposition* of Hun semigroups, including "the existence of a decomposition [of an arbitrary element]...into irreducibles and an anti-irreducible, or into more general kinds of factors," and many other results. However, we do not exploit their results further here, so we refer the reader to their book for details.

2.3. *Means and regular variation.* We now consider the mean of $g_{\ell,t}$ as a function of $t \in H_\ell$.

PROPOSITION 3. *Let $\ell \in \mathscr{L}$.*

(a) *For all $t \in H_\ell$, $g_{\ell,t}$ has mean $g'_{\ell,t}(1-) = e^{\eta(t)}$, where η: $H_\ell \to [0,\infty)$, $0 \le \eta(t) \le t\ \forall\, t \in H_\ell$, and η is additive and strictly increasing.*
(b) *If $1 - \ell \in RV_\alpha^{0+}$ for some $\alpha \in [0,1]$, then $\eta(t) = \alpha t$.*
(c) *If H_ℓ contains a set of positive Lebesgue measure, then $1 - \ell \in RV_\alpha^{0+}$ for some $\alpha \in (0,1]$. In this case $\eta(t) = \alpha t$.*

(d) *If* $H_\ell = \mathscr{H}$, *then* $\ell = \ell^*_\alpha$ *for some* $\alpha \in (0,1]$ *and* $\ell^* \in \mathscr{L}$ *with* $1-\ell^* \in RV_1^{0+}$. *In this case* $1-\ell \in RV_\alpha^{0+}$ *and* $\eta(t) = \alpha t$.

Thus the mean of $g_{\ell,t}$ need not be e^t. On the other hand, by Fact 1, for each $t \in H_\ell$ there is a unique $\ell_t \in \mathscr{L}$ (up to scale) such that

$$\ell(e^t \ell^{\leftarrow}(s)) = \ell_t(e^{\eta(t)} \ell_t^{\leftarrow}(s)), \qquad 0 < s \le 1.$$

If $1-\ell \in \mathrm{RV}_1^{0+}$, then Proposition 3(b) implies that $\ell_t \equiv \ell$; if $H_\ell = \mathscr{H}$, then Proposition 3(d) implies that

$$\ell(e^t \ell^{\leftarrow}(s)) = \ell^*_\alpha(e^t \ell^{*\leftarrow}_\alpha(s)) = \ell^*(e^{\alpha t} \ell^{*\leftarrow}(s)), \qquad 0 < s \le 1, \ t \in \mathscr{H},$$

that is, $\ell_t \equiv \ell^*$. However, it seems that in general ℓ_t may depend on t.

2.4. *Existence and approximation.* Next we consider the set of all p.g.f.'s of the form $g_{\ell,t}$, that is, $\bigcup_{\ell \in \mathscr{L}} G_\ell$. First, from Fact 1 and Proposition 3(a) we have

$$\bigcup_{\ell \in \mathscr{L}} G_\ell = \{g \in \mathscr{G}\colon \ g \text{ has finite mean}\}.$$

Observe that $\bigcup_{\ell \in \mathscr{L}} G_\ell$ is not closed in $\mathscr{G}$: for example, let g_n denote the p.g.f. of the probability measure

$$p_n\{j\} := \frac{c_n}{j(j+1)^{1+1/n}}, \qquad j \in \mathscr{N},$$

where $n \in \mathscr{N}$ and c_n is the appropriate normalizing constant. Then $\{g_n, \ n \in \mathscr{N}\} \subset \bigcup_{\ell \in \mathscr{L}} G_\ell$, but $\lim_{n\to\infty} g_n \in \mathscr{G} \setminus \bigcup_{\ell \in \mathscr{L}} G_\ell$. However, the next result shows that if $\lim_n \ell_n \in \mathscr{L}$, then $\lim_n g_{\ell_n, t_n} \in \bigcup_{\ell \in \mathscr{L}} G_\ell$.

