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A STRONG INVARIANCE PRINCIPLE FOR ASSOCIATED SEQUENCES[1]

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By combining the Berkes–Philipp blocking technique and the Csörgő–Révész quantile transform methods, we find that partial sums of an associated sequence can be approximated almost surely by partial sums of another sequence with Gaussian marginals. A crucial fact is that this latter sequence is still associated with covariances roughly bounded by the covariances of the original sequence, and that one can approximate it by an iid Gaussian process using the Berkes–Philipp method. We require that the original sequence has finite $(2 + r)$th moments, $r > 0$, and a power decay rate of a coefficient $u(n)$ which describes the covariance structure of the sequence. Based on this result, we obtain a strong invariance principle for associated sequences if $u(n)$ exponentially decreases to 0.

1. Introduction. Random variables, $X_1, \dots, X_n$, are associated if, for any two coordinatewise nondecreasing functions $f, g\colon R^n \to R$,

$$\operatorname{Cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0$$

whenever the covariance is defined. A sequence $\{X_n,\ n \geq 1\}$ is associated if every finite subcollection is associated. This definition was introduced by Esary, Proschan and Walkup (1967) and has found several applications, for example, in reliability theory [Barlow and Proschan (1981)], in mathematical physics [Newman (1980, 1983)] and in percolation theory [Cox and Grimmett (1984)].

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables on some probability space $(\Omega, \mathscr{F}, P)$ with $EX_n = 0$ and $EX_n^2 < \infty$. For each $n \geq 1$, let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = ES_n^2$. Under appropriate covariance conditions, a number of limit theorems have been proved for associated sequences. The first important result was the central limit theorem (CLT) proved by Newman (1980). He stated that if $\{X_n,\ n \geq 1\}$ is strictly stationary, associated and

$$0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} EX_1X_i < \infty, \tag{1.1}$$

then

$$\frac{S_n}{\sigma_n} \to_{\mathscr{D}} N(0,1) \quad \text{as } n \to \infty.$$

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Later Newman and Wright (1981) obtained the functional central limit theorem (FCLT) under the same covariance condition (1.1). They showed that

$$\frac{S_{[nt]}}{\sigma_n} \to_{\mathscr{D}} W(t) \quad \text{in } D[0,1],$$

where $\{W(t),\ t \in [0,1]\}$ is the standard Wiener process and $D[0,1]$ is the usual D space on $[0,1]$ with the Skorokhod J_1-topology [cf. Billingsley (1968)]. Further results are the law of the iterated logarithm (LIL) [Yu (1986)], the functional law of the iterated logarithm (FLIL) [Dabrowski and Dehling (1988)], the Berry–Esseen inequality [Wood (1983); Dabrowski and Dehling (1988); Birkel (1988a)], local limit theorems [Wood (1985)], the Glivenko–Cantelli lemma and weak convergence for empirical processes [Yu (1993)], extensions to nonstationary cases [Cox and Grimmett (1984); Yu (1985); Birkel (1987)] and extensions to weakly associated sequences [Burton, Dabrowski and Dehling (1986)].

Note that in the above-mentioned limit theorems, the strong invariance principle, or strong FCLT, is missing. It is well known that the CLT and FCLT, as well as the FLIL and other asymptotic fluctuation results, can be derived from a strong invariance principle for partial sums of a sequence [for the details see Theorems A–E in Section 1 of Philipp and Stout (1975)]. In addition, one can also obtain the Chung type of LIL from it, that is,

$$\liminf_{n \to \infty} \left[\frac{8 \log \log n}{\pi^2 \sigma_n^2} \right]^{1/2} \sup_{1 \le i \le n} |S_i| = 1 \quad \text{a.s.} \tag{1.2}$$

Hence a natural question is whether the strong invariance principle holds for a sequence of associated random variables under appropriate covariance conditions. Normally, the Berkes–Philipp blocking technique is used for dependent sequences such as mixing sequences [cf. Berkes and Philipp (1979)]. Then strong invariance principles can be obtained by approximation theorems based on estimates of the Prohorov distance and the Strassen–Dudley theorem, and conditional expectation inequalities [see the detailed discussions in Philipp (1986)]. Unfortunately, this approach is not suitable for associated sequences because of their special dependence structure implied by association. The main reason, in our opinion, is due to the lack of powerful conditional expectation inequalities for associated sequences. This prevents us from using the Skorohod embedding technique directly for associated sequences as well.

The purpose of this paper is to provide a different approach to obtaining a strong invariance principle for associated sequences. While the blocking technique is still used, we choose the quantile transform method so that we do not need to find a conditional expectation inequality for the constructed block sequence. The quantile transform method was first developed by Csörgő and Révész (1975a, b) and later refined by Kómlos, Major and Tusnady (1975, 1976). This method provides a powerful tool for establishing many optimal strong invariance pricniples for independent sequences. However, as Philipp

(1986) points out, this method seems to work with independent random variables (vectors) only. Fortunately, we find that this method is well suited for associated random variables because of their unique positive dependence structure. Indeed, with the quantile transform, an approximation of the original sequence by a sequence with Gaussian marginals is achieved (cf. Theorem 2.4). A crucial fact is that this latter sequence is still associated with covariances roughly bounded by the covariances of the original sequence, and that one can approximate it by an iid Gaussian process using the Berkes–Philipp method. Hence, a strong invariance principle is obtained for associated sequences (cf. Theorem 2.5) under the exponential decay rate of the original covariances.

The exact results are stated in Section 2. Some open problems are also discussed in Section 2. The proofs of our theorems and some lemmas will be given in Section 3.

2. Results. The covariance coefficient of an associated sequence $\{X_n, n \geq 1\}$, according to Cox and Grimmett (1984), is defined as

$$u(n) = \sup_{k \geq 1} \sum_{j: |j-k| \geq n} \operatorname{Cov}(X_j, X_k).$$

To use the blocking technique, we define blocks H_k and I_k of consecutive positive integers, leaving no gaps between the blocks. The order is $H_1, I_1, H_2, I_2, \dots$. The lengths of the blocks are defined by

$$\operatorname{card}\{H_k\} = [k^\alpha], \qquad \operatorname{card}\{I_k\} = [k^\beta]$$

for some suitable chosen real numbers $\alpha > \beta > 0$ with $\operatorname{card}\{K\}$ standing for the number of integers in K. Put

$$N_k = \sum_{i=1}^{k} \operatorname{card}\{H_i \cup I_i\} = \sum_{i=1}^{k} ([i^\alpha] + [i^\beta]) \sim \frac{1}{1+\alpha} k^{1+\alpha}, \tag{2.1}$$

$$\begin{aligned} u_k &= \sum_{i \in H_k} X_i, \qquad \lambda_k^2 = E u_k^2, \\ v_k &= \sum_{i \in I_k} X_i, \qquad \tau_k^2 = E v_k^2, \qquad k \geq 1, \end{aligned} \tag{2.2}$$

where u_k and v_k are called the long blocks and the short blocks, respectively.

