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THE STRONG LAW OF LARGE NUMBERS FOR A BROWNIAN POLYMER

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We prove the strong law of large numbers for a continuous-time version of reinforced random walk. This extends previous results of Durrett and Rogers.

Introduction. The purpose of this paper is to prove an asymptotic for the behavior of

$$X(t) = W(t) + \int_0^t ds \int_0^s f(X(s) - X(u))\, du,$$

where W is one-dimensional Brownian motion. We will be interested in the case where f satisfies the following assumption.

ASSUMPTION A. Let f be a nonnegative, Lipschitz continuous function with compact support. Assume supp $f \subseteq [-k, k]$ and that $f(x) > c > 0$ for $|x - x_0| < 5\delta$ for some $x_0 \in [-k/2,\ k/2]$.

For convenience, we will take $k = 1$, $\|f\|_\infty \leq 1$ and assume $2/\delta$ is an integer. Trivial modifications of our proof give the full result. Our interest has been inspired by the article of Durrett and Rogers (1991) in which this and other similar models were treated. Under pretty much the same assumptions as our own, they proved the existence of positive constants c and C such that

$$c \leq \liminf_{t\to\infty} \frac{X(t)}{t} \leq \limsup_{t\to\infty} \frac{X(t)}{t} \leq C \quad \text{a.s.}$$

We shall prove the following result.

THEOREM 1. *Under Assumption* A, *there is a strictly positive constant* c *such that*

$$\lim_{t\to\infty} \frac{X(t)}{t} = c \quad a.s.$$

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We are indebted to Rick Durrett for the following observation which forms the foundation of our proof. Define for T a large number

$$X^T(t) = W(t) + \int_0^t ds \int_{(s-T)\vee 0}^s f(X^T(s) - X^T(u))\, du.$$

Then

$$\lim_{t\to\infty} \frac{X^T(t)}{t} = c_T \quad \text{a.s.}$$

for c_T a positive constant.

To see that

$$\frac{X^T(t)}{t} \to c_T \quad \text{a.s.},$$

first define

$$Y_n(s) \equiv X^T((n-1)T + s) - X^T((n-1)T), \qquad 0 \le s \le T.$$

Then Y_n is a Markov process on the state space $C[0, T]$. This is apparent from the following, which is valid for $n \ge 2$,

$$\begin{aligned} Y_n(s) = W((n-1)T + s) - W((n-1)T) + \textstyle\int_0^s dr \int_0^r f(Y_n(r) \\ - Y_n(u))\, du + \int_0^s dr \int_r^T f(Y_n(r) + (Y_{n-1}(T) - Y_{n-1}(u)))\, du. \end{aligned}$$

Moreover, setting, for $w,\ y \in C[0, T]$,

$$b(r, w, y) = \int_0^r f(w(r) - w(u))\, du + \int_r^T f(w(r) + (y(T) - y(u)))\, du,$$

the transition probability for $\{Y_n\}$ has a density with respect to the Wiener measure, ν_T, on $C[0, T]$ given by

$$\frac{p(y, dw)}{\nu_T(dw)} = \exp\left\{\int_0^T b(r, w, y)\, dw(r) - \frac{1}{2}\int_0^T b^2(r, w, y)\, dr\right\}.$$

Furthermore, the key point is that Y_n is a Harris-recurrent chain on $C[0, T]$. This will follow [see Revuz (1975)] provided $E_y(\sum_0^\infty 1_K(Y_n)) = \infty$ for any $y \in C[0, T]$ and all compact subsets K of $C[0, T]$ with $\nu_T(K) > 0$.

However, if $p^k(y, dz)$ denotes the k-step transition probability for $\{Y_n\}$, then

$$\begin{aligned}
P_n 1_K(y) &\equiv E_y(1_K(Y_n)) \\
&= \int_{C[0,T]} p^{n-1}(y, dz) \int_K \exp\biggl\{\int_0^T b(r, w, z)\, dw(r) \\
&\qquad\qquad - \frac{1}{2}\int_0^T b^2(r, w, z)\, dr\biggr\} \nu_T(dw) \\
&\geq \nu_T(K) \inf_{z \in C[0,T]} \exp\biggl\{\frac{1}{\nu_T(K)} \int_K \biggl(\int_0^T b(r, w, z)\, dw(r) \\
&\qquad\qquad - \frac{1}{2}\int_0^T b^2(r, w, z)\, dr\biggr) \nu_T(dw)\biggr\} \\
&\geq \nu_T(K) \exp\biggl\{-\frac{T^3}{2}\biggr\} \\
&\quad \times \inf_{z \in C[0,T)} \exp\biggl\{\frac{1}{\nu_T(K)} \int_K \biggl(\int_0^T b(r, w, z)\, dw(r)\biggr) \nu_T(dw)\biggr\} \\
&\geq \nu_T(K) \exp\biggl\{-\frac{T^3}{2}\biggr\} \\
&\quad \times \inf_{z \in C[0,T]} \exp\biggl\{-\frac{1}{\nu_T(K)} \biggl(E\int_0^T b^2(r, w, z)\, dr\biggr)^{1/2} \nu_T(K)^{1/2}\biggr\} \\
&\geq \nu_T(K) \exp\biggl\{-\frac{T^3}{2} - \frac{T^{3/2}}{\nu_T(K)^{1/2}}\biggr\}
\end{aligned}$$

so Harris recurrence follows.

Since $p(y, dw) \ll \nu_T(dw)$ and Y_n is Harris recurrent, there is [see Revuz (1975)] an invariant measure μ_T for the process, $\mu_T \ll \nu_T$, and, for $f \in L^1(\mu_T)$,

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) \to \int f(w)\, d\mu_T(w) \quad \text{a.s.}$$

Selecting $f(Y) = Y(T)$, it arises that a.s.

$$\begin{aligned}
\lim_{n\to\infty} \frac{X^T(nT)}{nT} &= \frac{1}{T} \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} Y_k(T) \\
&= \int \biggl[\frac{w(T)}{T}\biggr] d\mu_T(w) \\
&= c_T.
\end{aligned}$$

This suffices for

$$\lim_{t\to\infty} \frac{X^T(t)}{t} = c_T$$

providing, of course, we can demonstrate the selected f is in $L^1(\mu_T)$. To do this, we first show $\mu_T(C[0,T]) < \infty$.

If μ_T is infinite and f is a bounded μ_T-integrable function, then one must have [again see Revuz (1975)] $P_n f(y) \to 0$ as $n \to \infty$ for $y \in C[0,T]$. For our f we select $f(y) = 1_K(y)$, where $K \subset C[0,1]$ is a compact set such that $\mu_T(K) < \infty$ and $\nu_T(K) > 0$ (note $\mu_T \ll \nu_T$). Our computation demonstrating recurrence now shows $P_n 1_K(y) \geq C(K,T) > 0$ and so μ_T is finite. Finally, we show $f(y) = y(T)$ is in $L^1(\mu_T)$:

$$
\begin{aligned}
&\int_{C[0,1]} |y(T)|\mu_T(dy) \\
&\quad = \int_{C[0,T]^2} \int |y(T)|\, p(w, dy)\mu_T(dw) \\
&\quad = \int_{C[0,T]^2} \int |y(T)| \exp\biggl\{\int_0^T b(r, y, w)\, dy(r) \\
&\qquad\qquad\qquad - \tfrac{1}{2}\int_0^T b^2(r, y, w)\, dr\biggr\}\nu_T(dy)\mu_T(dw) \\
&\quad \leq \int_{C[0,T]} \biggl(\int_{C[0,T]} |y(T)|^2 \nu_T(dy)\biggr)^{1/2} \\
&\qquad \times \biggl(\int_{C[0,T]} \exp\biggl\{2\int_0^T b(r, y, w)\, dy(r) \\
&\qquad\qquad\qquad - \int_0^T b^2(r, y, w)\, dr\biggr\}\nu_T(dy)\biggr)^{1/2} \mu_T(dw) \\
&\quad = T^{1/2} \int_{C[0,T]^2} \biggl(\int \nu_T(dy) \exp\biggl\{2\int_0^T b(r, y, w)\, dy(r) \\
&\qquad\qquad\qquad - \int_0^T b^2(r, y, u)\, dr\biggr\}\biggr)^{1/2} \mu_T(dw) \\
&\quad \leq T^{1/2} e^{T^3/2} \int_{C[0,T]} \biggl(\int_{C[0,T]} \nu_T(dy) \exp\biggl\{\int_0^T 2b(r, y, w)\, dy(r) \\
&\qquad\qquad\qquad - \tfrac{1}{2}\int_0^T (2b)^2(r, y, w)\, dr\biggr\}\biggr)^{1/2} \mu_T(dw) \\
&\quad = T^{1/2} e^{T^3/2} \mu_T(C[0,T]) < \infty,
\end{aligned}
$$

as desired.