PROPOSITION 4. (a) *Let* $\{\ell_n, n \in \mathscr{N}\} \subset \mathscr{L}$ *and suppose that* $\ell_n \to \ell \in \mathscr{L}$ *as* $n \to \infty$. *Let* $t_n \in H_{\ell_n}$ *and suppose that* $g_{\ell_n, t_n} \to g \in \mathscr{G}$ *as* $n \to \infty$. *Then* $g = g_{\ell,t}$ *for some* $t \in H_\ell$.

(b) *Let* $\Lambda \subset \mathscr{L}$ *denote a tight set of LSTs, that is, the set of probability measures corresponding to LSTs in* Λ *is tight, and suppose that* $\overline{\Lambda} \subset \mathscr{L}$, *where* $\overline{\Lambda}$ *denotes the closure of* Λ *with respect to weak convergence in the set of all LSTs of probability measures on* $[0,\infty)$. *Then*

$$\overline{\bigcup_{\ell \in \Lambda} G_\ell} \subset \bigcup_{\ell \in \overline{\Lambda}} G_\ell,$$

where the closure on the left-hand side is in $\mathscr{G}$ (*with respect to weak convergence*).

In particular, (a) implies that no sequence of LSTs corresponding to $\{g_n\}$ in the example above can converge to an LST in $\mathscr{L}$.

3. Random infinite divisibility and stability. Up to now we have only considered $g_{\ell,t}(s)$ for real $s \in (0,1]$. However, in the equations of $N_{\ell,t}$-divisibility and stability the argument of $g_{\ell,t}(\cdot)$ will be a characteristic function, so we must define $g_{\ell,t}(s)$ for s in the closed complex unit disk. To avoid specifying a complex inverse for ℓ, we write

$$g_{\ell,t}(s) = \sum_{n \geq 1} p_{\ell,t}\{n\} s^n, \qquad s \in (0,1],\ t \in H_\ell. \tag{3}$$

If $t \in H_\ell$, then $g_{\ell,t} \in \mathscr{G}$, so in this case we take (3) as the definition of $g_{\ell,t}(s)$ for s in the closed unit disk. Also, for the logarithm of a ch.f. we use the "distinguished logarithm" specified by Chung (1974), Section 7.6. We can now define G_ℓ-infinite divisibility and G_ℓ-stability [cf. Definition 2.1 in Melamed (1992)].

DEFINITION 2. Let $\ell \in \mathscr{L}$ with $H_\ell \neq \{0\}$, and let X denote a (real-valued) random variable with ch.f. f.

(a) We say that X (or f) is *G_ℓ-infinitely divisible* if X is $N_{\ell,t}$-divisible for every $t \in H_\ell$. In other words,

$$X =_D \sum_{n=1}^{N_{\ell,t}} X_n^{(t)} \qquad \forall\ t \in H_\ell,$$

where $\{X_n^{(t)}, n \in \mathscr{N}\}$ is an i.i.d. sequence independent of $N_{\ell,t}$. In terms of ch.f.'s, for every $t \in H_\ell$ there is a ch.f. f_t such that

$$f(u) = g_{\ell,t}(f_t(u)), \qquad u \in \mathscr{R}.$$

(b) We say that X (or f) is *G_ℓ-stable* if X is $N_{\ell,t}$-stable for every $t \in H_\ell$. In other words, there is a function c: $H_\ell \to \mathscr{R}$ such that

$$X =_D c(t) \sum_{n=1}^{N_{\ell,t}} X_n \qquad \forall\ t \in H_\ell,$$

where $\{X_n, n \in \mathscr{N}\}$ is an i.i.d. sequence with common ch.f. f independent of $N_{\ell,t}$. In terms of ch.f.'s,

$$f(u) = g_{\ell,t}(f(c(t)u)), \qquad u \in \mathscr{R},\ \forall\ t \in H_\ell.$$

