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THE COMPLETE CONVERGENCE THEOREM OF THE CONTACT PROCESS ON TREES[1]

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Consider the contact process on a homogeneous tree with degree $d \geq 3$. Denote by

$$\lambda_c = \inf\{\lambda: P(o \in \xi_t^o \text{ i.o.}) > 0\}$$

the critical value of local survival probability, where o is the root of the tree. Pemantle and Durrett and Schinazi both conjectured that the complete convergence theorem should hold if $\lambda > \lambda_c$. Here we answer the conjecture affirmatively. Furthermore, we will show that

$$P(o \in \xi_t^o \text{ i.o.}) = 0 \quad \text{at } \lambda_c.$$

Therefore, the conclusion of the complete convergence theorem cannot hold at λ_c.

1. Introduction and statement of results. Let T be an infinite homogeneous tree with $d \geq 2$ branches for each vertex in T. Note that T is a line if $d = 2$. The distance $|v_1 - v_2|$ between two vertices v_1 and v_2 is defined to be the number of vertices in the unique path of T from v_1 to v_2. A nominated vertex of T is called the root and labeled o. For simplicity, let $|v - o| = |v|$ for any $v \in T$. Also, for any collection A of vertices, $|A|$ denotes the number of vertices in A. Let S be any connected infinite subgraph of T. Consider the contact process on S as follows. We first set a continuous-time Markov process $\{\xi_t^A(S): t \geq 0\}$ as the collection of finite subsets of vertices in S such that $\xi_0^A(S) = A$ for some $A \subset S$. The vertices in $\xi_t^A(S)$ are thought of as occupied and the system evolves as follows:

1. If $x \in \xi_t^A(S)$, then x becomes vacant at rate 1.
2. If $x \notin \xi_t^A(S)$, then x becomes occupied at rate λ times the number of occupied neighbors.

If $S = T$, we denote

$$\xi_t^A(T) = \xi_t^A.$$

More specifically, we are interested in the processes ξ_t^1 and ξ_t^o, where

$$\xi_t^1 \text{ is the process with } \xi_0^1 = T$$

and

$$\xi_t^o \text{ is the process with } \xi_0^o = o.$$

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One of the most important questions in the contact process is to investigate the stationary measures. We start with two extreme measures. First, let δ_0 be the measure concentrated on the empty configuration. Clearly, δ_0 is a stationary measure. Second, it follows from a simple argument (see [2] or [6]) that $\xi_t^1 \Rightarrow \xi_\infty^1$, where ξ_∞^1 is called the *upper invariant measure*. It is another stationary measure. For a large λ, it was shown in [10] that

$$\xi_t^A \Rightarrow P(\tau^A < \infty)\delta_0 + P(\tau^A = \infty)\xi_\infty^1 \quad \text{as } t \to \infty \tag{1}$$

for any $A \subset T$, where

$$\tau^A = \inf\{t\colon \xi_t^A = \varnothing\}.$$

Equation (1) is often called the *complete convergence theorem*. Set

$$\lambda_s = \inf\{\lambda\colon P(|\xi_t^o| > 0 \text{ for all } t) > 0\},$$

$$\lambda_c = \inf\{\lambda\colon P(o \in \xi_t^o \text{ i.o.}) > 0\},$$

where λ_s and λ_c are the critical values for the survival and the local survival of the contact process, respectively. Clearly,

$$\lambda_s \le \lambda_c.$$

The most interesting phenomenon of the contact process on T, found by [10] and [7], is the difference between its two critical values, that is,

$$\lambda_s < \lambda_c, \tag{2}$$

when $d \ge 3$. For $d = 2$, it has been proved that $\lambda_c = \lambda_s$ (see [2] or [6]).

Let us return to the discussion of the stationary measures of the contact process. Clearly, if $\lambda < \lambda_s$,

$$\xi_t^A \Rightarrow \delta_0.$$

Furthermore, it follows from [9] and [1] that

$$\xi_t^A \Rightarrow \delta_0 \quad \text{at } \lambda_s.$$

When $d \ge 3$ and $\lambda_s < \lambda < \lambda_c$, it was proved in [3] that there are infinitely many extremal stationary measures. On the other hand, by (1), the complete convergence theorem holds for large λ. Then there are only two extremal stationary measures for large λ. It is natural to ask how many extremal stationary measures there are when λ is equal or near from the right-hand side of λ_c. In fact, both [10] and [4] conjectured that the complete convergence theorem should hold for $\lambda > \lambda_c$. Then it will imply that there are only two extremal stationary measures for $\lambda > \lambda_c$. Here we answer this question affirmatively as follows.

THEOREM 1. *For any homogeneous tree with $d \ge 2$, the complete convergence theorem holds if $\lambda > \lambda_c$.*

REMARKS. (a) When $d = 2$, the complete convergence theorem holds if $\lambda > \lambda_s$ (see [2] or [6]). This implies that $\lambda_c = \lambda_s$. Furthermore, the argument that the complete theorem holds when $\lambda > \lambda_c$ is also known for Z^d (see [1]). Therefore, Theorem 1 holds for $d = 2$.

(b) We can also consider the contact process on any homogeneous graph G. Note that if G is homogenous, then we can pick a vertex o as the origin of G. Clearly, we can also let

$$\lambda_c(G) = \inf\{\lambda: P(o \in \xi_t^o(G) \text{ i.o.}) > 0\}.$$

For both $G = Z^d$ and $G = T$, the complete convergence theorem holds if and only if $\lambda > \lambda_c$ by (a), the remark after Theorem 3 and Theorem 1 above. Here we conjecture that the result should hold for any graph as follows.

CONJECTURE. *For any homogeneous graph G, the complete convergence theorem holds iff $\lambda > \lambda_c(G)$.*

In general, the so-called critical case, that is, $\lambda = \lambda_c$, is more complicated. However, the method developed in Theorem 1 allows us also to prove the following theorem.

THEOREM 2. *For $\lambda = \lambda_c$,*

$$P(o \in \xi_t^o \text{ i.o.}) = 0.$$

By using the argument in [4], Theorem 2 will imply that there exist infinitely many extremal stationary distributions at λ_c. More precisely, we have the following theorem.

THEOREM 3. *If $\lambda = \lambda_c$, there are infinitely many extremal stationary distributions.*

REMARK. By Theorem 3, the complete convergence theorem cannot hold at λ_c.

The proofs of the theorems are organized as follows. We collect the preliminary results of the contact process on trees in Section 2. Then we complete the proofs of Theorems 1–3 in Section 3.

Since the proof of Theorem 1 is involved, we would like to outline its proof. To show the complete convergence theorem, one of the useful methods is to check the hypotheses of the following lemma.

LEMMA (Griffeath's lemma). *For any subsets A and B of T, if $\bar{\xi}_t^B$ is an independent copy of the contact process and*

$$P\left(\xi_t^A \cap \bar{\xi}_t^B = \emptyset, \xi_t^A \neq \emptyset, \bar{\xi}_t^B \neq \emptyset\right) \to 0 \quad \text{as } t \to \infty,$$

then the complete convergence theorem holds.

PROOF. See the same proof in Chapter 11 of [2]. □

To verify the hypotheses of Griffeath's lemma, we first show that if $\lambda > \lambda_c$, then for any $\varepsilon > 0$, there exist two positive numbers C and R such that

$$P(x \in \xi_t^o \text{ for } t \le R|x|) \ge C \exp(-\varepsilon |x|) \tag{3}$$

for any $x \in T$. To show (3), we will renormalize T to another tree with "bigger edges" (see Figure 2). Then we show in Proposition 1 that if $\lambda > \lambda_c$, a vertex in any bigger edge is occupied infinitely often in the bigger edge and some of its branches with a positive probability. By using the renormalized edges and standard ergodic and percolation results (see Propositions 2 and 3), we prove (3) in Proposition 4. By (3) we then show

$$\liminf_t P(o \in \xi_t^o(U)) > 0, \tag{4}$$

where U is a branch of o [see the definition of U after (6)]. Equation (4) is proved in Proposition 5 (see the intuitive explanation before the proof of Proposition 5). Finally, by using (3) and (4), we will verify the hypotheses of Griffeath's lemma (see the heuristic argument of the proof of Theorem 1 before the proof of Theorem 1).

2. Preliminaries. Now we only focus on the case $d > 2$. We start with the graphical representation of the contact process (see [2] and [9] for more details). Consider $T \times \mathscr{T}$, where T is a tree and $\mathscr{T}$ is the time interval $[0, \infty)$. We often denote by $\langle x, t \rangle$ and $A \times B$ an element and a subset of $T \times \mathscr{T}$, respectively. We associate each site of T with $d + 1$ independent Poisson processes, one with rate 1 and the d others with rate λ. Assume that these Poisson processes are independent from site to site in T. For each v, let $\{T_n^{v,k}: n \ge 1\}$, $k = 0, 1, 2, \dots, d$, be the arrival times of these $d + 1$ processes, respectively, where v represents the vertices in T. The process $\{T_n^{v,0}: n \ge 1\}$ has rate 1, the others rate λ. For each v and $n \ge 1$ we write a δ mark at the point $\langle v, T_n^{v,0} \rangle$ for $n \ge 1$ while if $k \ge 1$ we draw arrows from $\langle v, T_n^{v,k} \rangle$ to $\langle v_k, T_n^{v,k} \rangle$, where v_k, $k = 1, 2, \dots, d$, are the neighbors of v. We say that there is a path from $\langle v, s \rangle$ to $\langle u, t \rangle$ if there is a sequence of times $s_0 = s < s_1 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x_0 = v, x_1, \dots, x_n = u$ so that for $i = 1, 2, \dots, n$ there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ for $i = 1, \dots, n$ do not contain any δ. For any two sets A and B, we use the notation $A \times \{s\} \to B \times \{t\}$ to denote the event that there is a path from $\langle x, s \rangle$ to $\langle y, t \rangle$ for $x \in A$ and $y \in B$. Specifically, we say that $A \times \{s\} \to B \times \{t\}$ inside D for some set $D \subset T$ if the path mentioned above stays inside $D \times [0, \infty)$. We denote by B_A the subset of $T \times \mathscr{T}$ such that for any $\langle x, t \rangle \in B_A$ there exists a path from $\langle y, 0 \rangle$ to $\langle x, t \rangle$ for some $y \in A$. Clearly, B_x is a connected component in the sense of our graph construction for $x \in T$. We refer to B_x as a *cluster*.

We pick a line in the tree (a self-avoiding path of vertices $\{v_n: n \in Z\}$) which contains the root. We write L for the line and simply denote the vertices in L by $\{-\infty, \dots, -n, \dots, o, \dots, n, \dots, \infty\}$. We consider the segment $[-k, k]$ contained in L. For each vertex of T, there are d disjoint subgraphs connected to the vertex. These subgraphs are called the *branches* of the vertex.

Next we consider the following special sets. Let $H(-k, k)$ be the subgraph of T by:

1. removing $d-1$ branches from $\{k\}$ but leaving the branch that contains the segment $[-k, k]$;
2. removing $d-1$ branches from $\{-k\}$ but leaving the branch that contains the segment $[-k, k]$ (see Figure 1).

Clearly, $\lim_{k\to\infty} H(-k, k) = T$. In general, for any vertices $x, y \in T$, note that there is only one segment in T which can connect x and y. Let $S_{x,y}$ be the segment. By shifting the graph $H(-k, k)$ [or $H(-k, k+1)$] such that the segment $[-k, k]$ (or $[-k, k+1]$) matches the segment $S_{x,y}$ for some k, we can define $H(x, y)$ as the subgraph which contains the segment connecting x and y. Furthermore, if $|x-y|$ is an even number, let $c_{x,y}$ be the center of $S_{x,y}$. Then let (see Figure 1)

$$\mathscr{D}(x, y) = \{v: |v - c(x, y)| < |x-y|/2\} \cup \{x\} \cup \{y\}.$$

If $|x-y|$ is an odd number, let $y' \in S_{x,y}$ be the vertex next to y. Then $|x-y'| = |x-y| - 1$ is an even number. Let $c_{x,y}$ be the center of $S_{x,y'}$ and let (see Figure 1)

$$\mathscr{D}(x, y) = \{v: |v - c(x, y)| < |x-y'|/2\} \cup \{x\} \cup \{y'\} \cup \{y\}.$$

Since $H(-k, k)$ is an infinite graph, we can consider the contact process on $H(-k, k)$. Set

$$\lambda(k) = \inf\{\lambda: P(o \in \xi_t^o(H(-k, k)) \text{ i.o.}) > 0\}.$$

Clearly,

$$\lambda(k) \geq \lambda(k+1) \geq \lambda_c. \tag{5}$$

Let

$$\lim_{k\to\infty} \lambda(k) = \mu.$$

Then we have the following proposition.

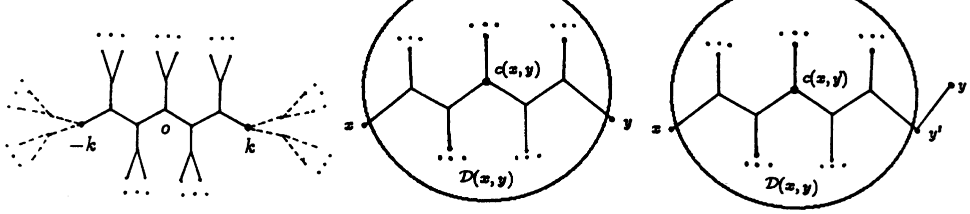

FIG. 1. *The left solid graph is $H(-k, k)$ with $k = 6$ and $d = 3$; the middle and the right graphs are $\mathscr{D}(x, y)$ with $d = 3$ and with $|x-y| = 6$ and 7, respectively.*

PROPOSITION 1. $\lambda_c = \mu$.

Before the proof of Proposition 1, we need to introduce a lemma. Clearly, if $\lambda_s < \lambda$, then

$$P(|\xi_t^o| > 0 \text{ for all } t) > 0. \tag{6}$$

Note that $d - 2$ of the branches of o do not contain any edges of L. We pick such a branch which contains o and denote it by U. Then we will show the following lemma which is stronger than (6).