Finally, note that we shall often invoke a version of the Borel–Cantelli lemma due to Dubins and Freedman (1965). This states that if G_n are F_n adapted and $p_n = P(G_n | F_{n-1})$, then

$$
\lim_{n\to\infty} \sum_{m=1}^{n} 1_{G_m} \bigg/ \sum_{m=1}^{n} p_m = 1 \quad \text{a.s. on } \biggl\{\sum_{m=1}^{\infty} p_m = \infty\biggr\}.
$$

In particular, $P(G_n|F_{n-1}) \le c$ a.s. $\forall n$ implies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} 1_{G_m} \le c \quad \text{a.s.}$$

1. For a process Z and $a, b \in \mathbb{R}^+$, $t > 0$, define

$$T(b, a, t; Z) = \inf\{s > t: Z(s) = Z(t) + b \text{ or } Z(s) = Z(t) - a\}.$$

When using this stopping time for the process X, we shall abbreviate

$$T(b, a, t) \equiv T(b, a, t; X).$$

PROPOSITION 1.1.

(i) *There exists an* $\varepsilon > 0$ *such that, for* $t > 0$,

$$P(X(T(1, 1, t)) - X(t) = 1|F_t) \ge \tfrac{1}{2} + \varepsilon.$$

(ii) *There exist* c, $C > 0$ *such that, for* $y > 0$,

$$P(T(1, 1, t) - t \ge y|F_t) \le Ce^{-cy}.$$

The proof of this proposition will be by means of several elementary lemmas.

LEMMA 1.2. *Given* a, $b > 0$, *for every* $t > 0$,

$$P(X(T(b, a, t)) - X(t) = b|F_t) \ge \frac{a}{a+b}.$$

PROOF. Notice that

$$P(W(T(b, a, t; W)) - W(t) = b) = \frac{a}{a+b}$$

and, since $f \ge 0$,

$$\{W(T(b, a, t; W)) - W(t) = b\} \subset \{X(T(b, a, t)) - X(t) = b\}. \qquad \square$$

The following lemma is a simple consequence of a comparison result [cf. Ikeda and Watanabe (1981)] and the scale function for Brownian motion [cf. Karatzas and Shreve (1987)].

LEMMA 1.3. *Consider the time-inhomogeneous* Y *with*

$$dY(t) = dW(t) + h(t)\,dt,$$

where W *is* $\mathrm{BM}(\mathbb{R}^1)$ *and* h *is a predictable, locally bounded process. If* $h(s) \ge h$ *for* $s \in [t, T(d, d, t; Y)]$, *then*

$$P(Y(T(d, d, t; Y)) - Y(t) = d) \ge (1 + e^{-2hd})^{-1}.$$

REMARK. We shall use $(1+e^{-2hd})^{-1} = \frac{1}{2}(1+\tanh(hd))$.

Define, for $A \subseteq \mathbb{R}$,

$$\mathrm{Occ}(t, A) = \int_0^t ds\, 1_A(X(s)).$$

LEMMA 1.4. *There is a* $c > 0$ *such that on the set* $G = \{X(t) = a, \mathrm{Occ}(t, (a - x_0 - 4\delta, a - x_0 + 4\delta)) \geq \delta^2\}$ *one has, with* c *and* δ *as in Assumption* A,

$$P(X(T(\delta, \delta, t)) = a + \delta | F_t) \geq \left(\tfrac{1}{2} + \tanh(c\delta^3)\right).$$

PROOF. The result follows immediately from Lemma 1.3 and Assumption A. □

PROOF OF PROPOSITION 1.1. Set

$$A = \{X(T(1, 1, t)) - X(t) = 1\},$$
$$B = \{T(\delta/2, \delta/2, t; W) > \delta^2 + t\},$$
$$C = \{T(2\delta, 2\delta, t) < T(\delta/2, \delta/2, t; W)\}.$$

We will use $P_t(\cdot)$ to denote the conditional probability $P(\cdot | F_t)$. Notice that there is a positive constant p, independent of δ, so that $P_t(B^c) = p$ and a positive λ such that $P_t(C|B) = \lambda$; λ is random but takes values in $[0, 1]$.

Now

$$P_t(A) = P_t(A|B^c)p + P_t(A|B)(1-p)$$

and, by the symmetry of Brownian paths and the positivity of f,

$$P_t(A|B^c)p \geq \tfrac{1}{2}p.$$

For the second term,

$$P_t(A|B) = P_t(A|B \cap C)\lambda + P_t(A|B \cap C^c)(1-\lambda).$$

We claim

$$P_t(A|B \cap C) \geq \tfrac{1}{2} + \delta$$

since, if X has left $[X(t) - 2\delta, X(t) + 2\delta]$ before $T(\delta/2, \delta/2, t; W)$, it must have done so by passing through $X(t) + 2\delta$ (f is nonnegative.) Then by Lemma 1.2 we get the result.

For the term $P_t(A|B \cap C^c)$ we first remark that, by the symmetry of Brownian paths and the positivity of f,

$$E_t[X(T(\delta/2, \delta/2, t; W) \wedge T(2\delta, 2\delta, t)) - X(t); B] \geq 0.$$

However, as previously observed, on C one has

$$X(T(\delta/2, \delta/2, t; W) \wedge T(2\delta, 2\delta, t)) - X(t) = 2\delta.$$

Thus

$$E_t[X(T(\delta/2, \delta/2, t; W)) - X(t); B \cap C^c] \geq -2\delta P_t(B \cap C)$$

or

$$\eta \equiv E_t[X(T(\delta/2, \delta/2, t; W)) - X(t)|B \cap C^c] \geq \frac{-2\delta\lambda}{(1-\lambda)}.$$

We also note, however, that $X(T(\delta/2, \delta/2, t; W) \wedge T(2\delta, 2\delta, t)) - X(t) \geq -\delta/2$. Therefore,

$$\eta \geq \left(-\frac{2\delta\lambda}{1-\lambda}\right) \vee \left(-\frac{\delta}{2}\right).$$

Now suppose $-\frac{1}{2} \leq x_0 < -2\delta$ and write $\zeta = X(T(\delta/2, \delta/2, t; W)) - X(t)$, $S = T(1-\zeta, \zeta - x_0, T(\delta/2, \delta/2, t; W))$. Notice that S is well defined so long as $1 > \zeta > -x_0$, which is satisfied on C^c:

$$\begin{aligned}
&P_t(A|B \cap C^c)\\
&\quad = P_t(X(S) - X(t) = 1|B \cap C^c)\\
&\qquad + P_t(X(S) - X(t) = x_0,\ X(T(\delta, \delta, S)) - X(t) = x_0 + \delta,\\
&\qquad\quad X(T(1 - (x_0 + \delta), x_0 + \delta + 1, T(\delta, \delta, S))) - X(t) = 1|B \cap C^c)\\
&\qquad + P_t(X(S) - X(t) = x_0,\ X(T(\delta, \delta, S)) - X(t) = x_0 - \delta,\\
&\qquad\quad X(T(1 - (x_0 - \delta), x_0 - \delta + 1, T(\delta, \delta, S))) - X(t) = 1|B \cap C^c).
\end{aligned}$$

