The Annals of Probability
1996, Vol. 24, No. 1, 532–540

A ZERO–ONE LAW APPROACH TO THE CENTRAL LIMIT THEOREM FOR THE WEIGHTED BOOTSTRAP MEAN[1]

BY EUSEBIO ARENAL-GUTIÉRREZ AND CARLOS MATRÁN

Universidad de Valladolid, Spain

We develop a new technique to prove the conditional CLT for the weighted bootstrap mean. Through 0–1 laws, we show that this conditional CLT can be derived from an unconditional one which easily arises (conditioning with respect to the weights) from the standard Lindeberg CLT.

1. Introduction. This paper follows our works [Arenal-Gutiérrez, Cuesta-Albertos and Matrán (1995a, b)] exploring the unconditional properties of the bootstrap. The study of these properties is motivated by the following simple observation. "Bootstrap convergence" is, roughly, a tail event for the sequence of data so, appealing to the Kolmogorov 0–1 law, we can produce conditional results from easier unconditional results.

To prove an unconditional bootstrap central limit theorem (CLT in the sequel), we use conditioning with respect to the weights first. This produces a triangular array of row-independent variables, and the asymptotic behavior of their sum can be easily obtained (see Theorem 3.1) through the standard Lindeberg CLT. On the other hand a 0–1 law argument obtained in Section 2 (from some lemmas of independent interest) gives the conditional statement from the unconditional one.

The use of 0–1 law arguments for the bootstrap was pioneered by Giné and Zinn (1989) (see Lemma 2.2 below). Also let us note in passing that the use of unconditional properties to obtain conditional ones has already been implicitly used in connection with the application of the Ledoux–Talagrand–Zinn inequality to the exchangeably weighted bootstrap general empirical process, in Praestgaard and Wellner (1993). Another 0–1 law for the bootstrap is implicit in Csörgő (1992) and Arenal-Gutiérrez, Cuesta-Albertos and Matrán (1995a); both are in relation to the strong law of large numbers (SLLN) for Efron's bootstrap mean.

The interest of using different resampling schemes for bootstrap has been often made apparent, as in Rubin (1981), Efron (1982), Wu (1986) and Lo (1993), and in the last few years its study has received a considerable impulse by means of a unified general treatment, as in Mason and Newton (1992), Einhmahl and Mason (1992), Haeusler, Mason and Newton (1992),

Received August 1994; revised March 1995.
[1]Research partially supported by DGICYT Grant PB91-0306-C02-00, 02.
AMS 1991 *subject classifications*. Primary 60F05, 60F17; secondary 62E20.
Key words and phrases. Bootstrap mean, zero–one laws, exchangeable weights, non-i.d. variables.

Praestgaard and Wellner (1993), Hušková and Janssen (1993), Hall and Mammen (1992) or Barbe (1994).

Note that our conditions E1–E5 are less restrictive than in the available results on weighted bootstrap. In particular, they are less restrictive than conditions A1–A5 in Praestgaard and Wellner (1993) (although the main reason for those more restrictive conditions arises from the Banach space setting, searching for tightness properties for the weighted bootstrap empirical process). Therefore this avoids the use of Hájek's CLT, which is [since Mason and Newton (1992)] the key technique for the weighted bootstrap CLT.

Given a sequence $\{X_n\}_{n=1}^{\infty}$ of independent random variables with common law $\mathscr{L}(X_1)$ and distribution function F, we consider a sequence $\{\mathbf{w}_n\}_{n=1}^{\infty} = \{(w_n(1), \dots, w_n(n))'\}_{n=1}^{\infty}$ of independent vectors of random weights, independent of the data sequence and satisfying the following, for a given sequence of numbers $\{m(n)\}_{n=1}^{\infty}$ such that $m(n) \to \infty$:

E1. The components of the vectors $\mathbf{w}_n$ are exchangeable for every n;
E2. $w_n(j) \geq 0$ for all n, j, and $\sum_{j=1}^{n} w_n(j) = 1$, for all n;
E3. $\operatorname{Var} w_n(1) = O(1/m(n)n)$;
E4. $\max_{1 \leq j \leq n} \sqrt{m(n)}\, |w_n(j) - 1/n| \to_P 0$;
E5. $m(n) \sum_{j=1}^{n} (w_n(j) - 1/n)^2 \to_P c^2$.