LEMMA 1. *If* $\lambda > \lambda_s$,

$$P(|\xi_t^o(U)| > 0 \text{ for all } t) > 0. \tag{7}$$

PROOF. Lemma 1 was proved by Morrow, Schinazi and Zhang (see [8]). However, the paper is unpublished and the method is involved. We prefer to give another proof which relies on a method in [9] as follows. Let

$$\lambda_U = \inf\{\lambda: P(|\xi_t^o(U)| > 0 \text{ for all } t) > 0\}.$$

To show Lemma 1, we only need to show that

$$\lambda_s = \lambda_U. \tag{8}$$

Clearly,

$$\lambda_s \le \lambda_U. \tag{9}$$

It follows from (6) in [9] that

$$\exp(c(\lambda)t) \le t\,dE(|\xi_t^o(U)|), \tag{10}$$

where $c(\lambda)$ is a function of λ such that $c(\lambda) > 0$ if and only if $\lambda > \lambda_s$. Furthermore, by a standard result in the theory of branching processes (see the proof of Theorem 2 in [9]) it can also be proved that

$$P(|\xi_t^o(U)| > 0 \text{ for all } t) > 0 \quad \text{if } \exists\, t_0 \text{ such that } E|\xi_{t_0}^o(U)| > K \tag{11}$$

for some large constant K. Clearly, if $\lambda > \lambda_s$, then by (10) there exists t_0 such that $E|\xi_{t_0}^o(U)| > K$. It follows from (11) that $\lambda > \lambda_U$. Therefore,

$$\lambda_s \ge \lambda_U. \tag{12}$$

Lemma 1 is proved by (9) and (12). □

PROOF OF PROPOSITION 1. It follows from (5) that

$$\mu \ge \lambda_c.$$

To show Proposition 1, we only need to show the other direction. Suppose that

$$\mu > \lambda_c. \tag{13}$$

Then we pick a λ such that

$$\lambda_c < \lambda < \mu.$$

Clearly, for such λ,

$$P(o \in \xi_t^o \text{ i.o.}) \geq \beta,$$

where β is a positive constant. If we use the graphical language of the contact process, we let

$$\{k \in \xi_t^o \text{ i.o}\} = \{\langle k, t\rangle \in B_o \text{ i.o}\}$$

and

$$\{k \in \xi_t^o \text{ finitely often for all } t\} = \{\langle k, t\rangle \in B_o \text{ f.o}\}.$$

Then

$$P(\langle o, t\rangle \in B_o \text{ i.o.}) \geq \beta, \tag{14}$$

where B_o is the cluster of o. On the other hand, note that $\lambda < \mu$ and $\mu \leq \lambda(k)$ so that, for any integer k,

$$P(\langle o, t\rangle \in B_o \text{ i.o.}, B_o \subset H(-k, k) \times (0, \infty)) = 0. \tag{15}$$

For each sample point in $\{\langle o, t\rangle \in B_o \text{ i.o.}\}$, by (15) the sample point is not in

$$\{B_o \subset H(-k, k) \times (0, \infty)\}.$$

In other words, for each such sample point, it is either in

$$\{B_o \cap \{k\} \times (0, \infty) \neq \varnothing\}$$

or in

$$\{B_o \cap \{-k\} \times (0, \infty) \neq \varnothing\}$$

for any positive integer k. By symmetry and (14),

$$P(B_o \cap \{k\} \times (0, \infty) \neq \varnothing) \geq \frac{\beta}{2}. \tag{16}$$

By (16), for any positive integer k there exists a real number $J(k)$ such that

$$P(B_o \cap \{k\} \times (0, J(k)) \neq \varnothing) \geq \frac{\beta}{3}. \tag{17}$$

On the other hand, by Lemma 1, with probability $\alpha > 0$, $|\xi_t^o(U)| > 0$ for all t. Note that $\lambda < \mu \leq \lambda(1)$ and $B_o(U) \subset H(-1, 1) \times (0, \infty)$ so that, by (15),

$$P(\langle o, t\rangle \in B_o(U) \text{ i.o.}) = 0. \tag{18}$$

In contrast to (18), on the assumption $\lambda_c < \lambda < \mu$, we will show that

$$P(\langle o, t\rangle \in B_o(U) \text{ f.o.}, |\xi_t^o(U)| > 0 \text{ for all } t) = 0. \tag{19}$$

Since

$$P(|\xi_t^o(U)| > 0 \text{ for all } t) = \alpha > 0, \tag{20}$$

(18) and (19) cannot both hold. The contradiction tell us that assumption (13) is wrong. This is

$$\lambda_c \leq \mu.$$

Therefore, Proposition 1 is proved if (19) holds.

Now we begin to show (19). Intuitively, for each time t, there exists x such that $\langle x, t\rangle \in B_o(U)$ if $|\xi_t^o(U)| > 0$ for all t. By (17) and translation invariance, with probability $\beta/3$ there exists a path connecting x to o with edges in U in the time interval $(t, t + J(|x|))$. On the event $|\xi_\xi^o(U)| > 0$ for all t, we can find infinitely many such $\langle x, t\rangle$, and with probability $\beta/3$ each $\langle x, t\rangle$ can be connected by a path to $\langle o, s\rangle$ for some $s \in (t, t + J(|x|))$. It would imply

$$\langle o, t\rangle \in B_o(U) \quad \text{i.o.}$$

on the event $|\xi_t^o(U)| > 0$ for all t. Then (19) can be shown.

Now we will give a formal proof of (19) as follows. Suppose that

$$P\big(\langle o, t\rangle \in B_o(U) \text{ f.o.}, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq 3r \tag{21}$$

for some $r > 0$. By (21), there exists M such that

$$P\big(\langle o, t\rangle \in B_o(U) \text{ at most } M \text{ times}, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq 2r. \tag{22}$$

Then we can find I large such that

$$P\big(B_o(U) \cap \{o\} \times (I, \infty) = \varnothing, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq r. \tag{23}$$

We take n large and then η small such that

$$\left(1 - \frac{\beta}{3}\right)^n < \frac{r}{4} \quad \text{and} \quad 1 - (1 - \eta)^n < \frac{r}{4}. \tag{24}$$

Let

$$U(M) = \{v \in U: |v| \leq M\}.$$

Note that, for any $s > 0$,

$$P\big(B_o(U) \cap U \times \{s\} \neq \varnothing \,\big|\, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) = 1$$

so that we can choose M_0 large such that

$$P\big(B_o(U) \cap U(M_0) \times \{I\} \neq \varnothing \,\big|\, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq 1 - \eta. \tag{25}$$

Note also that

$$\begin{aligned} &P\big(B_o(U) \cap U \times \{I + J(M_0)\} \neq \varnothing, \\ &\quad B_o(U) \cap U(M_0) \times \{I\} \neq \varnothing \,\big|\, \big|\xi_T^o(U)\big| > 0 \text{ for all } t\big) \geq 1 - \eta \end{aligned} \tag{26}$$

so that we can choose M_1 large such that

$$\begin{aligned} &P\big(B_o(U) \cap U(M_1) \times \{I + J(M_0)\} \neq \varnothing, \\ &\quad B_o(U) \cap U(M_0) \times \{I\} \neq \varnothing \,\big|\, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq (1 - \eta)^2, \end{aligned}$$

where $J(k)$ was defined in (17). Consequently, we choose $M_2, M_3, \dots, M_n$ large such that

$$P\big(E_n \,\big|\, \big|\xi_t^o(U)\big| > 0 \text{ for all } t\big) \geq (1 - \eta)^n, \tag{27}$$

where

$$E_n = \bigcap_{i=0}^{n-1} \{B_o(U) \cap U(M_i) \times \{I + J(M_0) + \cdots + J(M_{i-1})\} \neq \varnothing\}.$$

On the other hand,

$$\begin{aligned}
&P\big(B_o(U) \cap \{o\} \times (I,\infty) = \varnothing, |\xi_t^o(U)| > 0 \text{ for all } t\big) \\
&\quad \le P\big(B_o(U) \cap \{o\} \times (I,\infty) = \varnothing, E_n, |\xi_t^o(U)| > 0 \text{ for all } t\big) \\
&\qquad + P\big(E_n^C \,|\, |\xi_t^o(U)| > 0 \text{ for all } t\big) \\
&\quad \le P\big(B_o(U) \cap \{o\} \times (I,\infty) = \varnothing, E_n, |\xi_t^o(U)| > 0 \text{ for all } t\big) \\
&\qquad + 1 - (1-\eta)^n \\
&\quad = \sum_\Gamma P\big(B_o(U) \cap \{o\} \times (I,\infty) = \varnothing, E_n, |\xi_t^o(U)| > 0 \text{ for all } t, \\
&\qquad\qquad B_o(U) \cap U \times \big(I + J(M_0) + \cdots + J(M_n)\big) = \Gamma \\
&\qquad\qquad\quad \times \big(I + J(M_0) + \cdots + J(M_n)\big)\big) + 1 - (1-\eta)^n \\
&\quad = \sum_\Gamma P\big(B_o(U) \cap \{o\} \times \big(I, I + J(M_0) + \cdots + J(M_n)\big) = \varnothing, \\
&\qquad\qquad E_{n-1}, B_o(U) \cap U \times \big(I + J(M_0) + \cdots + J(M_n)\big) = \Gamma \\
&\qquad\qquad\quad \times \big(I + J(M_0) + \cdots + J(M_n)\big), \\
&\qquad\qquad \Gamma \times \big(I + J(M_0) + \cdots + J(M_n)\big) \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0) \\
&\qquad\qquad\qquad + \cdots + J(M_n) < s < \infty\big) + 1 - (1-\eta)^n,
\end{aligned} \tag{28}$$

where the sum is taken over all possible Γ and Γ is a finite vertex set in U. Let $B_o^t(U)$ be the cluster of $B_o(U)$ inside time interval $(0,t)$. Then

$$\{B_o(U) \cap \{o\} \times (0,t) = \varnothing\} = \{B_o^t(U) \cap \{\{o\} \times (0,t)\} = \varnothing\}. \tag{29}$$

In other words, $\{B_o(U) \cap o \times (0,t) = \varnothing\}$ only depends on the time interval $(0,t)$. Clearly, by (29),

$$\begin{aligned}
&\{B_o(U) \cap \{o\} \times \big(I, I + J(M_0) + \cdots + J(M_n)\big) = \varnothing, E_{n-1}, \\
&\quad B_o(U) \cap U \times \big(I + J(M_0) + \cdots + J(M_n)\big) \\
&\qquad\qquad = \Gamma \times \big(I + J(M_0) + \cdots + J(M_n)\big)\}
\end{aligned}$$

and

$$\begin{aligned}
&\{\Gamma \times \big(I + J(M_n) + \cdots + J(M_n)\big) \\
&\quad \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0) + \cdots + J(M_n) < s < \infty\}
\end{aligned}$$

are independent since both events depend on different time intervals. Furthermore, by translation invariance,

$$
\begin{aligned}
&P\big(\Gamma \times (I + J(M_0) + \cdots + J(M_n))\\
&\quad \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0) + \cdots + J(M_n) < s < \infty\big)\\
&\quad \le P\big(\langle x, I + J(M_0) + \cdots + J(M_n)\rangle\\
&\qquad \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0) + \cdots + J(M_n) < s < \infty\big)\\
&\qquad\qquad (\text{where } x \text{ is a vertex of } \Gamma)\\
&\quad \le P\big(B_o \cap \{x\} \times (0,\infty) = \varnothing\big)\\
&\quad \le \left(1 - \frac{\beta}{2}\right) < \left(1 - \frac{\beta}{3}\right) \quad [\text{by translation invariance and (16)}].
\end{aligned}
\tag{30}
$$

Then, by (30), the first term on the right-hand side of (28) equals

$$
\begin{aligned}
&\sum_{\Gamma} P\big(B_o(U) \cap \{o\} \times (I, I + J(M_0) + \cdots + J(M_n)) = \varnothing,\\
&\quad E_{n-1}, B_o(U) \cap U \times (I + J(M_0) + \cdots + J(M_n)) = \Gamma\\
&\quad \times (I + J(M_0) + \cdots + J(M_n)), \Gamma \times (I + J(M_0) + \cdots + J(M_n))\\
&\qquad \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0) + \cdots + J(M_n) < s < \infty\big)\\
&\quad = \sum_{\Gamma} P\big(B_o(U) \cap \{o\} \times (I, I + J(M_0) + \cdots + J(M_n)) = \varnothing,\\
&\qquad E_{n-1}, B_o(U) \cap U \times (I + J(M_0) + \cdots + J(M_n)) = \Gamma\\
&\qquad\qquad \times (I + J(M_0) + \cdots + J(M_n))\big)\\
&\qquad P\big(\Gamma \times (I + J(M_0) + \cdots + J(M_n)) \nrightarrow \langle o, s\rangle \text{ for } I + J(M_0)\\
&\qquad\qquad + \cdots + J(M_n) < s < \infty\big)\\
&\quad \le P\big(B_o(U) \cap \{o\} \times (I, I + J(M_0) + \cdots + J(M_n)) = \varnothing, E_{n-1}\big)\\
&\qquad \times \left(1 - \frac{\beta}{3}\right).
\end{aligned}
\tag{31}
$$

By the definition of $J(M_i)$, (17) and the same method repeated above $n - 1$ times,

$$
\begin{aligned}
&P\big(B_o(U) \cap \{o\} \times (I, I + J(M_0) + \cdots + J(M_n)) = \varnothing, E_{n-1}\big)\left(1 - \frac{\beta}{3}\right)\\
&\quad \le \left(1 - \frac{\beta}{3}\right)^n.
\end{aligned}
\tag{32}
$$

Combining (29), (31) and (32),

$$(33)\quad \begin{aligned} &P\Big(B_o(U) \cap \{o\} \times (I, \infty) = \emptyset, \big|\xi_\xi^o(U)\big| > 0 \text{ for all } t\Big) \\ &\quad \le \left(1 - \frac{\beta}{3}\right)^n + 1 - (1-\eta)^n < \frac{r}{2}. \end{aligned}$$

This contradicts assumption (23). Therefore, (19) is proved. □

Let $I_x(k)$ be the indicator of the event that the vertex x becomes occupied infinitely often in $x + H(-k, k)$. By translation invariance and Proposition 1, if $\lambda > \lambda_c$, then there exists k such that

$$(34)\qquad P(I_x(k)) = \theta(\lambda) > 0,$$

where $\theta(\lambda)$ is a constant. Recall that L is the line defined before. It is easy to check that $I_0(k), \dots, I_n(k), \dots$ for $n \in L$ is a stationary sequence. By a standard ergodic theorem,

$$(35)\qquad P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} I_i(k) = \theta(\lambda)\right) = 1.$$

Due to (35) we have the following fact.