We now show that any infinitely divisible (respectively strictly stable) random variable has a G_ℓ-infinitely divisible (resp. G_ℓ-stable) counterpart. First, on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, define a positive random variable W with LST $\ell \in \mathscr{L}$, where $H_\ell \neq \{0\}$. On another probability space $(\Omega', \mathscr{F}', \mathscr{P}')$, define a sequence of i.i.d. copies of W, $\{W_n,\ n \in \mathscr{N}\}$, and a random variable $N_{\ell,t}$ independent of $\{W_n,\ n \in \mathscr{N}\}$. Since W is G_ℓ-stable with $c(t) = e^{-t}$, we have

$$W =_D e^{-t} \sum_{n=1}^{N_{\ell,t}} W_n \qquad \forall\ t \in H_\ell.$$

Now let ϕ denote an infinitely divisible ch.f. and define a Lévy process $\{Y(\tau),\ \tau \in [0,\infty)\}$ on $(\Omega, \mathscr{F}, \mathscr{P})$, with distribution determined by $\{\phi^\tau,\ \tau \in [0,\infty)\}$. Also, define a sequence of i.i.d. copies of $Y(\cdot)$, $\{Y_n(\tau),\ \tau \in [0,\infty),\ n \in \mathscr{N}\}$, on $(\Omega', \mathscr{F}', \mathscr{P}')$. Now fix a nonzero $t \in H_\ell$ and write $\tau_n := e^{-t}\sum_{j=1}^n W_j$, $n \in \mathscr{N}$, $\tau_0 := 0$. A conditioning argument then yields

$$\begin{aligned} Y(W) &=_D Y_1(\tau_{N_{\ell,t}}) = \sum_{n=1}^{N_{\ell,t}} (Y_1(\tau_n) - Y_1(\tau_{n-1})) \\ &=_D \sum_{n=1}^{N_{\ell,t}} Y_n(\tau_n - \tau_{n-1}) = \sum_{n=1}^{N_{\ell,t}} Y_n(e^{-t}W_n). \end{aligned}$$

Since the ch.f. of $Y(W)$ is $\ell(-\log\phi)$, we have the following result (the stable case follows by an analogous argument).

PROPOSITION 5. *Let $\ell \in \mathscr{L}$ with $H_\ell \neq \{0\}$.*

(a) *If $\phi(\cdot)$ is an infinitely divisible ch.f., then $\ell(-\log\phi(\cdot))$ is a G_ℓ-infinitely divisible ch.f.*

(b) *If $\phi(\cdot)$ is a strictly stable ch.f., then $\ell(-\log\phi(\cdot))$ is a G_ℓ-stable ch.f.*

Klebanov, Maniya and Melamed (1985) proved Proposition 5 directly for ℓ with finite mean [without using the process $Y(\cdot)$], and they also proved the converse in this case. We can extend the converse to the case where $1 - \ell \in \mathrm{RV}_1^{0+}$ (as in Fact 1), when X is symmetric or one-sided.

THEOREM 1. *Let $\ell \in \mathscr{L}$ with $H_\ell \neq \{0\}$ and $1 - \ell \in RV_1^{0+}$, and suppose that f is the ch.f. of a measure that is either symmetric* (*about* 0) *or concentrated on* $[0,\infty)$.

(a) *If f is G_ℓ-infinitely divisible, then*

$$f(u) = \ell(-\log\phi(u)), \qquad u \in \mathscr{R}, \tag{4}$$

for some infinitely divisible ch.f. ϕ.

(b) *Suppose further that $\{\log n, n \in \mathscr{N}\} \subset H_\ell$. If f is G_ℓ-stable, then* (4) *holds for some strictly stable ch.f. ϕ.*

In either case, if f is symmetric or concentrated on $[0,\infty)$, then so is ϕ.

By combining Proposition 5(b) and Theorem 1(b) with information about transforms of symmetric or positive stable laws, we get the following result.