Before we use the quantile transform method, we need to introduce a sequence of Gaussian random variables which are used to smooth the long block sequence $\{u_k, k \geq 1\}$. Let $\{w_k, k \geq 1\}$ be a sequence of independent $N(0, \tau_k^2/2)$-distributed random variables which is also independent of $\{u_k, k \geq 1\}$. Put

$$\xi_k = (u_k + w_k)/(\lambda_k^2 + \tau_k^2/2)^{1/2}, \qquad k \geq 1. \tag{2.3}$$

Let F_k denote the distribution function of ξ_k. Note that F_k is continuous since the smooth random variable w_k is used. Now we can use the quantile transform method to construct a new associated sequence with Gaussian marginals. Define

$$\eta_k = \Phi^{-1}(F_k(\xi_k)), \qquad k \geq 1, \tag{2.4}$$

where Φ^{-1} is the inverse of the standard Gaussian distribution function Φ. Since $P\{\eta_k \leq x\} = P\{F_k(\xi_k) \leq \Phi(x)\} = \Phi(x)$, we conclude that each η_k is a standard Gaussian random variable. The following theorem verifies that $\{\eta_k, k \geq 1\}$ is an associated sequence and its covariances are controlled by that of the original sequence.

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be an associated sequence satisfying $EX_n = 0$,*

$$\sup_{n \geq 1} E|X_n|^{2+r+\delta} < \infty \quad \text{for some } r, \delta > 0 \tag{2.5}$$

and

$$\inf_{n \geq 1,\, k \geq 0} E(S_{n+k} - S_k)^2/n > 0. \tag{2.6}$$

Assume

$$u(n) = O(n^{-\gamma}), \qquad \gamma = r(2 + r + \delta)/2\delta > 1. \tag{2.7}$$

Then the sequence $\{\eta_k, k \geq 1\}$ constructed from (2.4) is associated with a common distribution $N(0,1)$. If moreover $5\beta/3 > \alpha > \beta > 0$, we get for any $0 < \theta < 1/2$ and all $i \neq j$ that

$$0 \leq E\eta_i\eta_j \leq C\Big((ij)^{-\alpha/2} Eu_i u_j\Big)^{\theta/(1+\theta)}, \tag{2.8}$$

where C is a constant not depending on i, j.

REMARK 2.2. The association of $\{\eta_k, k \geq 1\}$ is purely based on the facts that $\{\xi_k, k \geq 1\}$ constructed from (2.3) is associated by applying (P_2) and (P_4) of Esary, Proschan and Walkup (1967) and the fact that $\Phi^{-1}(F_k(\cdot))$ is an increasing function.

REMARK 2.3. Conditions (2.5) and (2.7), based on Theorem 1 of Birkel (1988b), are mainly used to get moment bounds:

$$\sup_{k \geq 0} E|S_{n+k} - S_k|^{2+r} = O(n^{(2+r)/2}). \tag{2.9}$$

The requirement $\gamma > 1$ is for $\sum_{n=1}^{\infty} u(n) < \infty$. By using the nonnegative covariance property of association, (2.6) can be replaced by a simple condition, $\inf_{n \geq 1} EX_n^2 > 0$, which is used by Cox and Grimmett (1984) for their CLT.

THEOREM 2.4. *Let* $\{X_n, n \geq 1\}$ *be an associated sequence satisfying* $EX_n = 0$ *and* (2.5)–(2.7). *Then there exist real numbers* $\alpha > \beta > 1$ *and some* $\varepsilon > 0$ *such that, for k satisfying* $N_k < N \leq N_{k+1}$,

$$\left| S_N - \sum_{i=1}^{k} \left(\lambda_i^2 + \tau_i^2/2\right)^{1/2} \eta_i \right| \leq CN^{1/2-\varepsilon} \quad a.s.,$$

where C is a constant not depending on N.

Based on Theorem 2.4, we can establish the following strong invariance principle for associated sequences.

THEOREM 2.5. *Let* $\{X_n, n \geq 1\}$ *be an associated sequence satisfying* $EX_n = 0$, (2.5) *and* (2.6). *Assume*

$$u(n) = O(e^{-\lambda n}) \quad \text{for some } \lambda > 0. \tag{2.10}$$

Then without changing its distribution we can redefine the sequence $\{X_n, n \geq 1\}$ *on a richer probability space together with a standard Wiener process* $\{W(t), t \in [0,\infty)\}$ *such that, for some* $\varepsilon > 0$,

$$S_N - W(\sigma_N^2) = O(N^{1/2-\varepsilon}) \quad a.s.$$

REMARK 2.6. The condition (2.10) in Theorem 2.5 can be slightly weakened to the condition

$$u(n) = O(e^{-\lambda n^{\mu}}) \quad \text{for some } \lambda > 0 \text{ and } \mu > 0. \tag{2.11}$$

Theorem 2.5 implies that Chung's LIL (1.2) holds for associated sequences.

COROLLARY 2.7. *Under the assumptions of Theorem* 2.5, *we have*

$$\liminf_{n\to\infty} \left[\frac{8 \log\log n}{\pi^2 \sigma_n^2}\right]^{1/2} \sup_{1 \leq i \leq n} |S_i| = 1 \quad a.s.$$

By comparing the decay rates of $u(n)$ in Theorems 2.4 and 2.5, one immediately notices their huge differences, even when the condition (2.11) is used. The reason is that in proving Theorem 2.5 we use Theorem 5 of Berkes and Philipp (1979), the only way so far to compute conditional expectations based on characteristic functions. Association indeed has some decent tools, for example, the inequality of characteristic functions [Newman (1980)] and the maximal inequality [Newman and Wright (1981)], but they are not sufficient to estimate the conditional expectation $E[\eta_{n+k} | \eta_1, \dots, \eta_k]$ directly under a power decay rate. If, however, $\{\eta_k, k \geq 1\}$ is a jointly Gaussian sequence, then we can at once get the conclusion that the strong invariance principle holds for associated sequences with a power decay rate of covariances [cf. Theorem 5.1 of Philipp and Stout (1975)].

Pitt (1982) proves that positively correlated random variables with jointly Gaussian distributions are associated. Hence one may guess that the associated sequence $\{\eta_k, k \geq 1\}$ constructed by the quantile transform method must have Gaussian joint distributions. Unfortunately, this is not true. Pitt (personal communication) has given a counterexample to show that there exist associated random variables with Gaussian marginals which do not have jointly Gaussian distributions. Here is his example.

Let $\{W_1(t),\ t \in [0,\infty)\}$ and $\{W_2(t),\ t \in [0,\infty)\}$ be two independent Wiener processes with $W_1(0) = W_2(0) = 0$. Let $\tau = \inf\{t > 0: W_1(t) = W_2(t)\}$ and define two new processes

$$X(t) = W_1(t) \quad \text{for all } t \ge 0,$$

$$Y(t) = \begin{cases} W_2(t), & \text{for } t < \tau, \\ W_1(t), & \text{for } t \ge \tau. \end{cases}$$

By the strong Markov property one can see that $\{Y(t),\ t \in [0,\infty)\}$ is still a Wiener process. Thus, by the coupling arguments that are used to prove association, it is easy to see that $X(1)$ and $Y(1)$ are associated, while both $X(1)$ and $Y(1)$ are Gaussian distributed. However, $X(1)$ and $Y(1)$ do not have jointly Gaussian distributions since $P\{X(1) = Y(1)\} = P\{\tau < 1\} > 0$.

