*The Annals of Probability*
1996, Vol. 24, No. 1, 320–345

# LARGE DEVIATIONS FOR A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS[1]

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We consider the random fields $X^\varepsilon(t, q)$, $t \geq 0$, $q \in \mathscr{O}$, governed by stochastic partial differential equations driven by a Gaussian white noise in space–time, where $\mathscr{O}$ is a bounded domain in $\mathbb{R}^d$ with regular boundary. To study the continuity of the random fields $X^\varepsilon$ in space and time variables, we prove an analogue of Garsia's theorem. We then derive the large deviation results based on the methods used by the second author in another paper. This article provides an alternative proof of Sowers' result for the case of $d = 1$.

**1. Introduction.** The large deviation principle (LDP) has been studied extensively by many authors. Most of the earlier work dealt with random variables or stochastic processes in finite-dimensional spaces. The LDP was derived based on some estimates for probabilities of different random events. The dimension of the underlying state spaces of the random variables or stochastic processes plays a key role in these inequalities. This becomes a major obstacle for the generalization to infinite-dimensional setups.

Making use of the properties of subadditive functions, large deviations for the empirical measure of independent identically distributed random variables taking values in infinite-dimensional spaces were obtained. These results were then applied to Gaussian measures on general Banach spaces with various applications [see the book by Stroock (1984) for details].

One of the major applications is to the investigation of the LDP for reaction-diffusion stochastic differential equations, which has been studied by various authors [e.g., see Faris and Jona-Lasinio (1982), Freidlin (1988) and Zabczyk (1988)]. Most of these works dealt essentially with linear equations, where the solution can be obtained by a continuous transformation of a Gaussian process.

Now let us mention some work for which nonlinear techniques are needed. Dawson and Gärtner (1987) considered the convergence rate for the empirical processes of interacting particle systems when the sizes of the system tend to infinity. They derived a large deviation result by approximating the interacting system by a system of independent components by freezing the interaction term. Sowers (1992) studied the LDP for a nonlinear reaction-diffusion stochastic differential equation with the space variable in a one-dimensional

Received July 1994; revised April 1995.

[1]Research partially supported by the NSF, by Air Force Office of Scientific Research Grant F49620-92-J-0154 and by Army Research Office Grant DAAL03-02-G-0008.

*AMS* 1991 *subject classifications.* Primary 60F10; secondary 60H15, 60G60.

*Key words and phrases.* Large deviation principle, SPDE, white noise.

bounded interval. Xiong (1994) derived the large deviations for a class of random transformations in abstract Wiener space and then applied them to diffusion processes taking values in the duals of countably Hilbertian nuclear spaces.

The existence and uniqueness of solution for a large class of stochastic partial differential equations (SPDE's) with the space variable in a $d$-dimensional bounded or unbounded domain has been established by Kotelenez (1993). In this paper, we study the large deviations for such equations when the domain of the space variable is bounded, by applying to the present setup the results obtained in Xiong (1994).

This paper is organized as follows: in Section 2, we introduce the results obtained by Xiong (1994) on large deviations in a general setting for later use. In Section 3, we state some conditions imposed by Kotelenez (1993) to establish the existence and uniqueness of solutions for SPDE's. Section 4 is the main body of this article, and we proceed to derive the LDP for SPDE's by verifying the assumptions of Section 2. Throughout this paper, we use two systems to denote constants: those appearing in the assumptions will be denoted by $K(T)$, $K(\gamma)$ and so forth (if they depend on a parameter) or simply by $K$; those arising from the proofs will be denoted by $K$ with an integer subscript in a consecutive way.

The papers of Peszat (1994) and Chow (1992) were brought to our attention by an anonymous referee. In these papers, the authors study the LDP for stochastic evolution equations. The present paper differs from theirs in the following aspects:

1. Their methods are similar to the finite-dimensional case; that is, they resort to a sequence of approximate solutions for which the LDP are satisfied and then show that the LDP is preserved in the limit. Our method is to approximate the probability that the solution lies in a small neighborhood by the probability that a Gaussian process, obtained by freezing the right-hand side of the SPDE, lies in the same neighborhood.
2. The stochastic integral on the right-hand side of the equation in our paper is different from the one in their papers. They consider it to be the integral of a Hilbert–Schmidt-valued process with respect to a Wiener process. We regard it as the integral of a real-valued random field (with both the time and space variables as parameters) with respect to a Brownian sheet in space–time. The advantage of this point of view is that the Hilbert–Schmidt property is not required and hence, the condition (C.2) in Peszat's paper and conditions (A.2) and (A.3) in Chow's paper can be relaxed. For a further comparison, see the end of this paper.

**2. Large deviations for a class of Banach space-valued random variables.** In this section we briefly introduce the large deviation results of Xiong (1994) for a class of random transformations in abstract Wiener space.

Let $(i, \mathscr{H}, \Omega)$ be an abstract Wiener space and let $P$ be the standard Wiener measure on $(\Omega, \mathscr{B}(\Omega))$. Let $\mathscr{X} \subset \mathscr{Y}$ be two separable Banach spaces.

Suppose that $A$ is a map from $\mathscr{X}$ to $\mathscr{Y}$ such that the following assumption holds:

ASSUMPTION A1. There exists a constant $K$ such that for any $x_1, x_2 \in \mathscr{X}$,

$$\|A(x_1) - A(x_2)\|_{\mathscr{Y}} \le K\|x_1 - x_2\|_{\mathscr{X}}. \tag{2.1}$$

Let $\mathscr{A}_{\mathscr{X}}$ be a class of measurable maps from $(\Omega, \mathscr{B}(\Omega))$ to $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$.

ASSUMPTION A2. (i) $\mathscr{A}_{\mathscr{X}}$ is a linear space; that is, for any $a, b \in \mathbb{R}$, $X_1, X_2 \in \mathscr{A}_{\mathscr{X}}$, we have $aX_1 + bX_2 \in \mathscr{A}_{\mathscr{X}}$.

(ii) $\mathscr{X} \subset \mathscr{A}_{\mathscr{X}}$; that is, for any $x \in \mathscr{X}$ fixed, the constant map $X(\omega) \equiv x$, $\forall\, \omega \in \Omega$, is in $\mathscr{A}_{\mathscr{X}}$.

(iii) For any $h \in \mathscr{H}$, $X \in \mathscr{A}_{\mathscr{X}}$, let $(T_h X)(\omega) = X(\omega - h)$. Then $T_h X \in \mathscr{A}_{\mathscr{X}}$.

DEFINITION 2.1. Let $l \in L(\mathscr{H}, \mathscr{Y})$. We say that the lifting of $l$ exists if for any sequence $\{l_n\} \subset L(\Omega, \mathscr{Y})$ such that $l_n \to l$ in $L(\mathscr{H}, \mathscr{Y})$, we have that $\{l_n(\omega)\}$ converges in probability to a $\mathscr{Y}$-valued random variable $\tilde{l}(\omega)$. We call $\tilde{l}$ the lifting of $l$.

Let $\mathscr{A}_{\mathscr{Y}}$ be a class of measurable maps from $(\Omega, \mathscr{B}(\Omega))$ to $(\mathscr{Y}, \mathscr{B}(\mathscr{Y}))$ satisfying (i) and (ii) of Assumption A2 with $\mathscr{X}$ replaced by $\mathscr{Y}$. Let $B$ be a map from $\mathscr{A}_{\mathscr{X}}$ to $\mathscr{A}_{\mathscr{Y}}$ satisfying the following:

ASSUMPTION A3. There exists a continuous map $\hat{B}: \mathscr{X} \times \mathscr{H} \to \mathscr{Y}$ such that:

(i) For each $x \in \mathscr{X}$, $\hat{B}(x, \cdot): \mathscr{H} \to \mathscr{Y}$ is linear and the lifting $\tilde{B}(x, \cdot)$: $\Omega \to \mathscr{Y}$ exists and is an element of $\mathscr{A}_{\mathscr{Y}}$.

(ii) There exists a constant $K$ such that, for any $x_1, x_2 \in \mathscr{X}$, $h \in \mathscr{H}$, we have

$$\|\hat{B}(x_1, h) - \hat{B}(x_2, h)\|_{\mathscr{Y}} \le K\|h\|_{\mathscr{H}}\|x_1 - x_2\|_{\mathscr{X}}.$$

For each $x \in \mathscr{X}$, as the constant map $X(\omega) \equiv x$ is in $\mathscr{A}_{\mathscr{X}}$, we have that $B(x) \equiv B(X)$ is an element of $\mathscr{A}_{\mathscr{Y}}$ and hence, $B(x)$ is a $\mathscr{Y}$-valued random variable. On the other hand, by Definition 2.1, the lifting $\tilde{B}(x, \cdot)$ of the linear map $\hat{B}(x, \cdot)$ is also a $\mathscr{Y}$-valued random variable. We make the following additional assumptions:

(iii) For any $x \in \mathscr{X}$, we have $B(x) = \tilde{B}(x, \cdot)$.

(iv) For any $h \in \mathscr{H}$, $X \in \mathscr{A}_{\mathscr{X}}$, the map $B_h(X)$: $\Omega \to \mathscr{Y}$ given by $B_h(X)\omega = \hat{B}(X(\omega), h)$ is in $\mathscr{A}_{\mathscr{Y}}$ and

$$B(X) - B_h(X) = T_h(B(T_{-h}X)).$$

(v) $B$ is exponentially continuous, that is, for any $L > 0$, there exists $\delta > 0$ such that for any $\{X_1(\varepsilon)\}, \{X_2(\varepsilon)\} \subset \mathscr{A}_{\mathscr{X}}$, we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\Big(\sqrt{\varepsilon}\|B(X_1(\varepsilon)) - B(X_2(\varepsilon))\|_{\mathscr{Y}} > \sqrt{\delta},\ \|X_1(\varepsilon) - X_2(\varepsilon)\|_{\mathscr{X}} < \delta\Big) \le -L. \tag{2.2}$$

We consider a family of $\mathscr{X}$-valued random variables $\{X^\varepsilon\}$ in the probability space $(\Omega, \mathscr{B}(\Omega), P)$ under the following:

ASSUMPTION A4. (i) $\{X^\varepsilon\}$ is exponentially tight; that is, for any $L > 0$, there exists a compact subset $C_L$ of $\mathscr{X}$ such that

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \notin C_L) \leq -L. \tag{2.3}$$

(ii) For any $\varepsilon > 0$, $X^\varepsilon \in \mathscr{A}_{\mathscr{X}}$ and satisfies the equation

$$X^\varepsilon = A(X^\varepsilon) + \sqrt{\varepsilon} B(X^\varepsilon) \tag{2.4}$$

as $\mathscr{Y}$-valued random variables.