COROLLARY 1. *Let $\ell \in \mathscr{L}$ with $1 - \ell \in RV_1^{0+}$ and $\{\log n, n \in \mathscr{N}\} \subset H_\ell$.*

(a) *The function $f(u)$, $u \in \mathscr{R}$, is the ch.f. of a symmetric G_ℓ-stable distribution if and only if* (*up to scale*)

$$f(u) = \ell(|u|^\alpha), \qquad u \in \mathscr{R},$$

for some $\alpha \in (0,2]$.

(b) *The function* $f(s)$, $s \geq 0$, *is the LST of a* G_ℓ-*stable distribution on* $[0, \infty)$ *if and only if* (*up to scale*)

$$f(s) = \ell(s^\alpha) = \ell_\alpha(s), \qquad s \geq 0,$$

for some $\alpha \in (0, 1]$.

We conclude by returning to the topic of thinned renewal processes. Let $\{R(\tau),\ \tau \in [0, \infty)\}$ denote a renewal process with (i.i.d.) intervals $X_1, X_2, \ldots$, and consider the thinned renewal process $R_{\ell, t}$ formed by retaining the points of R with indices $N_{\ell,t}^{(1)}, N_{\ell,t}^{(1)}+N_{\ell,t}^{(2)}, \ldots$, where $1-\ell \in \mathrm{RV}_1^{0+}$, $t \in H_\ell \supset \{\log n, n \in \mathscr{N}\}$ and $N_{\ell,t}^{(1)}, N_{\ell,t}^{(2)}, \ldots$ are i.i.d. The intervals of $R_{\ell,t}$ are then

$$X_1 + \cdots + X_{N_{\ell,t}^{(1)}},\ X_{N_{\ell,t}^{(1)}+1} + \cdots + X_{N_{\ell,t}^{(1)}+N_{\ell,t}^{(2)}}, \ldots.$$

Corollary 1(b) then says that

$$\{R(\tau),\ \tau \in [0, \infty)\} =_D \{R_{\ell,t}(c(t)\tau),\ \tau \in [0, \infty)\}, \qquad t \in H_\ell,$$

for some function $c(\cdot)$ if and only if X_1 has LST ℓ_α for some $\alpha \in (0, 1]$. Thus we have characterized renewal processes that are invariant up to change of time scale under a broad class of "regenerative" thinning mechanisms.

4. Proofs.

PROOF OF PROPOSITION 1. (a) Clearly, H_ℓ is an additive subsemigroup of $\mathscr{R}$. Furthermore, if there is some $t^* < 0$ in H_ℓ, then $\{g_{\ell, nt^*},\ n \in \mathscr{N}_0\} \subset \mathscr{G}$ and g_{ℓ, nt^*} converges to the p.g.f. of the unit mass at 0 as $n \to \infty$. However, this is impossible by the continuity theorem for p.g.f.'s, so $H_\ell \subset \mathscr{H}$. To see that H_ℓ is closed, suppose that $\{t_n,\ n \in \mathscr{N}\} \subset H_\ell$ with $t_n \to t \in \mathscr{H}$ as $n \to \infty$. By the continuity of ℓ,

$$g_{\ell, t_n}(s) = \ell(e^{t_n}\ell^{\leftarrow}(s)) \to \ell(e^t \ell^{\leftarrow}(s)) = g_{\ell,t}(s), \qquad n \to \infty,\ 0 < s \leq 1.$$

Then $g_{\ell,t} \in \mathscr{G}$ by the continuity theorem for p.g.f.'s, and hence $t \in H_\ell$.

(b) Clearly, G_ℓ is a commutative subsemigroup of $\mathscr{G}$, so it remains to show that G_ℓ is closed in $\mathscr{G}$. Suppose that

$$g_{\ell, t_n}(s) = \ell(e^{t_n}\ell^{\leftarrow}(s)) \to g(s), \qquad n \to \infty,\ 0 < s \leq 1,$$

where $\{t_n,\ n \in \mathscr{N}\} \subset H_\ell$ and $g \in \mathscr{G}$. Then, by the continuity of $\ell^{\leftarrow}$, t_n converges to a limit t, that is, $g = g_{\ell,t}$.