Therefore we should look for some different ways to overcome this setback for association. First of all, it is natural to ask whether a jointly Gaussian sequence can be constructed directly by refining the quantile transform method and whether its covariances can be estimated by that of the original sequence. Second, it may be possible, based on $\{\eta_k,\ k \ge 1\}$, to construct a new (jointly) Gaussian sequence $\{\eta'_k,\ k \ge 1\}$ with the same covariances so that $E(\eta_k - \eta'_k)^2$ converges to zero with a certain speed as $k \to \infty$. If so, the strong invariance principle holds for associated sequences with a power decay rate of covariances. Nevertheless, this is an open problem for association.

3. Proofs. We shall first give an estimate for the difference of the characteristic function of S_n/σ_n and that of the standard Gaussian distribution $N(0,1)$. This is essential for us to use the quantile transform method successfully. Although Birkel (1988a) obtains the sharpest rates for this difference under bounded third moments and exponential decay rate of $u(n)$, his results cannot be applied in our case since we may not have these conditions satisfied. So we present the following proposition. Since its proof is quite routine, we leave the proof to the Appendix, as well as the proof of Lemma 3.2.

From now on, without loss of generality, we assume that $0 < r \le 1$ and C stands for a generic positive constant, independent of t and n. It may, however, take different values in each appearance.

PROPOSITION 3.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\begin{aligned} &|E \exp\{itS_n/\sigma_n\} - \exp\{-t^2/2\}| \\ &\qquad \le C\big(t^2/p_n + (p_n/n)^{r/2}(|t| + t^2 + |t|^{2+r})\exp\{-t^2/12\}\big) \end{aligned}$$

for all $|t| \le C(n/p_n)^{1/2}$, *where* $\{p_n,\ n \ge 1\}$ *is an integer sequence satisfying* $0 < p_n < n$ *and* $p_n \to \infty$ *as* $n \to \infty$.

Recall that F_k is the distribution function ξ_k defined in (2.3). Since

$$\int_{-\infty}^{\infty} |E\exp\{it\xi_k\}|\,dt \le \int_{-\infty}^{\infty} \exp\left\{-\frac{\tau_k^2 t^2}{2(2\lambda_k^2+\tau_k^2)}\right\} dt < \infty,$$

the density function of ξ_k exists and is given by

$$f_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-itx\}E\exp\{it\xi_k\}\,dt. \tag{3.1}$$

LEMMA 3.2. *Under the assumptions of Theorem* 2.1, *we have*

$$\sup_{-\infty<x<\infty} |F_k(x) - \Phi(x)| \le Ck^{-r\beta/(2+r)} \tag{3.2}$$

and

$$\sup_{-\infty<x<\infty} |f_k(x) - \phi(x)| \le C \quad \textit{if } 5\beta/3 > \alpha > \beta > 0, \tag{3.3}$$

where $\phi(x)$ *is the density function of* $\Phi(x)$.

Before we give the proof of Theorem 2.1, we need the following lemma, which is a special case of Theorem 2.3 in Yu (1993).

LEMMA 3.3 [Yu (1993)]. *Let* g_1 *and* g_2 *be absolutely continuous in any finite interval of* R. *Then*, *for any random variables* Y_1 *and* Y_2, *we have*

$$\operatorname{Cov}(g_1(Y_1), g_2(Y_2)) = \int_{R^2} g_1'(x)g_2'(y)P_{xy}(Y_1,Y_2)\,dx\,dy$$

if the right-hand side of the equation is absolutely integrable, *where*

$$P_{xy}(Y_1,Y_2) = P\{Y_1 \le x, Y_2 \le y\} - P\{Y_1 \le x\}P\{Y_2 \le y\}.$$

PROOF OF THEOREM 2.1. The association of $\{\eta_k,\ k \ge 1\}$ is already verified in Remark 2.2. So we just need to show (2.8) holds. By (2.4), Lemma 3.3 and Hölder's inequality,

$$\begin{aligned}
E\eta_i\eta_j &= \int_{R^2} \frac{f_i(x)f_j(y)}{\phi\big(\Phi^{-1}(F_i(x))\big)\phi\big(\Phi^{-1}(F_j(y))\big)} P_{xy}(\xi_i,\xi_j)\,dx\,dy \\
&\le \left\{\int_{R^2}\left\{\frac{f_i(x)f_j(y)}{\phi\big(\Phi^{-1}(F_i(x))\big)\phi\big(\Phi^{-1}(F_j(y))\big)}\right\}^{1+\theta} P_{xy}(\xi_i,\xi_j)\,dx\,dy\right\}^{1/(1+\theta)} \\
&\quad \times\{E\xi_i\xi_j\}^{\theta/(1+\theta)}.
\end{aligned}$$

Since (2.6) implies $\lambda_k^2 + \tau_k^2/2 \ge Ck^{\alpha}$, we have by (2.3),

$$E\xi_i\xi_j \le C(ij)^{-\alpha/2}Eu_iu_j.$$

It is easy to see now that (2.8) follows if we can show that

$$\int_{R^2}\left\{\frac{f_i(x)f_j(y)}{\phi(\Phi^{-1}(F_i(x)))\phi(\Phi^{-1}(F_j(y)))}\right\}^{1+\theta} P_{xy}(\xi_i,\xi_j)\,dx\,dy \le C. \tag{3.4}$$

To prove (3.4), we define the differentiable function

$$g_k(x) = \left(\exp\left\{\theta\left(\Phi^{-1}(F_k(x))\right)^2/2\right\} - 1\right)\operatorname{sgn}\left(\Phi^{-1}(F_k(x))\right)$$

so that

$$\begin{aligned} g_k'(x) &= \theta \exp\left\{\theta\frac{\left(\Phi^{-1}(F_k(x))\right)}{2}\right\}|\Phi^{-1}(F_k(x))|\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))} \\ &\ge 0 \quad \text{for all } x. \end{aligned}$$

On the other hand, $f_k(x)$ is bounded by (3.3), which in turn implies

$$\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))} \le C\exp\left\{\frac{\left(\Phi^{-1}(F_k(x))\right)^2}{2}\right\} \quad \text{for all } x.$$

Thus, by treating the cases $|\Phi^{-1}(F_k(x))| \le 1$ or > 1 separately, we obtain

$$\begin{aligned} \left\{\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))}\right\}^{1+\theta} &= \left\{\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))}\right\}^{\theta}\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))} \\ &\le C\exp\left\{\theta\frac{\left(\Phi^{-1}(F_k(x))\right)^2}{2}\right\}\frac{f_k(x)}{\phi(\Phi^{-1}(F_k(x)))} \\ &\le C(1+g_k'(x)) \quad \text{for all } x. \end{aligned}$$