Let $h \in \mathscr{H}$. We say that $h \in \mathscr{D}(\gamma)$ if there exists $x \in \mathscr{X}$ such that

$$x = A(x) + \hat{B}(x, h).$$

In this case, we denote $x$ by $\gamma(h)$. We shall need the following assumption:

ASSUMPTION A5. Let $x \in \mathscr{X}$ be given by $\gamma(h)$ for an $h \in \mathscr{D}(\gamma)$. If $Z^\varepsilon \in \mathscr{A}_{\mathscr{X}}$ such that

$$Z^\varepsilon = A(Z^\varepsilon + x) - A(x) + B_h(Z^\varepsilon + x) - B_h(x) + \sqrt{\varepsilon} B(Z^\varepsilon + x), \tag{2.5}$$

then, for any $\delta > 0$,

$$P(\omega: \|Z^\varepsilon\|_{\mathscr{X}} < \delta) \to 1 \quad \text{as } \varepsilon \to 0.$$

THEOREM 2.1. *Under Assumptions* A1–A5, $\{X^\varepsilon\}$ *satisfies the large deviation principle on* $\mathscr{X}$ *with rate function I given by*

$$I(x) = \inf\left\{\tfrac{1}{2}\|h\|_{\mathscr{H}}^2 : h \in \mathscr{D}(\gamma) \text{ such that } x = \gamma(h)\right\}.$$

*In other words*:

(i) $I(x)$ *is a lower-semicontinuous function from* $\mathscr{X}$ *to* $[0, \infty]$ *and, for any* $c > 0$, *the level set* $L_c = \{x \in \mathscr{X}: I(x) \leq c\}$ *is compact.*

(ii) *For any open set G of* $\mathscr{X}$, *we have*

$$\liminf_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in G) \geq -\inf\{I(x): x \in G\}.$$

(iii) *For any closed subset C of* $\mathscr{X}$, *we have*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in C) \leq -\inf\{I(x): x \in C\}.$$

**3. Stochastic partial differential equations.** In this section we state the result of Kotelenez (1993) on the existence and uniqueness of solutions for a class of stochastic partial differential equations for the convenience of the reader. Throughout the rest of this paper, $\mathscr{O}$ denotes a bounded open domain in $\mathbb{R}^d$.

DEFINITION 3.1. Let $\gamma \in \mathbb{R}$. An operator $L$ on $C_0^\infty(\mathscr{O})$, the collection of smooth functions with compact support in $\mathscr{O}$, is called a pseudodifferential operator of order $\gamma$ if

$$Lu(x) = (2\pi)^{-d} \iint e^{i(x-y)v} a(x, y, v) u(y)\, dy\, dv, \qquad u \in C_0^\infty(\mathscr{O}),$$

where $a$ is a symbol of order $\gamma$ in the following sense: $a$ is a complex-valued smooth function on $\mathscr{O} \times \mathscr{O} \times \mathbb{R}^d$ with compact support and, for any compact set $C \subset \mathscr{O} \times \mathscr{O}$ and multiindices $\beta_1$, $\beta_2$, $\beta_3$, there exists a constant $K(C, \beta_1, \beta_2, \beta_3) > 0$ such that

$$|\partial_x^{\beta_1} \partial_y^{\beta_2} \beta_v^{\beta_3} a(x, y, v)| \le K(C, \beta_1, \beta_2, \beta_3)(1 + |v|)^{\gamma - |\beta_3|},$$
$$(x, y) \in C,\ v \in \mathbb{R}^d,$$

and $\partial$ denotes the derivative.

For details about pseudodifferential operators and the corresponding heat equations, we refer the reader to the book of Treves (1982).

REMARK 3.1. We give here an example of a pseudodifferential operator. Let $\gamma$ be a positive integer and

$$a(x, y, v) = \sum_{|\beta| \le \gamma} a_\beta(x) v^\beta, \qquad (x, y, v) \in \mathscr{O} \times \mathscr{O} \times \mathbb{R}^d,$$

where $a_\beta$ is a smooth function on $\mathscr{O}$ with compact support, $\forall\ |\beta| \le \gamma$. Then $a$ is a symbol of order $\gamma$ and $L$ is a differential operator of order $\gamma$ given by

$$L = \sum_{|\beta| \le \gamma} a_\beta(x)(-1)^{|\beta|/2} \partial_x^\beta.$$

In fact, for any $u \in C_0^\infty(\mathscr{O})$ we have

$$\begin{aligned} Lu(x) &= (2\pi)^{-d} \iint e^{i(x-y)v} \sum_{|\beta| \le \gamma} a_\beta(x) v^\beta u(y)\, dy\, dv \\ &= (2\pi)^{-d/2} \int e^{ixv} \sum_{|\beta| \le \gamma} a_\beta(x)(-1)^{|\beta|/2} \mathscr{F}^{-1}(\partial^\beta u)(v)\, dv \\ &= \sum_{|\beta| \le \gamma} a_\beta(x)(-1)^{|\beta|/2} \partial_x^\beta u(x), \end{aligned}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation.

Let $\{L(t): t \ge 0\}$ be a family of pseudodifferential operators on $\mathscr{O}$ and let $F$ and $R$ be two functions on $[0, T] \times \mathscr{O} \times \mathbb{R}$. Let $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$ be a stochastic basis satisfying the usual condition. Let $W(dr\,dt)$ be the standard Gaussian random measure on $\mathscr{O} \times [0, T]$ adapted to $\mathscr{F}_t$; that is, $\{W(\mathscr{O}_r \times [0, t]): r \in \mathscr{O}, t \in [0, T]\}$ is a centered Gaussian system such that (i) for any $t \in [0, T]$ and $r \in \mathscr{O}$, $W(\mathscr{O}_r \times [0, t])$ is $\mathscr{F}_t$-measurable; (ii) for any $t, s \in [0, T]$ and $r_1, r_2 \in \mathscr{O}$, we have

$$EW(\mathscr{O}_{r_1}\times[0,t])W(\mathscr{O}_{r_2}\times[0,s]) = |\mathscr{O}_{r_1}\cap\mathscr{O}_{r_2}|(t\wedge s),$$

where

$$\mathscr{O}_r = \{q\in\mathscr{O}: q_j\le r_j,\, j=1,2,\dots,d\}$$

and $|C|$ denotes the Lebesgue measure of the set $C$.

To study the stochastic partial differential equation

$$\begin{aligned} dX^\varepsilon(t,r) &= \big(L(t)X^\varepsilon(t,r) + R(t,r,X^\varepsilon(t,r))\big)\,dr\,dt \\ &\quad + \sqrt{\varepsilon}F(t,r,X^\varepsilon(t,r))W(dr\,dt) \end{aligned} \tag{3.1}$$

with initial condition

$$X^\varepsilon(0,r) = \xi(r),$$

we make the following assumptions:

(RD1) For $t\ge 0$, $L(t)$ is a pseudodifferential operator of order $\gamma > d$.

(RD2) $\{L(t)\}$ generates a two-parameter evolution semigroup $\{U(t,s)\colon 0\le s\le t\}$ on $\mathbb{C}(\mathscr{O})$, the space of all real-valued continuous functions in $\mathscr{O}$, which has a kernel function $G(t,s,r,q)$, $0\le s\le t$, $r,q\in\mathscr{O}$. That is,

$$(U(t,s)f)(r) = \int_{\mathscr{O}} G(t,s,r,q)f(q)\,dq.$$

For simplicity, we extend $G(t,s,r,q)$ to $t,s\in[0,T]$, $r,q\in\mathscr{O}$, by defining it to be zero when $t<s$. Under the above assumptions, (3.1) is understood as the equation

$$\begin{aligned} X^\varepsilon(t,r) &= \int_{\mathscr{O}} G(t,0,r,q)\xi(q)\,dq \\ &\quad + \int_0^t\int_{\mathscr{O}} G(t,s,r,q)R(s,q,X^\varepsilon(s,q))\,dq\,ds \\ &\quad + \sqrt{\varepsilon}\int_0^t\int_{\mathscr{O}} G(t,s,r,q)F(s,q,X^\varepsilon(s,q))W(dq\,ds). \end{aligned} \tag{3.2}$$

The solution of (3.2) is called the "mild solution" of the SPDE (3.1).

To solve (3.2) and derive large deviation results, we need some additional conditions:

(RD3) For any $T>0$, there exists a constant $K(T)<\infty$ such that

$$\int_{\mathscr{O}} |G(t,s,r,q)|\,dq \le K(T), \qquad \forall\ 0\le s\le t\le T,\ r,q\in\mathscr{O}. \tag{3.3}$$

(RD4) There is a symbol $g(t,s,q,\bar{q})$, $0 \le s \le t$, $q \in \mathscr{O}$, $\bar{q} \in \mathbb{R}^d$, such that

$$G(t,s,r,q) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(i(r-q)\cdot\bar{q}) g(t,s,q,\bar{q})\, d\bar{q},$$

and

$$\int_{\mathbb{R}^d} |x|^\alpha \left| g\left(t,s,q,(t-s)^{-1/\gamma} x\right)\right| dx \le K(T),$$

for all $0 \le \alpha \le \gamma - d$, $0 \le s \le t \le T$, $r, q \in \mathscr{O}$.

(RD5) There is a constant $K(\gamma) < \infty$ such that

$$\left| \int_{\mathbb{R}^d} \exp(i(r-q)\cdot\bar{q}) [g(t+u, t-uv, q, \bar{q}) - g(t, t-uv, q, \bar{q})]\, d\bar{q} \right|$$
$$\le K(\gamma) u^{-d/\gamma} v^{-1},$$

for all $u > 0$, $0 < uv \le t \le T$, $r, q \in \mathscr{O}$.