FACT 1. Assume that $\lambda > \lambda_c$. Given $\varepsilon > 0$, we can pick N large such that

$$(36)\qquad P(\exists\ i \in [0, n] \text{ such that } I_i(k,) = 1) > 1 - \varepsilon \quad \text{for } n \ge N.$$

For fixed λ and k' we can choose N such that $N > 4k'$. For the integer N let J_N be the indicator of the event that there exists $i \in [N/4, 3N/4]$ such that $I_i(k') = 1$ for some $k' < N/4$. By Fact 1 and translation invariance, we have the following fact.

FACT 2. Assume that $\lambda > \lambda_c$. For given $\varepsilon > 0$, we can pick N large such that

$$(37)\qquad P(J_N = 1) > 1 - \varepsilon.$$

Now we renormalize T as the following new graph (see Figure 2). We first choose $[0, N]$ as an edge called L_0. There are $d - 1$ branches that connect to N and which do not contain $[0, N)$. We pick two branches and select two segments from the two branches such that each of the segments has a length N (containing N vertices), and each is next to $\{N\}$. Denote these by $L_{0,1}$ and $L_{0,2}$. Then both $L_{0,1}$ and $L_{0,2}$ have two end vertices: the common one is N and the others are denoted by $l_{0,1}$ and $l_{0,2}$, respectively. Continuing, we pick $L_{0,1,1}$ and $L_{0,1,2}$ to be the two edges with length N next to the end vertex $l_{0,1}$ and $L_{0,2,1}$ and $L_{0,2,2}$ to be the other two edges with length N next to the end vertex $l_{0,2}$. With this construction, we get a new three-branch tree but with only one branch connecting the root (see Figure 2).

On the event J_N, $[0, N]$ is occupied by a particle infinitely often. Once $[0, N]$ is occupied by a particle, then with positive probability the particle can

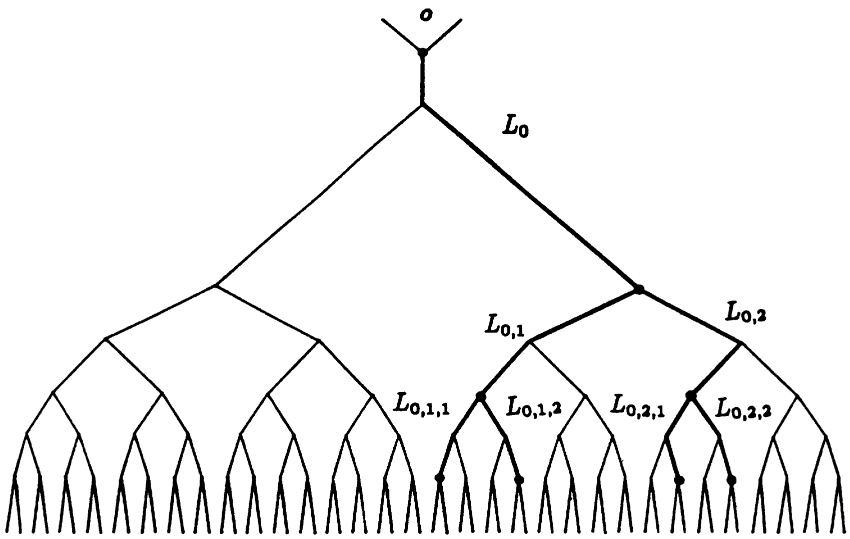

FIG. 2. *The bold graph is a renormalized tree with* $|N| = 2$ *from* T.

generate particles in each site of $L_{0,1}$. More precisely, if we write $H(K)$ for $H(x, y)$ for any segment K with two end vertices x and y, then with positive probability $L_{0,1} \subset \xi_i^{[0,N]}(H(L_0) \cup H(L_{0,1}))$ for some $t > 0$. Let $F_t(N)$ be this event; that is, each site of $L_{0,1}$ is occupied by $\xi_t^{[0,N]}(H(L_0) \cup H(L_{0,1}))$. Then, on the event J_N, $\bigcup_{t \in (0,\infty)} F_t(N)$ should occur with probability 1. More precisely, we have the following proposition.

PROPOSITION 2. *Assume that* $\lambda > \lambda_c$. *There is N which only depends on* λ *such that*

$$P(\exists\, 0 < t < \infty \text{ such that } F_t(N) \text{ occurs} | J_N) = 1. \tag{38}$$

PROOF. Note that $\lambda > \lambda_c$ so that, by Proposition 1,

$$\lambda > \lambda(k') \tag{39}$$

for some k'. We can pick N such that $N > 4k'$ as we did before. On the event J_N, let

$$\begin{aligned} \eta_1 &= \inf\{\infty > t > 1: [0, N] \cap \xi_t^{[0,N]}(H(0, N)) \neq \emptyset\}, \\ \eta_2 &= \inf\{\infty > t > \eta_1 + 1: [0, N] \cap \xi_t^{[0,N]}(H(0, N)) \neq \emptyset\}, \\ &\vdots \\ \eta_n &= \inf\{\infty > t > \eta_{n-1} + 1: [0, N] \cap \xi_t^{[0,N]}(H(0, N)) \neq \emptyset\}. \end{aligned} \tag{40}$$

Clearly, on the event J_N, with probability 1 there exists $\eta_1 < \eta_2 < \cdots < \eta_n < \infty$ for any integer n. Furthermore, let $q_N(x)$ be the probability of the event that

$$L_{0,1} \subset \xi_1^x(H(L_0) \cup H(L_{0,1}))$$

for $x \in [0, N]$. Let

$$q_N = \max_{x \in [0,N]} q_N(x).$$

Clearly,

$$q_N > 0.$$

With these definitions and translation invariance,

$$\begin{aligned}
&P(\nexists\, 0 < t < \infty \text{ such that } F_t(N) \text{ occurs} | J_N)\\
&\quad = P(\nexists\, 0 < t < \infty \text{ such that } F_t(N) \text{ occurs}, \exists\, \eta_1, \eta_2, \dots, \eta_n | J_N)\\
&\quad = \int_0^\infty P(\nexists\, 0 < t < \infty \text{ such that } F_t(N) \text{ occurs},\\
&\qquad\qquad \exists\, \eta_1, \eta_2, \dots, \eta_n, \eta_n = s | J_N)\, ds\\
&\quad \le \int_0^\infty (1 - q_N) P(\nexists\, 0 < t < s \text{ such that } F_t(N) \text{ occurs},\\
&\qquad\qquad \exists\, \eta_1, \eta_2, \dots, \eta_n, \eta_n = s | J_N)\, ds.
\end{aligned} \tag{41}$$

By iterating (41),

$$P(\nexists\, 0 < t < \infty \text{ such that } F_t(N) \text{ occurs} | J_N) \le (1 - q_N)^n. \tag{42}$$

By (42), we note that n can be arbitrarily large so that Proposition 2 is proved. □

Similarly, on the event J_N each of the vertices of $L_{0,2}$ is occupied by $\xi_t^{L_0}(H(L_0) \cup H(L_{0,2}))$ for some t with probability 1. Let

$$\tau_0 = \inf\{\infty > t > 1 \colon L_0 \subset \xi_t^{L_0}(H(L_0))\}.$$

By Fact 2 and the same proof of Proposition 2, for $\lambda > \lambda_c$ and any given $\varepsilon > 0$, we can pick N and R large such that

$$P(\tau_0 < R | L_0 \text{ is occupied by particles at time } 0) \ge 1 - \varepsilon. \tag{43}$$

Similarly, on the event $\tau_0 < R$, let

$$\begin{aligned}
\tau_{0,1} &= \inf\{\infty > t > \tau_0 \colon L_{0,1} \subset \xi_t^{L_0, \tau_0}(H(L_0 \cap L_{0,1}))\},\\
\tau_{0,2} &= \inf\{\infty > t > \tau_0 \colon L_{0,2} \subset \xi_t^{L_0, \tau_0}(H(L_0 \cap L_{0,2}))\},
\end{aligned} \tag{44}$$

where ξ_t^{L,τ_0} is the contact process for $t \ge \tau_0$ such that $\xi_{\tau_0}^{L,\tau_0} = L$. Then, by Fact 2 and Proposition 2, for large N and R,

$$P(\tau_{0,i} - t < R | \tau_0 = t) \ge 1 - \varepsilon \tag{45}$$

for $i = 1, 2$. Consequently, on the event that $\tau_{0, i_1, \dots, i_j} - \tau_{0, i_1, \dots, i_{j-1}} < R, \dots, \tau_0 < R$, for the N and the R in (45), let

$$\begin{aligned}
&\tau_{0, i_1, \dots, i_{j+1}}\\
&\quad = \inf\Big\{\infty > t > \tau_{0, i_1, \dots, i_j} \colon L_{0, i_1, \dots, i_{j+1}}\\
&\qquad\quad \subset \xi_t^{L_{0, i_1, \dots, i_j}, \tau_{0, i_1, \dots, i_j}}\big(H\big(L_{0, i_1, \dots, i_j}\big) \cup H\big(L_{0, i_1, \dots, i_{j+1}}\big)\big)\Big\},
\end{aligned} \tag{46}$$

where $i_1 = 1$ or $2, \dots, i_{j+1} = 1$ or 2. Then, by translation invariance and (45),

$$P\left(\tau_{0, i_1, \dots, i_j, i_{j+1}} - t < R | \tau_{0, i_1, \dots, i_j} = t\right) \geq 1 - \varepsilon. \tag{47}$$

Now, on the condition that every vertex in L_0 is occupied at time 0, we say L_0 is open if $\tau_0 < R$. Continuing, on the event that $L_{0, i_1, \dots, i_j}$ is open, we say $L_{0, i_1, \dots, i_{j+1}}$ is open if $\tau_{0, i_1, \dots, i_{j+1}} - \tau_{0, i_1, \dots, i_j} < R$. With this definition, on the condition that every vertex in L_0 is occupied at time 0, we define $C(N, R)$ as the open cluster of the root o with open edges in the edge set $\{L_{0, i_1, \dots, i_j}\}$. Now we show the following result.

PROPOSITION 3. *If $\lambda > \lambda_c$, we can pick N and R large such that*

$$P\left(|C(N, R)| = \infty \mid L_0 \textit{ is occupied at } 0\right) > \tfrac{1}{2}. \tag{48}$$

PROOF. By the Markov property, on the event that $L_{0, i_1, \dots, i_j}$ and $L_{0, l_1, \dots, l_k}$ are first occupied (for each vertex) by particles at t_1 and t_2, respectively, for $i_j \neq l_k$, then the events that $L_{0, i_1, \dots, i_j, i_{j+1}}$ and $L_{0, l_1, \dots, l_k, l_{k+1}}$ are open or not only depend on the Poisson processes on edges of $H(L_{0, i_1, \dots, i_j}) \cup H(L_{0, i_1, \dots, i_j, i_{j+1}})$ and $H(L_{0, l_1, \dots, l_k}) \cup H(L_{0, l_1, \dots, l_k, l_{k+1}})$, respectively. Note that

$$\left\{H(L_{0, i_1, \dots, i_j}) \cup H(L_{0, i_1, \dots, i_j, i_{j+1}})\right\} \cap \left\{H(L_{0, l_1, \dots, l_k}) \cup H(L_{0, l_1, \dots, l_k, l_{k+1}})\right\} = \varnothing$$

so that $L_{0, i_1, \dots, i_j, i_{j+1}}$ and $L_{0, l_1, \dots, l_k, l_{k+1}}$ are open or not independently on the event that $L_{0, i_1, \dots, i_j}$ and $L_{0, l_1, \dots, l_k}$ are first occupied at t_1 and t_2, respectively, for $i_j \neq l_k$. By a standard Peierls argument (see the proof of (8.12) in [5]) and the Markov property, for any $\delta > 0$ if ε is small enough in (45) and (47), then

$$P\left(|C(N, R)| = \infty | L_0 \text{ is occupied at } 0\right) > 1 - \delta. \tag{49}$$

Proposition 3 is proved. □

With Proposition 3, we have the following proposition.

PROPOSITION 4. *Suppose that $\lambda > \lambda_c$. Given any $\varepsilon > 0$, there exist M and G which may depend on ε such that*

$$P(\exists\, t \leq M|x| \textit{ such that } \langle o, 0\rangle \to \langle x, t\rangle \textit{ inside } H(o, x)) \geq \exp(-\varepsilon|x|)$$

for all $|x| > G$.

