The Annals of Probability
1996, Vol. 24, No. 4, 2057–2064

TRANSFORMING RANDOM ELEMENTS AND SHIFTING RANDOM FIELDS

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Consider a locally compact second countable topological transformation group acting measurably on an arbitrary space. We show that the distributions of two random elements X and X' in this space agree on invariant sets if and only if there is a random transformation Γ such that ΓX has the same distribution as X'. Applying this to random fields in d dimensions under site shifts, we show further that these equivalent claims are also equivalent to site-average total variation convergence. This convergence result extends to amenable groups.

1. Introduction. Consider a locally compact second countable topological transformation group acting measurably on an arbitrary space. The aim of this paper is to show that the distributions of two random elements in this space agree on invariant sets if and only if each can be randomly transformed into a copy of the other (Theorem 1). Applying this to random fields in d dimensions under site shifts, we show further that these equivalent claims are also equivalent to site-average total variation convergence over quite general averaging sets (Theorem 2). This convergence result in fact holds for amenable groups (Remark 2).

This extends results for one-sided stochastic processes on a Polish state space proved by Berbee [2], Greven [9], Aldous and Thorisson [1] and Thorisson [13], saying that the distributions of two processes agree on site-shift invariant sets if and only if the two processes can be represented in such a way that their paths eventually coincide modulo a random time shift, and if and only if the processes converge in time-average total variation over intervals. The assumption that the state space is Polish is needed in these papers because the one-sided shifts only form a semigroup, but we do not need this assumption here.

In a forthcoming paper, Georgii [7] extends the results of these papers in a slightly different direction. He considers a semigroup acting measurably on a Polish space and assumes that either the semigroup is countable normal or a compact metric group or composed of finitely many such building blocks.

There is a considerable body of classical theory involving σ-algebras and measures invariant under transformation groups; see [6] and [12]. The emphasis here is rather different; see, however, Remark 6 on ergodic theory.

Received October 1994; revised March 1996.

AMS 1991 *subject classifications*. Primary 60B99, 60G60.

Key words and phrases. Topological transformation group, random field, invariant σ-algebra, total variation, coupling.

The results of this paper are applied to Palm theory in [14]. In fact, in the ergodic case the distribution of a stationary point process in d dimensions agrees on invariant sets with the distribution of its Palm version. Moreover, in the non-ergodic case the distribution agrees on invariant sets with the distribution of a certain modified Palm version. Thus the stationary point process and its (modified) Palm version are really the same point process with different centers.

The plan of the paper is as follows. The main result on transforming random elements is stated in Section 2, and Section 3 contains remarks. The consequences for random fields are stated and proved in Section 4, and Section 5 contains remarks. After a minor preparation in Section 6, we finally prove the main result in Section 7.

2. Transforming random elements. Let $(H,\mathscr{H})$ be an arbitrary measurable space. Let G be a topological group of measurable mappings (transformations) from $(H,\mathscr{H})$ to $(H,\mathscr{H})$ which is locally compact and second countable. Let $\mathscr{G}$ be the Borel subsets of G. Let the mapping from $H \times G$ to H taking (x, γ) to γx be $\mathscr{H} \otimes \mathscr{G}/\mathscr{H}$ measurable. Define the invariant σ-algebra on $(H,\mathscr{H})$ under G by

$$\mathscr{I} = \{ A \in \mathscr{H}\colon \gamma A = A, \gamma \in G\}.$$

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a common probability space supporting all the random elements in this paper. This means that whenever we claim the existence of a random element, we have to show that it can coexist on the same probability space with those previously introduced (this is the case if the random element is assumed to be independent of the others or if it does not have a specified joint distribution with the others or if its conditional distribution given the others is regular). Let $=_D$ denote identity in distribution.

THEOREM 1. *Let X and X' be random elements in $(H,\mathscr{H})$. The distributions of X and X' agree on $\mathscr{I}$ if and only if*

$$\Gamma X =_D X'$$

for some random transformation Γ in $(G, \mathscr{G})$.