Note that, by E1 and E2, $\operatorname{Cov}(w_n(j), w_n(i)) = -\operatorname{var} w_n(1)/(n-1)$.

Some additional notation follows. By $\overline{X}_n$ and $\overline{X}_n^*$ we denote, respectively, the usual and bootstrap sample means,

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \overline{X}_n^* = \sum_{j=1}^{n} w_n(j) X_j.$$

A superscript ω indicates that the sequence of data $\{X_n^{\omega}\}_{n=1}^{\infty}$ is considered fixed. We use this notation also for the means $\overline{X}_n^{\omega}$ and $\overline{X}_n^{*\,\omega}$. Weak convergence of probability measures is denoted by $\to_w$.

As a final remark, we want to point out the additional advantage of the technique which applies to non-i.d. random variables under some general conditions regarding the uniform square-integrability of the sequence $\{X_n\}_{n=1}^{\infty}$ (Theorem 3.3).

2. Zero–one laws for the bootstrap mean.

Our interest is to get 0–1 laws to obtain conditional properties from unconditional ones in the line of the following lemma.

LEMMA 2.1. *Let $\{X_n\}_{n=1}^{\infty}$ be an arbitrary sequence of random variables and $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a sequence of random weights fulfilling conditions* E1–E3. *Then the event $\Gamma = \{\omega\colon \mathscr{L}(\sqrt{m(n)}\,(\overline{X}_n^{*\,\omega} - \overline{X}_n^{\omega})) \to_w\}$ is a tail event with respect to $\{X_n\}_{n=1}^{\infty}$.*

PROOF. Applying Chebyshev's inequality, we get

$$P\left(\sqrt{m(n)}\left|w_n(j) - \frac{1}{n}\right| > \varepsilon\right) \leq \frac{m(n)\mathrm{Var}\, w_n(j)}{\varepsilon^2} = O\left(\frac{1}{\varepsilon^2 n}\right)$$

so, for a fixed $k \geq 1$,

$$\lim_{n\to\infty}\left(\max_{1\leq j\leq k}\sqrt{m(n)}\left|w_n(j) - \frac{1}{n}\right|\right) = 0 \quad \text{in probability.}$$

Note that there are no hypotheses about the increasing order of the sequence $\{m(n)\}_{n=1}^{\infty}$. If this order is "small," then $\sqrt{m(n)}\,\max_{1\leq j\leq k}|w_n(j)|$ and $\sqrt{m(n)}\,/n$ will converge to zero in probability, but if it is "large," none of these two convergences is true.

Now we have

$$\mathscr{L}\left(\sqrt{m(n)}\sum_{j=1}^{k}\left(w_n(j) - \frac{1}{n}\right)X_j^{\omega}\right) \to_w \delta_0,$$

where δ_0 denotes the probability measure concentrated at 0. Hence if we write $\mu(\omega)$ as the limit law for ω in Γ,

$$\Gamma = \left\{\omega: \mathscr{L}\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j) - \frac{1}{n}\right)X_j^{\omega}\right) \to_w \mu(\omega)\right\}$$
$$= \left\{\omega: \mathscr{L}\left(\sqrt{m(n)}\sum_{j=k+1}^{n}\left(w_n(j) - \frac{1}{n}\right)X_j^{\omega}\right) \to_w \mu(\omega)\right\},$$

so that Γ is a tail event. □

The following 0–1 law, which is extracted from the proof of Theorem 1 in Giné and Zinn (1989), shows that the asymptotic distribution, if it exists, is the same for almost every (a.e.) ω.

LEMMA 2.2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables and $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a sequence of random weights fulfilling conditions* E1–E3. *If*

$$\mathscr{L}\left(\sqrt{m(n)}\left(\bar{X}_n^{*\omega} - \bar{X}_n^{\omega}\right)\right) \to_w \mu(\omega) \quad a.s.,$$

then there exists a measure μ such that $\mu(\omega) = \mu$ a.s.

PROOF. Let $\mathfrak{I}$ be a countable measure-determining set of bounded continuous functions. For example, $\mathfrak{I} = \{x \to \exp(itx): t \in Q\}$.