(c) It is readily shown that Φ_ℓ is an isomorphism, and the continuity of Φ_ℓ and $\Phi_\ell^{\leftarrow}$ follows from the continuity of ℓ and $\ell^{\leftarrow}$, respectively. □

PROOF OF PROPOSITION 2. Let $H \in \langle \mathscr{H} \rangle$; then clearly H satisfies the first part of the definition. For the second part it suffices to show that $\{u\colon\ u|t\}$ is closed $\forall\ t \in H$, since $\{u\colon\ u|t\} \subset [0, t]$. Fix $t \in H$ and let $\{u_n,\ n \in \mathscr{N}\}$ be a convergent sequence of divisors of t; then $u := \lim_{n\to\infty} u_n \in H$, $\{t - u_n,\ n \in \mathscr{N}\} \subset H$ and $t - u \in H$. Hence $u|t$. □

PROOF OF PROPOSITION 3. (a) Assume that $t \in H_\ell$. First, it is clear that $g'_{\ell,t}(1-) = 1$ iff $t = 0$, and otherwise $g'_{\ell,t}(1-) > 1$. The first result then follows from the fact that $g'_{\ell,t}(s)$ is increasing in s and

$$g'_{\ell,t}(s) = \frac{d}{ds}\ell(e^t \ell^{\leftarrow}(s)) = e^t \frac{-\ell'(e^t \ell^{\leftarrow}(s))}{-\ell'(\ell^{\leftarrow}(s))} \le e^t \qquad \forall\, s \in (0,1).$$

It is readily shown that η is strictly increasing and additive.

(b) If $1 - \ell \in \mathrm{RV}_\alpha^{0+}$, then

$$\begin{aligned} g'_{\ell,t}(1-) &= \lim_{s\uparrow 1} \frac{d}{ds}\ell(e^t \ell^{\leftarrow}(s)) \\ &= \lim_{s\uparrow 1} e^t \frac{-\ell'(e^t \ell^{\leftarrow}(s))}{-\ell'(\ell^{\leftarrow}(s))} \\ &= \lim_{s\downarrow 0} e^t \frac{-\ell'(e^t s)}{-\ell'(s)} = e^t e^{(\alpha-1)t} = e^{\alpha t}, \end{aligned}$$

where the next-to-last equality follows from Theorem 1.7.2b, page 39, in Bingham, Goldie and Teugels (1987).

(c) If $t \in H_\ell$, then

$$\lim_{s\downarrow 0} \frac{1-\ell(e^t s)}{1-\ell(s)} = \lim_{s\downarrow 0} e^t \frac{-\ell'(e^t s)}{-\ell'(s)} = e^{\eta(t)}.$$

If H_ℓ contains a set of positive Lebesgue measure, then Theorem 1.4.1, page 17, in Bingham, Goldie and Teugels (1987) implies that $1 - \ell \in \mathrm{RV}_\alpha^{0+}$, and necessarily $\alpha \in (0,1]$.

(d) By (c), if $H_\ell = \mathscr{H}$, then $1 - \ell \in \mathrm{RV}_\alpha^{0+}$ for some $\alpha \in (0,1]$ and $\eta(t) = \alpha t$. By Theorem 2.7, page 124, in Asmussen and Hering (1983), if $H_\ell = \mathscr{H}$, then there exists a unique $\ell^{**} \in \mathscr{L}$ (up to scale), with $1 - \ell^{**} \in \mathrm{RV}_1^{0+}$, such that

$$\ell(e^t \ell^{\leftarrow}(s)) = \ell^{**}(e^{\alpha t} \ell^{**\leftarrow}(s)), \qquad 0 < s \le 1,\ t \in \mathscr{H}. \tag{5}$$