See Section 7 for the proof.

3. Remarks on Theorem 1.

REMARK 1. A locally compact first countable topological group has an invariant metric inducing the topology (see [11], page 34) and a locally compact second countable metric space is separable and topologically complete (see [3], page 25, and [4], page 27). Thus $(G, \mathscr{G})$ is Polish.

Moreover, a locally compact topological group possesses an invariant measure—the Haar measure—and a locally compact second countable topological group is σ-compact. Thus the Haar measure is σ-finite (see [10], pages 254 and 256).

REMARK 2. Let λ be the Haar measure on $(G, \mathscr{G})$ and suppose there are sets $B_h \in \mathscr{G}$, $0 < h < \infty$, such that $0 < \lambda(B_h) < \infty$ and, for all $\gamma \in G$,

$$\lambda(B_h \cap \gamma B_h)/\lambda(B_h) \to 1, \qquad h \to \infty.$$

Repeating the proof of Theorem 2 below in this abstract setting and without the function g yields the following result. The equivalent claims in Theorem 1 are also equivalent to

$$\|\mathbf{P}(U(B_h)X \in \cdot) - \mathbf{P}(U(B_h)X' \in \cdot)\| \to 0, \qquad h \to \infty,$$

where $U(B_h)$ is a random transformation in $(G, \mathscr{G})$ which is independent of X and X' and has the distribution $\lambda(\cdot \mid B_h) = \lambda(\cdot \cap B_h)/\lambda(B_h)$ and $\|\cdot\|$ denotes the total variation norm defined for bounded signal measures ν on $(H, \mathscr{H})$ by

$$\|\nu\| = \sup_{A \in \mathscr{H}} \nu(A) - \inf_{A \in \mathscr{H}} \nu(A) \qquad \left[= 2 \sup_{A \in \mathscr{H}} \nu(A) \text{ when } \nu(H) = 0\right].$$

In particular, if the distribution of X' is invariant under G, then this limit result can be written as

$$U(B_h)X \to X' \quad \text{in total variation as } h \to \infty.$$

A family like B_h, $0 < h < \infty$, is called a Følner family. The existence of such a family is equivalent to G being amenable; see [8].

REMARK 3. Here are some examples where Theorem 1 applies:

(i) G countable;

(ii) countable site-space random fields under finite site permutations. In this case $\mathscr{I}$ is the exchangeable σ-algebra and the invariance property is exchangeability;

(iii) one-sided continuous time real-valued processes under space-and-time scaling. In this case $\mathscr{I}$ is the self-similar σ-algebra and the invariance property is self-similarity;

(iv) canonically measurable random fields (and random measures), with site space forming a group like G above, under the site shifts (generated by the group shifts). In this case $\mathscr{I}$ is the ordinary (site-shift) invariant σ-algebra and the invariance property is stationarity.

The last example covers random fields with site space $(Z^d, +)$, where Z denotes the integers and d is a positive integer. This case treated in the next section.

4. Shifting random fields. In this section let $(H, \mathscr{H})$ have the following structure: with R the real line, d a positive integer and $(E, \mathscr{E})$ an arbitrary measurable space, assume that H is a shift-invariant subset of E^{R^d}, that $\mathscr{H}$ is the σ-algebra on H generated by the projection mappings taking $x =$

$(x_s)_{s \in R^d}$ in H to x_t in E, $t \in R^d$, and (canonical measurability) that the mapping taking (x, t) in $H \times R^d$ to x_t in E is $\mathscr{H} \otimes \mathscr{B}^d / \mathscr{E}$ measurable, where $\mathscr{B}^d$ denotes the Borel subsets of R^d.