Using the linearity of the scale function and Lemma 1.2, the first term satisfies

$$P_t(X(S) - X(t) = 1|B \cap C^c) \geq \frac{\eta - x_0}{1 - x_0}.$$

For the last two terms we use Lemma 1.2 and the linearity of the scale function. Also, note that, on $B \cap C^c$, $|X(s) - X(t)| \leq 2\delta$ for $t \leq s \leq t + \delta^2$. So, using Lemma 1.4 at time S,

$$\begin{aligned}
&P_t(X(S) - X(t) = x_0, X(T(\delta, \delta, S)) - X(t) = x_0 + \delta, X(T(1 - (x_0 + \delta),\\
&\qquad\qquad x_0 + \delta + 1, T(\delta, \delta, S))) - X(t) = 1|B \cap C^c)\\
&\qquad \geq \frac{1-\eta}{1-x_0}\left(\frac{1}{2} + \alpha\right)\frac{x_0 + \delta + 1}{2},
\end{aligned}$$

$$\begin{aligned}
&P_t(X(S) - X(t) = x_0, X(T(\delta, \delta, S)) - X(t) = x_0 - \delta, X(T(1 - (x_0 - \delta),\\
&\qquad\qquad x_0 - \delta + 1, T(\delta, \delta, S))) - X(t) = 1|B \cap C^c)\\
&\qquad \geq \frac{1-\eta}{1-x_0}\left(\frac{1}{2} - \alpha\right)\frac{x_0 - \delta + 1}{2},
\end{aligned}$$

where α is random but, by Lemma 1.4, $\alpha \geq \frac{1}{2}\tanh(c\delta^3)$. Combining the last four inequalities, we arrive at

$$\begin{aligned} &P_t(A|B \cap C^c) \\ &\quad \geq \frac{\eta - x_0}{1 - x_0} + \frac{1-\eta}{1-x_0}\left(\frac{x_0+1}{2} + (\delta/2)\tanh(c\delta^3)\right) \\ &\quad = \frac{1}{2} + \frac{\eta}{2} + \frac{(\delta/2)\tanh(c\delta^3)}{1-x_0} \\ &\quad \geq \frac{1}{2} + \frac{1}{2}\left(\left(-\frac{2\delta\lambda}{1-\lambda}\right) \vee \left(-\frac{\delta}{2}\right)\right) + \frac{(\delta/2)\tanh(c\delta^3)}{1-x_0}. \end{aligned}$$

Thus

$$\begin{aligned} P_t(A) &= P_t(A|B^c)p + P_t(A|B \cap C)(1-p)\lambda + P_t(A|B \cap C^c)(1-p)(1-\lambda) \\ &\geq \frac{p}{2} + (1-p)\left(\left(\frac{1}{2}+\delta\right)\lambda + \left(\frac{1}{2} + \frac{1}{2}\left(\left(-\frac{2\delta\lambda}{1-\lambda}\right) \vee \left(-\frac{\delta}{2}\right)\right)\right.\right. \\ &\qquad\qquad \left.\left. + \frac{(\delta/2)\tanh(c\delta^3)}{1-x_0}\right)(1-\lambda)\right) \\ &= \frac{1}{2} + (1-p)\left(\delta\lambda + \frac{1}{2}\left(\left(-\frac{2\delta\lambda}{1-\lambda}\right) \vee \left(-\frac{\delta}{2}\right)\right)(1-\lambda)\right. \\ &\qquad\qquad \left. + \frac{(\delta/2)\tanh(c\delta^3)}{1-x_0}(1-\lambda)\right). \end{aligned}$$

However,

$$\left(-\frac{2\delta\lambda}{1-\lambda}\right) \vee \left(-\frac{\delta}{2}\right) = \begin{cases} -\dfrac{\delta}{2}, & \dfrac{1}{5} \leq \lambda \leq 1, \\ -\dfrac{2\delta\lambda}{1-\lambda}, & 0 \leq \lambda < \dfrac{1}{5}. \end{cases}$$

Thus

$$P_t(A) \geq \frac{1}{2} + \frac{4(1-p)(\delta/2)\tanh(c\delta^3)}{5(1-x_0)} \quad \text{when } 0 \leq \lambda < \frac{1}{5}$$

and

$$P_t(A) \geq \frac{1}{2} + (1-p)\left(\frac{\delta(5\lambda-1)}{4} + \frac{(\delta/2)\tanh(c\delta^3)(1-\lambda)}{1-x_0}\right) \quad \text{when } \frac{1}{5} \leq \lambda < 1.$$

In either case, there is a constant $\varepsilon = \varepsilon(p, c, \delta, x_0) > 0$ such that

$$P_t(A) \geq \tfrac{1}{2} + \varepsilon.$$

If $-2\delta < x_0 < 2\delta$, set

$$T_1 = T\left(\frac{5\delta}{2}, \frac{5\delta}{2}, t\right), \qquad T_2 = T(\delta, \delta, T_1).$$

Then

$$\begin{aligned}
P_t(A|B\cap C^c) &= P_t\Big(X(T_1)-X(t)=\frac{5\delta}{2},\ X(T_2)-X(t)=\frac{7\delta}{2},\\
&\qquad X\Big(T\Big(1-\frac{7\delta}{2},\ 1+\frac{7\delta}{2},\ T_2\Big)\Big)-X(t)=1\Big|B\cap C^c\Big)\\
&\quad+P_t\Big(X(T_1)-X(t)=\frac{7\delta}{2}, X(T_2)-X(t)=\frac{3\delta}{2},\\
&\qquad X\left(T\left(1-\frac{3\delta}{2},1+\frac{3\delta}{2}\right)\right)-X(t)=1\Big|B\cap C^c\Big)\\
&\quad+P_t\Big(X(T_1)-X(t)=\frac{-5\delta}{2},\ X\Big(T\Big(1+\frac{5\delta}{2},\ 1-\frac{5\delta}{2},\ T_1\Big)\Big)\\
&\qquad -X(t)=1\Big|B\cap C^c\Big)\\
&=\left(\frac{1}{2}+\mu\right)\left(\frac{1}{2}+\alpha\right)\frac{(7\delta/2)+1}{2}+\left(\frac{1}{2}+\mu\right)\left(\frac{1}{2}-\alpha\right)\frac{(3\delta/2)+1}{2}\\
&\quad+\left(\frac{1}{2}-\mu\right)\frac{-(5\delta/2)+1}{2},
\end{aligned}$$

with random variables $\mu \geq 0$, $\alpha \geq \frac{1}{2}\tanh(c\delta^3)$, by Lemmas 1.2 and 1.4 applied at time T_1. [Notice that, on $B\cap C^c$, $-\delta/2 \leq X(s)-X(t) \leq 2\delta$ for $t \leq s \leq t+\delta^2$.] However, this is bounded below by

$$\frac{1}{2}+\left(\frac{1}{2}+\mu\right)\delta\alpha+\mu\frac{10\delta}{4} \geq \frac{1}{2}+\frac{\delta}{4}\tanh(c\delta^3).$$

If $x_0 > 2\delta$, the argument is analogous to that for $x_0 < -2\delta$. For the exponential tail of $T(1,1,t)-t$, note that

$$\begin{aligned}
&P(T(1,1;t)-t > y|F_t)\\
&\quad\leq P(\inf\{r>t:\exists u\in[t,r]\text{s.t.}W(r)-W(u)\geq 2\}-t>y)\\
&\quad= P\left(\inf\left\{r>0: W(r)-\inf_{0\leq v\leq r}W(v)\geq 2\right\}>y\right)\\
&\quad\leq Ce^{-cy},
\end{aligned}$$

since $W(r)-\inf_{0\leq v\leq r}W(v)$ is reflecting Brownian motion. □

COROLLARY 1.5. *For* $0 \leq s < t$, *set*

$$A_{s,t} = \{X(w)-X(v)\leq 1 \textit{ for some } v\in[s,s+1], w\in[t,\infty)\}.$$

Then there exist positive constants C, d *such that*

$$P(A_{s,t}|F_s)\leq Ce^{-d(t-s)}.$$

If c_1 *is sufficiently small and* $B_{s,t} = \{X(w)-X(v)\leq 1+c_1(t-s)$ *for some* $v\in[s,s+1]$, $w\in[t,\infty)\}$, *then there exist* C_1, $d_1>0$ *such that* $P(B_{s,t}|F_s)\leq C_1e^{-d_1(t-s)}$.

PROOF. Define $\{T_{i,s}\}$, $i \geq 0$, by $T_{0,s} = s$, $T_{i,s} = T(1, 1, T_{i-1,s})$. Using Proposition 1.1, couple $X_i \equiv X(T_{i,s})$ and a random walk $\{Z_i : i \geq 0, Z_0 = 0\}$ with transition probabilities $p(k, k+1) = \frac{1}{2} + \varepsilon$, $p(k, k-1) = \frac{1}{2} - \varepsilon$ in such a way that $X(T_{i,s}) - X(s) \geq Z_i$ for all i a.s.