PROOF. For any large $|x|$ consider the graph $\{v \in T\colon |v| \leq |x|\}$. We construct the graph $\{L_0, L_{0, i_1, \dots, i_j}\}$ as we did in the proof of Proposition 3, where $|L_0| = N$ for some N which is large enough such that Proposition 3 holds, and j is the largest integer such that $j|L_0| \leq |x|$. We also choose our $\{L_0, \dots, L_{0, i_1, \dots, i_j}\}$ such that x can be connected by y directly, where y is one of the end vertices of $\{L_{0, i_1, \dots, i_j}\}$ (see Figure 3). Clearly, $|x - y| \leq N$. By Proposition 3, on the event that L_0 is occupied by particles at time 0, there exists an open path from o to one of $\{L_{0, i_1, \dots, i_j}\}$ with a probability larger than

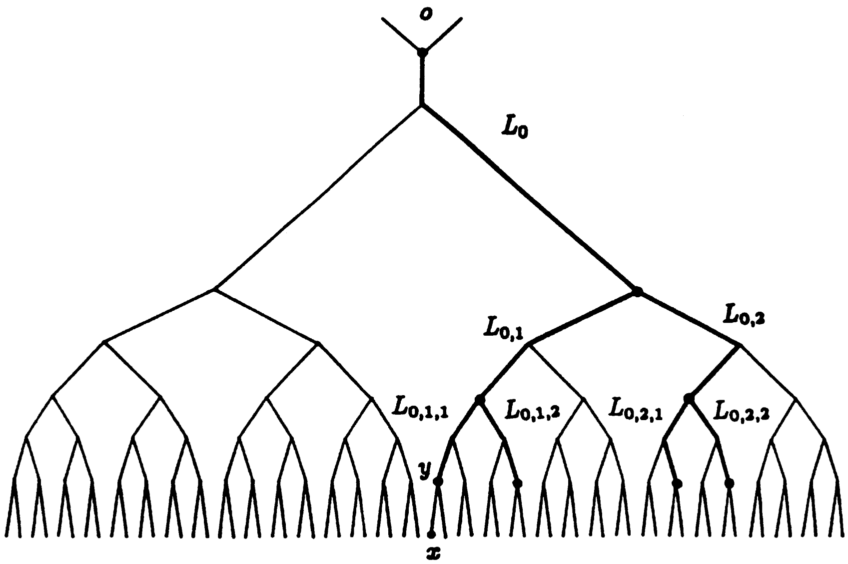

FIG. 3. *The bold graph is a special renormalized tree with $|N| = 2$ such that y can connect to x directly.*

$\frac{1}{2}$. Note that, on the event that there is such an open path, each of its open bonds $L_{0, i_1, \dots, i_m}$ for $0 \le m \le j$ has to allow all its vertices to be occupied by a particle at some time t_m with $t_m - t_{m-1} < R$ so that

$$(50)\quad \begin{aligned} &P\Big((L_0 \times \{0\}) \to \big(\{L_{0, i_1, \dots, i_j}\} \times \{t\}\big) \text{ in } U \\ &\qquad \text{with } t < jR | L_0 \text{ is occupied at } 0\Big) \ge \tfrac{1}{2}, \end{aligned}$$

where U is the branch of o which contains the segment from o to x. Let $q(L_0)$ be the probability that, starting with one particle at $(o, 0)$, each vertex of L_0 is occupied by particles at time 1. Then

$$(51)\quad P\Big(\langle o, 0\rangle \to \big(\{L_{0, i_1, \dots, i_j}\} \times \{t\}\big) \text{ in } U \text{ with } t < jR + 1\Big) \ge \tfrac{1}{2} q(L_0).$$

On the other hand, the number of end vertices of $\{L_{0, i_1, \dots, i_j}\}$ equals $2^j \le 2^{|x|/(|L_0| - 1)}$. By symmetry and (51),

$$(52)\quad \begin{aligned} &P(\exists\, t \le Rj + 1 \text{ such that } \langle o, 0\rangle \to \langle y, t\rangle \text{ in } U) \\ &\qquad \ge \frac{1}{2} q(L_0) \exp\left(-\frac{|x|}{(|L_0| - 1)}\right). \end{aligned}$$

We also let $q_1(L_0)$ be the probability that, on the event that y is first occupied at time τ by a particle, x is occupied by a particle at time $\tau + 1$.

Since N does not depend on x, $q_1(L_0)$ has a positive lower bound that does not depend on x:

$$P(\exists\, t \le Rj + 2 \text{ such that } \langle o, 0\rangle \to \langle x, t\rangle \text{ in } U) \tag{53}$$
$$\ge q_1(L_0)\frac{1}{2}q(L_0)\exp\left(-\frac{|x|}{(|L_0|-1).}\right).$$

For a given ε, we take $|L_0|$ large enough and then $|x|$ large such that

$$P(\exists\, t \le Rj + 2 \text{ such that } \langle o, 0\rangle \to \langle x, t\rangle \text{ in } U) \ge \exp(-\varepsilon|x|). \tag{54}$$

Note that T is a tree so that if there exists a path in $U \times [0, \infty)$ from $\langle o, 0\rangle$ to $\langle x, t\rangle$ with $t \le Rj + 2$, then there exists a path in $H(o, x) \times [0, \infty)$ from $\langle o, 0\rangle$ to $\langle x, t\rangle$ with $t \le Rj + 2$. Finally, by (54),

$$P(\exists\, t \le M|x| + 2 \text{ such that } \langle o, 0\rangle \to \langle x, t\rangle \text{ in } H(o, x))$$
$$\ge P(\exists\, t \le M|x| + 2 \text{ such that } \langle o, 0\rangle \to \langle x, t\rangle \text{ in } U)$$
$$\ge \exp(-\varepsilon|x|)$$

for $M = R/N$. Proposition 4 is proved. □

By adapting the proof of Proposition 4, we can show the following corollary.

COROLLARY. *For any $\lambda > \lambda_c$ and $\varepsilon > 0$, there exists M and G such that*

$$P(\exists\, t \le M|y| \text{ such that } \langle o, 0\rangle \to \langle y, t\rangle \text{ inside } \mathscr{D}(o, y)) \ge \exp(-\varepsilon|y|)$$

for any $|y| \ge G$.

PROOF. For a large integer f, let $J_N(f)$ be the indicator of the event that $i \in [N/4, 3N/4]$ is occupied more than f times in $H(o, N)$ for some i. By Fact 2, for some large N,

$$P(J_N(f) = 1) \ge P(J_N = 1) \ge 1 - \varepsilon/3.$$

Then, for each f, we take K large such that

$$P(J_N(f) = 1 \text{ inside } H(o, N) \cap \{|v| \le K\}) \ge 1 - \varepsilon/2. \tag{55}$$

On the event $J_N(f) = 1$, let

$$\tau_0' = \inf\{\infty \ge t > 1 \colon L_0 \subset \xi_t^{L_0}(H(L_0) \cap \{|v| \le K\})\}.$$

It follows from (55) for large f and the same proof of Proposition 2 that we can pick N, R and K such that

$$P(\tau_0' < R | L_0 \text{ is occupied by particles at } 0) \ge 1 - \varepsilon.$$

Similarly, on the event $\tau_0' < R$, let

$$\tau_{0,1}' = \inf\{\infty \ge t > \tau_0' \colon L_{0,1} \subset \xi_t^{L_0, \tau_0'}(H(L_0 \cap L_{0,1}) \cap \{|v - N| \le K\})\}$$

and

$$\tau_{0,2}' = \inf\{\infty \ge t > \tau_0' \colon L_{0,2} \subset \xi_t^{L_0, \tau_0'}(H(L_0 \cap L_{0,2}) \cap \{|v - N| \le K\})\}.$$

Then, by (55) and the same proof of Proposition 2, for large N, R and K,

$$P(\tau_{0,i}' - t < R | \tau_0' = t) \ge 1 - \varepsilon \tag{56}$$

for $i = 1, 2$. Consequently, on the event that $\tau'_0 < R, \tau'_{i_1} - \tau'_0 < R, \dots, \tau'_{0, i_1, \dots, i_j} - \tau'_{0, i_1, \dots, i_{j-1}} < R$ for the K, N and R in (56), let

$$\tau'_{0, i_1, \dots, i_{j+1}} = \inf\Big\{\infty \ge t > \tau'_{0, i_1, \dots, i_j}: L_{0, i_1, \dots, i_{j+1}} \subset \xi_t^{L_{0, i_1, \dots, i_j}, \tau'_{0, i_1, \dots, i_j}}\Big(\Big(H\big(L_{0, i_1, \dots, i_j}\big) \cup H\big(L_{0, i_1, \dots, i_{j+1}}\big)\Big) \cap \big\{|v - l_{0, i_1, \dots, i_j}| \le K\big\}\Big)\Big\},$$

where $i_1 = 1$ or $2, \dots, i_{j+1} = 1$ or 2, and $l_{0, i_1, \dots, i_j}$ is the common vertex of $H(L_{0, i_1, \dots, i_j})$ and $H(L_{0, i_1, \dots, i_{j+1}})$. Then, by translation invariance and (56),

$$P\Big(\tau'_{0, i_1, \dots, i_{j+1}} - t < R | \tau'_{0, i_1, \dots, i_j} = t\Big) \ge 1 - \epsilon. \tag{57}$$

Now, on the condition that every vertex in L_0 is occupied at time 0, we say L_0 is open if $\tau'_0 < R$. Continuing, on the event that $L_{0, i_1, \dots, i_j}$ is open, we say $L_{0, i_1, \dots, i_{j+1}}$ is open if $\tau'_{0, i_1, \dots, i_{j+1}} - \tau'_{0, i_1, \dots, i_j} < R$. With this definition, on the condition that every vertex in L_0 is occupied at time 0, let $C'(N, R, K)$ be the corresponding open cluster with open edges on $\{L_{0, i_1, \dots, i_j}\}$ defined above. By the same proofs of Propositions 3 and 4, we can show that, for a large N, R and K,

$$P((L_0 \times \{0\}) \text{ is connected to } (\{L_{0, i_1, \dots, i_j}\} \times \{t\}) \text{ by open edges in } U \text{ with } t < jR | L_0 \text{ is occupied at } 0) \ge \tfrac{1}{2}.$$

Note that the renormalized graph $\{L_{0, i_1, \dots, i_j}\}$ is a tree so that if there is an open path from o to x for some $x \in T$, then the open path is the unique path. By this observation and the same argument of (51), there exists $C > 0$ such that

$$P(\exists\ t \le Rj + 1 \text{ such that } \langle o, 0\rangle \to \langle y, t\rangle \text{ in } U \cap \{|v| \le |y| + K\}) \ge C \exp\left(-\frac{|x|}{(|L_0| - 1)}\right).$$

By the same argument from (52) to (53), note that K is a finite number which does not depend on y so that there exist M, G and K such that

$$P(\exists\ t \le M|y| \text{ such that } \langle o, 0\rangle \to \langle y, t\rangle \text{ in } \mathscr{D}(0, y)) \ge \exp(-\varepsilon|y|)$$

for all $|y| \ge G$. The corollary is proved. □

PROPOSITION 5. *For any $\lambda > \lambda_c$, there exists $\delta > 0$ (which may depend on λ) such that*

$$\liminf_t P\big(o \in \xi_t^o(U)\big) \ge \delta. \tag{58}$$

Before the proof of Proposition 5, we first prove the following lemma.

LEMMA 2. *If $\lambda > \lambda_s$, there exist α, β and $\delta > 0$ such that, for any $t \ge 0$,*

$$P\big(\big|\xi_t^o(U) \cap \{v: \alpha t \le |v| \le \beta t\}\big| \ge \exp(c_\lambda t)\big) \ge \delta, \tag{59}$$

where c_λ is a positive number which may depend on λ.

PROOF. It follows from Lemma 1 in [9] that, on the event that $\xi_t^o(U)$ survives,

$$\lim_{t\to\infty} |\xi_t^o(U)| = \infty. \tag{60}$$

Note that $\lambda > \lambda_s$ so that

$$P(\forall t,\ \xi_t^o(U) \neq \varnothing) = \eta > 0.$$

Then, by (60),

$$P\Big(\lim_{t\to\infty} |\xi_t^o(U)| = \infty\Big) = \eta. \tag{61}$$

Let us consider that i.i.d. sequence $\{X_i\}$ which has common distribution:

$$X_i = 1 \text{ with probability } r \text{ and } X_i = 0 \text{ with probability } 1 - r.$$

Let $\mathscr{P}$ be a probability measure corresponding to $\{X_i\}$ and let $S_n = \sum_{i=1}^n X_i$. By a standard large deviation result (see [3]),

$$\mathscr{P}\left(S_k < k\left(\frac{r}{2}\right)\right) \le \exp(-a(r)k) \tag{62}$$

for some constant $a(r) > 0$ which may depend on r but not k. Now, for any $S \subset T$, we define the *border set* of S as follows. We say that a vertex in S is in the border if at least one of the d branches emanating from x has no vertex in S except x. Denote by $\mathscr{B}(S)$ and $N(x)$ the border of S and one of the empty branches of x for $x \in S$, respectively. It is known (see [10]) that

$$|\mathscr{B}(S)| \ge \left(\frac{d-1}{d}\right)|S|. \tag{63}$$

Now, for any k with

$$k\left(\frac{\eta}{2}\right)\left(\frac{d-1}{d}\right) > 2,$$

by (61) we can take t_0 such that

$$P\Big(|\xi_{t_0}^o(U)| > k\Big) > \tfrac{3}{4}\eta. \tag{64}$$

Then we also can take α small and β large such that

$$P\Big(\big|\xi_{t_0}^o(U) \cap \{v\colon \alpha t_0 \le |v| \le \beta t_0\}\big| \ge k\Big) > \tfrac{1}{2}\eta. \tag{65}$$