(RD6) For each $T > 0$ there is an integrable positive function $p(T, \cdot)$ in $\mathbb{R}^d$ such that

$$p(T,x) \le K(T), \qquad \forall\, x \in \mathbb{R}^d, \tag{3.4}$$

and

$$(t-s)^{d/\gamma} |G(t,s,r,q)| \le p\left(T,(q-r)(t-s)^{-1/\gamma}\right), \quad \forall\, 0 \le s \le t \le T,\ r,q \in \mathscr{O}. \tag{3.5}$$

(RD7) There exists a constant $K(R,F,T)$ such that, for all $x, y \in \mathbb{R}$, $r \in \mathscr{O}$ and $0 \le t \le T$,

$$|R(t,r,x) - R(t,r,y)| + |F(t,r,x) - F(t,r,y)| \le K(R,F,T)|x-y| \tag{3.6}$$

and

$$|R(t,r,x)| \le K(R,F,T)(1+|x|), \qquad |F(t,r,x)| \le K(R,F,T). \tag{3.7}$$

For $\alpha > 0$, let

$$\mathbb{B}_\alpha = \{\psi \in \mathbb{C}(\mathscr{O}) : \|\psi\|_\alpha < \infty\}$$

be the Banach space with norm

$$\|\psi\|_\alpha = \sup_{r \in \mathscr{O}} |\psi(r)| + \sup_{r,q \in \mathscr{O}} \frac{|\psi(r) - \psi(q)|}{|r-q|^\alpha}.$$

We shall need the following assumption:

(RD8) For $\alpha > 0$ and $\xi \in \mathbb{B}_\alpha$, we have

$$\int_{\mathscr{O}} G(\cdot, 0, \cdot, q)\xi(q)\, dq \in \mathbb{C}([0,\infty), \mathbb{B}_\alpha).$$

The following theorem has been proved by Kotelenez (1993).

THEOREM 3.1. (i) *Under the assumptions* (RD1)–(RD8), *the SPDE* (3.1) *has a unique mild solution adapted to* $\mathscr{F}_t$; *that is, for any* $t \geq 0$ *and* $r \in \mathscr{O}$, $X^{\varepsilon}(t, r)$ *is* $\mathscr{F}_t$*-measurable.*

(ii) *Let* $0 < \mu < (\gamma - d)/2\gamma$. *If* $\xi \in \mathbb{B}_{\mu}$, *then, regarded as a stochastic process taking values in function space,* $X^{\varepsilon} \in \mathbb{C}([0, \infty), \mathbb{B}_{\mu})$ *almost surely.*

REMARK 3.2. The assumptions (RD1)–(RD8) with the exception of (3.7) are made by Kotelenez (1993) for the existence and uniqueness of solution. He assumes that the function $F$ can be of linear growth in $x$,

$$|F(t, r, x)| \leq K(1 + |x|), \qquad \forall\ x \in \mathbb{R}, r \in \mathscr{O} \text{ and } 0 \leq t \leq T,$$

where $K$ is a constant. We are not able to derive large deviation results under this weaker condition on $F$.

**4. Large deviations of stochastic partial differential equations.** In this section, we fix $T > 0$ and consider random fields $X^{\varepsilon}(t, r)$, $0 \leq t \leq T$ and $r \in \mathscr{O}$. Let $0 < \mu < (\gamma - d)/2\gamma$ be fixed. We regard $X^{\varepsilon}$ as a $\mathbb{B}_{\mu}$-valued stochastic process and denote $P^{\varepsilon}$ by its probability measure in $C([0, T], \mathbb{B}_{\mu})$. We now study the LDP for $\{P^{\varepsilon}\}$ by applying the results of Section 2 to the present setup. The most important step in verifying conditions (A1)–(A5) of Section 2 is Lemma 4.3 below, which is an analogue of a theorem of Garsia about the sample Hölder continuity of random fields.

Now we proceed to verify assumptions (A1)–(A5) of Section 2. As the solution $X^{\varepsilon}$ is a function of the Brownian sheet $\{W(t, r) \equiv W(\mathscr{O}_r \times [0, t])\}$, we may assume that $\Omega = \mathbb{C}(\mathscr{O} \times [0, T])$, $\mathscr{F} = \mathscr{B}(\mathbb{C}(\mathscr{O} \times [0, T]))$, $P$ is the probability measure induced by the Brownian sheet $W$ and $\mathscr{F}_t$ is the sub-$\sigma$-field of $\mathscr{F}$ generated by $\{w_s \colon s \leq t\}$, where $w_s(\omega) = \omega(\cdot, s)$. Let $\mathscr{H} \subset \Omega$ be the space of all $h \in \Omega$ with the following property: there exists $\hat{h} \in L^2(\mathscr{O} \times [0, T])$ such that

$$h(r, t) = \int_{\mathscr{O}_r} \int_0^t \hat{h}(q, s)\, dq\, ds, \qquad \forall\ r \in \mathscr{O}, t \in [0, T].$$

For $h_1, h_2 \in \mathscr{H}$, let

$$\langle h_1, h_2 \rangle_{\mathscr{H}} = \int_{\mathscr{O}} \int_0^T \hat{h}_1(r, t) \hat{h}_2(r, t)\, dr\, dt.$$

Then $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ is an inner product on $\mathscr{H}$ under which $\mathscr{H}$ becomes a separable Hilbert space.

PROPOSITION 4.1. *The triple* $(i, \mathscr{H}, \Omega)$ *is an abstract Wiener space and* $P$ *is the standard Wiener measure on* $\Omega$, *where* $i$ *is the canonical injection from* $\mathscr{H}$ *to* $\Omega$.

PROOF. We identify $\mathscr{H}'$ with $\mathscr{H}$ by the Riesz representation theorem and let $\Omega'$ be the dual of $\Omega$ such that

$$\langle l, h \rangle = \langle l, h \rangle_{\mathscr{H}}, \qquad \forall\ l \in \Omega', h \in \mathscr{H},$$

where $\langle \cdot , \cdot \rangle$ is the pairing between $\Omega'$ and $\Omega$. Let

$$\tilde{\Omega}' = \{ l \in \mathscr{H} \colon \hat{l} \in \mathbb{C}_0^2(\mathscr{O} \times [0,T]) \}.$$

Then $\tilde{\Omega}'$, contained in $\Omega'$, is a dense subset of $\mathscr{H}$. For $l \in \tilde{\Omega}'$, we have that

$$\langle l, W \rangle = \int_{\mathscr{O}} \int_0^T \hat{l}(r,t) W(dr\,dt)$$

is a Gaussian random variable with mean zero and variance $\|l\|_{\mathscr{H}}^2$. Hence $(i, \mathscr{H}, \Omega)$ is an abstract Wiener space and $P$ is the standard Wiener measure on $\Omega$. □

The following lemma (although not the proof given here) appears in Kotelenez (1993) in connection with his proof of the existence of a unique solution to the SPDE.

LEMMA 4.1. (i) *There exists a constant* $K_1$ *such that, for all* $t \in [0, T]$ *and* $r \in \mathscr{O}$,

$$\int_0^t \int_{\mathscr{O}} |G(t,s,r,q)|^2 \, dq\, ds \le K_1. \tag{4.1}$$

(ii) *For any* $\alpha < (\gamma - d)/2\gamma$, *there exists a constant* $K_2$ *such that, for all* $0 \le t_1 \le t_2 \le T$ *and* $r_1, r_2 \in \mathscr{O}$, *we have*

$$\begin{aligned} &\int_0^T \int_{\mathscr{O}} |G(t_1,s,r_1,q) - G(t_2,s,r_2,q)|^2 \, dq\, ds \\ &\qquad \le K_2\, \rho((t_1,r_1),(t_2,r_2))^{2\alpha}, \end{aligned} \tag{4.2}$$

*where* $\rho$ *is the Euclidean distance in* $[0,T] \times \mathscr{O} \subset \mathbb{R}^{d+1}$.

PROOF. Statement (i) follows from (4.7) below by letting $t_2 = t$, $t_1 = 0$ and $r_1 = r_2 = r$.

(ii) By (RD4), we have that for $t, s \in [0,T]$ and $r_1, r_2, q \in \mathscr{O}$,

$$\begin{aligned} |G(t,s,r_1,q) - G(t,s,r_2,q)| &\le 2(2\pi)^{-d} \int_{\mathbb{R}^d} |r_1 - r_2|^{2\alpha} |\bar{q}|^{2\alpha} |g(t,s,q,\bar{q})|\, d\bar{q} \\ &\le 2K(T)(2\pi)^{-d} |r_1 - r_2|^{2\alpha} (t-s)^{-(d+2\alpha)/\gamma}. \end{aligned}$$

Hence, by (RD3),

$$\begin{aligned} &\int_0^t \int_{\mathscr{O}} |G(t,s,r_1,q) - G(t,s,r_2,q)|^2 \, dq\, ds \\ &\quad \le \int_0^t \int_{\mathscr{O}} |G(t,s,r_1,q) - G(t,s,r_2,q)| \\ &\qquad\qquad \times 2K(T)(2\pi)^{-d} |r_1 - r_2|^{2\alpha} (t-s)^{-(d+2\alpha)/\gamma}\, dq\, ds \\ &\quad \le 4K(T)^2 (2\pi)^{-d} |r_1 - r_2|^{2\alpha} \int_0^T s^{-(d+2\alpha)/\gamma}\, ds \equiv K_2' |r_1 - r_2|^{2\alpha}. \end{aligned} \tag{4.3}$$

Taking $u = t_2 - t_1$ and $v = (t_1 - s)/(t_2 - t_1)$ in (RD5), it is easy to see that for $s < t_1$,

$$|G(t_1, s, r_1, q) - G(t_2, s, r_1, q)| \leq (2\pi)^{-d} K(\gamma)(t_2 - t_1)^{1-d/\gamma}(t_1 - s)^{-1}. \tag{4.4}$$

Hence, for any $\lambda \in (0, 1)$,

$$\begin{aligned} &\int_0^{t_1}\int_{\mathscr{O}} |G(t_1, s, r_1, q) - G(t_2, s, r_1, q)|^2 \, dq\, ds \\ &\quad \leq \left\{(2\pi)^{-d} K(\gamma)(t_2 - t_1)^{1-d/\gamma}\right\}^{\lambda} \\ &\qquad \times \int_0^{t_1}\int_{\mathscr{O}} |G(t_1, s, r_1, q) - G(t_2, s, r_1, q)|\, dq \\ &\qquad\quad \times \sup_{q \in \mathscr{O}} \left(|G(t_1, s, r_1, q)| + |G(t_2, s, r_1, q)|\right)^{1-\lambda}(t_1 - s)^{-\lambda}\, ds. \end{aligned} \tag{4.5}$$

By (4.5), (RD3) and (RD6), taking $\lambda \in (0, 1)$ such that $(1 - d/\gamma)\lambda > 2\alpha$, it is clear that there exists a constant $K_2''$ such that

$$\int_0^{t_1}\int_{\mathscr{O}} |G(t_1, s, r_1, q) - G(t_2, s, r_1, q)|^2 \, dq\, ds \leq K_2''(t_2 - t_1)^{2\alpha}. \tag{4.6}$$

Further, by (RD3) and (RD6) again, we have

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathscr{O}} |G(t_1, s, r_1, q) - G(t_2, s, r_1, q)|^2 \, dq\, ds \\ &\quad = \int_{t_1}^{t_2}\int_{\mathscr{O}} |G(t_2, s, r_1, q)|^2 \, dq\, ds \\ &\quad \leq \int_{t_1}^{t_2}\int_{\mathscr{O}} |G(t_2, s, r_1, q)|(t_2 - s)^{-d/\gamma} \\ &\qquad\quad \times p\left(T, (q - r_1)(t_2 - s)^{-1/\gamma}\right) dq\, ds \\ &\quad \leq K(T)^2(1 - d/\gamma)^{-1}(1 \vee T)(t_2 - t_1)^{2\alpha} \equiv K_2'''(t_2 - t_1)^{2\alpha}. \end{aligned} \tag{4.7}$$

The inequality (4.2) then follows from (4.3), (4.6), (4.7) and the triangle inequality. □

Next, we prove an analogue of Garsia's theorem [Garsia, (1970)] for a general bounded open domain $\mathscr{O}$ satisfying the cone condition. The latter condition which we assume throughout this paper means that there exist two positive constants $a$ and $a_0$ such that, for any $r \in \mathscr{O}$, there exists a cone $C_r$ with vertex at $r$ with height $a$ and base radius $a_0$.