Now select c_1 and c_2 with $0 < c_1 < \frac{4}{3}\varepsilon c_2$, $0 < c_1 < c_2 < 1$. Then $B_{s,t} \subset A_1 \cup A_2 \cup A_3$, where

$$A_1 = \left\{ \sup_{s-1 \leq u \leq s} W(u) - W(s) \geq \tfrac{1}{2} c_1 (t-s) \right\},$$

$$A_2 = \left\{ Z_n \leq \tfrac{3}{2} c_1 (t-s) + 3 \text{ for some } n \geq c_2(t-s) \right\},$$

$$A_3 = \{ T_{[c_2(t-s)]+1, s} \geq t \}.$$

This may be seen as follows: if A_1 fails, then, for $B_{s,t}$ to occur, $X(T_{n,s}) - X(s)$ must be smaller than $\frac{3}{2}c_1(t-s) + 3$ for some n such that $T_{n,s} \geq t$. A weaker restriction is that Z_n must be smaller than $\frac{3}{2}c_1(t-s)+3$ for some n such that $T_{n,s} \geq t$. However,

$$\begin{aligned} &\{Z_n \leq \tfrac{3}{2}c_1(t-s) + 3 \text{ for some } n \text{ such that } T_{n,s} \geq t\} \\ &\qquad \subset \{Z_n \leq \tfrac{3}{2}c_1(t-s) + 3 \text{ for some } n \text{ such that } T_{n,s} \geq t, \\ &\qquad\qquad\qquad\qquad n > [c_2(t-s)] + 1\} \\ &\qquad \cup \{T_{[c_2(t-s)]+1, s} \geq t\}. \end{aligned}$$

Thus the inclusion $B_{s,t} \subset A_1 \cup A_2 \cup A_3$ a.s. follows.

Now A_1 and A_2 have conditional probabilities that decay exponentially in $(t-s)$ (for A_2 this depends on our choice of c_1 and c_2). As for A_3, note that, by Proposition 1.1(ii), $T_{[c_2(t-s)]+1, s}$ is stochastically bounded by a sum of $[c_2(t-s)] + 1$ independent random variables, each having an exponential tail. The exponential decay of the conditional probability of $B_{s,t}$ follows by selecting sufficiently small c_1, depending on c and C from Proposition 1.1(ii). Since, for $c_1 > 0$, $A_{s,t} \subset B_{s,t}$ the claim about $A_{s,t}$ follows immediately. □

REMARK. Corollary 1.5 also holds for X^T. This requires a version of Proposition 1.1 for X^T. The proof for X^T follows the same lines as for X. One gets the upward bias from paths which move more than one by time $T + t$ on the set B.

COROLLARY 1.6. *There exist* $\delta > 0$, $C < \infty$ *such that*, *for all* $k \geq 1$, *all* $t, s \geq 0$,

$$P(T(k, \infty, t) > t + s | F_t) \leq C e^{-\delta s / k}.$$

PROOF. Fix k, t and note that the constants C_i, δ_i and δ introduced below do not depend on k or t. Define

$$R_0 = t$$

and, for $n \geq 1$,

$$R_n = T(1, 1, R_{n-1}).$$

Proposition 1.1(ii) and standard large deviations imply the existence of positive δ_1, C_1 such that

$$P(R_n \geq C_1 n + t | F_t) \leq C_1 e^{-\delta_1 n} \qquad \forall\, n.$$

Now consider X_{R_n}, $n \geq 0$. Proposition 1.1(i) says there is an $\varepsilon > 0$ such that, $\forall\, n$,

$$P(X_{R_n} = X_{R_{n-1}} + 1 \big| F_{R_{n-1}}) \geq \tfrac{1}{2} + \varepsilon,$$
$$P(X_{R_n} = X_{R_{n-1}} - 1 \big| F_{R_{n-1}}) \leq \tfrac{1}{2} - \varepsilon \qquad \forall\, n \geq 1.$$

Thus, from elementary bounds on tail probabilities for binomial random variables, there exist positive δ_2, C_2 such that

$$P(X_{R_n} - X_t \leq \varepsilon n \mid F_t) \leq C_2 e^{-\delta_2 n} \qquad \forall n \geq 1.$$

Suppose now that s satisfies $\varepsilon[s/c_3] \geq k$, where $c_3 = \max\{C_1, C_2\}$. Notice that, for any n,

$$\{T(k, \infty, t) > t + s\} \subseteq \{R_n > t + s\} \cup \{X_{R_n} - X_t < k\}.$$

So taking $n = [s/c_3]$ so that $\varepsilon n \geq k$, we have

$$\begin{aligned} P(X_{R_n} - X_t < k \mid F_t) &\leq P(X_{R_n} - X_t < \varepsilon n \mid F_t) \\ &\leq C_2 e^{-\delta_2 n} \end{aligned}$$

and

$$\begin{aligned} P(R_n > t + s \mid F_t) &\leq P(R_n > t + C_1 n | F_t) \\ &\leq C_1 e^{-\delta_1 n}. \end{aligned}$$

Thus, for $\varepsilon[s/c_3] \geq k$, there is a $c' > 1$ such that

$$\begin{aligned} P(T(k, \infty, t) > t + s \mid F_t) &\leq C_2 e^{-\delta_2 n} + C_1 e^{-\delta_1 n} \\ &\leq c' e^{-\delta s} \\ &\leq c' e^{-\delta s/k}. \end{aligned}$$

This gives the desired inequality for $\varepsilon[s/c_3] \geq k$, and so, for $s \geq 0$,

$$P(T(k, \infty, t) > t + s \mid F_t) \leq C e^{-\delta s/k},$$

with $C = c' \big/ \inf_k [\inf_s \{e^{-\delta s/k} : 0 \leq [s/c_3] \leq k/\varepsilon\}]$ (which is independent of k). □

2. In order to make a comparison with the process X^T, we write

$$X(t) = W(t) + \int_0^t ds \int_{(s-T)\vee 0}^{s} f(X(s) - X(u))\, du + \int_0^t r_T(s)\, ds.$$

In this section we prove

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t r_T(u)\,du = o(1) \quad \text{as } T \to \infty.$$

Note that this is made entirely plausible by Corollary 1.5 which implies

$$\begin{aligned}
\frac{1}{t}E\left[\int_0^t r_T(u)\,du\right] &= \frac{1}{t}\int_0^t ds \int_0^{(s-T)\vee 0} du\, E(f(X(s)-X(u))) \\
&\le \frac{1}{t}\int_0^t ds \int_0^{(s-T)\vee 0} du \|f\|_\infty P(X(s)-X(u) \le 1) \\
&\le C\frac{1}{t}\int_0^t ds \int_0^{(s-T)\vee 0} du e^{-d(s-u)} \\
&\le \tfrac{C}{d}e^{-dT}.
\end{aligned}$$

LEMMA 2.1. *There exists a positive constant c so that for T and t sufficiently large*, $\log t \gg T$,

$$P\left(\frac{1}{t}\int_0^t r_T(u)\,du \ge \frac{1}{T}\right) \le \frac{c}{t^{1/3}}.$$

PROOF. Write, assuming $\log t \gg T$,

$$\begin{aligned}
r_T(s) &= \int_0^{(s-\log^2 t)\vee 0} f(X(s)-X(u)\,du + \int_{(s-\log^2 t)\vee 0}^{(s-T)\vee 0} f(X(s)-X(u))\,du \\
&\equiv V_{T,t}(s) + r_{T,t}(s).
\end{aligned}$$

Now we claim that, with $A_{s,t}$ as in Corollary 1.5,

$$\left\{\int_0^t V_{T,t}(s)\,ds \ne 0\right\} \subset \bigcup_{i=1}^{[t-\log^2 t]+1} A_{i,\log^2 t+i-1}.$$

This follows since $\int_0^t V_{T,t}(s)\,ds \ne 0$ only when $X(s)-X(u) \le 1$ for some $s \in (\log^2 t, t)$, $u \in (0, s-\log^2 t)$. Now if i is such that $i-1 \le u < i$, then $i-1+\log^2 t \le s$ which proves the claim. Consequently,

$$\begin{aligned}
P\left(\int_0^t V_{T,t}(s)\,ds \ne 0\right) &\le \sum_{i=1}^{[t-\log^2 t]+1} P(A_{i,\log^2 t+i-1}) \\
&\le Cte^{-d\log^2 t} \\
&\le t^{-1} \quad \text{for } t \text{ large.}
\end{aligned}$$

We now turn our attention to $\int_0^t r_{T,t}(s)\,ds$. Partition $[0,t]$ into $[t^{1/2}]$ intervals of length $t/[t^{1/2}]$ and call these $I_1, \dots, I_{[t^{1/2}]}$.