Clearly, by (63), on the event that $|\xi_{t_0}^o(U) \cap \{v\colon\ \alpha t_0 \le |v| \le \beta t_0\}| > k$, the number of border vertices of $\xi_{t_0}^o(U)$ is at least $((d-1)/d)k$. By the same argument of (63), there exist at least $((d-1)/d)k$ border vertices $\{x\}$ of $\xi_{t_0}^o(U) \cap \{v\colon\ \alpha t_0 \le |\nu| \le \beta t_0\}$ such that $N(x) \cap \{v\colon |v| < \alpha t_0\} = \varnothing$. Note that, for two such border vertices x and y, $N(x) \cap N(y) = \varnothing$ so that each such border vertex x can also generate another k particles with probability $\frac{1}{2}\eta$ independently inside

$$N(x) \cap \{v\colon \alpha t_0 \le |v - x| \le \beta t_0\} \subset N(x) \cap \{v\colon 2\alpha t_0 \le |v| \le 2\beta t_0\}$$

by repeating the step in (65). Therefore, by (62), (65) and the Markov property,

$$P\left(\left|\xi^o_{2t_0}(U)\cap\{v: 2\alpha t_0\le|v|\le 2\beta t_0\}\right|\right.$$

$$< \frac{k^2(d-1)}{d}\left(\frac{\eta}{2}\right)\left|\,\left|\xi^o_{t_0}(U)\cap\{v:\ \alpha t_0\le|v|\le\beta t_0\}\right|\ge k\right) \tag{66}$$

$$\le \exp\left(-a\left(\frac{\eta}{2}\right)\frac{k(d-1)}{d}\right).$$

Iterating by using the argument in (66),

$$P\left(\left|\xi^o_{mt_0}(U)\cap\{v: m\alpha t_0\le|v|m\beta t_0\}\right|\right.$$

$$>\left(\frac{k(d-1)}{d}\left(\frac{\eta}{2}\right)\right)^m\left|\,\left|\xi^o_{t}(U)\cap\{v:\ \alpha t_0\le|v|\le\beta t_0\}\right|>k\right)$$

$$\ge P\left(\left|\xi^o_{mt_0}(U)\cap\{v: m\alpha t_0\le|v|\le m\beta t_0\}\right|>\left(\frac{k(d-1)}{d}\left(\frac{\eta}{2}\right)\right)^m\right|$$

$$\left|\xi^o_{(m-1)t_0}(U)\cap\{v:\ \alpha(m-1)t_0\le|v|\le\beta(m-1)t_0\}\right|$$

$$\left.>\left(\frac{k(d-1)\eta}{2d}\right)^{m-1}\right)$$

$$\times P\left(\left|\xi^o_{(m-1)t_0}(U)\cap\{v: (m-1)\alpha t_0\le|v|\le(m-1)\beta t_0\}\right|\right.$$

$$>\left(\frac{k(d-1)}{d}\left(\frac{\eta}{2}\right)\right)^{m-1}\Bigg| \tag{67}$$

$$\left|\xi^o_{(m-2)t_0}(U)\cap\{v:\ \alpha(m-2)t_0\le|v|\le\beta(m-2)t_0\}\right|$$

$$\left.>\left(\frac{k(d-1)\eta}{2d}\right)^{m-2}\right)$$

$$\vdots$$

$$\times P\left(\left|\xi^o_{2t_0}(U)\cap\{v: 2\alpha t_0\le|v|\le 2\beta t_0\}\right|\right.$$

$$>\left(\frac{k(d-1)}{d}\left(\frac{\eta}{2}\right)^2\right)\left|\,\left|\xi^o_{t_0}(U)\cap\{v:\ \alpha t_0\le|v|\le\beta t_0\}\right|\ge k\right)$$

(by the Markov property)

$$\ge\prod_{i=1}^{m-1}\left[1-\exp\left(-a\left(\frac{\eta}{2}\right)\left(\frac{k(d-1)}{d}\right)^i\right)\right].$$

Note that

$$k\left(\frac{\eta}{2}\right)\left(\frac{d-1}{d}\right) > 2$$

and (67) so that

$$P\left(\left|\xi^{o}_{mt_0}(U) \cap \{v : m\alpha t_0 \le |v| \le m\beta t_0\}\right| > 2^m\right) \ge \left(\frac{\eta}{2}\right)\sigma, \tag{68}$$

where

$$\sigma = \prod_{i=1}^{\infty}\left[1 - \exp\left(-a\left(\frac{\eta}{2}\right)\left(\frac{k(d-1)}{d}\right)^i\right)\right] > 0.$$

Lemma 2 is proved. □

Since the proof of Proposition 5 is involved, we would like to present an intuitive explanation first. In fact, if a particle at $(o, 0)$ survives for a long time t_1, by Lemma 2 and (63), there should exist $C\exp(c_\lambda t_1)$ vertices $\{x_1\}$ inside $\{\alpha t_1 \le |v| \le \beta t_1\}$ such that each of them is occupied by a particle η_{x_1} and no other particles occupy $N(x_1)$, where C is a constant. For each x_1, at time $2t_1$, there should exist $C\exp(c_\lambda t_1)$ vertices $\{x_2\}$ in $N(x_1) \cap \{\alpha t_1 \le |v - x_1| \le \beta t_1\}$ such that each of them is occupied by a particle η_{x_2} generated from η_{x_1} inside $N(x_1)$ (see Figure 4). After doing this step m times, at time mt_1, there exist $C\exp(c_\lambda t_1)$ vertices $\{x_m\}$ inside $N(x_{m-1})$ such that each of them is occupied by a particle η_{x_m} generated from $\eta_{x_{m-1}}$ inside $N(x_{m-1}) \cap \{\alpha t_1 \le |v - x_m| \le \beta t_1\}$. Now we consider a backward process, that is, to generate the particles $\{\eta_{x_m}\}$ to o. For each x_m, by Proposition 4, η_{x_m} can generate a particle to x_{m-1} inside $N(x_{m-1})$ with a probability $\exp(-\varepsilon|x_m - x_{m-1}|) \ge \exp(-\varepsilon\beta m t_1)$, where β, defined in Lemma 2, is a constant which does not depend on t_1 and m, and ε can be very small if t_1 is large. However, there are at least $C\exp(c_\lambda(m-1)t_1)$ such $N(x_{m-1})$ as we discussed above. Note that $N(x) \cap N(y) = \emptyset$ if $x \ne y$ so that, by a standard probability estimate, there are at least $D\exp[(c_\lambda(m-1) - \varepsilon\beta)t_1]$ such x_{m-1} that are occupied by a particle from $\{\eta_{x_m}\}$, where D is a constant. We denote by $\{\eta_{x_{m-1}}\}$ these particles. Subsequently, by the same argument there are at least

$$D\exp(c_\lambda(m-1)t_1 - \varepsilon\beta t_1 - \varepsilon\beta t_1)$$

such x_{m-2} that are occupied by a particle from $\{\eta_{m-1}\}$. Note that ε can be very small so that we can repeat this method m times to generate a particle from x_m back to o again with a positive probability.

Now we give a formal proof as follows. In the following proof, we first give a probability estimate for $m = 3$. Note that, except the first time, each time we only generate particles in $N(x_i)$ from η_i to η_{i+1} for $i = 2, 3, \dots$ and consider the backward generation from η_{i+1} to η_i also in $N(x_i)$ so that we can repeat the same method as $m = 2$ and 3 for a general m.

Proof of Proposition 5. We divide the proof into three parts. The first part, part A, is to show that a particle from o generates particles in $\{N(x_2)\}$

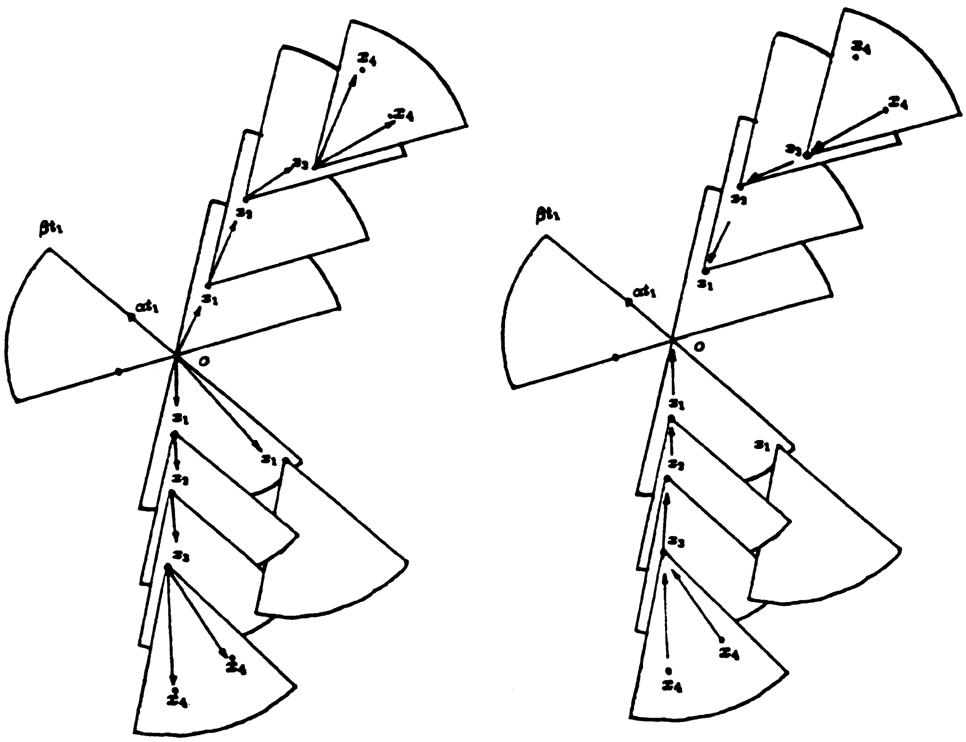

FIG. 4. *The left figure is the event that the particle from o comes to* $\{x_4\}$ *and the right one is the event that o is reoccupied by the particles from* $\{x_4\}$.

as mentioned in the intuitive explanation. The second part, part B, is to show that, with a uniform positive probability, o is reoccupied by the particles in $\{N(x_2)\}$. The third part, part C, is to give a general probability estimate for any m.

Now we prove part A. By Lemma 2, we may take t_1 large such that

$$P\Big(\Big|\,\xi^o_{t_1}(U) \cap \{v\colon \alpha t_1 \le |v| \le \beta t_1\}\Big| > \exp(c_\lambda t_1)\Big) \ge \delta. \tag{69}$$

Now we consider the border of $\xi^o_{t_1}(U)$. Let Y_1 be the border set of these particles. Then, by (63), the number of its vertices has to be larger than

$$\frac{d-1}{d}\exp(c_\lambda t_1)$$

if

$$\Big\{\Big|\,\xi^o_{t_1}(U) \cap \{v\colon \alpha t_1 \le |v|\, \beta t_1\}\Big| > \exp(c_\lambda t_1)\Big\}.$$

For such a particle at x, at time $2t_1$, by Lemma 2 with a probability larger than δ, there exist more than $\exp(c_\lambda t_1)$ particles generated from x which stay in $N(x) \cap \{v\colon \alpha t_1 \le |v - x| \le \beta t_1\}$. Here by a particle generated from x inside $N(x) \cap \{v\colon \alpha t_1 \le |v - x| \le \beta t_1\}$ we mean that at $2t_1$ it can be connected by a path from $\langle x, t_1\rangle$ inside $N(x) \cap \{v\colon \alpha t_1 \le |v-x| \le \beta t_1\}$. If the border vertex x has the above property, $N(x)$ is called a *good branch* or *good*. By (62) and the same argument of Lemma 2 again, there exist

$$\frac{(d-1)}{d}\left(\frac{\delta}{2}\right)\exp(c_\lambda t_1)$$

good $N(x)$ with a probability larger than

$$1-\exp\left[-a(\delta)\frac{(d-1)}{d}\exp(c_\lambda t_1)\right].$$

Now we consider the border vertices of the particles generated by such x at time $3t_1$. Let Y_2 denote these particles. We also call $N(y)$ good if there exist $\exp(c_\lambda t_1)$ particles generated from y in $N(y) \cap \{v\colon \alpha t_1 \le |v-y| \le \beta t_1\}$. Similarly, on the event that there exist

$$\frac{(d-1)}{d}\left(\frac{\delta}{2}\right)\exp(c_\lambda t_1)$$

good $N(x)$, at time $3t_1$ there exist

$$\left[\frac{(d-1)}{d}\left(\frac{\delta}{2}\right)\right]^2\exp(c_\lambda 2t_1)$$

good $N(y)$ with a probability larger than

$$1-\exp\left[-a(\delta)\exp(c_\lambda 2t_1)\left(\frac{(d-1)}{d}\right)^2\frac{\delta}{2}\right]$$

(see Figure 4), where $y \in Y_2$. Note that δ does not depend on t_1 so that we may choose t_1 large enough such that, for all $m \ge 1$,

$$\left[\frac{(d-1)}{d}\left(\frac{\delta}{2}\right)\right]^m\exp(c_\lambda m t_1) \ge \exp(mbt_1)$$

and

$$\exp\left(-a(\delta)\exp(c_\lambda m t_1)\left[\frac{(d-1)}{d}\right]^m\left[\frac{\delta}{2}\right]^{m-1}\right) \le \exp(-a(\delta)\exp(bmt_1))$$

for some $0 < b < c_\lambda$. Thus part A is proved.

Next we show part B; that is, o is reoccupied by a particle generated by these particles in $\bigcup_y N(y)$ with a uniform positive probability. By Proposition 4, for $\varepsilon > 0$ and any $z \in \{v\colon \alpha t_1 \le |v| \le \beta t_1\}$, there exist M and G such that, for all $t_1 > G$,

$$\begin{aligned}P(\exists\, t(z) \le M\beta t_1 \text{ such that } \langle o,0\rangle \to \langle z,t(z)\rangle \text{ in } H(o,z))\\ \ge \exp(-\varepsilon\beta t_1).\end{aligned}\tag{70}$$

Clearly, for each $z \in \bigcup_y N(y)$,

$$\begin{aligned}P(\exists\, t(z) \le M\beta t_1 \text{ such that } \langle z,0\rangle \to \langle y,t(z)\rangle \text{ in } H(z,y))\\ \ge \exp(-\varepsilon\beta t_1).\end{aligned}\tag{71}$$

On the event that there exist $\exp(bt_1)$ good $N(x)$ and $\exp(2bt_1)$ good $N(y)$, for each $N(y)$ let $z \in N(y)$ be a vertex such that z is occupied by the particle generated from y. Then, by (71) and translation invariance,

$$\begin{aligned} P(\exists\ t(z) \le M\beta t_1 \text{ such that } \langle z, 3t_1 \rangle \to \langle y, 3t_1 + t(z) \rangle \text{ in } H(z,y)) \\ \ge \exp(-\varepsilon\beta t_1). \end{aligned} \tag{72}$$

Specifically, z is called *excellent* if z satisfies the condition above. Let Z be the set of all excellent vertices. Let us consider the i.i.d. sequence $\{X_i\}$ with a common distribution

$$X_i = \begin{cases} 1, & \text{with probability } \exp(-\varepsilon\beta t_1), \\ 0, & \text{with probability } 1 - \exp(-\varepsilon\beta t_1). \end{cases}$$

Let $S_n = \sum_{i=1}^n X_i$. By Chebyshev's inequality, for $n = \exp(2bt_1)$,

$$\mathscr{P}\left(S_n \le \frac{n\exp(-\varepsilon t_1)}{2}\right) \le 4\exp(-(2b - \varepsilon\beta)t_1) \tag{73}$$

for the probability measure $\mathscr{P}$, where $\mathscr{P}$ is the product probability measure for $\{X_i\}$. Note that if $y_1 \ne y_2$, then $N(y_1) \cap N(y_2) = \varnothing$ so that

$$\{\exists\ t(z_1) \le M\beta t_1 \text{ such that } (z_1, 3t_1) \to (y_1, 3t_1 + t(z_1)) \text{ in } H(z_1, y_1)\}$$

and

$$\{\exists\ t(z_2) \le M\beta t_1 \text{ such that } (z_2, 3t_1) \to (y_2, 3t_1 + t(z_2)) \text{ in } H(z_2, y_2)\}$$

are independent events since

$$H(z_1, y_1) \cap H(z_2, y_2) = \varnothing.$$

Furthermore, each event has a probability larger than $\exp(-\varepsilon\beta t_1)$. Then, by (72), (73), and the independence of two such events, with a probability larger than

$$1 - 4\exp(-(2b - \varepsilon\beta)t_1),$$

there exists $\frac{1}{2}\exp((2b - \varepsilon\beta)t_1)$ such $N(y)$ so that each of them contains such excellent z.