Call a random element X in $(H, \mathscr{H})$ a canonically measurable random field in d dimensions and a random element T in $(R^d, \mathscr{B}^d)$ a random site. Call $(H, \mathscr{H})$ the path space and $(R^d, \mathscr{B}^d)$ the site space. Define the shift maps θ_t, $t \in R^d$, by

$$\theta_t x = (x_{t+s})_{s \in R^d}, \qquad x \in H.$$

The invariant σ-algebra is $\mathscr{I} = \{A \in \mathscr{H}\colon \theta_t A = A, t \in R^d\}$ and λ is Lebesgue measure on $(R^d, \mathscr{B}^d)$.

THEOREM 2. *Let X and X' be canonically measurable random fields in d dimensions. Let $B_h \in \mathscr{B}^d$, $0 < h < \infty$, be a family of sets satisfying $0 < \lambda(B_h) < \infty$ and, for all $t \in R^d$,*

$$\lambda(B_h \cap (t + B_h))/\lambda(B_h) \to 1, \qquad \text{as } h \to \infty, \tag{1}$$

and let $U(B_h)$ be a random site which is uniform on B_h and independent of X and X'. Let $g \in \mathscr{I}$ be a strictly positive and finite function. The following claims are equivalent:

(a) *The distributions of X and X' agree on $\mathscr{I}$.*
(b) *$\theta_T X =_D X'$ for some random site T.*
(c) $\|\mathbf{P}(\theta_{g(X)U(B_h)} X \in \cdot) - \mathbf{P}(\theta_{g(X')U(B_h)} X' \in \cdot)\| \to 0$ *as $h \to \infty$.*

PROOF. The equivalence of (a) and (b) follows from Theorem 1. In order to show that (b) implies (c), note that for any random site T it holds that

$$\begin{aligned} &\lambda(B_h)^{-1} \left(\int_{B_h} 1\{\theta_{g(X)s} X \in A\}\, ds - \int_{B_h} 1\{\theta_{g(X)s} \theta_T X \in \cdot\}\, ds \right) \\ &\qquad \le 1 - \lambda\big(B_h \cap (T/g(X) + B_h)\big)/\lambda(B_h) \end{aligned}$$

and that $g(\theta_T X) = g(X)$. If (b) holds, this yields

$$\begin{aligned} &\|\mathbf{P}\big(\theta_{g(X)U(B_h)} X \in \cdot\big) - \mathbf{P}\big(\theta_{g(X')U(B_h)} X' \in \cdot\big)\| \\ &\qquad \le 2 - 2\mathbf{E}\big[\lambda\big(B_h \cap (T/g(X) + B_h)\big)/\lambda(B_h)\big] \end{aligned}$$

and (c) follows by bounded convergence due to (1). Conversely, note that

$$\mathbf{P}(X \in \cdot) - \mathbf{P}(X' \in \cdot) = \mathbf{P}\big(\theta_{g(X)U(B_h)} X \in \cdot\big) - \mathbf{P}\big(\theta_{g(X')U(B_h)} X' \in \cdot\big) \quad \text{on } \mathscr{I}.$$

If (c) holds, then the right-hand side tends to 0 as $h \to \infty$ and (a) follows. □

5. Remarks on Theorem 2.

REMARK 4. If X' is stationary, that is, if $\theta_t X' =_D X'$, $t \in R^d$, then we can write (c) as

$$\theta_{g(X)U(B_h)} X \to X' \quad \text{in total variation as } h \to \infty.$$

If we choose $g \equiv 1$, then this becomes $\theta_{U(B_h)}X \to X'$ in total variation as $h \to \infty$.