Define the random variables

$$R_i = \int_{I_i} r_{T,t}(s)\,ds$$

and note that

$$\int_0^t r_{T,t}(s)\,ds = \sum_i R_i$$

and the σ-fields

$$G_i = \sigma\left\{X(u): u \in \bigcup_{j=1}^{i-1} I_j\right\}.$$

Then, setting $G_0 = G_1$,

$$\begin{aligned}
E(R_1|G_0) &= E(R_1)\\
&= E\left(\int_T^{\log^2 t} ds \int_0^{s-T} f(X(s)-X(u))\,du\right)\\
&\quad + E\left(\int_{\log^2 t}^{t/[t^{1/2}]} ds \int_{s-\log^2 t}^{s-T} f(X(s)-X(u))\,du\right)\\
&\le \tfrac{1}{2}(\log^2 t - T)^2 + \int_{\log^2 t}^{t/[t^{1/2}]} ds \int_{s-\log^2 t}^{s-T} P(A_{u,s})\,du\\
&\le \tfrac{1}{2}\log^4 t + C\int_{\log^2 t}^{t/[t^{1/2}]} ds \int_{s-\log^2 t}^{s-T} e^{-d(s-u)}\,du, \quad \text{by Corollary 1.5}\\
&\le C\left(\log^4 t + t^{1/2}\left(e^{-dT} - e^{-d\log^2 t}\right)\right),\\
&\qquad\qquad \text{with a possibly new value of } C,
\end{aligned}$$

and, for $i > 1$,

$$\begin{aligned}
E(R_i|G_{i-1}) &= E\left(\int_{(i-1)(t/[t^{1/2}])}^{i(t/[t^{1/2}])} ds \int_{s-\log^2 t}^{s-T} f(X(s)-X(u))\,du \Big| G_{i-1}\right)\\
&\le \int_{(i-1)(t/[t^{1/2}])}^{i(t/[t^{1/2}])} ds \int_{s-\log^2 t}^{s-T} P(P(A_{u,s}|F_u)|G_{i-1})\,du\\
&\le ct^{1/2}(e^{-dT} - e^{-d\log^2 t}).
\end{aligned}$$

In other words, for $i \ge 1$,

$$E(R_i|G_{i-1}) \le ct^{1/2}e^{-dT} \quad \text{for } t \text{ large enough.}$$

Now an elementary estimate gives

$$\begin{aligned}
&E\left(\left(\sum_i R_i - E(R_i|G_{i-1})\right)^2\right)\\
&\qquad = E\left(\sum_{|i-j|\le 1} (R_i - E(R_i|G_{i-1}))(R_j - E(R_j|G_{j-1}))\right)\\
&\qquad \le ct^{3/2}\log^4 t,
\end{aligned}$$

since there are no more than $3t^{1/2}$ terms and the "inner integral" in the definition of R_i is over an interval of length less than $\log^2 t$. So by Chebychev,

$$P\Big(\sum_i R_i > cte^{-dT}\Big) \le P\Big(\sum_i R_i - E(R_i|G_{i-1}) > cte^{-dT}\Big)$$
$$\le c\frac{\log^4 t}{t^{1/2}} e^{2dT}$$
$$\le ct^{-1/3}$$

for t large since $\log t \gg T$. Thus, for T large enough so that $ce^{-dT} < 1/T$, we see the proof is complete. □

PROPOSITION 2.2. *With $r_T(s) = \int_0^{(s-T)\vee 0} f(X(s)-X(u))\,du$, we have, for T sufficiently large,*

$$P\bigg(\limsup_{t\to\infty} \frac{1}{t}\int_0^t r_T(s)\,ds \le \frac{1}{T}\bigg) = 1.$$

PROOF. Define $A(n) = \{n^{-4}\int_0^{n^4} r_T(s)\,ds \ge 1/T\}$. Then, by Lemma 2.1 and the Borel–Cantelli lemma, $P(A(n)\text{ i.o.}) = 0$. However,

$$\{A(n)\text{ i.o.}\} \supset \bigg\{\limsup \frac{1}{t}\int_0^t r_T(u)\,du > \frac{1}{T}\bigg\}$$

and the proposition is proved. □

3. The guiding thoughts of this section are, first, given X is "like a random walk with drift" in the sense of Proposition 1.1, X should spend most of its time near its maximum, and, second, when X is close to its maximum we have sufficient control on its behavior to make a successful comparison with X^T.

We first want to make some remarks of an elementary nature concerning the first point. Let Z_n be the random walk of Section 1 with transition probabilities $p(k, k+1) = \frac{1}{2} + \varepsilon$, $p(k, k-1) = \frac{1}{2} - \varepsilon$. Setting $Z_n^* = \max_{0\le k\le n} Z_k$, the process $Y_n = Z_n^* - Z_n$ is a birth and death chain on $\mathbb{N}$ with transition probabilities

$$p(x, y) = \begin{cases} \frac{1}{2} - \varepsilon, & \text{if } y = x+1, x \ge 0,\\ \frac{1}{2} + \varepsilon, & \text{if } y = x-1, x \ge 1,\\ \frac{1}{2} + \varepsilon, & \text{if } y = x = 0.\end{cases}$$

Then $\{Y_n\}$ has an invariant probability distribution given by

$$\pi(y) = \frac{4\varepsilon}{1+2\varepsilon}\bigg(\frac{1-2\varepsilon}{1+2\varepsilon}\bigg)^y, \qquad y \in \mathbb{N}.$$

Thus, if $\tau_y = \inf\{n > 0: Y_n = y\}$,

$$E_0\tau_0 = \frac{1+2\varepsilon}{4\varepsilon} < \infty,$$

so thinking of $\{Y_n\}$ as being composed of independent excursions from 0, the expected excursion length is $(1+2\varepsilon)/4\varepsilon$. Moreover, letting H denote the maximum height of an excursion, one has

$$\begin{aligned} P(H = x) &= \left(\frac{1}{2} - \varepsilon\right) P_1(\tau_x < \tau_0) P_x(\tau_0 < \tau_{x+1}) \\ &= \left(\frac{1}{2} - \varepsilon\right) \frac{1}{\sum_{y=0}^{x-1}(((1/2)+\varepsilon)/((1/2)-\varepsilon))^y} \\ &\quad \times \frac{1}{\sum_{y=0}^{x}(((1/2)-\varepsilon)/((1/2)+\varepsilon))^y} \\ &\cong c(\varepsilon)\left(\frac{1-2\varepsilon}{1+2\varepsilon}\right)^x, \qquad x \to \infty. \end{aligned}$$

Or $P(H \geq x) \cong Ce^{-cx}$ for some positive constants c and C. This leads to the estimate, using T^8 as an upper bound for the number of excursions before T^8,

$$\begin{aligned} &P(Y_n > \sqrt{T} \text{ for some } n \leq T^8) \\ &\quad = 1 - P(Y_n \leq \sqrt{T}, \forall\, n \leq T^8) \\ &\quad \leq 1 - (1 - Ce^{-c\sqrt{T}})^{T^8} \\ &\quad \leq Ce^{-c\sqrt{T}} \quad \text{for } T \text{ large with a change of constants.} \end{aligned}$$

Define the stopping times $T_0 = 0$, and, for $i > 0$, $T_i = T(1, 1, T_{i-1}) = \inf\{t > T_{i-1}: |X(t) - X(T_{i-1})| = 1\}$ and the random variables $M_i = \sup_{0\leq j\leq i} X(T_j)$. The next lemma follows easily from Proposition 1.1.

LEMMA 3.1. *Let Z_n be the random walk described above. We may couple $\{Z_n\}$ and $\{X(T_n)\}$ so that*:

(i) *For each n, $X(T_n) \geq Z_n$, a.s.*
(ii) *For each n, $M_n - X(T_n) \leq Z_n^* - Z_n$, a.s.*

As a consequence, for some $c > 0$,

(iii) $$P\left(\liminf_{t\to\infty} \frac{X(t)}{t} > c\right) = 1.$$

PROOF. We only show (iii). It follows directly from the coupling of $X(T_i)$ and Z_i that

$$\liminf_{i\to\infty} \frac{X(T_i)}{i} \geq 2\varepsilon \quad \text{a.s.}$$

Also,

$$\limsup_{i\to\infty} \frac{T_i}{i} \le c < \infty \quad \text{a.s.}$$

follows from the fact that T_i is stochastically bounded by a sum of i independent random variables distributed like the ones in Proposition 1.1(ii).

