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WEAK LIMITS OF PERTURBED RANDOM WALKS AND THE EQUATION $Y_t = B_t + \alpha \sup\{Y_s : s \le t\} + \beta \inf\{Y_s : s \le t\}$[1]

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Let α and β be real numbers and $f \in C_0[0,\infty)$. We study the existence and uniqueness of solutions g of the equation $g(t) = f(t) + \alpha \sup_{0 \le s \le t} g(s) + \beta\, inf_{0 \le s \le t}\, g(s)$. Carmona, Petit, Le Gall, and Yor have shown existence (or nonexistence) and uniqueness for some α, β. We settle the remaining cases. We study the nearest neighbor walk on the integers, which behaves just like fair random walk unless one neighbor has been visited and the other has not, when it jumps to the unvisited neighbor with probability p. If $p < 2/3$, we show these processes, scaled, converge to the solution of the equation above for Brownian paths, with $\alpha = \beta = (2p - 1)/p$.

1. Introduction. If f is a real-valued function on $[0,\infty)$, we put $f^*(t) = \sup_{0 \le s \le t} f(s)$ and $f^\#(t) = \inf_{0 \le s \le t} f(s)$, and also we use * and $^\#$ to denote maxima and minima of sequences. We study the existence and uniqueness of solutions g of the equation

$$g(t) = f(t) + \alpha g^*(t) + \beta g^\#(t), \qquad t \ge 0. \tag{1.1}$$

Here α and β are real numbers and f is a continuous function vanishing at 0, an assumption always in force whenever (1.1) is discussed, without further mention. This equation was first studied by Le Gall (1986), and more recently, in a paper that will hereafter be referred to as CPY, by Carmona, Petit and Yor (1994). Let $\rho = \alpha\beta/((1-\alpha)(1-\beta))$. It is shown in CPY that if either $\alpha \ge 1$ or $\beta \ge 1$, there are f for which (1.1) has no solution, while if $\alpha < 1$, $\beta < 1$ and $|\rho| < 1$, there is a unique solution for every f. From now on we assume $\alpha < 1$ and $\beta < 1$. Le Gall and Yor (1992) study a closely related equation, essentially (1.1) with $\beta = -\infty$. Their methods adapt to prove existence of solutions of (1.1) for all α, β, f. This is more carefully explained before the statement of our Lemma 2.3. In Section 2 we prove results which, when combined with those mentioned, yield the following theorem.

THEOREM 1.1. *If* $|\rho| \le 1$, (1.1) *has a unique solution for each* f. *If* $|\rho| > 1$, *there is at least one solution of* (1.1) *for each* f *and there are functions* $f = f_{\alpha,\beta}$ *with more than one solution.*

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Our contribution to this theorem is the uniqueness when $|\rho| = 1$ and the nonuniqueness when $|\rho| > 1$. Our methods in the $|\rho| = 1$ case adapt to settle, affirmatively, the $\alpha = 1/2$ case of the question asked at the end of Le Gall and Yor (1992).

If $|\rho| < 1$, we show, with the aid of a result from CPY, that there is a constant $C = C_{\alpha,\beta}$ such that if g_1 and g_2 solve (1.1) for f_1 and f_2, respectively, then $\sup_{s \le t}|g_1(s) - g_2(s)| \le C \sup_{s \le t}|f_1(s) - f_2(s)|$. This is not true if $|\rho| = 1$, when there can be a "butterfly effect." There are functions f_1 and f_2, which agree in $[\varepsilon, \infty)$ and which never differ by more than ε, such that $g_1 - g_2$ is unbounded. This is discussed further at the end of Section 2.

CPY shows that if $|\rho| < 1$, then the solution of (1.1) for Brownian paths, that is, the process $\mathbf{Y}^{\alpha,\beta} = \mathbf{Y} = Y_t$, $t \ge 0$, defined by

$$Y_t = B_t + \alpha Y_t^* + \beta Y_t^{\#}, \tag{1.2}$$

where $\mathbf{B} = B_t$, $t \ge 0$, is Brownian motion started at 0, is adapted to the filtration of $\mathbf{B}$. It is easy to extend this result to the cases $|\rho| = 1$, using the proof of the existence and uniqueness of solutions of (1.1) for these α, β.