Hence by (2.3), (2.4), Lemma 3.3 and Cauchy's inequality, the left-hand side of (3.4) is bounded by

$$\begin{aligned} &C\int_{R^2}(1+g_i'(x))(1+g_j'(y))P_{xy}(\xi_i,\xi_j)\,dx\,dy \\ &\quad = C\Big\{E\xi_i\xi_j + E\left(\exp\{\theta\eta_i^2/2\} - 1\right)\operatorname{sgn}(\eta_i)\xi_j \\ &\qquad + E\xi_i\left(\exp\{\theta\eta_j^2/2\} - 1\right)\operatorname{sgn}(\eta_j) \\ &\qquad + E\left(\exp\{\theta\eta_i^2/2\} - 1\right)\operatorname{sgn}(\eta_i)\left(\exp\{\theta\eta_j^2/2\} - 1\right)\operatorname{sgn}(\eta_j)\Big\} \\ &\quad \le C. \end{aligned}$$

The last inequality follows by the facts that $E\exp\{\theta\eta_k^2\} < \infty$ for $0 < \theta < 1/2$ since η_k is $N(0,1)$-distributed and $E\xi_k^2 = 1$ for all $k \ge 1$. This completes the proof of Theorem 2.1. □

To begin the proof of Theorem 2.4, we shall check the small block sequence $\{v_k,\ k \ge 1\}$; the smooth sequence $\{w_k,\ k \ge 1\}$ can be neglected in the partial sum of S_N and the distance between η_k and ξ_k can be estimated. For these purposes we break the proof of Theorem 2.4 into several lemmas.

LEMMA 3.4. *Under the assumptions of Theorem* 2.1, *we have*

$$\begin{aligned}0 \le E\left(\sum_{i=k+1}^{k+n} X_i\right)\left(\sum_{j=k+n+l+1}^{k+n+l+m} X_j\right) &\le C \sum_{i=1}^{\min\{m,n\}} u(l+i) \\ &\le C \frac{\min\{\min\{m,n\}, l+1\}}{(l+1)^{\gamma}}\end{aligned} \tag{3.5}$$

for any integers $m, n \ge 1$, $k, l \ge 0$,

$$Eu_i u_{i+k} \le \begin{cases} Ci^{-(\gamma-1)\beta}, & \text{if } k = 1, \\ Ci^{-(\gamma-1)\alpha} k^{-\gamma}, & \text{if } 2 \le k \le i, \\ Ci^{\alpha} k^{-(1+\alpha)\gamma}, & \text{if } k > i, \end{cases} \tag{3.6}$$

and

$$Ev_i v_{i+k} \le Ci^{\beta} k^{-\gamma}. \tag{3.7}$$

PROOF. The proof of (3.5) is trivial, so we omit it. The case $k = 1$ in (3.6) is special since the gap between u_i and u_{i+1} is only $[i^{\beta}]$. Nevertheless it follows directly from (3.5). Using (3.5) again, we have for $k \ge 2$,

$$\begin{aligned} Eu_i u_{i+k} &\le Ci^{\alpha}\left((i+1)^{\alpha} + \cdots + (i+k-1)^{\alpha}\right)^{-\gamma} \\ &\le Ci^{\alpha}\left((i+k-1)^{1+\alpha} - i^{1+\alpha}\right)^{-\gamma}. \end{aligned}$$

Thus (3.6) follows easily by looking at the cases $2 \le k \le i$ and $k > i$, respectively. Equation (3.7) follows similarly. □

LEMMA 3.5. *Under the assumptions of Theorem* 2.1, *we have for* λ_k^2 *and* τ_k^2 *defined in* (2.2),

$$Ck^{\alpha} \le \lambda_k^2 \le Ck^{\alpha}, \qquad Ck^{\beta} \le \tau_k^2 \le Ck^{\beta} \tag{3.8}$$

and

$$0 \le \sigma_{N_k}^2 - \sum_{i=1}^{k} \left(\lambda_i^2 + \tau_i^2\right) \le Ck. \tag{3.9}$$

PROOF. It is easy to prove (3.8) by (2.6) and (2.7). Now by (2.1) and (2.2),

$$0 \le \sigma_{N_k}^2 - \sum_{i=1}^{k} \left(\lambda_i^2 + \tau_i^2\right) = 2\sum_{i=1}^{k} Eu_i v_i + 2 \sum_{1 \le i < j \le k} E(u_i + v_i)E(u_j + v_j).$$

Thus, similarly to the proof of (3.6), (3.9) follows easily by (3.5). □

LEMMA 3.6. *Under the assumptions of Theorem* 2.1, *we have*

$$\left|\sum_{i=1}^{k} v_i\right| \le Ck^{(1+\beta)/2}\log^3 k \quad a.s.$$

PROOF. By (3.7) and (3.8), for all $m, n \ge 0$,

$$\begin{aligned} E\left(\sum_{i=m+1}^{m+n} v_i\right)^2 &= \sum_{i=m+1}^{m+n} Ev_i^2 + 2\sum_{m<i<j\le m+n} Ev_iv_j \\ &\le C\left(\sum_{i=m+1}^{m+n} i^{\beta} + \sum_{i=m+1}^{m+n} i^{\beta}\sum_{k=1}^{\infty} k^{-\gamma}\right) \\ &\le C\left((m+n)^{1+\beta} - m^{1+\beta}\right). \end{aligned}$$

Hence by the Gaal–Koksma strong law of large numbers [page 134, Philipp and Stout (1975)] we get the conclusion of our lemma. □

LEMMA 3.7. *We have*

$$\left|\sum_{i=1}^{k} w_i\right| \le k^{(1+\beta)/2}\log^{1/2} k \quad a.s.$$

PROOF. Since $\{w_k, k \ge 1\}$ is an independent Gaussian sequence, it follows easily from the Borel–Cantelli lemma. □

LEMMA 3.8. *Under the assumptions of Theorem* 2.1, *we have*

$$\max_{N_k<N\le N_{k+1}}\left|\sum_{i=N_k+1}^{N} X_i\right| \le CN_k^{1/2-\varepsilon_1} \quad a.s.,$$

where $0 < \varepsilon_1 < r/(2(2+r)(1+\alpha))$.

PROOF. Noting by (3.8) that $E(u_{k+1}+v_{k+1})^2 \le Ck^{\alpha}$, we have by the maximal inequality of Newman and Wright (1981), (2.1) and (2.9),

$$\begin{aligned} &P\left\{\max_{N_k<N\le N_{k+1}}\left|\sum_{i=N_k+1}^{N} X_i\right| \ge N_k^{1/2-\varepsilon_1}\right\} \\ &\quad\le \left(1 - \frac{E(u_{k+1}+v_{k+1})^2}{4Ck^{\alpha}}\right)^{-1} P\left\{|u_{k+1}+v_{k+1}| \ge N_k^{1/2-\varepsilon_1} - 2C^{1/2}k^{\alpha/2}\right\} \\ &\quad\le CP\left\{|u_{k+1}+v_{k+1}| \ge \frac{N_k^{1/2-\varepsilon_1}}{2}\right\} \\ &\quad\le CN_k^{-(1/2-\varepsilon_1)(2+r)}E|u_{k+1}+v_{k+1}|^{2+r} \\ &\quad\le Ck^{-(1/2-\varepsilon_1)(2+r)(1+\alpha)+(2+r)\alpha/2} \\ &\quad\le Ck^{-1-(r/2-\varepsilon_1(2+r)(1+\alpha))}. \end{aligned}$$

Our lemma now follows from the Borel–Cantelli lemma. □

LEMMA 3.9. *Under the assumptions of Theorem* 2.1, *we have*

$$|\eta_k - \xi_k| \le Ck^{-(r\beta/(2+r) - K^2/2)},$$

provided that $|\xi_k| \le K(\log k)^{1/2}$ *and* $0 < K < (2r\beta/(2+r))^{1/2}$.