COROLLARY 3.2. *If* $A(j) = \{\exists\, T_i \in [jT^8, (j+1)T^8]: M_i - X(T_i) > \sqrt{T}\}$, *then there exist positive constants c and C such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{A(j)} \le Ce^{-c\sqrt{T}} \quad a.s.$$

PROOF. Given $\{T_i\}$, consider the subsequence of stopping times defined as follows:

$$\begin{aligned} S_0 &= 0, \\ S_{2n+1} &= \inf\{T_i > S_{2n}: M_i - X(T_i) > \sqrt{T}\}, \qquad n \ge 0, \\ S_{2n} &= \inf\{T_i > S_{2n-1}: M_i = X(T_i) > X(T_j), \forall\, j < i\}, \qquad n \ge 1. \end{aligned}$$

Now a problem in dealing with the times $T_{i+1} - T_i$ is that the drifts which influence their size can be arbitrarily large and thus $T_{i+1} - T_i$ can, given F_{T_i}, be arbitrarily stochastically small. However, by Proposition 1.1(ii), we have exponential upper bounds on this distribution uniformly over F_{T_i}. Thus, if

$$V_n = \#\{j: [jT^8, (j+1)T^8] \cap [S_{2n-1}, S_{2n}] \ne \varnothing\},$$

then Proposition 1.1(i) and (ii) and basic large deviations estimates yield

$$P(V_n \ge k | F_{S_{2n-1}}) \le 2^{-(k-2)}, \qquad k \ge 1,$$

for T sufficiently large. This is because, for $k \ge 3$,

$$\{V_n \ge k\} \subset \{T(\sqrt{T} + 2, \infty, S_{2n-1}) \ge S_{2n-1} + (k-2)T^8\}.$$

By Corollary 1.6,

$$P(V_n \ge k \mid F_{S_{2n-1}}) \le Ce^{-\delta(k-2)T^8 / \sqrt{T}+2}.$$

As δ is fixed, this upper bound is less than $2^{-(k-2)}$ for all $k > 2$ when T is sufficiently large.

On the other hand, if $M_i = X(T_i) > X(T_j)$, $\forall\, j < i$, then there is a $p > 0$ such that if

$$\begin{aligned} A_1(i) &= \{X(T_{i+1}) - X(T_i) = 1\,,\ T_{i+1} - T_i \le 1\}, \\ A_2(i) &= \{X(T_{i+2}) - X(T_{i+1}) = 1\,,\ T_{i+2} - T_i \ge 1\}, \end{aligned}$$

then $P(A_1(i) \cap A_2(i) \mid F_{T_i}) \geq p$ on $\{M_i = X(T_i) > X(T_j),\ \forall j < i\}$. Thus we have (with suitable adjustment of constants) from the remarks preceding Lemma 3.1 that if

$$U_n = \#\{j: [jT^8, (j+1)T^8] \subseteq [S_{2n}, S_{2n+1}]\},$$

then $P(U_n \geq Ce^{c\sqrt{T}} \mid F_{S_{2n}}) \geq \frac{1}{2}$ for some strictly positive c, C not depending on T, so long as T is sufficiently large. This can be seen as follows.

Define, for fixed n,

$$R_0 = S_{2n}$$

and, for $v \geq 0$,

$$R_{v+1} = \inf\{T_i > R_v: M_i = X(T_i) > X(T_j), \forall j < i\}.$$

Then, by Lemma 3.1 and the remarks preceding it, for suitable c, C, c', C', independent of T,

$$P(R_{[Ce^{c\sqrt{T}}]} \geq S_{2n+1} \mid F_{S_{2n}}) \leq C'e^{-c'\sqrt{T}}.$$

By our choice of p and the strong Markov property of Brownian motion,

$$P(R_{j+2} - R_j > 1 \mid F_{R_j}) \geq p \qquad \forall j.$$

Thus, by standard binomial tail probabilities,

$$P(R_{[Ce^{c\sqrt{T}}]} \geq \frac{p}{2}Ce^{c\sqrt{T}} + R_0 \mid F_{R_0}) \geq 1 - C''e^{-c''\sqrt{T}}$$

for positive c'', C'' not depending on T. On replacing C by $pC/2$, we obtain the desired inequality.

These inequalities and the fact that

$$V_1, \dots, V_{j-1} \in F_{S_{2j-1}}, \qquad U_1, \dots, U_{j-1} \in F_{S_{2j}}$$

yield i.i.d. Bernoulli random variables I_j such that

$$U_j \geq Ce^{c\sqrt{T}} I_j \qquad \forall j,$$

and i.i.d. geometric with parameter $\frac{1}{2}$ random variables H_j such that

$$V_j \leq 1 + H_j \qquad \forall j.$$

Thus

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U_j \geq Ce^{c\sqrt{T}} \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} I_j$$
$$= \frac{C}{2}e^{c\sqrt{T}}.$$

Also,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} V_j \le 1 + \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H_j$$
$$\le 3.$$

Now, assuming without loss of generality that $S_{2n+1} < \infty$, $\forall n$, if $(n+1)T^8 \in [S_{2m}, S_{2m+2}]$, then

$$\sum_{j=1}^{n} 1_{A(j)} \le \sum_{k=1}^{m+1} V_k$$

and

$$\sum_{k=1}^{m-1} U_k \le n.$$

Thus, with the new value of C we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{A(j)} \le \limsup_{m\to\infty} \left(\frac{\sum_{k=1}^{m+1} V_k}{\sum_{k=1}^{m-1} U_k} \right)$$
$$\le Ce^{-c\sqrt{T}}$$

and the proof is complete. □

COROLLARY 3.3. *If* $B(j) = \{\exists\, t \in [jT^8, (j+1)T^8] \colon X(t) \le 1 + \sup_{s\le t-T} X(s)\}$, *then there exist positive constants* c *and* C *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{B(j)} \le Ce^{-c\sqrt{T}}.$$

PROOF. Set $L(j) = \{\exists\, t \in [jT^8, (j+1)T^8] \colon \inf_{r\ge t} X(r) - X(t-T) \le \sqrt{T} + 4\}$. First, we make the claim: $B(j) \subseteq L(j) \cup A(j-1) \cup A(j)$. Now on $B(j) \setminus L(j)$ we have for some $t \in [jT^8, (j+1)T^8]$ that

$$X(t) \le 1 + X^*(t-T) \equiv \sup_{s\le t-T} X(s)$$

and

$$X(t) - X(t-T) \ge \sqrt{T} + 4.$$

Now take T_i so that $T_i \le t - T < T_{i+1}$. Then $T_i \in [(j-1)T^8, (j+1)T^8]$ except on a set of probability not exceeding $Ce^{-c(T^8-T)}$ by Proposition 1.1(ii) and

$$\begin{aligned} M_i - X(T_i) &= (M_i - X^*(t-T)) + (X^*(t-T) - X(t)) \\ &\quad + (X(t) - X(t-T)) + (X(t-T) - X(T_i)) \\ &\ge (-1) + (-1) + (\sqrt{T} + 4) + (-1) \\ &\ge \sqrt{T}. \end{aligned}$$

Thus $B(j)\setminus L(j)\subset A(j-1)\cup A(j)$ and the claim is proved. Next set $t_{j,i}=jT^8+iT^{-9}$ for $i=0,1,\dots,T^{17}$, and define $L(j,i)=\{\inf_{r\ge t_{j,i}} X(r)-X(t_{j,i}-T)\le\sqrt{T}+5\}$.

By Corollary 1.5, $P(L(j,i)|F_{jT^8})\le C_1e^{-d_1T}$ provided T is sufficiently large. By an argument similar to that used in Corollary 3.2, $P(L(j)\setminus\bigcup_{i=0}^{T^{17}}L(j,i)|F_{jT^8})\le Ce^{-c\sqrt{T}}$ (the probability X moves more than one between $t_{j,i}$ and $t_{j,i+1}$ is small). Since $B(j)\subseteq L(j)\cup A(j-1)\cup A(j)$, the corollary now follows from Corollary 3.2 and Dubins and Freedman (1965). □

The next corollary follows easily from Corollary 3.3. It is crucial for our attempt to couple X with X^T as it gives many times in $[jT^8,(j+1)T^8]$ at which X has "almost forgotten" its history.