Then, taking $s = \ell(1)$ in (5), we get

$$\ell(e^t) = \ell^{**}(e^{\alpha t} \ell^{**\leftarrow}(\ell(1))), \qquad t \in \mathscr{H}.$$

However, two LSTs that coincide on $[1, \infty)$ coincide on $[0, \infty)$ [Feller (1971), page 432], so

$$\ell(s) = \ell^{**}(\ell^{**\leftarrow}(\ell(1)) s^\alpha) =: \ell^*(s^\alpha) = \ell_\alpha^*(s), \qquad s \ge 0. \qquad \square$$

PROOF OF PROPOSITION 4. (a) By hypothesis,

$$g_{\ell_n,t_n}(s) = \ell_n(e^{t_n} \ell_n^{\leftarrow}(s)) \to g(s), \qquad n \to \infty,\ 0 < s \le 1,$$

where $g \in \mathscr{G}$. Now $g_{\ell_n,t_n}(s) \to g(s)$ locally uniformly as $n \to \infty$, and also $\ell_n^{\leftarrow}(s) \to \ell^{\leftarrow}(s)$ locally uniformly. Then, since $\ell_n(s) \to \ell(s)\ \forall\, s \ge 0$, we have

$$\ell_n^{\leftarrow}(g_{\ell_n,t_n}(\ell_n(s))) = e^{t_n} s \to \ell^{\leftarrow}(g(\ell(s))), \qquad n \to \infty,\ s \ge 0.$$

Hence $\{t_n,\ n \in \mathscr{N}\}$ converges to a limit $t \in \mathscr{H}$, and therefore

$$g_{\ell_n, t_n}(s) = \ell_n(e^{t_n}\ell_n^{\leftarrow}(s)) \to \ell(e^t \ell^{\leftarrow}(s)) = g(s), \qquad n \to \infty,\ 0 < s \le 1.$$

(b) Let $\{\ell_n, n \in \mathscr{N}\} \subset \Lambda$, let $t_n \in H_{\ell_n}$ and suppose that $g_{\ell_n, t_n} \to g \in \mathscr{G}$ as $n \to \infty$. By Prohorov's theorem, we can use the tightness of Λ to select a convergent subsequence $\{\ell_{n_k},\ k \in \mathscr{N}\}$ with $\ell := \lim_{k\to\infty}\ell_{n_k} \in \overline{\Lambda} \subset \mathscr{L}$. Then, by part (a),

$$g = \lim_{n\to\infty} g_{\ell_n, t_n} = \lim_{k\to\infty} g_{\ell_{n_k}, t_{n_k}} = g_{\ell, t}$$

for some $t \in H_\ell$. □

Proof of Theorem 1. (a) In the following arguments we need a real-valued transform f. For concreteness we take f to be the ch.f. of a symmetric measure, in which case f is real. To prove the theorem for a measure concentrated on $[0, \infty)$, take f to be the (real-valued) LST of the measure; the assertion of the theorem remains true for the corresponding ch.f.

First, we need to know that f does not vanish in the symmetric case.

Lemma 1. *Let $\ell \in \mathscr{L}$ with $H_\ell \neq \{0\}$. If f is the ch.f. of a symmetric G_ℓ-infinitely divisible law, then $f(u) > 0$ for all $u \in \mathscr{R}$.*

Proof 1. Fix a nonzero $t \in H_\ell$, so that $\{nt,\ n \in \mathscr{N}\} \subset H_\ell$. We have

$$f(u) = g_{\ell, nt}(f_{nt}(u)), \qquad u \in \mathscr{R},\ n \in \mathscr{N}, \tag{6}$$

for some sequence of ch.f.'s $\{f_{nt},\ n \in \mathscr{N}\}$. Clearly, f_{nt} is real for every n. Furthermore, there is an $\epsilon > 0$ such that, for all $n \in \mathscr{N}$, $f_{nt}(u) > 0$ for $u \in (-\epsilon, \epsilon)$. Therefore,

$$f(u) = \ell(e^{nt}\ell^{\leftarrow}(f_{nt}(u))), \qquad u \in (-\epsilon, \epsilon),\ n \in \mathscr{N},$$

and

$$f_{nt}(u) = \ell(e^{-nt}\ell^{\leftarrow}(f(u))), \qquad u \in (-\epsilon, \epsilon),\ n \in \mathscr{N}. \tag{7}$$