Then, for every $f \in \mathfrak{I}$,

$$\int f\, d\mathscr{L}\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j) - \frac{1}{n}\right)X_j^{\omega}\right) \to \int f\, d\mu(\omega) \quad \text{for a.e. } \omega.$$

Applying the same scheme as in the previous lemma, we get

$$\int f\, d\mathscr{L}\left(\sqrt{m(n)}\sum_{j=k+1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_j^{\omega}\right)\to\int f\, d\mu(\omega)\quad\text{for a.e. }\omega;$$

so $\int f\, d\mu(\omega)$ is a tail variable, hence a constant. Since $\mathfrak{I}$ is measure determining, there is a fixed deterministic measure μ such that $\mu(\omega)=\mu$ for a.e. ω. □

The following lemmas give the key of our argument. The first proves the a.s. tightness for the laws of the normalized sums, and the second proves the weak equivalence ($\sim_w$) of these laws for two different realizations of the data sequence. This weak equivalence means that the two sequences have the same weak limits for the same subsequences of subscripts.

LEMMA 2.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with finite variance, and $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a sequence of random weights fulfilling conditions* E1–E3. *Then in a probability*-1 *set the sequence $\{\mathscr{L}(\sqrt{m(n)}\,(\overline{X}_n^{*\,\omega}-\overline{X}_n^{\omega}))\}_{n=1}^{\infty}$ is tight.*

PROOF. Note that the set Ω_0 where the sequence is tight can be expressed as

$$\Omega_0=\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=1}^{\infty}\left\{\omega\colon P\left(\sqrt{m(n)}\left(\overline{X}_n^{*\,\omega}-\overline{X}_n^{\omega}\right)\geq N\right)<\frac{1}{k}\right\}.$$

On the other hand, by Markov's inequality (and taking into account the elementary inequality $2ab\leq a^2+b^2$),

$$\begin{aligned}
&P\left(\sqrt{m(n)}\left|\overline{X}_n^{*\,\omega}-\overline{X}_n^{\omega}\right|>C\right)\\
&\quad\leq\frac{m(n)}{C^2}\left(\sum_{i=1}^{n}(X_i^{\omega})^2\operatorname{Var}w_n(i)+2\sum_{i<j}X_i^{\omega}X_j^{\omega}\operatorname{Cov}(w_n(j),w_n(i))\right)\\
&\quad\leq\frac{K}{C^2}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i^{\omega})^2+\left|\frac{2}{n(n-1)}\sum_{i<j}X_i^{\omega}X_j^{\omega}\right|\right)\\
&\quad\leq\frac{K'}{C^2}\frac{1}{n}\sum_{i=1}^{n}(X_i^{\omega})^2,
\end{aligned}$$

where the right-hand side converges a.s. (by the SLLN) to $K'EX_1^2/C^2$.

Therefore, by enlarging C if necessary we get that P-a.e. ω belongs to Ω_0. □

LEMMA 2.4. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with finite variance, and let $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a sequence* *of random weights satisfying conditions* E1–E3. *There exists a probability*-1 *set* Ω_0 *such that, for every* ω, $\omega' \in \Omega_0$,

$$\left\{\ell\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_j^{\omega}\right)\right\}_{n=1}^{\infty} \sim_w \left\{\ell\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_j^{\omega'}\right)\right\}_{n=1}^{\infty}$$

(*the two sequences of laws are weakly equivalent*).

PROOF. Exchangeability of the weights immediately leads to the equivalence of bootstrapping the sample or the ordered sample. In particular, letting $X_{(1)},\dots,X_{(n)}$ denote the ordered sample, we have

$$\ell\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_{(j)}^{\omega}\right)=\ell\left(\sqrt{m(n)}\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_j^{\omega}\right),$$

so it is sufficient to show that, for every pair ω, ω' in a probability-1 set,

$$\lim_{n\to\infty} m(n)E\left(\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_{(j)}^{\omega}-\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)X_{(j)}^{\omega'}\right)^2=0.$$

Let us now consider the set

$$\Omega_0=\left\{\omega:\frac{1}{n}\sum_{i=1}^{n}(X_i^{\omega})^2\to EX_1^2 \text{ and } \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^{\omega}}\to\ell(X_1)\right\}.$$