COROLLARY 3.4. *Define* $\sigma_1^j=\inf\{t\ge jT^8: X(t)=\sup_{s\le t}X(s)\}$, *and, for* $i>0$, $\sigma_{i+1}^j=\inf\{t\ge\sigma_i^j+T+4: X(t)=\sup_{s\le t}X(s)\}$. *Let* τ_i^j *be the analogously defined times for* X^T. *Define* $C(j)=\{\sigma_{T+1}^j>jT^8+T^3\}$, $D(j)=\{\tau_{T+1}^j>jT^8+T^3\}$. *Then there exist positive constants c and C such that*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}1_{C(j)\cup D(j)}\le Ce^{-c\sqrt{T}}.$$

PROOF. By Corollary 3.2, it suffices to show $P(C(j)\cup D(j)\cap A(j)^c|F_{jT^8})\le Ce^{-c\sqrt{T}}$. By definition, on $A(j)^c$, one has $X^*(t)<X(t)+\sqrt{T}+2$, $\forall\, t\in[jT^8,(j+1)T^8]$. Thus we need to estimate probability bounds for the time to achieve the maximum when the maximum is not greater than $\sqrt{T}+2$ plus the present value. Define, for $jT^8\le t<(j+1)T^8$, $A(j;t)=\{\exists\, T_i\in[t,(j+1)T^8]: M_i-X(T_i)>\sqrt{T}\}$. From Corollary 3.2 it is clear that $P(A(j;t)|F_t)\le Ce^{-c\sqrt{T}}$. By Corollary 1.5, $P(X(u)-X(t)\le c_1(u-t)|F_t)\le Ce^{-c(u-t)}$ so that if $u=T+t$, one has

$$\begin{aligned}&P(X(T+t)<X^*(t)|F_t)\\&\quad\le P(X(T+t)<X(t)+\sqrt{T}+2|F_t)+P(A(j;t)|F_t)\\&\quad\le P(X(T+t)<X(t)+c_1T|F_t)+Ce^{-c\sqrt{T}}\\&\quad\le Ce^{-c\sqrt{T}}.\end{aligned}$$

This implies $P(\sigma_{i+1}^j-\sigma_i^j>2T+4|F_{\sigma_i^j})\le Ce^{-c\sqrt{T}}$ and so $P(\sigma_{T+1}^j>jT^8+T^3|F_{jT^8})\le Ce^{-c\sqrt{T}}$. The same argument applies to τ_i^j and X^T (see the remark following Corollary 1.5). These estimates and Dubins and Freedman (1965) complete the proof. □

The next result contains the main step in the proof of Theorem 1. The proof will be broken up into a series of lemmas.

PROPOSITION 3.5. *There is a coupling of* X *and* X^T *and an event* $V(j) \in F_{(j+1)T^8}$ *satisfying*:

(i) $P(V(j)|F_{jT^8}) \le Ce^{-cT}$ *for some positive* c, C;

(ii) $V(j)^c \subset B(j) \cup C(j) \cup D(j) \cup E(j)$, *where*

$$E(j) = \left\{ \left| \int_{jT^8}^{(j+1)T^8} ds \int_{(s-T)\vee 0}^{s} f(X(s) - X(u))\,du - \int_{jT^8}^{(j+1)T^8} ds \int_{(s-T)\vee 0}^{s} f(X^T(s) - X^T(u))\,du \right| \le 3T^4 \right\}.$$

Note that $B(j)$ *is as defined in Corollary* 3.3, $C(j)$ *and* $D(j)$ *as in Corollary* 3.4.

We can now give the proof of Theorem 1.

PROOF OF THEOREM 1. We have that, as $t \to \infty$ a.s.,

$$\frac{1}{t}\int_0^t ds \int_{(s-T)\vee 0}^{s} f(X^T(s) - X^T(u))\,du \to C_T.$$

Using the coupling of Proposition 3.5 and Corollary 3.4 applied to both X and X^T and defining $b_T(s) = \int_{(s-T)\vee 0}^{s} f(X(s) - X(u))\,du$,

$$\begin{aligned}
\limsup_{t\to\infty} & \left| C_T - \frac{1}{t}\int_0^t ds\, b_T(s) \right| \\
&= \limsup_{t\to\infty} \left| \frac{1}{t}\int_0^t ds \int_{(s-T)\vee 0}^{s} f(X^T(s) - X^T(u))\,du - \frac{1}{t}\int_0^t ds \int_{(s-T)\vee 0}^{s} f(X(s) - X(u))\,du \right| \\
&= \limsup_{n\to\infty} \left| \frac{1}{nT^8} \sum_{j=0}^{n-1} \int_{jT^8}^{(j+1)T^8} ds \int_{(s-T)\vee 0}^{s} (f(X(s) - X(u)) - f(X^T(s) - X^T(u)))\,du \right| \\
&\le \limsup_{n\to\infty} \frac{T}{n} \sum_{j=0}^{n-1} 1_{V(j)} + \limsup_{n\to\infty} \frac{T}{n} \sum_{j=0}^{n-1} 1_{B(j)\cup C(j)\cup D(j)} \\
&\quad + \limsup_{n\to\infty} \frac{cT^4}{nT^8} \sum_{j=0}^{n-1} 1_{E(j)} \\
&\le CTe^{-cT} + CTe^{-c\sqrt{T}} + \frac{c}{T^4} \quad \text{by Dubins and Freedman (1965)} \\
&< \frac{c}{T^4}.
\end{aligned}$$

Together with Proposition 2.2, this implies

$$\limsup_{t\to\infty}\left|C_T - \frac{1}{t}\int_0^t ds\int_0^s f(X(s)-X(u))\,du\right| \le \frac{c}{T^4}+\frac{1}{T}.$$

Letting $T\to\infty$ and using Lemma 3.1(iii) finishes the proof. □

By a coupling we mean a joint distribution and one such is called the maximal coupling for two random variables. Let X and Y be $C[0,1]$-valued random variables which have densities f_x and f_y with respect to the Wiener measure ν. Then there is a joint distribution such that $P(X=Y)=\int f_x(z)\wedge f_y(z)\nu(dz)$. See, for example, Pitman (1976).

In our next lemma we shall use the following device: given stopping times σ for X and τ for X^T, we say X and X^T are driven by the same Brownian motion, β_t, after (σ,τ) if

$$X(\sigma+t) = X(\sigma)+\beta_t+\int_0^{\sigma+t} ds\int_0^s f(X(s)-X(u))\,du,$$

$$X^T(\tau+t) = X^T(\tau)+\beta_t+\int_\tau^{\tau+t} ds\int_{(s-T)\vee 0}^s f(X^T(s)-X^T(u))\,du.$$

LEMMA 3.6. *Let σ and τ be stopping times for X and X^T, respectively. Let $A\in F_{\sigma,\tau}\equiv F_\sigma(X)\vee F_\tau(X^T)$ be such that the following hold on A*:

(i) $X(\sigma-s)-X(\sigma)=X^T(\tau-s)-X^T(\tau)$, $0\le s\le 1$;
(ii) $\sup_{s\le\sigma-1}X(s)<X(\sigma)-2$;
(iii) $\sup_{s\le\tau-1}X^T(s)<X^T(\tau)-2$.

Then if X and X^T are driven by the same Brownian motion after (σ,τ), on the set A one has $X(\sigma+s)-X(\sigma)=X^T(\tau+s)-X^T(\tau)$ for all $s\le S_0\equiv\inf\{t>0\colon X(\sigma+t)-X(\sigma)\le -1\}\wedge\inf\{t>0\colon X(\sigma+t)-1\le\sup_{r\le\sigma+t-T}X(r)\}$.

PROOF. Observe that S_0 is the first time after σ when X is influenced by times more than T units in its past; that is, the first time it differs from X^T. □

LEMMA 3.7. *Suppose σ and τ are stopping times for X and X^T, respectively, and that, on a set $A\in F_{\sigma+T,\tau+T}$, $X(\sigma+s)-X(\sigma)=X^T(\tau+s)-X^T(\tau)$ for $0\le s\le T$. If X and X^T are driven by the same Brownian motion after $(\sigma+T,\tau+T)$, then, on the set A,*

$$X(\sigma+s)-X(\sigma)=X^T(\tau+s)-X^T(\tau)$$

$$\textit{for } 0\le s\le \tilde S_0\equiv\inf\left\{t>T\colon X(\sigma+t)-1\le\sup_{r\le\sigma+t-T}X(r)\right\}.$$

PROOF. As in the previous proof, $\sigma+\tilde S_0$ is the first time X stops behaving as if it were X^T. □