But (7) implies that

$$f_{nt}(u) \to 1, \qquad u \in (-\epsilon, \epsilon),\ n \to \infty,$$

and hence $f_{nt}(u) \to 1$ for all $u \in \mathscr{R}$. By taking n large enough in (6), we see that $f(u) > 0$ for every $u \in \mathscr{R}$. □

Proof of Theorem 1 (Continued). (a) As above, fix a nonzero $t \in H_\ell$. Then $\{nt,\ n \in \mathscr{N}\} \subset H_\ell$, and the function $f_{nt}(u) = \ell(e^{-nt}\ell^{\leftarrow}(f(u)))$, $u \in \mathscr{R}$, is a ch.f. for all n. Observe now that, for any $c > 0$, $\ell(c(1-s))$, $0 < s \le 1$, is the p.g.f. of a mixed Poisson distribution (on $\mathscr{N}_0$) with mixing distribution corresponding to $\ell(cs)$. Hence $\ell(c(1-f_{nt}))$ is a ch.f. for all c and n, and so is the function

$$\ell\left(\frac{1}{1-\ell(e^{-nt})}(1-f_{nt}(u))\right) = \ell\left(\frac{1}{1-\ell(e^{-nt})}(1-\ell(e^{-nt}\ell^{\leftarrow}(f(u))))\right), \qquad u \in \mathscr{R}.$$

Since $1-\ell \in \mathrm{RV}_1^{0+}$ and f is real, we have

$$\lim_{n\to\infty} \ell\left(\frac{1}{1-\ell(e^{-nt})}(1-f_{nt}(u))\right) = \lim_{n\to\infty} \ell\left(\frac{1}{1-\ell(e^{-nt})}(1-\ell(e^{-nt}\ell^{\leftarrow}(f(u))))\right) = f(u), \qquad u \in \mathscr{R}.$$

Hence

$$\begin{aligned}\phi(u) &:= \exp(-\ell^{\leftarrow}(f(u))) \\ &= \lim_{n\to\infty} \exp\left(-\ell^{\leftarrow}\left(\ell\left(\frac{1}{1-\ell(e^{-nt})}(1-f_{nt}(u))\right)\right)\right) \\ &= \lim_{n\to\infty} \exp\left(\frac{1}{1-\ell(e^{-nt})}(f_{nt}(u)-1)\right) \\ &= \lim_{n\to\infty} \exp(c_n(f_{nt}(u)-1)), \qquad u \in \mathscr{R},\end{aligned} \tag{8}$$

where $c_n := 1/(1-\ell(e^{-nt}))$. According to (8), ϕ is the limit of compound Poisson ch.f.'s. Clearly, ϕ is continuous at 0, so ϕ is a ch.f. by the continuity theorem for ch.f.'s [Chung (1974), Theorem 6.3.2, page 161], and ϕ is infinitely divisible, being the limit of compound Poisson ch.f.'s [Feller (1971), Theorem 3, page 557].

(b) If f is G_ℓ-stable, then f is G_ℓ-infinitely divisible, and (a) then implies that $\phi = \exp(-\ell^{\leftarrow}(f))$ is an infinitely divisible ch.f. Writing the defining equation of G_ℓ-stability in terms of ϕ, we have

$$\phi(u) = (\phi(c(t)u))^{e^t}, \qquad u \in \mathscr{R},\ t \in H_\ell, \tag{9}$$

and if $\{\log n,\ n \in \mathscr{N}\} \subset H_\ell$, then (9) implies that ϕ is strictly stable. □

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