By Lemma 3.2, the proof is basically the same as that of Lemma 2.5.1 of Csörgő and Révész (1981). So the proof is omitted.

LEMMA 3.10. *Under the assumptions of Theorem* 2.1, *we have*

$$\left| \sum_{i=1}^{k} \left(\lambda_i^2 + \tau_i^2/2\right)^{1/2} (\eta_i - \xi_i) \right| \le Ck^{(1+\alpha-\varepsilon_2)/2} \log^3 k \quad a.s.,$$

provided that $5\beta/3 > \alpha > \beta > 0$ *and*

$$\gamma > \max\left\{1 + \frac{1}{(1+\alpha-\beta)\theta}, \frac{\alpha + 2(1+\theta)}{2(1+\alpha)\theta}\right\}, \tag{3.10}$$

where

$$\varepsilon_2 = \min\left\{\frac{2r^2\beta}{(2+r)(4+3r)}, \frac{(\alpha + (\gamma-1)\beta)\theta}{1+\theta}\right\}.$$

PROOF. Let $e_k = (\lambda_k^2 + \tau_k^2/2)^{1/2}(\eta_i - \xi_k)$. Then, by Lemma 3.9, Hölder's inequality, (2.3), (2.9), (3.2) and (3.8),

$$\begin{aligned} Ee_k^2 &= Ee_k^2 I\left(|\xi_k| \le K(\log k)^{1/2}\right) + Ee_k^2 I\left(|\xi_k| > K(\log k)^{1/2}\right) \\ &\le Ck^{\alpha - 2(r\beta/(2+r) - K^2/2)} \\ &\quad + Ck^{\alpha}\left(E|\eta_k - \xi_k|^{2+r}\right)^{2/(2+r)} \left(P\left\{|\xi_k| > K(\log k)^{1/2}\right\}\right)^{r/(2+r)} \\ &\le Ck^{\alpha - 2(r\beta/(2+r) - K^2/2)} + Ck^{\alpha}\left(k^{-K^2/2} + k^{-r\beta/(2+r)}\right)^{r/(2+r)}, \end{aligned}$$

where $I(E)$ is the indicator function of the set E. Next by choosing $K = 2(r\beta/(4+3r))^{1/2}$, we have

$$Ee_k^2 \le Ck^{\alpha - \varepsilon_2} \quad \text{for all } k \ge 1.$$

On the other hand, since η_i $[= \Phi^{-1}(F_i(\xi_i))]$ and ξ_j are associated for all $i \ne j$, $E\eta_i \xi_j \ge 0$. This, together with (3.8) and Theorem 2.1, gives us

$$\begin{aligned} E\left(\sum_{i=m+1}^{m+n} e_i\right)^2 &= \sum_{i=m+1}^{m+n} Ee_i^2 + 2 \sum_{m+1 \le i < j \le m+n} Ee_i e_j \\ &\le \sum_{i=m+1}^{m+n} Ee_i^2 \\ &\quad + C \sum_{i=m+1}^{m+n} \sum_{k=1}^{m+n-i} (i(i+k))^{\alpha/(2(1+\theta))} (Eu_i u_{i+k})^{\theta/(1+\theta)}. \end{aligned}$$

To estimate the second part in last inequality above, we use (3.6) to find its upper bound by

$$
\begin{aligned}
C\Bigg\{ & \sum_{i=m+1}^{m+n} i^{(\alpha+(1-\gamma)\theta\beta)/(1+\theta)} + \sum_{i=m+1}^{m+n} i^{(\alpha+(1-\gamma)\theta\alpha)/(1+\theta)} \sum_{k=2}^{i} k^{-\theta\gamma/(1+\theta)} \\
& + \sum_{i=m+1}^{m+n} i^{((1+2\theta)\alpha)/(2(1+\theta))} \sum_{k=i+1}^{\infty} k^{(\alpha-2(1+\alpha)\gamma\theta)/(2(1+\theta))}\Bigg\} \\
& \le C \sum_{i=m+1}^{m+n} i^{(\alpha+(1-\gamma)\theta\beta)/(1+\theta)} \le C \sum_{i=m+1}^{m+n} i^{\alpha-\varepsilon_2}
\end{aligned}
$$

if (3.10) is satisfied. This proves our lemma by the Gaal–Koksma strong law of large numbers. □

PROOF OF THEOREM 2.4. The proof follows easily by Lemmas 3.6–3.8 and Lemma 3.10. The only thing that one has to verify is that (3.10) holds for some $\alpha > \beta > 0$ and $0 < \theta < 1/2$. Since $\gamma > 1$, it is possible to choose α, β large enough and θ close to $1/2$. This completes the proof of Theorem 2.4. □

The following proposition, used in proving Theorem 2.5, is the improvement of the Berkes and Philipp (1979) approximation theorem.

PROPOSITION 3.11 [Berbee (1987)]. *Let $\{X_k,\ k \ge 1\}$ be a sequence of random variables and let $\{\mathscr{F}_k,\ k \ge 1\}$ be a sequence of nondecreasing σ-fields, such that X_k is $\mathscr{F}_k$-measurable. Suppose that for some sequence $\{\beta_k,\ k \ge 1\}$ of nonnegative numbers,*

$$E \sup_{A \in \mathscr{B}} |P\{X_k \in A | \mathscr{F}_{k-1}\} - P\{X_k \in A\}| \le \beta_k$$

for all $k \ge 1$, where $\mathscr{B}$ is the σ-field of Borel sets on R. Then without changing its distribution we can redefine the sequence of $\{X_k,\ k \ge 1\}$ on a richer probability space on which there exists a sequence $\{Y_k,\ k \ge 1\}$ of independent random variables with the same distribution as X_k, such that for all $k \ge 1$,

$$P\{|X_k - Y_k| > 0\} \le \beta_k.$$

PROOF OF THEOREM 2.5. Similarly to the proof of Lemma 3.4, we have for $k \ge 1$,

$$
Eu_{k-i}u_k \le C \begin{cases} \exp\{-\lambda(k-1)^{\beta}\}, & \text{if } i = 1, \\ \exp\{-\lambda((k-i+1)^{\alpha} + \cdots + (k-1)^{\alpha})\}, & \text{if } 2 \le i \le k-1. \end{cases} \tag{3.11}
$$

Observing that $\eta_1, \dots, \eta_k$ are associated, we have by Lemma 2.2 of Dabrowski and Dehling (1987),

$$\left| E\exp\left\{ i\sum_{j=1}^{k} t_j\eta_j \right\} - E\exp\left\{ i\sum_{j=1}^{k-1} t_j\eta_j \right\} E\exp\{it_k\eta_k\} \right| \le 2\sum_{i=1}^{k-1} |t_i t_k| E\eta_i\eta_k. \tag{3.12}$$