However, $t(z)$ varies from $3t_1$ to $3t_1 + M\beta t_1$. To fix a unique time, we will do the following work. We denote by $E(2t_1)$ the event that there exist $\frac{1}{2}\exp((2b - \varepsilon\beta)t_1)$ such $N(y)$ so that each of them contains an excellent z. Then, on the event $E(2t_1)$, for each z, there exists an integer $h(z)$ with $3t_1 \le h(z) \le 3t_1 + M\beta t_1$ such that

$$P(t(z) \in [h(z), h(z) + 1]) \ge \frac{1}{M\beta t_1}.$$

By Chebyshev's inequality [see (73)] with

$$X_i = \begin{cases} 1, & \text{with probability } \dfrac{1}{M\beta t_1}, \\ 0, & \text{with probability } 1 - \dfrac{1}{M\beta t_1}, \end{cases}$$

then there exist

$$\frac{1}{2}\exp[(2b - \varepsilon\beta)t_1]\left(\frac{1}{2M\beta t_1}\right)$$

such $h(z)$ defined above with a probability larger than

$$1 - 8M\beta t_1 \exp[-(2b - \varepsilon\beta)t_1].$$

Clearly, if $t(z)$ is fixed in the interval $[h(z), h(z)+1]$, with a unique strictly positive probability ρ for all z,

$$\langle y, t(z)\rangle \to \langle y, h(z) + 2\rangle.$$

By (62), the Markov property and the discussion above, with a probability larger than

$$[1 - 4\exp(-(2b - \varepsilon\beta)t_1)][1 - 8M\beta t_1 \exp(-(2b - \varepsilon\beta)t_1)] \times\left[1 - \exp\left(-\alpha(\rho)\frac{1}{2^2 M\beta t_1}\exp((2b - \varepsilon\beta)t_1)\right)\right],$$

there exist

$$\frac{\rho}{2^3 M\beta t_1}\exp((2b - \varepsilon\beta)t_1)$$

such z with

$$\langle z, 3t_1\rangle \to \langle y, h(z) + 2\rangle.$$

Let $\mathscr{N}$ be the number of such z so that

$$\langle z, 3t_1\rangle \to \langle y, m + 2\rangle$$

for some integer m with $3t_1 \le m \le 3t_1 + M\beta t_1$, where m does not depend on z. Note that $h(z)$ is an integer (nonrandom) and $3t_1 \le h(z) \le 3t_1 + M\beta t_1$ so that there exists m such that

$$\begin{aligned} P&\left(\mathscr{N} \ge \frac{1}{M\beta t_1}\frac{\rho}{2^3 M\beta t_1}\exp((2b - \varepsilon\beta)t_1)\right) \\ &\ge [1 - 4\exp(-(2b - \varepsilon\beta)t_1)][1 - 8M\beta t_1\exp(-(2b - \varepsilon\beta)t_1)] \\ &\quad\times\left[1 - \exp\left(-\alpha(\rho)\frac{1}{2^2 M\beta t_1}\exp((2b - \varepsilon\beta)t_1)\right)\right]. \end{aligned}$$

Since β does not depend on t_1, by Proposition 4, we may take t_1 large enough in (70) such that $|z|$ is large enough to make

$$\varepsilon\beta < \frac{b}{4}.$$

Also, note that M does not depend on t_1 so that we take t_1 large enough such that, for any integer $n > 0$,

$$\begin{aligned}&[1 - 4\exp(-(nb - \varepsilon\beta)t_1)][1 - 8M\beta t_1 \exp(-(nb - \varepsilon\beta)t_1)]\\ &\quad\times\left[1 - \exp\left(-a(\rho)\frac{1}{2^2 M\beta t_1}\exp((nb - \varepsilon\beta)t_1)\right)\right]\\ &\quad\geq 1 - \exp(-b(n-1)t_1)\end{aligned} \tag{74}$$

and

$$\frac{1}{M\beta t_1}\frac{\rho}{2^3 M\beta t_1}\exp((nb - \varepsilon\beta)t_1) \geq \exp(b(n-1)t_1). \tag{75}$$

Clearly, with $n = 2$, if t_1 satisfies (70), (74) and (75), with a probability larger than

$$[1 - \exp(-bt_1)],$$

there exist $\exp(bt_1)$ such $N(y)$ so that each of them contains a z which satisfies

$$\langle z, 3t_1\rangle \to \langle y, m + 2\rangle.$$

Similarly, for each y connected by a path from z, we consider the event $(y, m + 2) \to (x, t)$ for some time t, where $x \in Y_1$. By the same estimate we can show the following result. If t_1 satisfies (70), (74) and (75), then, with a probability larger than

$$1 - \exp(-(bt_1)/2),$$

there exist $m \leq m_1 \leq m + M\beta t_1$ and $\exp(c_\lambda t_1/2)$ such y with

$$\langle y, m + 2\rangle \to \langle x, m_1 + 2\rangle.$$

We can then choose a time S such that

$$\begin{aligned}&P(\langle x, 3t_1 + (m + 2) + (m_1 + 2)\rangle\\ &\quad\to \langle o, 3t_1 + (m + 2) + (m_1 + 2) + S\rangle) = \tau\end{aligned}$$

for $\tau > 0$ depending on t_1 and β only. Clearly,

$$\begin{aligned}&P(\langle o, 0\rangle \to \langle o, 3t_1 + (m + 2) + (m_1 + 2) + S\rangle)\\ &\quad\geq \delta\tau(1 - \exp(-a(\delta)\exp(bt_1)))\\ &\qquad\times(1 - \exp(-a(\delta)\exp(2bt_1)))\\ &\qquad\times(1 - \exp(-bt_1))\\ &\qquad\times(1 - \exp(-bt_1/2))\end{aligned}$$

Finally, we show part C. In general, if we do this step l times, we can construct good branches $Y_1, Y_2, \dots, Y_l$. Then we can find $m(l), m_1(l), \dots, m_l(l)$ such that

$$
\begin{aligned}
& lt_1 \le m(l) \\
& \le lt_1 + M\beta t_1, m(l) \le m_1(l) \qquad (76)\\
& \le m(l) + M\beta t_1, \dots, m_{l-1}(l) \le m_l(l) \le m_{l-1}(l) + M\beta t_1,
\end{aligned}
$$

where t_1 satisfies (70), (74) and (75). By the same discussion above, note that t_1 does not depend on l so that

$$
\begin{aligned}
& P(\langle o, 0\rangle \to \langle o, lt_1 + (m(l)+2) + (m_1(l)+2) + \cdots + (m_l(l)+2) + S\rangle) \\
& \quad \ge \delta\tau(1 - \exp(-a(\delta)\exp(bt_1))) \\
& \qquad \times (1 - \exp(-a(\delta)\exp(2bt_1))) \\
& \qquad \vdots \\
& \qquad \times (1 - \exp(-a(\delta)\exp(lbt_1))) \\
& \qquad \times (1 - \exp(-b(l-2)t_1)) \\
& \qquad \times (1 - \exp(-b(l-3)t_1)) \\
& \qquad \vdots \\
& \qquad \times (1 - \exp(-bt_1)) \\
& \qquad \times (1 - \exp(-bt_1/2)) \\
& \quad \ge \delta\tau\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma = & \left[\prod_{l=1}^{\infty} (1 - \exp(-a(\delta)\exp(lbt_1)))\right]\left[\prod_{l=3}^{\infty} (1 - \exp(-b(l-2)t_1))\right] \\
& \times [1 - \exp(-bt_1/2)] > 0.
\end{aligned}
$$

Note that t_1, $m(l), m_1(l), \dots, m_l(l)$ and S are fixed times. For any t, we choose l such that

$$t \le lt_1 + m(l) + 2 + m_1(l) + 2 + \cdots + m_l(l) + 2 + S.$$

Clearly, we have good branches $Y_1, \dots, Y_l$. Let X_{l-1} be the particles generated back from Y_l as we did before. Clearly, $X_{l-1} \subset Y_{l-1}$. Now we only consider doing $l-1$ steps instead of doing l steps. Clearly, we have the same good branches $Y_1, \dots, Y_{l-1}$ as above. If we only consider that o is reoccupied by the particles in X_{l-1} instead of the particles in Y_{l-1}, then we have the same $m_l, m_1(l), \dots, m_{l-1}(l)$ as above such that

$$
\begin{aligned}
P(\langle o, 0\rangle \to \langle o, & (l-1)t_1 + (m(l)+2) \\
& + (m_l(l)+2) + \cdots + (m_{l-1}(l)+2) + S\rangle) \ge \delta\tau\sigma.
\end{aligned}
$$

Similarly, let X_{l-2} be the particles generated back from $X_{l-1},\dots,$ and let X_2 be the particles generated back from X_3. By the same reasoning,

$$P\big(\langle o,0\rangle \to \langle o,(l-i)t_1 + (m(l)+2) + (m_1(l)+2) + \cdots + (m_{l-i}(l)+2) + S\rangle\big) \ge \delta\tau\sigma.$$

Clearly,

$$(l-i)t_1 + m_1(l) + 2 + \cdots + m_{l-i}(l) + 2 + S \le t \le (l-i+1)t_1 + m_1(l) + 2 + \cdots + m_{l-i+1}(l) + 2 + S$$

for some i. Let

$$\pi = P\big(\langle o,(l-i)t_1 + m_1(l) + 2 + \cdots + m_{l-i}(l) + 2 + S\rangle \to \langle o,t\rangle\big).$$

It follows from (76) that π only depends on t_1, M and β. By the Markov property for any t,

$$P(\langle o,0\rangle \to \langle o,t\rangle) \ge \delta\tau\sigma\pi.$$

Proposition 5 is proved. □

3. Proofs of the theorems. Before the proof of Theorem 1 we give the following lemma. We write $\langle x,t\rangle \leftrightarrow \langle y,t\rangle$ if there exist $z \in T$ and time $s > t$ such that $\langle x,t\rangle \to \langle z,s\rangle$ and $\langle y,t\rangle \to \langle z,s\rangle$. With the definition, we have the following lemma.

LEMMA 3. *Suppose that $\lambda > \lambda_c$. Given $\varepsilon > 0$, for any $x, y \in T$, there exists C which may depend on ε but not on x and y such that*

$$P(\langle y,0\rangle \leftrightarrow \langle x,0\rangle \text{ inside } \mathscr{D}\langle y,x\rangle) \ge C|x-y|^{-2}\exp(-\varepsilon|x-y|).$$

PROOF. By the corollary, we can show that there exist M and G such that

$$P(\exists\, t \le M|y| \text{ such that } \langle y,0\rangle \to \langle o,t\rangle \text{ inside } \mathscr{D}(o,y)) \ge \exp(-\epsilon|y|)$$

for any $|y| \ge G$. Therefore, there exist $C_1 > 0$, $C_2 > 0$ and $M > 0$ which are independent of y such that

$$\begin{aligned} &C_1\exp(-\varepsilon|y|)\\ &\quad\le P(\exists\, t \le M|y| \text{ such that } \langle y,0\rangle \to \langle o,t\rangle \text{ inside } \mathscr{D}(o,y))\\ &\quad\le \sum_{i=0}^{M|y|} P(\langle y,0\rangle \to \langle o,t\rangle \text{ inside } \mathscr{D}(o,y) \text{ for } i \le t \le i+1)\\ &\quad\le M|y|\max_i P(\langle y,0\rangle \to \langle o,t\rangle \text{ inside } \mathscr{D}(o,y) \text{ for } i \le t \le i+1)\\ &\quad= C_2M|y|P(\langle y,0\rangle \to \langle o,t_0(y)\rangle \text{ inside } \mathscr{D}(o,y)) \end{aligned} \tag{77}$$

for all y, where $t_0(y)$ is an integer time such that the probability that $\langle y,0\rangle \to \langle 0,t_0(y)\rangle$ inside $\mathscr{D}(o,y)$ is the largest among all i with $0 \le i \le M|y|$, and we assume that $M|y|$ is an integer without loss of generality. For each x

and y, let z be the center of $\mathscr{D}(x,y)$ and let $\bar{z}$ be a vertex such that $|\bar{z}| = \min\{|y-z|, |z-x|\}$. Note that $\bar{z}$ is the center of the segment connecting x and y if $|x-y|$ is an even number. Therefore, by translation invariance, the FKG inequality (see page 78 in [6] for more details) and (77),

$$\begin{aligned} &P(\langle y,0\rangle \leftrightarrow \langle x,0\rangle \text{ inside } \mathscr{D}(y,x)) \\ &\quad \ge P(\langle y,0\rangle \to \langle z, t_0(\bar{z})\rangle \text{ inside } \mathscr{D}(y,z), \\ &\qquad\quad \langle x,0\rangle \to \langle z, t_0(\bar{z})\rangle \text{ inside } \mathscr{D}(x,z)) \\ &\quad \ge P(\langle y,0\rangle \to \langle z, t_0(\bar{z})\rangle \text{ inside } \mathscr{D}(y,z)) \\ &\qquad \times P(\langle x,0\rangle \to \langle z, t_0(\bar{z})\rangle \text{ inside } \mathscr{D}(x,z)) \\ &\quad \ge C_3(C_1/C_2M)^2|y-x|^{-2}\exp(-\varepsilon 2|y-x|) \end{aligned}$$

for some constant $C_3 > 0$. Lemma 3 is proved. □

To show Theorem 1, we need to verify the condition of Griffeath's lemma as we said before; that is, for any subsets A and B of T,

$$P\Big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\ \xi_t^A \ne \varnothing,\ \bar{\xi}_t^B \ne \varnothing\Big) \to 0 \quad \text{as } t \to \infty, \tag{78}$$

where $\bar{\xi}_t^B$ is an independent copy of the contact process. To verify (78), we would like to present the following heuristic argument first. We will first show that

$$P\Big(\forall\ t \le \tau \text{ such that } \xi_t^A \cap \bar{\xi}_t^B = \varnothing,\ \xi_\tau^A \ne \varnothing,\ \bar{\xi}_\tau^B \ne \varnothing\Big) \to 0 \quad \text{as } \tau \to \infty. \tag{79}$$