For any hypercube $Q$ in $\mathbb{R}^d$, we denote by $Q'$ the hypercube in $\mathbb{R}^d$ such that $Q$ and $Q'$ have the same center with edges parallel to the coordinate axes and $e(Q) = 2e(Q')$, where $e(Q)$ is the common length of the edge of $Q$. For any set $C \subset \mathbb{R}^d$, let $|C|$ be its Lebesgue measure.

LEMMA 4.2. (i) *There exists a constant* $K_3$ *such that*

$$|Q \cap \mathscr{O}| \geq K_3 |Q|, \tag{4.8}$$

*for any hypercube* $Q$ *such that*

$$e(Q) \leq a\sqrt{d} \quad \text{and} \quad Q' \cap \mathscr{O} \neq \phi. \tag{4.9}$$

(ii) *Let* $Q$ *be a hypercube satisfying* (4.9). *For any* $r \in Q' \cap \mathscr{O}$ *and* $0 < \delta < e(Q)$, *there exists a hypercube* $Q_1 \subset Q$ *such that* $r \in Q_1'$ *and* $e(Q_1) = \delta$.

PROOF. (i) Let $r \in Q' \cap \mathscr{O}$. As $e(Q) \leq a/\sqrt{d}$, $C_r$ is not contained in $Q$. Otherwise

$$a \geq \text{diameter}(Q) \geq \text{diameter}(C_r) > a.$$

Let $C_r'$ be the maximal cone contained in $C_r \cap Q$ such that its base is parallel to the base of $C_r$. Then the base of $C_r'$ intersects with the boundary of $Q$. Let $q$ be a point in this intersection. Let $b$ (resp. $l$) be the slant edge of $C_r'$ (resp. $C_r$). Then

$$b = |rq| \geq \text{distance}(Q', Q^c) = \frac{e(Q)}{4},$$

where $Q^c$ denotes the complement of $Q$.

It is easy to see that the height and base radius of $C_r'$ are $ab/l$ and $a_0 b/l$, respectively. Hence

$$\frac{|C_r'|}{|C_r|} = \frac{d^{-1}V_{d-1}(a_0 b/l)^{d-1}(ab/l)}{d^{-1}V_{d-1}a_0^{d-1}a} = \left(\frac{b}{l}\right)^d \geq \left(\frac{e(Q)}{4l}\right)^d,$$

where $V_{d-1}$ is the volume of the unit sphere in $\mathbb{R}^{d-1}$. Therefore,

$$|Q \cap \mathscr{O}| \geq |C_r'| \geq \left\{\frac{e(Q)}{4l}\right\}^d |C_r| = \frac{|C_0|}{(4l)^d}|Q| \equiv K_3|Q|.$$

(ii) Extend $d$ segments through $r$ with the following properties: (a) they are orthogonal to each other and lie in $Q$; (b) each has length $\delta$ and is parallel to an edge of $Q$; (c) $r$ divides each segment into two parts, the length of each being not less than $\delta/4$.

Construct a hypercube $Q_1$ with edges parallel to those of $Q$ and all the endpoints of the $d$ segments mentioned above are in the surface of $Q_1$. It can be easily checked that $r \in Q_1'$ and $e(Q) = \delta$. □

LEMMA 4.3. *Let* $\psi$ *be a continuous function on* $\mathscr{O}$. *Let* $\Psi$ *and* $p$ *be increasing functions in* $x \geq 0$ *such that* $\Psi(0) = p(0) = 0$ *and* $\Psi$ *is convex. Let*

$$\eta = \int_{\mathscr{O}}\int_{\mathscr{O}} \Psi\left(\frac{|\psi(r) - \psi(q)|}{p(|r-q|)}\right) dr\, dq \leq \infty.$$

*Then, for any $r, q \in \mathscr{O}$ with $|r - q| \le a/2$, we have*

$$|\psi(r) - \psi(q)| \le 8\int_0^{2|r-q|} \Psi^{-1}\left(\frac{\eta}{K_4 u^{2d}}\right) p(du), \tag{4.10}$$

*where $K_4$ is a constant and $\Psi^{-1}$ denotes the inverse function.*

PROOF. If $\eta = \infty$, then (4.10) is obviously satisfied. Let $\eta$ be finite. For $r, q \in \mathscr{O}$, if $|r - q| \le a/2$, we have a hypercube $Q'_0$ such that $r, q \in Q'_0$ and $e(Q'_0) \le a/2\sqrt{d}$. Define $Q_0$ to be a hypercube having the same center as $Q'_0$ and $e(Q_0) = 2e(Q'_0)$. As $r \in Q'_0 \cap \mathscr{O}$, it follows from Lemma 4.2 that $|Q_0 \cap \mathscr{O}| \ge K_3 |Q_0|$. From here on we proceed similarly as in Garsia (1970). By Lemma 4.2 again, there exists a decreasing sequence $\{Q_n\}_{n \ge 0}$ of hypercubes such that $r \in Q'_n$ and $p(x_{n-1}) = 2p(x_n)$, for all $n \ge 1$, where $x_n = \sqrt{d}\, e(Q_n)$. Let

$$\tilde{Q} = Q \cap \mathscr{O} \quad \text{and} \quad \psi_Q = \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \psi(r)\, dr.$$

Then by (4.8) and Jensen's inequality, we have

$$\begin{aligned}
\Psi\left(\frac{|\psi_{Q_n} - \psi_{Q_{n-1}}|}{p(x_{n-1})}\right) &\le \Psi\left(\frac{1}{|\tilde{Q}_{n-1}|\,|\tilde{Q}_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n} \frac{|\psi(r) - \psi(q)|}{p(x_{n-1})}\, dr\, dq\right) \\
&\le \frac{1}{|\tilde{Q}_{n-1}|\,|\tilde{Q}_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n} \Psi\left(\frac{|\psi(r) - \psi(q)|}{p(x_{n-1})}\right) dr\, dq \\
&\le \frac{1}{K_3^2 |Q_{n-1}|\,|Q_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n} \Psi\left(\frac{|\psi(r) - \psi(q)|}{p(|r - q|)}\right) dr\, dq \\
&\le \frac{\eta}{K_4 x_n^{2d}}
\end{aligned}$$

since

$$|Q_{n-1}|\,|Q_n| = [e(Q_{n-1})e(Q_n)]^d = \left(\frac{x_{n-1}}{\sqrt{d}}\right)^d \left(\frac{x_n}{\sqrt{d}}\right)^d \ge \frac{x_n^{2d}}{d^d}.$$

Here we have that $K_4 = d^{-d}K_3^2$. Hence, noting that $p(x_{n-1}) = 2p(x_n) = 4p(x_{n+1})$ we have

$$\begin{aligned}
|\psi_{Q_n} - \psi_{Q_{n-1}}| &\le \Psi^{-1}\left(\frac{\eta}{K_4 x_n^{2d}}\right) p(x_{n-1}) \\
&= 4\Psi^{-1}\left(\frac{\eta}{K_4 x_n^{2d}}\right)(p(x_n) - p(x_{n+1})) \\
&\le 4\int_{x_{n+1}}^{x_n} \Psi^{-1}\left(\frac{\eta}{K_4 u^{2d}}\right) p(du).
\end{aligned}$$

Summing up over $n$ from 0 to $\infty$, we have

$$|\psi(r) - \psi_{Q_0}| \leq 4\int_0^{x_0} \Psi^{-1}\left(\frac{\eta}{K_4 u^{2d}}\right) p(du).$$

It is clear that the above procedure applies with $r$ replaced by $q$, and hence

$$|\psi(r) - \psi(q)| \leq 8\int_0^{x_0} \Psi^{-1}\left(\frac{\eta}{K_4 u^{2d}}\right) p(du).$$

Making $Q_0$ as small as possible, we have $x_0$ arbitrarily close to $2|r-q|$. This finishes the proof of (4.10). □

It is easy to check that $[0,T] \times \mathscr{O}$ also satisfies the cone condition. Therefore, Lemma 4.3 is applicable to continuous function $\psi$ defined on $[0,T] \times \mathscr{O}$. Since only Corollary 4.1 below will be used in the rest of this paper, we shall use the notation $a$ for the height of the new cone although its value has been changed.

COROLLARY 4.1. *Let $\psi$ be a continuous function on $[0,T] \times \mathscr{O}$. Let $\Psi$ and $p$ be increasing functions in $x \geq 0$ such that $\Psi(0) = p(0) = 0$ and $\Psi$ is convex. Let*

$$\eta = \int_0^T \int_0^T \int_{\mathscr{O}} \int_{\mathscr{O}} \Psi\left(\frac{|\psi(t,r) - \psi(s,q)|}{p(\rho((t,r),(s,q)))}\right) dr\, dq\, dt\, ds.$$

*Then, for any $t, s \in [0,T]$ and $r, q \in \mathscr{O}$ with $\rho \equiv \rho((t,r),(s,q)) \leq a/2$, we have*

$$|\psi(t,r) - \psi(s,q)| \leq 8\int_0^{2\rho} \Psi^{-1}\left(\frac{\eta}{K_5 u^{2(d+1)}}\right) p(du),$$

*where $K_5$ is a constant.*

Next we define some seminorms in $C([0,T], \mathbb{B}_\alpha)$. Let

$$[\psi]_\alpha = \sup\left\{\frac{|\psi(t,r) - \psi(s,q)|}{\rho((t,r),(s,q))^\alpha} : \rho((t,r),(s,q)) \leq \frac{a}{2}\right\}$$

and

$$\|\psi\|_{t,\alpha} = \sup_{s\in[0,t]} \left\{\sup_{r\in\mathscr{O}} |\psi(s,r)| + \sup_{r,q\in\mathscr{O}} \frac{|\psi(s,r) - \psi(s,q)|}{|r-q|^\alpha}\right\}$$

with the convention that $\|\psi\|_{t,0}$ is the usual supremum norm.