The Glivenko–Cantelli theorem and the SLLN assure that Ω_0 has probability 1. Then, if $\omega,\omega'\in\Omega_0$, from E3 we have

$$\begin{aligned}
&m(n)E\left(\sum_{j=1}^{n}\left(w_n(j)-\frac{1}{n}\right)\left(X_{(j)}^{\omega}-X_{(j)}^{\omega'}\right)\right)^2\\
&\quad\le K\left(\frac{1}{n}\sum_{j=1}^{n}\left(X_{(j)}^{\omega}-X_{(j)}^{\omega'}\right)^2-\frac{2}{n(n-1)}\sum_{i<j}\left(X_{(i)}^{\omega}-X_{(i)}^{\omega'}\right)\left(X_{(j)}^{\omega}-X_{(j)}^{\omega'}\right)\right)\\
&\quad\le\frac{K'}{n}\sum_{j=1}^{n}\left(X_{(j)}^{\omega}-X_{(j)}^{\omega'}\right)^2,
\end{aligned}$$

and the last term converges to zero because

$$\frac{1}{n}\sum_{j=1}^{n}\left(X_{(j)}^{\omega}-X_{(j)}^{\omega'}\right)^2=\int_0^1\left(F_n^{-1}(\omega,t)-F_n^{-1}(\omega',t)\right)^2dt,$$

where $F_n^{-1}(\omega,\cdot)$ is the quantile representation of $(1/n)\sum_{i=1}^{n}\delta_{X_i^{\omega}}$. In fact,

$$\lim_{n\to\infty}\left(F_n^{-1}(\omega,t)-F_n^{-1}(\omega',t)\right)^2=\left(F^{-1}(t)-F^{-1}(t)\right)^2=0 \quad \text{a.s.},$$

and in mean, because the uniform integrability which arises from

$$\left(F_n^{-1}(\omega,\cdot) - F_n^{-1}(\omega',\cdot)\right)^2 \leq 2\left(\left(F_n^{-1}(\omega,\cdot)\right)^2 + \left(F_n^{-1}(\omega',\cdot)\right)^2\right)$$

and the convergence in mean to $2(F^{-1})^2$ of the last term. □

We can now prove the following theorem, which makes it possible to obtain a conditional convergence from an unconditional one. Observe that in the theorem as well as in the preceding lemmas we only use conditions E1, E2 and E3 on the weights.

THEOREM 2.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with finite variance, and let $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a sequence of random weights fulfilling conditions* E1–E3. *If*

$$\mathscr{L}\left(\sqrt{m(n)}\left(\overline{X}_n^* - \overline{X}_n\right)\right) \to_w \mu, \quad \textit{then} \quad \mathscr{L}\left(\sqrt{m(n)}\left(\overline{X}_n^{*\omega} - \overline{X}_n^{\omega}\right)\right) \to_w \mu \quad \textit{a.s.}$$

PROOF. Since the sequence $\{\mathscr{L}(\sqrt{m(n)}\,(\overline{X}_n^{*\omega} - \overline{X}_n^{\omega}))\}_{n=1}^{\infty}$ is tight in a probability-1 set (Lemma 2.3), applying Prohorov's theorem we have that for every ω in that set there exists a weak convergent subsequence for each given subsequence. Lemmas 2.4 and 2.2 assure that the subsequence and its weak limit do not depend on ω.

On the other hand, for every x such that $\mu(\{x\}) = 0$,

$$\int P\left(\sqrt{m(n_{k_j})}\left(\overline{X}_{n_{k_j}}^{*\omega} - \overline{X}_{n_{k_j}}^{\omega}\right) \leq x\right) dP$$
$$= P\left(\sqrt{m(n_{k_j})}\left(\overline{X}_{n_{k_j}}^{*} - \overline{X}_{n_{k_j}}\right) \leq x\right) \to \mu(-\infty, x],$$

and, by the dominated convergence theorem,

$$\int P\left(\sqrt{m(n_{k_j})}\left(\overline{X}_{n_{k_j}}^{*\omega} - \overline{X}_{n_{k_j}}^{\omega}\right) \leq x\right) dP \to \nu(-\infty, x],$$

for every x such that $\nu(\{x\}) = 0$, so $\nu = \mu$. Then every subsequence converges to the same measure, μ, and the theorem is proved. □

3. Conditional CLT for the bootstrap mean. In this section we obtain a conditional CLT for the bootstrap mean and an extension for non-i.d. variables. Using the results given in the preceding section, we only need an unconditional CLT which is obtained conditioning not with respect to the data sequence but with respect to the weights.