In Section 3 we show that if $|\rho| < 1$, the solution $\mathbf{Y}$ of (1.2) can be identified as the weak limit of a discrete process. If $\mathbf{Z} = Z_0, Z_1, \dots$ is a discrete time stochastic process, we identify it with the continuous time process on $[0, \infty)$ which results from linearly interpolating: $Z_t = Z_n + (t - n)[Z_{n+1} - Z_n]$, if $n \le t \le n + 1$.

THEOREM 1.2. *Define the integer-valued stochastic process* $\mathbf{X}_{\alpha,\beta} = \mathbf{X} = X_0, X_1, X_2, \dots$ *by* $X_0 = 0$, $P[X_{n+1} = X_n + 1 | X_i, i \le n] = 1 - P[X_{n+1} = X_n - 1 | X_i, i \le n] = \frac{1}{2}$ *if* $n = 0$ *or* $X_n^{\#} < X_n < X_n^*$; $= 1/(2 - \alpha)$ *if* $n > 0$ *and* $X_n = X_n^*$; $= 1 - 1/(2 - \beta)$ *if* $n > 0$ *and* $X_n = X_n^{\#}$.

Then if $|\rho| < 1$, *the processes* $(1/\sqrt{n})\mathbf{X}_{nt}$, $t \ge 0$, *converge weakly to* $\mathbf{Y}^{\alpha,\beta}$.

The $\alpha = 0$ (and $\beta = 0$) cases of Theorem 1.2 have been proved by Werner (1994). It seems very likely that the analog of Theorem 1.2 for $|\rho| = 1$ holds, but our proof does not extend to this case. It is also likely that, for all α and β, the processes $\mathbf{X}_{\alpha,\beta}$ converge weakly, but it is not clear that the limit process can be constructed a.s. path by path by solving (1.2). Several people have suggested that the excursion theory of Perman (1995), for the solutions of (1.2) when $\alpha = 0$, may help settle the remaining weak convergence questions.

If $\alpha = \beta < 0$, the processes $\mathbf{X}_{\alpha,\alpha}$ can be realized as the simplest of the reinforced random walks: if we assign a weight of 1 to each "bond" $(i, i+1)$ which has not been crossed by $\mathbf{X}_{\alpha,\alpha}$ and assign weight $1 - \alpha$ to bonds which have been crossed, then $\mathbf{X}_{\alpha,\alpha}$ may be described as jumping up or down with probabilities proportional to the weights of the connecting bonds. See Davis (1990) for more details. Recent papers at least partly concerned with reinforced random walk include Diaconis (1988), Pemantle (1988, 1992), Davis (1989), Sellke (1994a, b), Tóth (1994, 1995, 1996) and Othmer and Stevens (1995). Bolthausen and Schmock (1994) proved weak convergence for a different kind of non-Markovian walk. Harrison and Shepp (1981) proved weak convergence of the (Markovian) walk which behaves like fair random walk except at zero, where it goes up with probability p.

Our study of the processes $\mathbf{X}_{\alpha,\beta}$ and $\mathbf{Y}^{\alpha,\beta}$ was motivated by Nester (1994), where stopping times for the processes $\mathbf{X}_{\alpha,\beta}$, when $\alpha=\beta$, were studied. Many of Nester's results translate immediately to results about the limiting processes $\mathbf{Y}^{\alpha,\alpha}$, of course only in the $\alpha<\frac{1}{2}$ case for now, since Theorem 1.2 does not cover other α. Nester's formulas, in common with the formulas in CPY, are very pretty and often involve beta densities. For example, Nester's results show the probability that $\mathbf{Y}^{\alpha,\alpha}$ equals a before it equals $-b$, if $a,b>0$, is $\int_0^{b/(a+b)}t^{-\alpha}(1-t)^{-\alpha}\,dt/\int_0^1 t^{-\alpha}(1-t)^{-\alpha}\,dt$.