LEMMA 4.4. *There exists a constant $K_6$ such that $\|\psi\|_{T,\alpha} \leq K_6[\psi]_\alpha$, for any $\psi \in \mathbb{C}([0,T] \times \mathscr{O})$ satisfying $\psi(0,r) = 0$, $\forall\ r \in \mathscr{O}$.*

PROOF. We only need to note that

$$\begin{aligned}\|\psi\|_{T,\alpha} &\le \sup\{|\psi(t,r)|: t\in[0,T],\, r\in\mathscr{O}\}\\ &\quad+\sup\left\{\frac{|\psi(t,r)-\psi(t,q)|}{|r-q|^\alpha}: t\in[0,T],\, r,q\in\mathscr{O},\, |r-q|>\frac{a}{2}\right\}\\ &\quad+\sup\left\{\frac{|\psi(t,r)-\psi(t,q)|}{|r-q|^\alpha}: t\in[0,T],\, r,q\in\mathscr{O},\, |r-q|\le\frac{a}{2}\right\}\\ &\le\left(1+2\left(\frac{2}{a}\right)^\alpha\right)\sup\{|\psi(t,r)-\psi(0,r)|: t\in[0,T],\, r\in\mathscr{O}\}+[\psi]_\alpha\\ &\le\left(1+2\left(\frac{2}{a}\right)^\alpha\right)\sup_{0\le t\le T}\sum_{j=1}^\infty\sup_{r\in\mathscr{O}}\left|\psi\left(t\wedge\frac{j}{2}a,r\right)-\psi\left(t\wedge\frac{j-1}{2}a,r\right)\right|\\ &\quad+[\psi]_\alpha\\ &\le\left(\left(1+2\left(\frac{2}{a}\right)^\alpha\right)\left(\frac{2T}{a}+1\right)\left(\frac{a}{2}\right)^\alpha+1\right)[\psi]_\alpha. \qquad\square\end{aligned}$$

The next lemma is useful for the verification of (2.2) and (2.3).

LEMMA 4.5. *Let $f(s,q)$ and $\tilde{f}(s,q)$ be two adapted random fields (may depend on $\varepsilon$) such that there exist constants $K_7$ and $K_8$ such that*

$$\int_0^T\int_{\mathscr{O}} f(s,q)^2\,dq\,ds\le K_7\quad a.s. \tag{4.11}$$

*and*

$$\sup\{|\tilde{f}(s,q)|: s\in[0,T] \text{ and } q\in\mathscr{O}\}\le K_8\quad a.s.$$

(i) *Let*

$$M_t=\int_0^t\int_{\mathscr{O}} f(s,q)W(dq\,ds).$$

*Then there exist constants $K_9, K_{10}>0$ such that*

$$E\big(\exp(K_9|M_T|^2)|K_{10}\big). \tag{4.12}$$

(ii) *Let*

$$\psi(t,r)=\int_0^t\int_{\mathscr{O}} G(t,s,r,q)\tilde{f}(s,q)W(dq\,ds)$$

*and $\alpha<(\gamma-d)/2\gamma$. Then $[\psi]_\alpha<\infty$ a.s. and, for any $L>0$, there exists $\delta>0$ such that*

$$\limsup_{\varepsilon\to0}\varepsilon\log P\big([\psi]_\alpha>1/\sqrt{\delta\varepsilon}\big)\le-L.$$

PROOF. (i) It is easy to see that $M_t$ is a continuous square integrable martingale with quadratic variation process

$$\langle M_t\rangle=\int_0^t\int_{\mathscr{O}} f(s,q)^2\,dq\,ds\le K_7.$$

There exists a one-dimensional Brownian motion $\hat{W}$ such that $M_t = \hat{W}_{\langle M_t \rangle}$ and hence

$$|M_T| \leq \sup\{\hat{W}_t : 0 \leq t \leq K_7\}.$$

Now (4.12) follows from Fernique's theorem [see Kuo (1975), page 159].

(ii) For $0 \leq t_1 < t_2 \leq T$, $r_1, r_2 \in \mathscr{O}$, let

$$f(s,q) = \frac{G(t_1, s, r_1, q) - G(t_2, s, r_2, q)}{\rho((t_1, r_1), (t_2, r_2))^{\alpha}} \tilde{f}(s,q), \qquad \forall\, s \in [0,T], q \in \mathscr{O}.$$

It is easy to see that

$$M_T = \frac{\psi(t_1, r_1) - \psi(t_2, r_2)}{\rho((t_1, r_1), (t_2, r_2))^{\alpha}}.$$

It follows from (4.2) that (4.11) holds with $K_7 = K_8 K_2$. Let

$$\eta = \int_0^T \int_0^T \int_{\mathscr{O}} \int_{\mathscr{O}} \exp\left( K_9 \frac{|\psi(t,r) - \psi(s,q)|^2}{\rho((t,r),(s,q))^{2\alpha}} \right) dr\, dq\, dt\, ds.$$

By (i), we have

$$E\eta \leq \int_0^T \int_0^T \int_{\mathscr{O}} \int_{\mathscr{O}} E \exp(K_9 M_T^2)\, dr\, dq\, dt\, ds \leq T^2 |\mathscr{O}|^2 K_{10} < \infty.$$

Let $\Psi(x) = \exp(K_9 x^2) - 1$ and $p(x) = x^{\alpha}$, for $x \geq 0$. Then $\Psi$ and $p$ are strictly increasing, $\Psi(0) = p(0) = 0$ and

$$\eta - T^2 |\mathscr{O}|^2 = \int_0^T \int_0^T \int_{\mathscr{O}} \int_{\mathscr{O}} \Psi\left( \frac{|\psi(t,r) - \psi(s,q|}{p(\rho((t,r),(s,q)))} \right) dr\, dq\, dt\, ds.$$

It follows from Corollary 4.1 that, for any $t, s \in [0,T]$ and $r, q \in \mathscr{O}$ with $\rho((t,r), (s,q)) \leq a/2$, we have

$$\begin{aligned}
|\psi(t,r) - \psi(s,q)| &\leq 8 \int_0^{2\rho} \Psi^{-1}\left( \frac{\eta - T^2 |\mathscr{O}|^2}{K_5 u^{2(d+1)}} \right) p(du) \\
&= \frac{8}{\sqrt{K_9}} \int_0^{2\rho} \sqrt{\log\left(1 + \frac{\eta - T^2|\mathscr{O}|^2}{K_5 u^{2(d+1)}}\right)}\, du^{\alpha} \\
&\leq \frac{8}{\sqrt{K_9}} \int_0^{2\rho} \left( \sqrt{\log\left( \frac{\eta}{K_5 u^{2(d+1)}} \vee 1 \right)} + \sqrt{\log 2} \right) du^{\alpha} \\
&\leq \frac{8}{\sqrt{K_9}} \int_0^{2\rho} \Big( \sqrt{\log((\eta/K_5) \vee 1)} \\
&\qquad + \sqrt{|2(d+1)\log u|} + \sqrt{\log 2} \Big)\, du^{\alpha}.
\end{aligned}$$

Also, for any $\alpha' < \alpha$, we have

$$\lim_{\rho\to 0} \rho^{-\alpha'} \int_0^{2\rho} \sqrt{|\log u|}\, du^{\alpha} = 0$$

and hence, there exists a constant $K_{11} > 0$ such that

$$(4.13) \qquad \begin{aligned} &|\psi(t,r) - \psi(s,q)| \\ &\quad \le K_{11}\Big(1 + \sqrt{\log((\eta/K_5) \vee 1)}\Big)\rho((t,r),(s,q))^{\alpha'}. \end{aligned}$$

As $\alpha$ is an arbitrary constant which is smaller than $(\gamma - d)/2\gamma$, we may assume that (4.13) holds for $\alpha$. Hence

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \varepsilon \log P\big([\psi]_\alpha > 1/\sqrt{\delta\varepsilon}\big) \\ &\quad \le \limsup_{\varepsilon\to 0} \varepsilon \exp P\Big(K_{11}\Big(1 + \sqrt{\log((\eta/K_5) \vee 1)}\Big) > 1/\sqrt{\delta\varepsilon}\Big) \\ &\quad = \limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\eta > K_5 \exp\Big[\big(1/\big(\sqrt{\delta\varepsilon}K_{11}\big) - 1\big)^2\Big]\Big) \\ &\quad \le \limsup_{\varepsilon\to 0} \varepsilon \log\Big\{K_5^{-1} \exp\Big[-\big(1/\big(\sqrt{\delta\varepsilon}K_{11}\big) - 1\big)^2\Big]E\eta\Big\} = -1/\delta K_{11}^2. \end{aligned}$$

Taking $\delta = 1/LK_{11}^2$, the lemma is proved. □

Let $0 < \mu < (\gamma - d)/2\gamma$. Using the notation of Section 2, Let $\mathscr{X} = \mathscr{Y} = C([0,T], \mathbb{B}_\mu)$ and let

$$(4.14) \qquad \begin{aligned} (A(x))(t,r) &= \int_{\mathscr{O}} G(t,0,r,q)\xi(q)\, dq \\ &\quad + \int_0^t \int_{\mathscr{O}} G(t,s,r,q) R(s,q,x(s,q))\, dq\, ds, \\ &\qquad\qquad \forall\, x \in \mathscr{X}. \end{aligned}$$

Let $\mathscr{A}_{\mathscr{X}}$ be the class of all $\mathbb{B}_\mu$-valued adapted continuous processes. It is easy to see that Assumption A2 holds. For $X \in \mathscr{A}_{\mathscr{X}}$, let

$$B(X)(t,r) = \int_0^t \int_{\mathscr{O}} G(t,s,r,q) F(s,q,X(s,q)) W(dq\, ds).$$

The next four propositions verify the conditions of Section 2 for the present setup.

Proposition 4.2. *The condition* (2.1) *holds for A.*

Proof. (i) For $x \in \mathscr{X}$, we denote the two terms on the right-hand side of (4.14) by $A_1$ and $A_2(x)$, respectively. It follows from (RD8) that $A_1 \in \mathscr{X}$. On the other hand, from (3.7) and (4.2) we can easily show that

$$(4.15) \qquad [A_2(x)]_\mu \le \sqrt{K_2 T|\mathscr{O}|}\, K(R,F,T)(1 + \|x\|_{T,0})$$

and hence by Lemma 4.4, $A_2(x) \in \mathscr{X}$ whenever $x \in \mathscr{X}$. Therefore, $A$ is a map from $\mathscr{X}$ to $\mathscr{X}$.