Now we follow the lines of the proof of Theorem 5 in Berkes and Philipp (1979). Let $\{\zeta_k,\ k \ge 1\}$ be a sequence of independent $N(0, \rho_k^2)$-distributed random variables, where $\rho_k^2 = \tau_k^2/(2\lambda_k^2 + \tau_k^2)$. Put

$$Z_k = \eta_k + \zeta_k \quad \text{for } k \ge 1.$$

Since $E\exp\{it\zeta_k\}$ is integrable as a function of t, the joint density $p_k(z_1, \dots, z_k)$ of $Z_1, \dots, Z_k$, is given by

$$p_k(z_1, \dots, z_k) = (2\pi)^{-k} \int_{R^k} \exp\left\{ -i\sum_{j=1}^{k} z_j t_j \right\} E\exp\left\{ i\sum_{j=1}^{k} t_j\eta_j \right\} \times \exp\left\{ -\sum_{j=1}^{k} \rho_j^2 t_j^2/2 \right\} dt_1 \cdots dt_k \tag{3.13}$$

and the density $p^{(k)}(z_k)$ of Z_k is given by

$$p^{(k)}(z_k) = (2\pi)^{-1} \int_R \exp\{-it_k z_k\} E\exp\{it_k\eta_k\} \exp\{-\rho_k^2 t_k^2/2\}\, dt_k. \tag{3.14}$$

Thus for some $U_k > 0$ we have

$$\begin{aligned} &E \sup_{A\in\mathscr{B}} |P\{Z_k \in A | Z_1, \dots, Z_{k-1}\} - P\{Z_k \in A\}| \\ &\quad \le \int_{-U_k}^{U_k} \cdots \int_{-U_k}^{U_k} \big| p_k(z_1, \dots, z_k) - p_{k-1}(z_1, \dots, z_{k-1}) p^{(k)}(z_k) \big|\, dz_1 \cdots dz_k \\ &\qquad + \sum_{i=1}^{k} P\{|Z_i| \ge U_k\} = I_1^{(k)} + I_2^{(k)}. \end{aligned} \tag{3.15}$$

By (3.11)–(3.14) and (2.8) of Theorem 2.1,

$$\begin{aligned} I_1^{(k)} &\le (U_k/\pi)^k \int_{R^k} \left| E\exp\left\{ i\sum_{j=1}^{k} t_j\eta_j \right\} - E\exp\left\{ i\sum_{j=1}^{k-1} t_j\eta_j \right\} E\exp\{it_k\eta_k\} \right| \\ &\qquad \times \exp\left\{ -\sum_{j=1}^{k} \rho_j^2 t_j^2/2 \right\} dt_1 \cdots dt_k \\ &\le C U_k^{2(1+k)} \exp\{-\lambda(k-1)^\beta \theta/(1+\theta)\} \\ &\qquad + 2U_k^k \int_{\sum_{j=1}^{k} t_j^2 \ge U_k^2} \exp\left\{ -\sum_{j=1}^{k} \rho_j^2 t_j^2/2 \right\} dt_1 \cdots dt_k. \end{aligned} \tag{3.16}$$

Then the last term in the above inequality, after we select $U_k = k^{(1+\alpha-\beta)/2}\log k$ and use (3.8), is bounded by

$$\begin{aligned} &CU_k^{2k}\int_{\sum_{j=1}^k t_j^2 > U_k^2\rho_k^2} \exp\left\{-\sum_{j=1}^k t_j^2 \Big/ 2\right\} dt_1 \cdots dt_k \\ &\quad \le C2^k U_k^{2k}\exp\{-3U_k^2\rho_k^2/8\} \\ &\quad \le Ck^{-2} \quad \text{for } k \text{ large enough.} \end{aligned}$$

The other remaining term in (3.16), after we choose $\beta > 1$, can also be bounded by Ck^{-2} for k large enough. In general, we obtain

$$I_1^{(k)} \le Ck^{-2} \quad \text{for } k \text{ large enough.}$$

Because of the normality of each Z_k, we can easily prove

$$I_2^{(k)} \le Ck^{-2} \quad \text{for } k \text{ large enough.}$$

Thus, based on the above proof and (3.15), we arrive at

$$E \sup_{A\in\mathscr{B}} |P\{Z_k \in A | Z_1, \dots, Z_{k-1}\} - P\{Z_k \in A\}| \le Ck^{-2}$$

for k large enough. Hence by Proposition 3.11 we can redefine $\{Z_k,\ k \ge 1\}$ on a new probability space together with a sequence $\{Y_k,\ k \ge 1\}$ of independent $N(0, 1+\rho_k^2)$-distributed random variables such that

$$P\{|Z_k - Y_k| > 0\} \le Ck^{-2}.$$

Obviously, based on the Borel–Cantelli lemma, we get

$$\text{(3.17)} \quad \sum_{i=1}^k \left(\lambda_i^2 + \tau_i^2/2\right)^{1/2} |Z_i - Y_i| \le CnN_k^{1/2-\varepsilon} \quad \text{a.s. for some } \varepsilon > 0.$$

Since $\{Y_k,\ k \ge 1\}$ is an independent Gaussian sequence, we assume without loss of generality that there exists a standard Wiener process $\{W(t),\ t \ge 0\}$ satisfying

$$\text{(3.18)} \quad Y_k = \left(\lambda_k^2 + \tau_k^2/2\right)^{-1/2}\left(W\left(\sum_{i=1}^k \left(\lambda_i^2 + \tau_i^2\right)\right) - W\left(\sum_{i=1}^{k-1}\left(\lambda_i^2 + \tau_i^2\right)\right)\right)$$

for all $k \ge 1$. On the other hand, by the Borel–Cantelli lemma, we have

$$\text{(3.19)} \quad \left|\sum_{i=1}^k \left(\lambda_i^2 + \tau_i^2/2\right)^{1/2} \zeta_i\right| \le Ck^{(1+\beta)/2}\log^{1/2} k \quad \text{a.s.}$$

Hence by Theorem 2.4 and (3.17)–(3.19), our theorem follows if we can show that

$$\sup_{N_k < N \le N_{k+1}} \left| W\left(\sigma_N^2\right) - W\left(\sum_{i=1}^k \left(\lambda_i^2 + \tau_i^2\right)\right)\right| \le CN_k^{1/2-\varepsilon} \quad \text{a.s.}$$

for some $\varepsilon > 0$, which in fact can be proved by applying Theorem 2.1 of Csörgő and Révész (1981) and the inequality

$$0 \le \sigma_N^2 - \sum_{i=1}^k \left(\lambda_i^2 + \tau_i^2\right) \le k^{\alpha} \quad \text{for } N_k < N \le N_{k+1}$$

by Lemma 3.5. This completes our proof. □

APPENDIX

PROOF OF PROPOSITION 3.1. Let $k_n = [n/p_n]$. We denote

$$X_i^{(n)} = X_{(i-1)p_n+1} + \cdots + X_{ip_n}, \qquad i = 1, \ldots, k_n, \tag{A.1}$$
$$X_{k_n+1}^{(n)} = X_{k_n p_n+1} + \cdots + X_n.$$