LEMMA 3.8. *Let σ, τ be stopping times for X and X^T, respectively, with $X(\sigma) = \sup_{s\le\sigma} X(s)$, $X^T(\tau) = \sup_{s\le\tau} X^T(s)$. Define stopping times $T_{\sigma,2}$, $S_{\tau,2}$ by $T_{\sigma,2} = \inf\{s > \sigma: X(s) - X(\sigma) = 2\}$, $S_{\tau,2} = \inf\{s > \tau: X^T(s) - X^T(\tau) = 2\}$. There exists a positive constant c, independent of T, and a coupling of $X(\sigma+\cdot)$ and $X^T(\tau+\cdot)$ such that, conditional on $F_{\sigma,\tau}$, the following event has probability at least c*:

$$\{X(T_{\sigma,2}+s) - X(T_{\sigma,2}) = X^T(S_{\tau,2}+s) - X^T(S_{\tau,2}), 0 \le s \le 1\}$$
$$\cap \{X(T_{\sigma,2}+1) - X(T_{\sigma,2}) = X^T(S_{\tau,2}+1) - X^T(S_{\tau,2}) > 2\}.$$

PROOF. On $[\sigma, T_{\sigma,2}]$, $[\tau, S_{\tau,2}]$ drive X and X^T by independent Brownian motions. Set $A_1 = \{T_{\sigma,2}-\sigma \le 1\}$, $A_2 = \{S_{\tau,2}-\tau \le 1\}$. By a simple comparison using the fact that X and X^T have positive drift and independence,

$$P(A_1 \cap A_2 | F_{\sigma,\tau}) \ge \left(\sqrt{\frac{2}{\pi}} \int_2^\infty \exp\left(-\frac{x^2}{2}\right) dx\right)^2.$$

On the event A_1, $Y(s) = X(T_{\sigma,2}+s) - X(T_{\sigma,2})$, $0 \le s \le 1$, is a process with drift bounded by 2 so long as Y does not reach -1. Now consider the space of paths of Y, namely $C[0, 1]$. The law of Y is absolutely continuous with respect to Wiener measure on $C[0, 1]$ which we denote by ν. The density given by the Cameron–Martin–Girsanov formula will shortly be seen to be "manageable" on the set $F = \{w \in C[0,1]: w(0) = 0, \inf w(s) > -1, w(1) > 2\}$. Similar considerations apply to $Y^T(s) = X^T(S_{\tau,2}+s) - X^T(S_{\tau,2})$, $0 \le s \le 1$.

To make this precise, for $w \in C[0,1]$, let

$$X^w(t) = \begin{cases} X(t), & t \le T_{\sigma,2}, \\ X(T_{\sigma,2}) + w(t - T_{\sigma,2}) & \\ \quad + \int_{T_{\sigma,2}}^t ds \int_0^s f(X^w(s) - X^w(u))\, du, & t \ge T_{\sigma,2}, \end{cases}$$

and

$$b(s,w) = 1_{\{\inf_{0\le r\le s} w(r) \ge -1\}} \int_0^{T_{\sigma,2}+s} f(X(s) - X(u))\, du.$$

Then, on A_1, $|b(s,w)| < 2$ for all w, and, on F, the law of Y has density with respect to ν given by

$$f_Y(w) = \exp\left\{\int_0^1 b(s,w)\, dw - \tfrac{1}{2}\int_0^1 b^2(s,w)\, ds\right\}.$$

Noticing that $E(\int_0^1 b(s,w)\, dw)^2 = E\int_0^1 b^2(s,w)\, ds \le 4$, it follows that we can select n sufficiently large to make

$$\nu\left(F \cap \left\{f_Y \ge \frac{1}{n}\right\}\right) \ge \nu\left(F \cap \left\{\int_0^1 b(s,w)\, dw \ge -\log n + 4\right\}\right)$$
$$\ge \frac{3}{4}\nu(F).$$

A similar argument for Y^T shows, for n large enough,

$$\nu\left(F \cap \left\{f_{Y^T} \geq \frac{1}{n}\right\}\right) \geq \frac{3}{4}\nu(F).$$

Thus, fixing an n which makes both inequalities hold yields a coupling (joint distribution) of Y and Y^T such that

$$P(Y = Y^T, Y \in F) \geq \frac{1}{2n}\nu(F). \qquad \square$$

PROOF OF PROPOSITION 3.5. Our object is to show that for most j, there are random times σ^j and τ^j in $[jT^8, (j+1)T^8]$ such that $\sigma^j, \tau^j \leq jT^8 + T^3$ and, for $s \leq T^8 - T^3$, $X(\sigma^j + s) - X(\sigma^j) = X^T(\tau^j + s) - X^T(\tau^j)$. We attempt to couple X and X^T at the times σ_i^j, τ_i^j as in Lemma 3.8. On $(C(j) \cup D(j))^c$ we will have T attempts over an interval of length T^3 to link X and X^T. It follows that outside of a set of exponentially small probability in T, either $C(j) \cup D(j)$ occurs or there are times $\sigma^j, \tau^j \leq jT^8 + T^3$ for which $X(\sigma^j + s) - X(\sigma^j) = X^T(\tau^j + s) - X^T(\tau^j)$, $0 \leq s \leq T$. By Lemma 3.7, this linkage will prevail for $0 \leq s \leq T^8 - T^3$ unless the event $B(j)$ of Corollary 3.3 occurs. We now give a more detailed account of this argument.

All of the following statements hold on $(B(j) \cup C(j) \cup D(j))^c$. There are stopping times for X and X^T, $\sigma_i^j, \tau_i^j \geq jT^8$ with $\sigma_{T+1}^j, \tau_{T+1}^j \leq jT^8 + T^3$, $\sigma_{i+1}^j - \sigma_i^j \geq T + 4$, $\tau_{i+1}^j - \tau_i^j \geq T + 4$, $0 \leq i \leq T$. By Lemma 3.8 there is a $C > 0$ such that, conditional on $F_{\sigma_1^j, \tau_1^j}$, with probability at least C,

$$(1)\quad X(T_{\sigma_1^j,2} + s) - X(T_{\sigma_1^j,2}) = X^T(S_{\tau_1^j,2} + s) - X^T(S_{\tau_1^j,2}), \qquad 0 \leq s \leq 1,$$

and

$$(2)\qquad X(T_{\sigma_1^j,2} + 1) - X(T_{\sigma_1^j,2}) = X^T(S_{\tau_1^j,2} + 1) - X^T(S_{\tau_1^j,2}) > 2.$$

Let the event described by (1) and (2) be denoted by L_1. On L_1, drive X and X^T after $T_{\sigma_1^j,2} + 1$, $S_{\tau_1^j,2} + 1$ by the same Brownian motion. Let $K_1 = \{\inf_{t \in [0,T]} X(T_{\sigma_1^j,2} + t) - X(T_{\sigma_1^j,2}) \geq -1\}$. By Lemma 3.7, on $L_1 \cap K_1$, $X(T_{\sigma_1^j,2} + s) - X(T_{\sigma_1^j,2}) = X^T(S_{\tau_1^j,2} + s) - X^T(S_{\tau_1^j,2})$, $0 \leq s \leq T^8 - T^3$. By Lemma 3.8 and Proposition 1.1, $P(L_1 \cap K_1 | F_{\sigma_1^j, \tau_1^j}) > \gamma$ for some strictly positive γ not depending on T. If $L_1 \cap K_1$ does not occur, we try again at the times σ_2^j, τ_2^j and so on. Setting $V_j = \bigcap_{i=1}^{[T]} (K_i \cap L_i)^c$, the set on which coupling in $[jT^8, (j+1)T^8]$ fails, $P(V_j | F_{jT^8}) \leq (1-\gamma)^{[T]} \leq Ce^{-\gamma T}$. $\square$

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REFERENCES

DUBINS, L. and FREEDMAN, D. (1965). A sharper form of the Borel–Cantelli lemma and the strong law. *Ann. Math. Statist.* **36** 800–807.

DURRETT, R. and ROGERS, L. C. G. (1991). Asymptotic behavior of Brownian polymers. *Probab. Theory Related Fields* **92** 337–349.

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam.

KARATZAS, I. and SHREVE, S. E. (1987). *Brownian Motion and Stochastic Calculus*. Springer, New York.

LEVY, P. (1965). *Processus Stochastiques et Mouvement Brownien*. Gauthiers-Villars, Paris.

NORRIS, J. R., ROGERS, L. C. G. and WILLIAMS, D. (1987). Self-avoiding random walk: a Brownian motion model with local time drift. *Probab. Theory Related Fields* **74** 271–287.

PITMAN, J. (1976). On coupling of Markov chains. *Z. Wahrsch. Verw. Gebiete* **35** 315–322.

REVUZ, D. (1975). *Markov Chains*. North-Holland.

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