(ii) For $x, y \in \mathscr{X}$, it follows from (3.3) and (3.6) that

$$|(A(x) - A(y))(t,r)| \leq TK(T)K(R,F,T)\|x-y\|_{\mathscr{X}}.$$

By (3.6) and (4.3), we have

$$\begin{aligned}|(A(x) &- A(y))(t,r) - (A(x) - A(y))(t,q)|\\ &\leq \int_0^t \int_{\mathscr{O}} |G(t,s,r,u) - G(t,s,q,u)|\\ &\quad \times |R(s,q,x(s,u)) - R(s,q,y(s,u))|\, du\, ds\\ &\leq \sqrt{K_2''T|\mathscr{O}|}K(R,F,T)\|x-y\|_{\mathscr{X}}|r-q|^{\mu}.\end{aligned}$$

Hence

$$\|A(x) - A(y)\|_{\mathscr{X}} \leq \left(TK(T) + \sqrt{K_2''T|\mathscr{O}|}\right)K(R,F,T)\|x-y\|_{\mathscr{X}}. \qquad \square$$

PROPOSITION 4.3. *Assumption* A3 *holds for* $B$.

PROOF. For $X \in \mathscr{A}_{\mathscr{X}}$, let $\tilde{f}(s,q) = F(s,q,X(s,q))$. Then, by (RD7), $|\tilde{f}(s,q)| \leq K(R,F,T)$, for all $\omega$. Hence, it follows from Lemma 4.5 that $[B(X)]_{\mu} < \infty$ a.s. Therefore $B(X) \in \mathscr{X}$ a.s. By definition of the stochastic integral, $B(X)$ is adapted. Hence, $B$ is a map from $\mathscr{A}_{\mathscr{X}}$ to $\mathscr{A}_{\mathscr{X}}$. Let

$$\hat{B}(x,h)(t,r) = \int_0^t \int_{\mathscr{O}} G(t,s,r,q)F(s,q,x(s,q))\hat{h}(s,q)\, dq\, ds \qquad \forall\ x \in \mathscr{X}, h \in \mathscr{H}.$$

It is easy to see that for each $x \in \mathscr{X}$, $\hat{B}(x,\cdot)$: $\mathscr{H} \to \mathscr{X}$ is linear and the lifting $\tilde{B}(x,\cdot)$: $\Omega \to \mathscr{X}$ is given by

$$\tilde{B}(x,\cdot)(t,r) = \int_0^t \int_{\mathscr{O}} G(t,s,r,q)F(s,q,x(s,q))W(dq\, ds), \tag{4.16}$$

which is an element of $\mathscr{A}_{\mathscr{X}}$. This verifies (i) of Assumption A3.

Note that, by (3.6), (4.1) and Hölder's inequality,

$$\begin{aligned}&\left|\left(\hat{B}(x_1,h) - \hat{B}(x_2,h)\right)(t,r)\right|\\ &\leq \int_0^t \int_{\mathscr{O}} |G(t,s,r,q)|\\ &\quad \times |F(s,q,x_1(s,q)) - F(s,q,x_2(s,q))||\hat{h}(s,q)|\, dq\, ds\\ &\leq K(R,F,T)\\ &\quad \times \sqrt{\int_0^t \int_{\mathscr{O}} |G(t,s,r,q)|^2 |x_1(s,q) - x_2(s,q)|^2\, dq\, ds}\, \|h\|_{\mathscr{H}}\\ &\leq \sqrt{K_1}K(R,F,T)\|h\|_{\mathscr{H}}\|x_1 - x_2\|_{\mathscr{X}},\end{aligned}$$

and by (4.2),

$$\begin{aligned}
&\left|\left(\hat{B}(x_1,h)-\hat{B}(x_2,h)\right)(t,r)-\left(\hat{B}(x_1,h)-\hat{B}(x_2,h)\right)(t,q)\right| \\
&\quad\le \int_0^t\int_{\mathscr{O}}|G(t,s,r,u)-G(t,s,q,u)| \\
&\quad\quad\times|F(s,q,x_1(s,u))-F(s,q,x_2(s,u))| \\
&\quad\quad\times|\hat{h}(s,u)|\,du\,ds \\
&\quad\le \sqrt{\int_0^t\int_{\mathscr{O}}|G(t,s,r,u)-G(t,s,q,u)|^2\,du\,ds} \\
&\quad\quad\times K(R,F,T)\|x_1-x_2\|_{\mathscr{X}}\|h\|_{\mathscr{H}} \\
&\quad\le \sqrt{K_2}\,K(R,F,T)\|x_1-x_2\|_{\mathscr{X}}\|h\|_{\mathscr{H}}|r-q|^{\mu}.
\end{aligned}$$

Hence, there exists a constant $K$ such that

$$\|\hat{B}(x_1,h)-\hat{B}(x_2,h)\|_{\mathscr{X}}\le K\|h\|_{\mathscr{H}}\|x_1-x_2\|_{\mathscr{X}}. \tag{4.17}$$

That is, (ii) of Assumption A3 holds. Similarly, we have

$$\|\hat{B}(x,h_1)-\hat{B}(x,h_2)\|_{\mathscr{X}}\le K\|h_1-h_2\|_{\mathscr{H}}. \tag{4.18}$$

It follows from (4.17) and (4.18) that $\hat{B}$ is continuous in both variables. Condition (iii) of Assumption A3 follows immediately from (4.16). Condition (iv) is obvious by the linearity of the stochastic integral. Finally we verify the exponential continuity of $B$. Let $X, Y \in \mathscr{A}_{\mathscr{X}}$ (which may depend on $\varepsilon$) and let $\psi$ be given by Lemma 4.5 with

$$\tilde{f}(s,1)=\delta^{-1}(F(s,q,X(s,q))-F(s,q,Y(s,q)))1_{|X(s,q)-Y(s,q)|<\delta}.$$

Then

$$|\tilde{f}(s,q)|\le\delta^{-1}K(R,F,T)|X(s,q)-Y(s,q)|1_{|X(s,q)-Y(s,q)|<\delta}\le K(R,F,T).$$

Note that

$$\begin{aligned}
&P\left(\sqrt{\varepsilon}\|B(X,\omega)-B(Y,\omega)\|_{\mathscr{X}}>\sqrt{\delta},\ \|X-Y\|_{\mathscr{X}}<\delta\right)\\
&\quad\le P\left(\|\psi\|_{\mathscr{X}}>1/\sqrt{\delta\varepsilon}\right)\le P\left(K_6[\psi]_{\alpha}>1/\sqrt{\delta\varepsilon}\right),
\end{aligned}$$

and hence our result follows from Lemma 4.5. □

It is clear that for any $\varepsilon>0$, $X^{\varepsilon}\in\mathscr{A}_{\mathscr{X}}$ satisfies (2.4). That is, the second condition of Assumption A4 holds. The next proposition verifies the first one.

PROPOSITION 4.4. *$\{X^{\varepsilon}: \varepsilon>0\}$ is exponentially tight on $\mathscr{X}$.*

PROOF. Let $\mu<\alpha<(\gamma-d)/2\gamma$ and $C_L=\{x\in\mathscr{X}: [x]_{\alpha}\le M,\ x(0,\cdot)=\xi\}$. It is well known that $C_L$ is a compact subset of $\mathscr{X}$. Let $\psi$ be defined by Lemma 4.5 with $\tilde{f}(s,q)$ given by $F(s,q,X^{\varepsilon}(s,q))$. It follows from (3.7) that $\tilde{f}$

satisfies the condition in Lemma 4.5 with $K_8 = K(R, F, T)$. As

$$X_t^\varepsilon = (A_1)_t + A_2(X^\varepsilon)_t + \sqrt{\varepsilon}\,\psi_t, \tag{4.19}$$

by (3.7) and (4.1), we have

$$\begin{aligned} \|X_t^\varepsilon\|_0 &\equiv \sup\{|X^\varepsilon(t,r)|: r \in \mathscr{O}\} \\ &\le \|A_1\|_{t,0} + \sqrt{\varepsilon}\|\psi\|_{t,0} + \sup_{r\in\mathscr{O}} \int_0^t \int_{\mathscr{O}} |G(t,s,r,q) R(s,q,X^\varepsilon(s,q))|\,dq\,ds \\ &\le \|A_1\|_{T,0} + \sqrt{\varepsilon}\|\psi\|_{T,0} + K(R,F,T)\sqrt{K_1}\sqrt{\int_0^t |\mathscr{O}|(1 + \|X_s^\varepsilon\|_0)^2\,ds}\,. \end{aligned}$$

It follows from Gronwall's inequality that, for some constant $K_{12}$,

$$1 + \|X^\varepsilon\|_{T,0} \le K_{12}\big(1 + \|A_1\|_{T,0} + \sqrt{\varepsilon}\|\psi\|_{T,0}\big). \tag{4.20}$$

Further, by (4.15), (4.19) and (4.20), we have

$$\begin{aligned} [X^\varepsilon]_\alpha &\le [A_1]_\alpha + \sqrt{\varepsilon}[\psi]_\alpha \\ &\quad + \sqrt{K_2 T|\mathscr{O}|}\,K(R,F,T)K_{12}\big(1 + \|A_1\|_{T,0} + \sqrt{\varepsilon}\|\psi\|_{T,0}\big). \end{aligned}$$

As $A_1$ is fixed, by Lemma 4.4, we have

$$[X^\varepsilon]_\alpha \le K_{13} + K_{14}\sqrt{\varepsilon}[\psi]_\alpha,$$

where $K_{13}$ and $K_{14}$ are two constants. Taking $M$ such that

$$(M - K_{13})/K_{14} = 1/\sqrt{\delta},$$

it follows from Lemma 4.5 that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(X^\varepsilon \notin C_L) \le \limsup_{\varepsilon\to 0} \varepsilon \log P\big([\psi]_\alpha > 1\sqrt{\varepsilon\delta}\big) \le -L. \qquad \square$$

PROPOSITION 4.5. *Let $Z^\varepsilon$ be given by* (2.5). *Then $Z^\varepsilon$ tends to* 0 *in probability as $\varepsilon \to 0$.*

PROOF. Let $\psi$ be given as in the proof of Proposition 4.4. Then

$$\begin{aligned} Z^\varepsilon(t,r) = \int_0^t \int_{\mathscr{O}} & G(t,s,r,q)\big(R(s,q,Z^\varepsilon(s,q) + x(s,q)) \\ &\qquad - R(s,q,x(s,q))\big)\,dq\,ds \\ &+ \int_0^t \int_{\mathscr{O}} G(t,s,r,q)\big(F(s,q,Z^\varepsilon(s,q) + x(s,q)) \\ &\qquad - F(s,q,x(s,q))\big)\hat{h}(s,q)\,dq\,ds + \sqrt{\varepsilon}\,\psi(t,r). \end{aligned}$$