By Lemma 3.4 and some calculations, it is easy to find that

$$0 \le E\left(\sum_{i=1}^{k_n} X_i^{(n)}\right)^2 - \sum_{i=1}^{k_n} E\big(X_i^{(n)}\big)^2 \le k_n \sum_{i=1}^{p_n} u(i) \le Ck_n \tag{A.2}$$

and

$$0 \le \sigma_n^2 - \sum_{i=1}^{k_n} E\big(X_i^{(n)}\big)^2 \le C(k_n + p_n). \tag{A.3}$$

By (A.1) we have

$$\begin{aligned} &|E\exp\{itS_n/\sigma_n\} - \exp\{-t^2/2\}| \\ &\quad \le \left| E\exp\{itS_n/\sigma_n\} - E\exp\left\{it\sum_{j=1}^{k_n} X_j^{(n)}/\sigma_n\right\} E\exp\left\{itX_{k_n+1}^{(n)}/\sigma_n\right\}\right| \\ &\qquad + \left| E\exp\left\{it\sum_{j=1}^{k_n} X_j^{(n)}/\sigma_n\right\} - \prod_{j=1}^{k_n} E\exp\left\{itX_j^{(n)}/\sigma_n\right\}\right| \\ &\qquad + \left|\prod_{j=1}^{k_n} E\exp\left\{itX_j^{(n)}/\sigma_n\right\} - \exp\{-t^2/2\}\right| \\ &\qquad + \exp\{-t^2/2\}|E\exp\left\{itX_{k_n+1}^{(n)}/\sigma_n\right\} - 1| \\ &\quad = I_1^{(k)} + I_2^{(k)} + I_3^{(k)} + I_4^{(k)}. \end{aligned} \tag{A.4}$$

By Newman's (1980) inequality for characteristic functions, (2.6)–(2.7) and (A.2),

$$I_1^{(k)} \le t^2 E\left(\sum_{j=1}^{k_n} X_j^{(n)} X_{k_n+1}^{(n)}\right) \Big/ \big(2\sigma_n^2\big) \le Ct^2/n, \tag{A.5}$$

$$I_2^{(k)} \le t^2\left(E\left(\sum_{i=1}^{k_n} X_i^{(n)}\right)^2 - \sum_{i=1}^{k_n} E\big(X_i^{(n)}\big)^2\right) \Big/ \big(2\sigma_n^2\big) \le Ct^2/p_n \tag{A.6}$$

and

$$I_4^{(k)} \le |t|\exp\{-t^2/2\}E|X_{k_n+1}^{(n)}|/\sigma_n \le C|t|\exp\{-t^2/2\}(p_n/n)^{1/2}. \tag{A.7}$$

Now we have [cf. Loéve (1977), page 212] for $j = 1, \ldots, k_n$,

$$E\exp\left\{itX_j^{(n)}/\sigma_n\right\} = 1 - t^2\int_0^1 (1-u)E\big(X_j^{(n)}/\sigma_n\big)^2 \exp\left\{ituX_j^{(n)}/\sigma_n\right\} du. \tag{A.8}$$

When $|t| \le \sigma_n / \max_{1\le j\le k_n}\{(E(X_j^{(n)})^2)^{1/2}\}$,

$$|E\exp\{itX_j^{(n)}/\sigma_n\} - 1| \le 1/2 \quad \text{for } j = 1,\dots,k_n \tag{A.9}$$

and

$$E^2\big(X_j^{(n)}\big)^2|t|^4/\big(4\sigma_n^4\big) \le E|X_j^{(n)}|^{2+r}|t|^{2+r}/\big(4\sigma_n^{2+r}\big) \quad \text{for } j = 1,\dots,k_n. \tag{A.10}$$

Since $\log(1-x) = -x + \theta|x|^2$, $|\theta| \le 1$ for all $|x| \le 1/2$, by (A.8)–(A.10) [cf. Loéve (1977), page 212], for $j = 1,\dots,k_n$,

$$\begin{aligned}
\log E\exp\left\{\frac{itX_j^{(n)}}{\sigma_n}\right\}
&= -t^2\int_0^1(1-u)E\left(\frac{X_j^{(n)}}{\sigma_n}\right)^2\exp\left\{\frac{itX_j^{(n)}}{\sigma_n}\right\}du + \theta_j\frac{E^2\big(X_j^{(n)}\big)^2|t|^4}{4\sigma_n^4}\\
&= -\frac{E\big(X_j^{(n)}\big)^2t^2}{2\sigma_n^2} + 2^{1-r}\theta_j\frac{E|X_j^{(n)}|^{2+r}|t|^{2+r}}{(1+r)(2+r)\sigma_n^{2+r}} + \theta_j\frac{E|X_j^{(n)}|^{2+r}|t|^{2+r}}{4\sigma_n^{2+r}}\\
&= -\frac{E\big(X_j^{(n)}\big)^2t^2}{2\sigma_n^2} + \theta_jK_r\frac{E|X_j^{(n)}|^{2+r}|t|^{2+r}}{\sigma_n^{2+r}}, \qquad |\theta_j| \le 1,
\end{aligned}$$

where $K_r = 1/4 + 2^{1-r}/((1+r)(2+r))$. Thus the inequality $|e^x - 1| \le |x|e^{|x|}$ for all x yields

$$\begin{aligned}
I_3^{(k)} &\le \exp\left\{-\frac{t^2}{2}\right\}\left|\exp\left\{\left(1-\frac{\sum_{i=1}^{k_n}E\big(X_j^{(n)}\big)^2}{\sigma_n^2}\right)\frac{t^2}{2} + \theta K_r\frac{\sum_{i=1}^{k_n}E|X_j^{(n)}|^{2+r}|t|^{2+r}}{\sigma_n^{2+r}}\right\} - 1\right|\\
&\le \left\{\left(1-\frac{\sum_{i=1}^{k_n}E\big(X_j^{(n)}\big)^2}{\sigma_n^2}\right)\frac{t^2}{2} + \theta K_r\frac{\sum_{i=1}^{k_n}E|X_j^{(n)}|^{2+r}|t|^{2+r}}{\sigma_n^{2+r}}\right\}\\
&\quad\times\exp\left\{-\frac{t^2}{2} + \left(1-\frac{\sum_{i=1}^{k_n}E\big(X_j^{(n)}\big)^2}{\sigma_n^2}\right)\frac{t^2}{2} + \theta K_r\frac{\sum_{i=1}^{k_n}E|X_j^{(n)}|^{2+r}|t|^{2+r}}{\sigma_n^{2+r}}\right\}.
\end{aligned} \tag{A.11}$$

Without loss of generality we can assume that $\sigma_n^2 - \sum_{j=1}^{k_n} E(X_j^{(n)})^2 \le \sigma_n^2/2$ by (A.3). Then when $|t| \le \sigma_n^{2+r}/(6K_r \sum_{i=1}^{k_n} E|X_j^{(n)}|^{2+r})^{1/r}$,

$$-\frac{t^2}{2} + \left(1 - \frac{\sum_{i=1}^{k_n} E\left(X_j^{(n)}\right)^2}{\sigma_n^2}\right)\frac{t^2}{2} + \theta K_r \frac{\sum_{i=1}^{k_n} E|X_j^{(n)}|^{2+r}|t|^{2+r}}{\sigma_n^{2+r}} \le -\frac{t^2}{12}. \tag{A.12}$$