Assuming that both $\xi_\tau^A \ne \varnothing$ and $\bar{\xi}_\tau^B \ne \varnothing$, then by Lemma 2 we can choose s large with $s < \tau$ such that

$$\Big|\, \xi_s^A \cap \{v : |v-u| \le 2\beta s \text{ for } u \in A\} \Big| \ge \exp(c_\lambda s)$$

and

$$\Big|\, \bar{\xi}_s^B \cap \{v : |v-u| \le 2\beta s \text{ for } u \in B\} \Big| \ge \exp(c_\lambda s).$$

We set

$$(X,Y) = \Big\{(x,y) : x \in \mathscr{B}\big(\xi_s^A\big),\ y \in \mathscr{B}\big(\bar{\xi}_x^B\big)\Big\},$$

where x and y are border vertices of ξ_s^A and $\bar{\xi}_s^B$, respectively. Clearly, $x \ne y$ for any pair (x,y) if $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for $t < \tau$. Furthermore, $\langle x,s\rangle \leftrightarrow \langle y,s\rangle$ cannot occur on the time interval (s,t) if $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for $t < \tau$. However, $\langle x,s\rangle \not\leftrightarrow \langle y,s\rangle$ in some time interval (s, s_1) will have a probability smaller than

$$1 - C|x-y|^{-2}\exp(-\varepsilon|x-y|)$$

by Lemma 3. Now, for another pair $(x_1, y_1) \in (X,Y)$, the particles in x_1 and y_1 can survive and hit x_1 and y_1 at time s_1 in $N(x_1)$ and $N(y_1)$, respectively, with a positive probability δ^2 by Proposition 5. Suppose that x_1 and y_1 are hit by s_1 by particles from $\langle x_1, s\rangle$ and $\langle y_1, s\rangle$, respectively. Then $\langle x_1, s_1\rangle \not\leftrightarrow \langle y_1, s_1\rangle$ in some time interval (s_1, s_2) for any $\tau > s_2 > s_1$ if $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for $s_2 < \tau$. It will have a probability smaller than $1 - \delta^2 C|x_1 - y_1|^{-1}\exp(-\varepsilon$

$|x_1 - y_1|$). By using the same argument for each pair in (X, Y), $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for $s < t < \tau$ has a probability smaller than

$$\Big(1 - \delta^2 C \max_i |x_i - y_i|^{-2} \exp\big(-\varepsilon \max_i |x_i - y_i|\big)^{\exp(c_\lambda s)}\Big).$$

Note that $|x_i - y_i| \le 2\beta s$ for each i so that the probability in (79) goes to 0 as s goes to ∞ if we take ε small. With the probability estimate above and the Markov property, we know that ξ_t^A and $\bar{\xi}_t^B$ intersect many times if t is large. Then, by Proposition 5, we can show that Lemma 4 holds if t is large.

Now we present a formal proof as follows.

PROOF OF THEOREM 1. Clearly, to show that the complete convergence theorem holds for $\lambda > \lambda_c$, we only need to check Griffeath's lemma for $\lambda > \lambda_c$. We first restrict our discussion to the finite sets A and B. Now we will show (79). We denote

$$M = \max_{v \in A,\, u \in B} \{|u - v|\}.$$

We also denote

$$D_s(A) = \mathscr{B}\bigg(\xi_s^A \cap \Big\{\bigcup_{u \in A} \{v : |v - u| \le \beta s\}\Big\}\bigg)$$

and

$$D_s(B) = \mathscr{B}\bigg(\bar{\xi}_s^B \cap \Big\{\bigcup_{u \in B} \{v : |v - u| \le \beta s\}\Big\}\bigg),$$

where $\mathscr{B}(S)$ is the border of S. Note that A is finite so that it follows from the same argument in Lemma 2 [see (67) and (68)] and (63) that, given $\varepsilon > 0$, we can take $s > M$ large such that

$$P\big(|D_s(A)| \ge \exp(c_\lambda s) \,|\, \xi_s^A \ne \varnothing\big) \ge 1 - \varepsilon \tag{80}$$

and

$$P\big(|D_s(B)| \ge \exp(c_\lambda s) \,|\, \bar{\xi}_s^B \ne \varnothing\big) \ge 1 - \varepsilon, \tag{81}$$

where β and c_λ are constants which do not depend on s. We write $Z(s)$ for the following event:

$$\big\{|D_s(A)| \ge \exp(c_\lambda s)\big\} \cap \big\{|D_s(B)| \ge \exp(c_\lambda s)\big\}.$$

Let $x_1 \in D(A)$ and $y_1 \in D_s(B)$ be the vertices such that

$$|x_1 - y_1| = \min_{x \in D_s(A),\, y \in D_s(B)} \{|x - y|\}.$$

Note that (x_1, y_1) may not be a unique pair and $\mathscr{D}(x_1, y_1)$ does not contain the other vertices of $D_s(A)$ and $D_s(B)$ except for x_1 and y_1. On the event that

$$\big\{\xi_t^A \cap \bar{\xi}_t^B = \varnothing, \forall\, t < \tau\big\}$$

for $x_1 \in D_s(A)$ and $y_1 \in D_s(B)$ and any $s < t_1 < \tau$, then

$$\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ cannot occur,}$$

where $\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle$ is the event that $\langle x_1, s\rangle \leftrightarrow \langle y_1, s\rangle$ at some t with $s < t < t_1$. However, by Lemma 3, for given $\eta > 0$ there exist t_1 and C which does not depend on x_1 and y_1 such that

$$P\Big(\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big) \geq C|x_1 - y_1|^{-2} \exp(-\eta|x_1 - y_1|). \tag{82}$$

On the event $Z(s)$, consider $x_2 \in D_s(A)$ and $y_2 \in D_s(B)$ with

$$|x_2 - y_2| = \min_{x \in D_s(A)\setminus\{x_1\},\ y \in D_s(B)\setminus\{y_1\}} \{|x - y|\}.$$

Note that $\mathscr{D}(x_2, y_2)$ does not contain vertices of $D_s(A) \setminus \{x_1, x_2\}$ and $D_s(B) \setminus \{y_1, y_2\}$. Note also that x_2 and y_2 are border vertices of ξ_s^A and $\bar{\xi}_s^B$, respectively, so that there exist $N(x_2)$ and $N(y_2)$ such that $N(x_2)$ and $N(y_2)$ do not contain particles of ξ_s^A and $\bar{\xi}_s^B$ except for x_2 and y_2. It follows from our definitions of x_i and y_i for $i = 1, 2$ that if $N(x_2) \cap \mathscr{D}(x_1, y_1) \neq \varnothing$, then $\mathscr{D}(x_1, y_1) \subset N(x_2)$. This is impossible since $N(x_2)$ does not contain any vertex of $D_s(A) \cup D_s(B)$ except for x_2. Similarly, we can show that $N(y_2) \cap \mathscr{D}(x_1, y_1) = \varnothing$. Then

$$[N(x_2) \cup N(y_2)] \cap \mathscr{D}(x_1, y_1) = \varnothing. \tag{83}$$

On the event $D_s(A) = \Gamma_1$ and $D_s(B) = \Gamma_2$ for some vertex set Γ_1 and Γ_2, by Proposition 5 and the independence of ξ_t^A and $\bar{\xi}_t^B$, there exists $s_1 > t_1$ such that $x_2 \in \xi_{s_1}^{x_2, s}(N(x_2))$ and $y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2))$ with probability δ^2 for some $\delta > 0$, $x_2 \in \Gamma_1$ and $y_2 \in \Gamma_2$, where $\xi_t^{x, s}(N(x))$ was defined in (44), that is, the contact process for $t \geq s$ on $N(x)$ such that $\xi_s^{x, s}(N(x)) = x$. Furthermore, by Lemma 3, for $\eta > 0$ in (82) there exist the C [in (82)] and t_2 such that

$$\begin{aligned} &P\Big(x_2 \in \xi_{s_1}^{x_2, x}(N(x_2)),\ y_2 \in \xi_{s_1}^{y_2, s}(N(y_2)), \\ &\qquad \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big) \\ &\quad \geq C\delta^2|x_2 - y_2|^{-2}\big(\exp(-\eta|x_2 - y_2|)\big) \end{aligned} \tag{84}$$

for any x_2 and y_2. However, on the event $Z(s)$, $D_s(A) = \Gamma_1$, $D_s(B) = \Gamma_2$ and $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for all $s < t < \tau$, then, if $t_2 < \tau$,

$$\Big\{x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)),\ y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)), \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big\}$$

cannot occur. Note that, by (83), $\{N(x_2) \cup N(y_2)\} \cap \mathscr{D}(x_1, y_1) = \varnothing$ and $s < t_1 < s_1 < t_2$ so that, on the event that $D_s(A) = \Gamma_1$ and $D_s(B) = \Gamma_2$,

$$\Big\{\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big\}$$

and

$$\Big\{x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)),\ y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)), \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big\}$$

are independent for $(x_i, y_i) \in (D_s(A), D_s(B))$ with $i = 1, 2$ since the first event only depends on the Poisson processes on $\mathscr{D}(x_1, y_1) \times (s, t_1)$ and the second event only depends on the Poisson processes on $N(x_2) \cup N(y_2) \times (s, s_1)$ and $\mathscr{D}(x_2, y_2) \times (s_1, t_2)$. Continuing, on the event $Z(s)$, we can construct k pairs as follows:

$$\{(x_3, y_3); \dots; (x_k, y_k)\} \subset (D_s(A), D_s(B))$$

for an integer k with $\exp(c_\lambda s) - 1 \le k \le \exp(c_\lambda s)$. Then, by the same argument on the event $Z(s)$, $D_s(A) = \Gamma_1$ and $D_s(B) = \Gamma_2$, we have the following independent events:

$$\begin{aligned}
&\Big\{\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big\},\\
&\Big\{x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)), y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)),\\
&\qquad \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\},\\
&\quad\vdots\\
&\Big\{x_k \in \xi_{s_{k-1}}^{x_k, s}(N(x_k)), y_k \in \bar{\xi}_{s_{k-1}}^{y_k, s}(N(y_k)),\\
&\qquad \langle x_k, s_{k-1}\rangle \overset{t_k}{\leftrightarrow} \langle y_k, s_{k-1}\rangle \text{ inside } \mathscr{D}(x_k, y_k)\}
\end{aligned} \tag{85}$$

for $s < t_1 < s_1 < t_2 < \cdots < s_{k-1} < t_k$. By the independence of ξ_t^A and $\bar{\xi}_t^B$, Proposition 5 and Lemma 3 again for the $\eta > 0$ in (82), there exist the C [in (82)] and t_i such that

$$\begin{aligned}
&P\Big(x_i \in \xi_{s_{i-1}}^{x_i, s}(N(x_i)),\\
&\qquad y_i \in \bar{\xi}_{s_{i-1}}^{y_i, s}(N(y_i)), \langle x_i, s_{i-1}\rangle \overset{t_i}{\leftrightarrow} \langle y_i, s_{i-1}\rangle \text{ inside } \mathscr{D}(x_i, y_i)\Big)\\
&\qquad \ge C\delta^2 |x_i - y_i|^{-2} \exp(-\eta|x_i - y_i|)
\end{aligned} \tag{86}$$

for any x_i and y_i. On the other hand, on the event $Z(s)$, $D_s(A) = \Gamma_1$, $D_s(B) = \Gamma_2$ and $\xi_t^A \cap \bar{\xi}_t^B = \varnothing$ for all $s < t < \tau$, then, if $t_k < \tau$,

$$\begin{aligned}
&\Big\{\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big\}\\
&\quad \cup \Big\{x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)), y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)),\\
&\qquad\quad \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big\}\\
&\quad\vdots\\
&\quad \cup \Big\{x_k \in \xi_{s_{k-1}}^{x_k, s}(N(x_k)), y_k \in \bar{\xi}_{s_{k-1}}^{x_k, s}(N(y_k)),\\
&\qquad \langle x_k, s_{k-1}\rangle \overset{t_k}{\leftrightarrow} \langle y_k, s_{k-1}\rangle \text{ inside } \mathscr{D}(x_k, y_k)\Big\}
\end{aligned} \tag{87}$$

cannot occur. Therefore, if we take τ large such that $t_k < \tau$,

$$P\big(\forall\ t < \tau \text{ with } \xi_t^A \cap \bar{\xi}_t^B = \varnothing,\ \xi_\tau^A \neq \varnothing,\ \xi_\tau^B \neq \varnothing\big)$$

$$\leq P\big(Z(s), \forall\ t < \tau \text{ with } \xi_t^A \cap \bar{\xi}_t^B = \varnothing \text{ for } s < t < \tau\big) + 2\varepsilon \qquad \text{[by (80) and (81)]}$$

$$\leq \sum_{\Gamma_1, \Gamma_2} P\big(Z(s), D_s(A) = \Gamma_1, D_s(B) = \Gamma_2,$$

$$\forall\ t < \tau \text{ with } \xi_t^{\Gamma_1, s} \cap \bar{\xi}_t^{\Gamma_2, s} = \varnothing \text{ for } s < t < \tau\big) + 2\varepsilon$$