REMARK 5. An example of sets satisfying (1) is

$$B_h = hB, \qquad 0 < h < \infty,$$

where $B \in \mathscr{B}^d$ and $0 < \lambda(B) < \infty$. In order to establish (1), note first that

$$\lambda(hB \cap (t + hB))/\lambda(hB) = \lambda(B \cap (t/h + B))/\lambda(B), \qquad t \in R^d.$$

Let f_n, $n \geq 1$, be a sequence of bounded continuous functions such that $\|1_B - f_n\|_2 \to 0$ as $n \to \infty$, where $\|\cdot\|_2$ denotes the L_2-norm w.r.t. λ. Now

$$\begin{aligned}\lambda(B) - \lambda(B \cap (t/h + B)) &= 2^{-1}\int (1_B - 1_{t/h+B})^2 \, d\lambda \\ &= 2^{-1}(\|1_B - 1_{t/h+B}\|_2)^2,\end{aligned}$$

which tends to 0 as $h \to \infty$ since, sending first $h \to \infty$ and then $n \to \infty$,

$$\|1_B - 1_{t/h+B}\|_2 \leq 2\|1_B - f_n\|_2 + \|f_n - f_n(\cdot - t/h)\|_2 \to 2\|1_B - f_n\|_2 \to 0.$$

REMARK 6. Theorem 2 has an immediate application in ergodic theory (see, e.g., [5]). Suppose X' is stationary and satisfies, for bounded $\mathscr{H}/\mathscr{B}$ measurable functions f and with B_h as in Theorem 2,

$$\lambda(B_h)^{-1} \int_{B_h} f(\theta_s X') \, ds \to \mathbf{E}\big[f(X') \mid X'^{-1}\mathscr{I}\big] \quad \text{a.s., } h \to \infty.$$

Then [due to the first inequality in the proof of Theorem 2 and the fact that (a) implies (b)] the following holds: for all X agreeing with X' in distribution on $\mathscr{I}$ it holds that

$$\lambda(B_h)^{-1} \int_{B_h} f(\theta_t X) \, ds \quad \text{tends a.s. to a limit as } h \to \infty$$

and the limit has the same distribution as $\mathbf{E}[f(X') \mid X'^{-1}\mathscr{I}]$. This remark of course also applies in the abstract group setting.

6. Preparing for the proof of Theorem 1. We shall need the following result.

LEMMA 1. *Let $(K, \mathscr{K})$ and $(L, \mathscr{L})$ be measurable spaces and f a measurable mapping from $(K, \mathscr{K})$ to $(L, \mathscr{L})$. If α is a finite measure on $(K, \mathscr{K})$ and β is a component of αf^{-1} (i.e., β is a measure satisfying $\beta \leq \alpha f^{-1}$ setwise), then there exists a component ν of α such that $\nu f^{-1} = \beta$.*

PROOF. Define a measure ρ on $f^{-1}\mathscr{L}$ by

$$\rho f^{-1} B = \beta B, \qquad B \in \mathscr{L}; \tag{2}$$

ρ is well defined since if B_1 and B_2 are in $\mathscr{L}$ and $f^{-1}B_1 = f^{-1}B_2$, then $\alpha f^{-1}(B_1 \triangle B_2) = 0$ and $\beta \le \alpha f^{-1}$ yields $\beta(B_1 \triangle B_2) = 0$ and thus $\beta(B_1) = \beta(B_2)$. Define a set function ν on $\mathscr{H}$ by

$$\nu(A) = \int \alpha(A \mid f^{-1}\mathscr{L})\, d\rho, \qquad A \in \mathscr{H}. \tag{3}$$

From (2) and $\beta \le \alpha f^{-1}$ we obtain $\rho \le \alpha$ on $f^{-1}\mathscr{L}$ and thus ν does not depend on the version of $\alpha(A \mid f^{-1}\mathscr{L})$. Thus for a given sequence of disjoint sets in $\mathscr{H}$ we can choose a version which is σ-additive for that particular sequence. Hence ν is σ-additive. Since $\rho \le \alpha$ on $f^{-1}\mathscr{L}$ we obtain, by replacing ρ by α in (3), that $\nu \le \alpha$ on $\mathscr{H}$. Since $\alpha(A \mid f^{-1}\mathscr{L})$ is α a.e. the indicator of A, for $A \in f^{-1}\mathscr{L}$, we have $\nu = \rho$ on $f^{-1}\mathscr{L}$ and (2) yields that $\nu f^{-1} = \beta$, and the lemma is established. □