It is clear by (2.5)–(2.7) and (2.10) that

$$C\left(\frac{n}{p_n}\right)^{1/2} \le \min\left\{\frac{\sigma_n}{\max_{1\le j\le k_n}\left\{\left(E\left(X_j^{(n)}\right)^2\right)^{1/2}\right\}}, \left(\frac{\sigma_n^{2+r}}{6K_r \sum_{i=1}^{k_n} E|X_j^{(n)}|^{2+r}}\right)^{1/r}\right\}. \tag{A.13}$$

Hence Proposition 3.1 follows now by (A.2)–(A.7) and (A.11)–(A.13). □

Proof of Lemma 3.2. By the smoothing lemma of Berry [cf. Feller (1971)] and the independence between $\{u_k, k \ge 1\}$ and $\{w_k, k \ge 1\}$, for any $T > 0$,

$$\begin{aligned}\sup_{-\infty<x<\infty} |F_k(x) - \Phi(x)| &\le \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\left(E\exp\{it\xi_k\} - \exp\{-t^2/2\}\right)}{t}\right| dt + \frac{24}{\pi T} \\ &\le C\int_{-T}^{T}\left|\frac{\left(E\exp\{itu_k/\lambda_k\} - \exp\{-t^2/2\}\right)}{t}\right| \exp\left\{-\frac{\tau_k^2 t^2}{4\lambda_k^2}\right\} dt + \frac{1}{T}.\end{aligned}$$

Then replacing S_n/σ_n by u_k/λ_k in Proposition 3.1, we have for $n = [k^{\alpha}]$ and $|T| \le C(n/p_n)^{1/2}$,

$$\sup_{-\infty<x<\infty} |F_k(x) - \Phi(x)| \le \lambda_k^2/\left(\tau_k^2 p_n\right) + (p_n/n)^{r/2} + 1/T.$$

Hence (3.2) holds by putting $p_n = [k^{\alpha - 2\beta/(2+r)}]$ and $T = C(n/p_n)^{r/2}$.

By (3.1) for any $T > 0$,

$$\begin{aligned}\sup_{-\infty<x<\infty} |f_k(x) - \phi(x)| \le{}& \frac{C}{2\pi}\int_{-T}^{T}\left|E\exp\left\{\frac{it\xi_k}{\left(\lambda_k^2 + \tau_k^2/2\right)^{1/2}}\right\} - \exp\left\{-\frac{t^2}{2}\right\}\right| dt \\ &+ \frac{C}{\pi}\int_{|t|\ge T}\exp\left\{-\frac{\tau_k^2 t^2}{2\left(2\lambda_k^2 + \tau_k^2\right)}\right\} dt.\end{aligned}$$

Similarly by Proposition 3.1 for $0 < T \le C(n/p_n)^{1/2}$,

$$\sup_{-\infty<x<\infty} |f_k(x) - \phi(x)| \le \frac{C\lambda_k^3}{\tau_k^3 p_n} + C\left(\frac{p_n}{n}\right)^{r/2} + \frac{C(2\lambda_k^2+\tau_k^2)}{\tau_k^2 T}\exp\left\{-\frac{\tau_k^2 T^2}{2(2\lambda_k^2+\tau_k^2)}\right\}.$$

Then putting $p_n = [k^{\alpha-(3\beta-\alpha)/2}]$ and $T = C(n/p_n)^{1/2}$, we obtain by (3.8) that (3.3) holds for $\alpha < 5\beta/3$. □

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REFERENCES

BARLOW, R. E. and PROSCHAN, F. (1981). *Statistical Theory of Reliability and Life Testing: Probability Models*. To Begin with Publisher, Silver Spring, MD.

BERBEE, H. (1987). Convergence rates in the strong law for bounded mixing sequences. *Probab. Theory Related Fields* **74** 255–270.

BERKES, I. and PHILIPP, W. (1979). Approximation theorems for independent and weakly dependent random variables. *Ann. Probab.* **7** 29–54.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

BIRKEL, T. (1987). The invariance principle for associated processes. *Stochastic Process. Appl.* **27** 57–71.

BIRKEL, T. (1988a). On the convergence rate in the central limit theorems for associated processes. *Ann. Probab.* **16** 1685–1698.

BIRKEL, T. (1988b). Moment bounds for associated sequences. *Ann. Probab.* **16** 1184–1193.

BURTON, R. M., DABROWSKI, A. R. and DEHLING, H. (1986). An invariance principle for weakly associated random variables. *Stochastic Process. Appl.* **23** 301–306.

COX, J. T. and GRIMMETT, G. (1984). Central limit theorems for associated random variables and the percolation model. *Ann. Probab.* **12** 514–528.

CSÖRGŐ, M. and RÉVÉSZ, P. (1975a). A new method to prove Strassen type laws of invariance principle. I. *Z. Wahrsch. Verw. Gebiete* **31** 255–260.

CSÖRGŐ, M. and RÉVÉSZ, P. (1975b). A new method to prove Strassen type laws of invariance principle. II. *Z. Wahrsch. Verw. Gebiete* **31** 261–269.

CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York.

DABROWSKI, A. R. and DEHLING, H. (1988). A Berry–Esseen theorem and a functional law of the iterated logarithm for weakly associated random variables. *Stochastic Process. Appl.* **30** 277–289.

ESARY, J., PROSCHAN, F. and WALKUP, D. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

FELLER, W. (1971). *An Introduction to Probability Theory and its Applications* **2**, 2nd ed. John Wiley, New York.

KÓMLOS, J., MAJOR, P. and TUSNADY, G. (1975). An approximation of partial sums of independent R.V.'s and the sample DF. I. *Z. Wahrsch. Verw. Gebiete* **32** 111–131.

KÓMLOS, J., MAJOR, P. and TUSNADY, G. (1976). An approximation of partial sums of independent R.V.'s and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 35–58.

LOÉVE, M. (1977). *Probability Theory* **1**, 4th ed. Springer, New York.

NEWMAN, C. M. (1980). Normal fluctuations and the FKG inequalities. *Comm. Math. Phys.* **74** 119–128.

NEWMAN, C. M. (1983). A general central limit theorem for FKG systems. *Comm. Math. Phys.* **91** 75–80.

NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675.

PHILIPP, W. (1986). Invariance principles for independent and weakly dependent random variables. In *Dependence in Probability and Statistics* (E. Eberlein and M. Taqque, eds.) 225–268. Birkhäuser, Boston.

PHILIPP, W. and STOUT, W. F. (1975). Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.* **161**.

PITT, L. D. (1982). Positively correlated normal variables are associated. *Ann. Probab.* **10** 496–499.

WOOD, T. E. (1983). A Berry–Esseen theorem for associated random variables. *Ann. Probab.* **11** 1042–1047.

WOOD, T. E. (1985). A local limit theorem for associated sequences. *Ann. Probab.* **13** 625–629.

YU, H. (1985). An invariance principle for associated sequences of random variables. *J. Engrg. Math.* **2** 55–60.

YU, H. (1986). The law of the iterated logarithm for associated random variables. *Acta Math. Sinica* **29** 507–511.

YU, H. (1993). A Glivenko–Cantelli lemma and weak convergence for empirical processes of associated sequences. *Probab. Theory Related Fields* **95** 357–370.

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