[the sum is taken over all Γ_1 and Γ_2 for $\Gamma_1 \subset \bigcup_{u \in A}\{v: |v - u| \leq \beta s\}$ and $\Gamma_2 \subset \bigcup_{u \in B}\{v: |v - u| \leq \beta s\}$ with $\Gamma_1 \cap \Gamma_2 = \varnothing$ and $|\Gamma_1| \geq \exp(c_\lambda s)$ and $|\Gamma_2| > \exp(c_\lambda s)$]

$$\leq \sum_{\Gamma_1, \Gamma_2} P\big(Z(s), D_s(A) = \Gamma_1, D_s(B) = \Gamma_2,$$

$$\exists\ (x_1, y_1), \dots, (x_k, y_k) \subset (\Gamma_1, \Gamma_2)$$

$$\text{with } k = \lfloor \exp(c_\lambda s) \rfloor \text{ such that } \Big\{\langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big\}^C,$$

$$\Big\{x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)),\ y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)),\ \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big\}^C$$

$$\vdots$$

$$\Big\{x_k \in \xi_{s_{k-1}}^{x_k, s}(N(x_k)),\ y_k \in \bar{\xi}_{s_{k-1}}^{y_k, s}(N(y_k)),\ \langle x_k, s_{k-1}\rangle \overset{t_k}{\leftrightarrow} \langle y_k, s_{k-1}\rangle \text{ inside } \mathscr{D}(x_k, y_k)\Big\}^C\Big) + 2\varepsilon \quad \text{[by (87)]} \tag{88}$$

$$\leq \sum_{\Gamma_1, \Gamma_2} P(Z(s), D_s(A) = \Gamma_1, D_s(B) = \Gamma_2)$$

$$\times \Big[1 - P\Big(\exists\ (x_1, y_1) \in (\Gamma_1, \Gamma_2) \text{ such that } \langle x_1, s\rangle \overset{t_1}{\leftrightarrow} \langle y_1, s\rangle \text{ inside } \mathscr{D}(x_1, y_1)\Big)\Big]$$

$$\times \Big[1 - P\Big(\exists\ (x_2, y_2) \in (\Gamma_1, \Gamma_2),\ x_2 \in \xi_{s_1}^{x_2, s}(N(x_2)),\ y_2 \in \bar{\xi}_{s_1}^{y_2, s}(N(y_2)),\ \langle x_2, s_1\rangle \overset{t_2}{\leftrightarrow} \langle y_2, s_1\rangle \text{ inside } \mathscr{D}(x_2, y_2)\Big)\Big]$$

$$\vdots$$

$$\times\Big[1 - P\Big(\exists\,(x_k, y_k) \in (\Gamma_1, \Gamma_2),\ x_k \in \xi_{s_{k-1}}^{x_k, s}(N(x_k)),$$
$$y_k \in \xi_{s_{k-1}}^{y_k, s}(N(y_k)),$$
$$l\langle x_k, s_{k-1}\rangle \overset{t_k}{\leftrightarrow} \langle y_k, s_{k-1}\rangle \text{ inside } \mathscr{D}(x_k, y_k)\Big)\Big] + 2\varepsilon$$
$$[\text{by the Markov property and (85)}]$$
$$\le \Big[1 - C\delta^2 \max_{1\le i\le k} |x_i - y_i|^{-2} \exp\Big(-\eta \max_{1\le i\le k} |x_i - y_i|\Big)\Big]^k + 2\varepsilon \quad [\text{by (86)}]$$
$$\le \Big[1 - C\delta^2 (\beta s)^{-2} \exp(-\eta\beta s)\Big]^{\exp(c_\lambda s) - 1} + 2\varepsilon.$$

Note that β and c_λ depend only on λ so that we may take η small such that $\eta\beta < c_\lambda$. On the other hand, δ only depends on λ and C does not depend on x_i and y_i so does not s. Hence, by taking η small, s large then τ large in (88), it follows that, for any finite A and B,

$$P\Big(\forall\ t \le \tau,\ \xi_t^A \cap \bar{\xi}_b^T = \varnothing,\ \xi_\tau^A \ne \varnothing,\ \bar{\xi}_\tau^B \ne \varnothing\Big) \to 0 \quad \text{as } \tau \to \infty. \tag{89}$$

Therefore, (79) is proved. With (79), we next show (78). By the Markov property and (79), for any m,

$$P\Big(\exists\ t_1 < t_2 < \cdots < t_m < \tau \text{ with } t_i + 1 < t_{i+1},\ i = 1, \dots, m-1, \text{ such that}$$
$$\xi_{t_1}^A \cap \bar{\xi}_{t_1}^B \ne \varnothing, \xi_{t_2}^A \cap \bar{\xi}_{t_2}^B \ne \varnothing, \dots, \xi_{t_m}^A \cap \bar{\xi}_{t_m}^B \ne \varnothing \mid \xi_\tau^A \ne \varnothing,\ \bar{\xi}_\tau^B \ne \varnothing\Big) \tag{90}$$
$$\to 1 \quad \text{as } \tau \to \infty.$$

For any N, let

$$P\Big(|\xi_{t+1}^A \cap \bar{\xi}_{t+1}^B| \ge N | \xi_t^A \cap \bar{\xi}_t^B \ne \varnothing\Big) = \rho(N).$$

By the Markov property again, (90) and the same proof as in Proposition 2, for any integer N,

$$P\Big(\exists\ t < \tau \text{ such that } |\xi_t^A \cap \bar{\xi}_t^B| \ge N | \xi_\tau^A \ne \varnothing,\ \bar{\xi}_\tau^t \ne \varnothing\Big) \to 1 \quad \text{as } \tau \to \infty. \tag{91}$$

If $|\xi_s^A \cap \bar{\xi}_s^B| \ge N$, then, by (62),

$$\Big|\mathscr{B}\big(\xi_s^A \cap \bar{\xi}_s^B\big)\Big| \ge \frac{d-1}{d} N.$$

For each $x \in \mathscr{B}(\xi_t^A \cap \bar{\xi}_t^B)$, by Proposition 5 and the independence of ξ_t^A and $\bar{\xi}_t^B$ for all s and t with $s < t$, there exists $\delta > 0$:

$$P\big(x \in \xi_t^A(N(x)) \cap \bar{\xi}_t^B(N(x)) | x \in \xi_s^A \cap \bar{\xi}_s^B\big)$$
$$\ge P\big(x \in \xi_t^A(N(x)),\ x \in \bar{\xi}_t^B(N(x)) | x \in \xi_s^A \cap \bar{\xi}_s^B\big)$$
$$\ge P\big(x \in \xi_{t-s}^x(N(x)),\ x \in \bar{\xi}_{t-s}^x(N(x))\big)$$
$$(\text{by the Markov property and translation invariance})$$
$$\ge P\big(x \in \xi_{t-s}^x(N(x))\big) P\big(x \in \bar{\xi}_{t-s}^x(N(x))\big)$$
$$\ge \delta^2.$$

Note that, on the event that $y, z \in \xi_s^A \cap \bar{\xi}_s^B$ if $t > s$ and $y \neq z$,

$$\{z \in \xi_t^A(N(z)) \cap \bar{\xi}_t^B(N(z))\} \text{ and } \{y \in \xi_t^A(N(y)) \cap \bar{\xi}_t^B(N(y))\}$$

are independent so that, given $\varepsilon > 0$, by the law of large numbers we can find N such that, for any s and t with $s < t$,

$$(92)\qquad P(\exists\, x \in \xi_s^A \cap \bar{\xi}_s^B \text{ such that } x \in \xi_t^A(N(x)) \cap \bar{\xi}_t^B(N(x)) \,|\, |\xi_s^A \cap \bar{\xi}_s^B| > N) > 1 - \varepsilon.$$

Finally,

$$\begin{aligned}
&P\big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\, \xi_t^A \neq \varnothing,\, \bar{\xi}_t^B \neq \varnothing\big)\\
&\quad = P\big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\, \xi_t^A \neq \varnothing,\, \bar{\xi}_t^B \neq \varnothing,\, \exists\, s < t \text{ such that } |\xi_s^A \cap \bar{\xi}_s^B| > N\big)\\
&\qquad + P\big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\, \xi_t^A \neq \varnothing,\, \bar{\xi}_t^B \neq \varnothing,\\
&\qquad\qquad \forall\, s < t \text{ such that } |\xi_s^A \cap \bar{\xi}_s^B| \le N\big)\\
&\quad \to 0 \quad \text{as } t \to \infty \quad [\text{by (91) and (92)}].
\end{aligned}$$

Then (78) is proved for finite A and B. Now we show that (78) holds for any sets A and B. Consider only A and B are infinite. The other cases can be shown by the same argument. Since A and B are both infinite sets, the borders of A and B are both infinite. Note that $\lambda > \lambda_s$ and recall Lemma 1 so that by the law of large numbers we can find finite $A_1 \subset A$ and $B_1 \subset B$ such that, for any $\varepsilon > 0$,

$$(93)\qquad P\big(\xi_t^{A_1} \neq \varnothing\big) \ge 1 - \varepsilon \quad \text{and} \quad P\big(\xi_t^{B_1} \neq \varnothing\big) \ge 1 - \varepsilon.$$

Therefore,

$$\begin{aligned}
&P\big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\, \xi_t^A \neq \varnothing,\, \bar{\xi}_t^B \neq \varnothing\big)\\
&\quad \le P\big(\xi_t^A \cap \bar{\xi}_t^B = \varnothing,\, \xi_t^{A_1} \neq \varnothing,\, \bar{\xi}_t^{B_1} \neq \varnothing\big) + 2\varepsilon \quad [\text{by (93)}]\\
(94)\quad &\quad \le P\big(\xi_t^{A_1} \cap \bar{\xi}_t^{B_1} = \varnothing,\, \xi_t^{A_1} = \varnothing,\, \bar{\xi}_t^{B_1} \neq \varnothing\big) + 2\epsilon\\
&\quad \to 2\varepsilon \quad \text{as } t \to \infty \quad [\text{by (78) and note that } A_1 \text{ and } B_1 \text{ are finite}].
\end{aligned}$$

Then, by Griffeath's lemma, Theorem 1 is proved. □

Proof of Theorem 2. We assume that

$$(95)\qquad P(o \in \xi_t^o \text{ i.o}) \ge \delta > 0 \quad \text{at } \lambda_c.$$

Then we will find a contradiction later. With assumption (95) we first show that there exists k such that

$$(96)\qquad P(o \in \xi_t^o(H(-k, k)) \text{ i.o}) > 0 \quad \text{at } \lambda_c,$$

where $H(-k, k)$ was defined in the proof of Proposition 1. The proof can be adapted directly from the proof of Proposition 1. Indeed, in the proof of Proposition 1, we only assumed that (14) holds, which is (95). Next we will prove that for any $\varepsilon > 0$ and large integer f there exist N and K such that

$$(97) \qquad P\big(J_N(f) = 1 \text{ inside } H(o,N) \cap \{|v| \le K\}\big) > 1 - \varepsilon \quad \text{at } \lambda_c,$$

where $J_N(f)$ is defined in the corollary. This can be proved by checking the proof of Fact 2 and the corollary directly. In fact, we only need to use (96) and a standard ergodic theorem to show (97). Note that the event that $J_N(f) = 1$ inside $H(o,N) \cap \{|v| \le K\}$ only depends on the Poisson processes in a finite set in T. By (2) we may take $\lambda_s < \lambda_0$ close to λ_c, but less than λ_c such that

$$(98) \qquad P\big(J_N(f) = 1 \text{ inside } H(o,N) \cap \{|v| \le K\}\big) > 1 - 2\varepsilon \quad \text{at } \lambda_0.$$

With a small ε in (98) and a large f, we will prove that for any $\eta > 0$ there exist M and G such that, at λ_0,

$$(99) \quad P\big(\exists\, t \le M|x| \text{ such that } \langle o,0\rangle \to \langle x,t\rangle \text{ in } H(o,x)\big) \ge \exp(-\eta|x|)$$

for any $|x| > G$. The proof of (99) is the same as the proof of Proposition 4. In fact, we only need (98) to show (99) in Proposition 4. Finally, we show that, at λ_0,

$$(100) \qquad \limsup P\big(o \in \xi_t^o(U)\big) \ge \beta > 0.$$

To prove (100), we just need to check the proof of Proposition 5 directly. The first part of the proof of Proposition 5, part A, depends only on $\lambda > \lambda_s$ (see the proof of Proposition 5, part A). Certainly, it holds for λ_0 since $\lambda_0 > \lambda_s$. The second part of the proof of Proposition 5, part B, depends on Proposition 4. Clearly, it also works for λ_0 since (99) holds. Then (100) is proved. However, (100) will imply that

$$(101) \qquad P\big(o \in \xi_t^o \text{ i.o.}\big) \ge \beta \quad \text{at } \lambda_0$$

which would contradict (95) since $\lambda_0 < \lambda_c$. Theorem 2 is proved. □

PROOF OF THEOREM 3. To show Theorem 3, we only need to show that (see (2.4) in [4])

$$(102) \qquad P\,(v \text{ is ever occupied by a particle}) \to 0 \quad \text{as } |v| \to \infty \text{ at } \lambda_c.$$

However, (102) is implied by Theorem 2 and the proof of Lemma 6.4 in [10]. □

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