7. Proof of Theorem 1. Let μ_1 and μ_1' be the distributions of X and X', respectively. Certainly $\Gamma X =_D X'$ implies that $\mu_1 = \mu_1'$ on $\mathscr{I}$. In order to prove the converse, assume that $\mu_1 = \mu_1'$ on $\mathscr{I}$, let λ be the Haar measure on $(G, \mathscr{G})$ and let π be a probability measure which has the same null sets as λ [such a π exists because λ is σ-additive: if $\lambda(H) = \infty$, let $A_n \in \mathscr{H}$, $n \ge 1$, be a countable partition of H with $0 < \lambda(A_n) < \infty$ and put $\pi(\cdot \cap A_n) = 2^{-n}\lambda(\cdot \cap A_n)/\lambda(A_n)$]. Let $\wedge$ denote infimum of measures (greatest common component) and put

$$f(x, \gamma) = \gamma x.$$

Apply Lemma 1 recursively to obtain that for each $n \ge 1$ there are subprobability measures ν_n and ν_n' on $(H, \mathscr{H}) \otimes (G, \mathscr{G})$ such that [in the lemma take $\alpha = \mu_n \otimes \pi$ and $\alpha = \mu_n' \otimes \pi$, respectively, and $\beta = (\mu_n \otimes \pi)f^{-1} \wedge (\mu_n' \otimes \pi)f^{-1}$]

$$\begin{aligned}
&\nu_n \le \mu_n \otimes \pi, \quad \text{where } \mu_n = \mu_1 - \nu_1(\cdot \times G) - \cdots - \nu_{n-1}(\cdot \times G),\\
&\nu_n' \le \mu_n' \otimes \pi, \quad \text{where } \mu_n' = \mu_1' - \nu_1'(\cdot \times G) - \cdots - \nu_{n-1}'(\cdot \times G),\\
&\nu_n f^{-1} = \nu_n' f^{-1} = (\mu_n \otimes \pi)f^{-1} \wedge (\mu_n' \otimes \pi)f^{-1}.
\end{aligned}$$

Thus $(\mu_n \otimes \pi - \nu_n)f^{-1}$ and $(\mu_n' \otimes \pi - \nu_n')f^{-1}$ are mutually singular. That is, for each $n \ge 1$ there is an $A_n \in \mathscr{H}$ such that

$$(\mu_n \otimes \pi)f^{-1}A_n = \nu_n f^{-1}A_n \quad \text{and} \quad (\mu_n' \otimes \pi)f^{-1}A_n^c = \nu_n' f^{-1}A_n^c. \tag{4}$$

Put

$$\begin{aligned}
\nu &= \nu_1 + \nu_2 + \cdots, & \nu' &= \nu_1' + \nu_2' + \cdots,\\
\mu_\infty &= \mu_1 - \nu(\cdot \times G), & \mu_\infty' &= \mu_1' - \nu'(\cdot \times G)
\end{aligned}$$

and note that

$$\nu f^{-1} = \nu' f^{-1}, \qquad \nu(\cdot \times G) \le \mu_1 \quad \text{and} \quad \nu'(\cdot \times G) \le \mu_1'.$$

From $\nu f^{-1} = \nu' f^{-1}$ we obtain that $\nu(\cdot \times G) = \nu'(\cdot \times G)$ on $\mathscr{I}$ and thus (since $\mu_1 = \mu_1'$ on $\mathscr{I}$)

$$\mu_\infty = \mu_\infty' \quad \text{on } \mathscr{I}. \tag{5}$$

Since $\mu_n \geq \mu_\infty$ and $\mu'_n \geq \mu'_\infty$, we obtain from (4) that for each $n \geq 1$,

$$\int \left(\int 1\{\gamma x \in A_n\} \pi(d\gamma) \right) \mu_\infty(dx) \leq \|\nu_n\|,$$

$$\int \left(\int 1\{\gamma x \in A_n^c\} \pi(d\gamma) \right) \mu'_\infty(dx) \leq \|\nu'_n\|.$$