THEOREM 3.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables such that* $\operatorname{Var} X_n = \sigma^2$. *If $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is a sequence of weights fulfilling conditions* E1–E5, *then*

$$\mathscr{L}\left(\sqrt{m(n)}\left(\overline{X}_n^* - \overline{X}_n\right)\middle|\{w_n(i)\}_{i=1}^{n}\right) \to_w N(0, c^2\sigma^2) \quad \textit{in probability}.$$

PROOF. It is straightforward to show that each increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of positive integers has some subsequence $\{n_{k_i}\}_{i=1}^{\infty}$ such that conditions E4 and E5 hold a.s., and the random variables

$$\sqrt{m(n_{k_i})}\left(w_{n_{k_i}}(j) - \frac{1}{n_{k_i}}\right)X_j, \qquad j = 1, 2, \dots, n_{k_i},$$

under the conditional distribution given $\{w_{n_{k_i}}(j)\}_{j=1}^{n_{k_i}}$, satisfy a.s. the Lindeberg condition for the CLT, so that

$$\mathscr{L}\left(\frac{\sqrt{m(n_{k_i})}\left(\bar{X}^*_{n_{k_j}} - \bar{X}_{n_{k_j}}\right)}{s_{n_{k_i}}}\middle|\left\{w_{n_{k_i}}(j)\right\}_{j=1}^{n_{k_i}}\right) \to_w N(0,1) \quad \text{a.s.}$$

Therefore the result follows from the fact that the sum of the conditional variances is

$$s^2_{n_{k_i}} = \sigma^2 m(n_{k_i}) \sum_{j=1}^{n_{k_i}} \left(w_{n_{k_i}}(j) - \frac{1}{n_{k_i}}\right)^2,$$

which converges a.s. to $c^2\sigma^2$. □

COROLLARY 3.1 (Unconditional CLT). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables such that* $\operatorname{Var} X_n = \sigma^2$. *If $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is a sequence of weights fulfilling conditions* E1–E5, *then*

$$\mathscr{L}\left(\sqrt{m(n)}\left(\bar{X}^*_n - \bar{X}_n\right)\right) \to_w N(0, c^2\sigma^2).$$

We can now get immediately the following conditional CLT as a consequence of this corollary and Theorem 2.1.

THEOREM 3.2 (Conditional CLT). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables such that* $\operatorname{Var} X_n = \sigma^2$. *If $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is a sequence of weights fulfilling conditions* E1–E5, *then*

$$\mathscr{L}\left(\sqrt{m(n)}\left(\bar{X}^{*\omega}_n - \bar{X}^{\omega}_n\right)\right) \to_w N(0, c^2\sigma^2) \ a.s.$$

The same idea can be used to prove a CLT for nonidentically distributed centered random variables. The only additional conditions we need are a kind of uniform integrability of the sequence $\{X_n^2\}_{n=1}^{\infty}$ and the following change of condition E5 on the weights [with $\operatorname{Var}(X_i) = \sigma_i^2$]:

E5′. $$\frac{m(n)\sum_{j=1}^{n}\left(w_n(j) - (1/n)\right)^2\sigma_j}{(1/n)\sum_{i=1}^{n}\sigma_i^2} \to c^2 \quad \text{in probability.}$$

Now, in order to check Lindeberg's condition, it is easy to show that

$$\frac{1}{s_{n_{k_i}}^2}\sum_{j=1}^{n_{k_i}}\int_{\{|\sqrt{m(n_{k_i})}(w_{n_{k_i}}(j)-1/n_{k_i})X_j|>\varepsilon s_{n_{k_i}}\}} m(n_{k_i}) \times\left(w_{n_{k_i}}(j)-\frac{1}{n_{k_i}}\right)^2 |X_j|^2\,dP$$

converges to zero by the uniform integrability of the sequence $\{X_n^2\}_{n=1}^{\infty}$.

If there exists a function $\varphi\colon \mathfrak{R}\to\mathfrak{R}$ such that $\varphi(t)/t\uparrow\infty$ when $t\uparrow\infty$, $\sup_{n\ge 1}E[\varphi(X_n^2)]<\infty$ and $\sum_{n=1}^{\infty}1/\varphi(n)<\infty$ [in particular, $\{X_n^2\}_{n=1}^{\infty}$ becomes uniformly integrable; see Landers and Rogge (1987)], then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2-EX_i^2\right)=0\quad\text{a.s.}$$

so we have an a.s. tightness condition as in Lemma 2.3.