By (3.6) and Hölder's inequality, we have

$$\begin{aligned}\|Z_t^\varepsilon\|_0 &\equiv \sup\{|Z^\varepsilon(t,r)|: r\in\mathscr{O}\}\\ &\le \sqrt{\varepsilon}\|\psi_t\|_0\\ &\quad + K(R,F,T)\sup\left\{\int_0^t\int_{\mathscr{O}}|G(t,s,r,q)||Z^\varepsilon(s,q)|\,dq\,ds: r\in\mathscr{O}\right\}\\ &\quad + K(R,F,T)\|h\|_{\mathscr{H}}\\ &\qquad\times\sup\left\{\left(\int_0^t\int_{\mathscr{O}}|G(t,s,r,q)|^2|Z^\varepsilon(s,q)|^2\,dq\,ds\right)^{1/2}: r\in\mathscr{O}\right\}\\ &\equiv \sqrt{\varepsilon}\|\psi_t\|_0 + K(R,F,T)I_1 + K(R,F,T)\|h\|_{\mathscr{H}}\sqrt{I_2}.\end{aligned}$$

Note that, by Hölder's inequality and (4.1),

$$I_1 \le \sqrt{K_1|\mathscr{O}|}K(R,F,T)\left(\int_0^t\|Z_s^\varepsilon\|_0^2\,ds\right)^{1/2}.$$

On the other hand, it follows from (3.3)–(3.5) that

$$\begin{aligned}I_2 &\le \sup\left\{\int_0^t\int_{\mathscr{O}}|G(t,s,r,q)|K(T)(t-s)^{-d/\gamma}|Z^\varepsilon(s,q)|^2\,dq\,ds: r\in\mathscr{O}\right\}\\ &\le K(T)^2\int_0^t(t-s)^{-d/\gamma}\|Z_s^\varepsilon\|_0^2\,ds.\end{aligned}$$

Hence, there exists a constant $K_{15}$ such that

$$\|Z_t^\varepsilon\|_0^2 \le 3\varepsilon\|\psi\|_{T,0}^2 + K_{15}\int_0^t(t-s)^{-d/\gamma}\|Z_s^\varepsilon\|_0^2\,ds. \tag{4.21}$$

Applying (4.21) to $\|Z_s^\varepsilon\|_0^2$ on the right-hand side of (4.21), it is easy to show that

$$\begin{aligned}\|Z_t^\varepsilon\|_0^2 &\le 3\varepsilon\|\psi\|_{T,0}^2\left(1+(K_{15}/(1-d/\gamma))T^{1-d/\gamma}\right)\\ &\quad + 2K_{15}^2\int_0^t\|Z_s^\varepsilon\|_0^2\left(\frac{t-s}{2}\right)^{1-2d/\gamma}ds/(1-d/\gamma).\end{aligned}$$

If $1-2d/\gamma\ge 0$, we stop here; otherwise, as $1-2d/\gamma > -d/\gamma$, continuing the above estimate we will find two constants $K_{16}$ and $K_{17}$ such that

$$\|Z_t^\varepsilon\|_0^2 \le \varepsilon K_{16}\|\psi\|_{T,0}^2 + K_{17}\int_0^t\|Z_s^\varepsilon\|_0^2\,ds.$$

It follows from the Gronwall inequality that

$$\|Z^\varepsilon\|_{T,0}^2 = \sup\left\{\|Z_t^\varepsilon\|_0^2: t\in[0,T]\right\} \le \varepsilon K_{18}\|\psi\|_{T,0}^2, \tag{4.22}$$

where $K_{18}$ is a constant. By (3.6), (4.2) and Hölder's inequality, we have

$$\left[Z^\varepsilon - \sqrt{\varepsilon}\,\psi\right]_\alpha \le \sqrt{K_2}\left(\sqrt{T|\mathscr{O}|} + \|h\|_{\mathscr{H}}\right)K(R,F,T)\|Z^\varepsilon\|_{T,0}. \tag{4.23}$$

From (4.22), (4.23) and Lemma 4.4, we have

$$\|Z^\varepsilon\|_{T,\alpha} \le \sqrt{\varepsilon} K_{19}[\psi]_\alpha \qquad (K_{19} \text{ a constant})$$

and our result then follows from Lemma 4.5. □

We summarize our results as the main result of this paper.

THEOREM 4.1. *Suppose that* $\mathscr{O}$ *satisfies the cone condition. Then, under the assumptions* (RD1)–(RD8), $\{X^\varepsilon\}$ *satisfies the large deviation principle on* $C([0,T],\mathbb{B}_\mu)$ *with rate function* $I$ *given by*

$$I(x) = \inf\left\{\frac{1}{2}\int_0^T \int_{\mathscr{O}} |\hat{h}(s,r)|^2 \, ds\, dr : h \in \mathscr{H} \text{ such that } x = \gamma(h)\right\}, \tag{4.24}$$

*where* $\gamma$ *is a map from* $\mathscr{H}$ *into* $C([0,T],\mathbb{B}_\mu)$ *given by*

$$\begin{aligned} x(t,r) = &\int_{\mathscr{O}} G(t,0,r,q)\xi(q)\,dq \\ &+ \int_0^t \int_{\mathscr{O}} G(t,s,r,q)\big(R(s,q,x(s,q)) \\ &\qquad + F(s,q,x(s,q))\hat{h}(s,q)\big)\,dq\,ds. \end{aligned} \tag{4.25}$$

PROOF. We only need to prove that $\gamma$ is well defined with $\mathscr{D}(\gamma) = \mathscr{H}$. First we consider $h \in \mathscr{H}$ such that $\|\hat{h}\|_{T,0} < \infty$. For $(t,r,x) \in [0,T] \times \mathscr{O} \times \mathbb{R}$, let

$$\tilde{R}(t,r,x) = R(t,r,x) + F(t,r,x)\hat{h}(t,r) \quad \text{and} \quad \tilde{F}(t,r,x) = 0.$$

Then

$$\begin{aligned} |\tilde{R}(t,r,x) - \tilde{R}(t,r,y)| &\le |R(t,r,x) - R(t,r,y)| \\ &\quad + |F(t,r,x) - F(t,r,y)|\,|\hat{h}(t,r)| \\ &\le K(R,F,T)\big(1 + \|\hat{h}\|_{T,0}\big)|x-y| \end{aligned}$$

and

$$\begin{aligned} |\tilde{R}(t,r,x)| &\le |R(t,r,x)| + |F(t,r,x)|\,|\hat{h}(t,r)| \\ &\le K(R,F,T)\big(1 + \|\hat{h}\|_{T,0}\big)(1 + |x|). \end{aligned}$$

It follows from Theorem 3.1 that (4.25) has a unique solution in $\mathscr{X}$.

For general $h$, let $h^n \in \mathscr{H}$ such that $\|\hat{h}^n\|_{T,0} < \infty$, $\forall\ n \ge 1$, and $\|h^n - h\|_{\mathscr{H}} \to 0$ as $n \to \infty$. Let $x^n = \gamma(h^n)$. Then

$$\begin{aligned} x^n(t,r) = &\int_{\mathscr{O}} G(t,0,r,q)\xi(q)\,dq \\ &+ \int_0^t \int_{\mathscr{O}} G(t,s,r,q)\big(R(s,q,x^n(s,q)) \\ &\qquad + F(s,q,x^n(s,q))\hat{h}^n(s,q)\big)\,dq\,ds. \end{aligned} \tag{4.26}$$

Similar to the proof of Proposition 4.5, there exists a constant $K_{20}$ such that $\|x^n\|_{\mathscr{X}} \le K_{20}$ $\forall\ n \ge 1$. As

$$\begin{aligned}(x^n - x^m)(t,r) &= \int_0^t \int_{\mathscr{O}} G(t,s,r,q)(R(s,q,x^n(s,q)) - R(s,q,x^m(s,q)))\,dq\,ds \\ &\quad + \int_0^t \int_{\mathscr{O}} G(t,s,r,q)(F(s,q,x^n(s,q)) \\ &\qquad\qquad -F(s,q,x^m(s,q)))\hat{h}^n(s,q)\,dq\,ds \\ &\quad + \int_0^t \int_{\mathscr{O}} G(t,s,r,q)F(s,q,x^m(s,q))\big(\hat{h}^n(s,q) - \hat{h}^m(s,q)\big)\,dq\,ds,\end{aligned}$$

similar to the proof of Proposition 4.5 again, there exists a constant $K_{21}$ such that

$$\|x^n - x^m\|_{\mathscr{X}} \le K_{21}\|h^n - h^m\|_{\mathscr{X}} \qquad \forall\ n,m \ge 1. \tag{4.27}$$

Hence $x^n$ converges, say to $x$, in $\mathscr{X}$. By (4.26), $x$ is a solution of (4.25). The uniqueness of the solution of (4.25) follows from (4.27) directly. □

**5. Reaction-diffusion SPDE's.** Now we apply our results to a class of reaction-diffusion SPDE's. In this case, $\{L(t)\}$ is a family of second order (i.e., $\gamma = 2$) differential operators. Let $d = 1$ and $\mathscr{O} = (0,l)$. Let $\{X^\varepsilon\}$ be the solution of (3.1).

THEOREM 5.1. *Suppose that $\{L(t)\}$ generates a two-parameter evolution semi-group $\{U(t,s)\colon\ 0 \le s \le t\}$ on $C([0,l])$ which has kernel function $G(t,s,r,q)$, $0 \le s < t$, $0 < r,q < l$, satisfying the following conditions*:

(i) *For all $\alpha < \frac{1}{4}$, $\exists$ a constant $K$ s.t. for all $0 \le t_1, t_2 \le T$ and $0 \le r_1, r_2 < l$, we have*

$$\int_0^T \int_0^l |G(t_1,s,r_1,q) - G(t_2,s,r_2,q)|^2\,dq\,ds \le K\big(|t_1 - t_2|^2 + |r_1 - r_2|^2\big)^\alpha.$$

(ii) *For all $0 \le s < t \le T$, we have*

$$\int_0^l |G(t,s,r,q)|^2\,dq \le K(t-s)^{-1/2}.$$

(iii) (RD8) *holds.*

*We also assume that $R$ and $F$ satisfy the condition* (RD7). *Then for any $\mu < \frac{1}{4}$, $\{X^\varepsilon\}$ satisfies* LDP *with rate function*

$$I(x) = \inf\left\{\tfrac{1}{2}\int_0^T \int_0^l |\hat{h}(t,r)|^2\,dt\,dr\colon\ \hat{h} \in L^2([0,T]\times(0,l)) \right.$$
$$\left. s.t.\ (4.25)\ holds\ with\ \mathscr{O} = (0,l)\right\}.$$

REMARK 5.1. The existence and uniqueness for the solution of the SPDE (3.1) under the conditions of Theorem 5.1 follow from the same arguments as those in Kotelenez's proof of Theorem 3.1. The LDP of $\{X^\varepsilon\}$ can be obtained by noticing that our arguments in the previous section are still true under the conditions of Theorem 5.1.