Put

$$A = \limsup_{n \to \infty} A_n.$$

Since $1\{\gamma x \in A\} \leq 1\{\gamma x \in A_n\} + 1\{\gamma x \in A_{n+1}\} + \cdots$ and $1\{\gamma x \in A^c\} \leq 1\{\gamma x \in A_n^c\} + 1\{\gamma x \in A_{n+1}^c\} + \cdots$, we obtain that, for each $n \geq 1$,

$$\int g \, d\mu_\infty \leq \|\nu_n + \nu_{n+1} + \cdots\| \quad \text{and} \quad \int (1-g) \, d\mu'_\infty \leq \|\nu'_n + \nu'_{n+1} + \cdots\|,$$

where

$$g(x) = \int 1\{\gamma x \in A\} \pi(d\gamma) \quad (\text{note that } 0 \leq g \leq 1).$$

Send $n \to \infty$ to obtain

$$\int g \, d\mu_\infty = 0 \quad \text{and} \quad \int (1-g) \, d\mu'_\infty = 0.$$

Put $B = \{x \in H: g(x) > 0\}$ and note that B^c is a subset of $\{x \in H: g(x) < 1\}$. Thus

$$\mu_\infty(B) = 0 \quad \text{and} \quad \mu'_\infty(B^c) = 0. \tag{6}$$

Now take $\psi \in G$ and note that

$$\begin{aligned}
\psi x \in B^c &\Leftrightarrow 1\{\gamma\psi x \in A\} = 0 \quad (\text{for } \pi \text{ a.e. } \gamma) \\
&\Leftrightarrow 1\{\gamma\psi x \in A\} = 0 \\
&\qquad\qquad [\text{for } \lambda \text{ a.e. } \gamma \ (\text{since } \pi \text{ and } \lambda \text{ have the same null sets})] \\
&\Leftrightarrow 1\{\gamma x \in A\} = 0 \quad [\text{for } \lambda \text{ a.e. } \gamma \ (\text{since } \lambda \text{ is invariant under } G)] \\
&\Leftrightarrow x \in B^c.
\end{aligned}$$

Thus $B \in \mathscr{I}$ and (5) yields $\mu_\infty(B) = \mu'_\infty(B)$ and $\mu_\infty(B^c) = \mu'_\infty(B^c)$. This and (6) imply that μ_∞ and μ'_∞ have mass 0. That is, ν and ν' are probability measures and

$$\mu_1 = \nu(\cdot \times G), \qquad \mu'_1 = \nu'(\cdot \times G). \tag{7}$$

Take Ψ and Ψ' such that (X, Ψ) has distribution ν and (X', Ψ') has distribution ν'. This is possible since (7) holds and $(G, \mathscr{G})$ is Polish, which implies that there exists a regular version of the conditional probabilities

$$\nu(H \times \cdot \mid \mathscr{H} \otimes \{\varnothing, G\}) \quad \text{and} \quad \nu'(H \times \cdot \mid \mathscr{H} \otimes \{\varnothing, G\}).$$

Now $\nu f^{-1} = \nu' f^{-1}$ implies that $\Psi X =_D \Psi' X'$, and since $(G, \mathscr{G})$ is Polish there exists a regular version of the conditional distribution of Ψ' given $\Psi' X'$. Thus there is a random transformation Φ in $(G, \mathscr{G})$ such that

$$(\Phi, \Psi X) =_D (\Psi', \Psi' X').$$

This implies that $\Phi^{-1}\Psi X =_D X'$, where Φ^{-1} denotes the pointwise group inverse of Φ [and not the inverse of Φ as a mapping from the underlying probability space to $(G, \mathscr{G})$]. Taking $\Gamma = \Phi^{-1}\Psi$ completes the proof of Theorem 1. $\square$

Acknowledgments. I wish to thank the referee for useful comments and Sigurdur Helgason for expert information on topological groups.

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