Finally, a result similar to Lemma 2.4 can be obtained using the previous condition and the Glivenko–Cantelli theorem for triangular arrays [Shorack (1979), Theorem 2.1], so we can state the following theorem.

THEOREM 3.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with zero means such that there exists a function $\varphi\colon\mathfrak{R}\to\mathfrak{R}$ verifying $\varphi(t)/t\uparrow\infty$ when $t\uparrow\infty$, $\sup_{n\ge1}E[\varphi(X_n^2)]<\infty$ and $\sum_{n=1}^{\infty}1/\varphi(n)<\infty$. Suppose that for some constant $K>0$, $K\le\sigma_i^2$ for $i=1,2,\dots$. If $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is a sequence of weights fulfilling conditions* E1–E4 *and* E5′, *then*

$$\mathscr{L}\left(\frac{\sqrt{m(n)}\left(\bar{X}_n^{*\omega}-\bar{X}_n^{\omega}\right)}{\sqrt{(1/n)\sum_{i=1}^{n}\sigma_i^2}}\right)\to_w N(0,c^2)\quad a.s.$$

Acknowledgments. We owe thanks to Evarist Giné for helpful conversations concerning limit theorems. Also we wish to thank J. A. Cuesta-Albertos for his helpful comments about the subject and especially in relation to the proof of Lemma 2.4. The criticism of the referees was very helpful in shaping the final version.

REFERENCES

ARENAL-GUTIÉRREZ, E., CUESTA-ALBERTOS, J. A. and MATRÁN, C. (1995a). On the unconditional strong law of large numbers for the bootstrap mean. *Statist. Probab. Lett.* To appear.

ARENAL-GUTIÉRREZ, E., CUESTA-ALBERTOS, J. A. and MATRÁN, C. (1995b). Unconditional Glivenko–Cantelli-type theorems and weak laws of large numbers for bootstrap. *Statist. Probab. Lett.* To appear.

BARBE, P. (1994). Strong laws for the generalized bootstrapped mean. Unpublished manuscript.

CSÖRGŐ, S. (1992). On the law of large numbers for the bootstrap mean. *Statist. Probab. Lett.* **14** 1–7.

EFRON, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*. SIAM, Philadelphia.

EINMAHL, U. and MASON, D. M. (1992). Approximations to permutation and exchangeable processes. *J. Theoret. Probab.* **5** 101–126.

GINÉ, E. and ZINN, J. (1989). Necessary conditions for the bootstrap of the mean. *Ann. Statist.* **17** 684–691.

HAEUSLER, E., MASON, D. M. and NEWTON, M. A. (1992). Weighted bootstrapping of means. *CWI Quarterly* 213–228.

HÁJEK, J. (1961). Some extensions of the Wald–Wolfowitz–Noether theorem. *Ann. Math. Statist.* **32** 506–523.

HALL, P. and MAMMEN, E. (1992). On general resampling algorithms and their performance in distribution estimation. Unpublished manuscript.

HUŠKOVA, M. and JANSSEN, P. (1993). Consistency of the generalized bootstrap for degenerate U-statistics. *Ann. Statist.* **21** 1811–1823.

LANDERS, D. and ROGGE, L. (1987). Laws of large numbers for pairwise independent uniformly integrable random variables. *Math. Nachr.* **130** 189–182.

LO, A. Y. (1993). A Bayesian method for weighted sampling. *Ann. Statist.* **21** 2138–2148.

MASON, D. M. and NEWTON, M. A. (1992). A rank statistics approach to the consistency of a general bootstrap. *Ann. Statist.* **20** 1611–1624.

PRAESTGAARD, J. and WELLNER, J. A. (1993). Exchangeably weighted bootstraps of general empirical process. *Ann. Statist.* **21** 2053–2086.

RUBIN, D. (1981). The Bayesian bootstrap. *Ann. Statist.* **9** 130–134.

SHORACK, G. R. (1979). The weighted empirical process of row independent random variables with arbitrary distributions functions. *Statist. Neerlandica* **33** 169–189.

WU, C. F. J. (1986). Jackknife, bootstrap and other resampling methods in regression analysis. *Ann. Statist.* **14** 1261–1295.

DEPARTAMENTO DE ESTADÍSTICA E I.O.
FACULTAD DE CIENCIAS
UNIVERSIDAD DE VALLADOLID
47002 VALLADOLID
SPAIN
E-mail: matran@mac.cie.uva.es