REMARK 5.2. The conditions (i)–(iii) hold for most parabolic operators $\partial/\partial t - L(t)$ [cf. Friedman (1964)].

Recently, Sowers (1992) derived the LDP for a class of reaction-diffusion equations. We shall now obtain his result as a special case of Theorem 5.1. Sowers (1992) considered the following stochastic reaction-diffusion equation:

$$\begin{aligned}\frac{\partial}{\partial t}v^\varepsilon(t,x) &= \left(D_1\frac{\partial^2}{\partial x^2} - D_2\right)v^\varepsilon(t,x) + f(x, v^\varepsilon(t,x)) \\ &\quad + \sqrt{\varepsilon}\,\sigma(x, v^\varepsilon(t,x))\frac{\partial^2}{\partial t\,\partial x}W,\end{aligned} \tag{5.1}$$

for $t \in [0,T]$ and $x \in (0, 2\pi)$ with a periodic boundary condition. Here $D_1$ and $D_2$ are two constants. The following conditions were imposed: there exist constants $F$, $\bar{f}$, $m$, $M$ and $\bar{\sigma}$ such that, for any $x \in [0, 2\pi]$ and $y, z \in \mathbb{R}$, we have

$$|f(x,y)| \le F(1 + |y|), \qquad |f(x,y) - f(x,z)| \le \bar{f}|Vy - z| \tag{5.2}$$

and

$$0 < m \le \sigma(x,y) \le M, \qquad |\sigma(x,y) - \sigma(x,z)| \le \bar{\sigma}|y - z|. \tag{5.3}$$

Sowers proved that $\{v^\varepsilon\}$ satisfies the large deviation principle in $C([0,T], \mathbb{B}_\mu)$ (for any $\mu < \frac{1}{4}$) with rate function $S_\xi$ given by

$$\begin{aligned}&S_\xi(\phi) \\ &= \frac{1}{2}\int_0^T\int_0^{2\pi}\left|\frac{\partial}{\partial t}\phi(t,x) - \left(D_1\frac{\partial^2}{\partial x^2} - D_2\right)\phi(t,x) - f(x, \phi(t,x))\right|^2 \frac{dt\,dx}{\sigma(x, \phi(t,x))^2},\end{aligned} \tag{5.4}$$

if $\phi \in W_2^{1,2}$ and $\phi(0, \cdot) = \xi$; otherwise, $S_\xi(\phi) = \infty$.

Now we discuss the conditions of Theorem 5.1 for this special case. The condition (RD7) follows from (5.2) and (5.3) directly. It is clear that the kernel function $G$ is given by

$$G(t,s,r,q) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{4\pi D_1(t-s)}}\exp\left(-\frac{(r - q - 2\pi k)^2}{4D_1(t-s)} - D_2(t-s)\right).$$

Condition (i) of Theorem 5.1 has been verified by Sowers [(1992), Appendix]. Note that

$$\int_0^{2\pi}|G(t,s,r,q)|\,dq = \exp(-D_2(t-s)) \le \exp(|D_2|T)$$

and

$$\sqrt{t-s}\,G(t,s,r,q) \le \frac{\exp(|D_2|T)}{\sqrt{4\pi D_1}}\left(1+\sum_{k=1}^{\infty}\exp\left(-\frac{(2\pi(k-1))^2}{4D_1T}\right)\right) \equiv C.$$

Hence

$$\int_0^{2\pi}|G(t,s,r,q)|^2\,dq \le C(t-s)^{-1/2}\exp(|D_2|T).$$

That is, condition (ii) holds. Finally, if we extend $\xi$ to $\mathbb{R}$ to be periodic, then

$$\int_0^{2\pi}G(t,0,r,q)\xi(q)\,dq = \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1}}\exp\left(-\frac{x^2}{4D_1}\right)\xi(\sqrt{t}\,x+r)\,dx,$$

and hence condition (RD8) follows.

Based on the discussion above, we see that $\{v^\varepsilon\}$ satisfies the LDP. It remains to show that the rate function $I$ given by Theorem 5.1 coincides with the function $S_\xi$ defined by (5.4).

PROPOSITION 5.1. *For any $\phi \in C([0,T],\mathbb{B}_\mu)$, we have $S_\xi(\phi) = I(\phi)$, which is given by Theorem* 4.1.

PROOF. If $S_\xi(\phi) < \infty$, then $\phi \in W_2^{1,2}$, $\phi(0,\cdot) = \xi$ and

$$\hat{h}(t,x) \equiv \frac{\{(\partial/\partial t)\phi(t,x) - (D_1(\partial^2/\partial x^2) - D_2)\phi(t,x) - f(x,\phi(t,x))\}}{\sigma(x,\phi(t,x))} \tag{5.5}$$

is in $L^2([0,T]\times\mathscr{O})$. Note that (5.5) implies that

$$\begin{aligned}\phi(t,r) &= \int_{\mathscr{O}}G(t,0,r,q)\xi(q)\,dq \\ &\quad + \int_0^t\int_{\mathscr{O}}G(t,s,r,q) \\ &\qquad \times\{f(q,\phi(s,q)) + \sigma(q,\phi(s,q))\hat{h}(s,q)\}\,dq\,ds.\end{aligned} \tag{5.6}$$

Hence $I(\phi) < \infty$.

On the other hand, if $I(\phi) < \infty$, then, for any $\delta > 0$, there exists $\hat{h} \in L^2([0,T]\times\mathscr{O})$ such that (5.6) holds and

$$\tfrac{1}{2}\int_0^T\int_{\mathscr{O}}|\hat{h}(s,r)|^2\,ds\,dr \in [I(\phi), I(\phi)+\delta].$$

It is easy to see that $\hat{h}$ is uniquely determined and coincides with the right-hand side of (5.5) [refer to Walsh (1984) for details] and hence,

$$\tfrac{1}{2}\int_0^T\int_{\mathscr{O}}|\hat{h}(s,r)|^2\,ds\,dr = I(\phi).$$

Therefore, $S_\xi(\phi) < \infty$ and $S_\xi(\phi) = I(\phi)$. □

**6. Concluding remarks.** Now we compare our conditions with those imposed by Peszat (1994). In that paper, Peszat considers the large deviation problem under a more abstract setup. Namely, he considers the stochastic evolution equation

$$dX^\varepsilon = (-LX^\varepsilon + F(X^\varepsilon, t))\,dt + \varepsilon R(X^\varepsilon)\,dW, \qquad X^\varepsilon(0) = x \in E,$$

where $-L$ is the generator of a $C_0$-semigroup $\{S(t)\}$ on a Hilbert space $H$, $F$: $E \times [0,\infty) \to H$, $G$: $E \to L(H,H)$ and $E$ is a Banach space densely embedded in $H$ [he used $G(X^\varepsilon)$ in the diffusion term instead of $R(X^\varepsilon)$; we change his notation to avoid confusion with our kernel function $G$].

Specialized to our case, $H = L^2(\mathscr{O})$, $E = \mathbb{B}_\mu$ and

$$(S(t)h)(r) = \int_{\mathscr{O}} G(t,0,r,q)h(q)\,dq.$$

Among other conditions, he assumed the following:

(E.3) There exist $\alpha_0 \in (0, \frac{1}{2})$ and $p_0 > 1$ such that

$$\int_0^T t^{(\alpha_0 - 1)p_0} \|S(t)\|_{L(H,E)}^{p_0}\,dt < \infty.$$

(C.2)
$$\int_0^T t^{-2\alpha_0} \sup_{h \in E} \|S(t)R(h)\|_2^2\,dt < \infty,$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm from $H$ to $H$.

Note that, in our setup, $R(h)$ is a multiplicative operator given by $(R(h)f)(r) = R(h(r))f(r)$,

$$\begin{aligned} &\|S(t)\|_{L(H,E)}^2 \\ &\quad = \sup_{r_1, r_2 \in \mathscr{O}} |r_1 - r_2|^{-2\mu} \int_{\mathscr{O}} |G(t,0,r_1,q) - G(t,0,r_2,q)|^2\,dq \end{aligned} \tag{6.1}$$

and

$$\|S(t)R(h)\|_2^2 = \int_{\mathscr{O}}\int_{\mathscr{O}} G(t,0,r,q)^2 R(h(q))^2\,dq\,dr. \tag{6.2}$$

When $L(t)$ is the closure of an elliptic operator of order $\gamma = 2m$, $G(t,s,r,q) \sim (t-s)^{-d/2m}$ as $t \to s$ [cf. Eidel'man (1969)]. Hence, the right-hand sides of both (6.2) and (6.1) are equivalent to $t^{-d/2m}$, as $t \to 0$. Therefore, the conditions (E.3) and (C.2) together are "equivalent" to

$$(\alpha_0 - 1)p_0 - \frac{d}{2\gamma}p_0 > -1 \quad \text{and} \quad -2\alpha_0 - \frac{d}{\gamma} > -1.$$

That is,

$$1 - \frac{1}{p_0} + \frac{d}{2\gamma} < \alpha_0 \le \frac{1}{2}\left(1 - \frac{d}{\gamma}\right).$$

Hence $\gamma$ has to be greater than $2d$, which is stronger than our assumption that $\gamma > d$. This explains why Sowers' case is not covered by Peszat's result since, in this case, $\gamma = 2$ and $d = 1$.

Peszat's condition (C.2) requires the Hilbert–Schmidt property for $R$ [composed with $S(t)$]. Chow assumes that $R$ ($\Sigma$ in his notation) composed with the covariance operator of the noise $W$, is Hilbert–Schmidt. Sowers' case is thus not covered by Chow's results since, in this case, the covariance operator is the identity and $R$ is a multiplicative operator which is not Hilbert–Schmidt.

Chow considers the Hilbert space case [$H = L^2(\mathscr{O})$] and works with a strong solution of (3.1), while we obtain large deviation results with respect to the Hölder continuity topology for the mild solution of (3.1). Moreover, the generator $A$ [corresponding to $L(t)$ of our paper] is assumed to be self-adjoint.

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