2. Proof of Theorem 1.1. Recall that α and β will both be assumed to be less than 1, often without mention. If g and f satisfy (1.1), that is, if g solves (1.1) for f, then our assumptions that f is continuous and vanishes at zero are easily seen to imply that g has these properties. Positive absolute constants which depend only on α and β are usually denoted by c and C; subscripts will be used to denote dependence on various quantities. We put $a^+=\max(a,0)$, $a\vee b=\max(a,b)$ and $a\wedge b=\min(a,b)$. If h is a function on $[a,b]$ we let $h^*[a,b]=\max_{a\le x\le b}h(x)$ and $h^\#[a,b]=\min_{a\le x\le b}h(x)$. Let $C_0[0,\infty)$ denote the continuous functions on $[0,\infty)$ which vanish at 0.

LEMMA 2.1. *Let g solve* (1.1) *for f and let* $0\le a<b<\infty$. *Then*

$$g^*[a,b]-g^\#[a,b]\le\max\left[1,(1-\alpha)^{-1},(1-\beta)^{-1}\right]\times\left(f^*[a,b]-f^\#[a,b]\right). \tag{2.1}$$

PROOF. Suppose, first, that g achieves its minimum in $[a,b]$ before it achieves its maximum; that is, there exist $a\le s<r\le b$ such that $g^\#[a,b]=g(s)$ and $g^*[a,b]=g(r)$. Since $g^\#(s)=g^\#(r)$, subtracting the version of (1.1) with $t=s$ from the version with $t=r$ gives

$$g(r)-g(s)=f(r)-f(s)+\alpha(g^*(r)-g^*(s)).$$

If $\alpha\le 0$, this gives $g(r)-g(s)\le f(r)-f(s)$; if $0<\alpha<1$, it gives $(g(r)-g(s))(1-\alpha)\le f(r)-f(s)$, since $g^*(r)-g^*(s)=g(r)-g^*(s)\le g(r)-g(s)$. These inequalities give (2.1) in this case. The case where g achieves its minimum before it achieves its maximum is similar. □

COROLLARY 2.2. *Let $f\in C_0[0,\infty)$. Suppose that f_n, $n\ge 1$, converges uniformly to f on compact subintervals of $[0,\infty)$ and that g_n solves* (1.1) *for f_n. Then there is a subsequence $m(n)$, $n\ge1$, of integers such that $g_{m(n)}$, $n\ge1$, converges uniformly on compact subintervals of $[0,\infty)$. The limit of $g_{m(n)}$ solves* (1.1) *for f.*

PROOF. Lemma 2.1 and the fact that f_n, $n\ge1$, is equicontinuous and uniformly bounded on compact subintervals of $[0,\infty)$ imply that g_n, $n\ge1$, is also equicontinuous and uniformly bounded. Thus the Arzela–Ascoli theorem and a diagonalization argument give the desired sequence $g_{m(n)}$, $n \geq 1$. Since the functions $g_{m(n)}$ converge uniformly, say to g, $g^*_{m(n)}(t)$ and $g^\#_{m(n)}(t)$ converge to $g^*(t)$ and $g^\#(t)$, respectively, for each $t \geq 0$. This implies g solves (1.1) for f. □

Our proof of the following lemma, the key for existence, resembles the proof of Proposition 6.2 of Le Gall and Yor (1992).

LEMMA 2.3. *Suppose either of the following two conditions hold*:

(a) *There are* $\delta > 0$, $c \neq 0$, *such that* $f(x) = cx$, $0 \leq x \leq \delta$.
(b) *There are* $\delta > 0$, $c \neq 0$, *such that* $f(x) = 0$, $0 \leq x \leq \delta/2$, *and* $f(x) = c(x - (\delta/2))$, $\delta/2 \leq x \leq \delta$.

Then (1.1) *has a solution for* f.

PROOF. We remark that in (a), the "$0 \leq x \leq \delta$" stands for "if $0 \leq x \leq \delta$." We omit "if's" throughout the paper. If $g \in C_0[0,\infty)$ solves (1.1) for f, then both the following hold:

(2.2) If $[a,b] \subset [0,\infty)$ and if $g(x) \geq g^\#(a)$, $x \in [a,b]$, and if $s = \inf\{t \geq a: g^*(a) = g(t)\}$, then $g(t) - g(a) = f(t) - f(a)$, $a \leq t \leq b$, if $s \geq b$, while if $s < b$, $g(t) - g(a) = f(t) - f(a)$, $a \leq t \leq s$, and $g(t) - g(s) = f(t) - f(s) + \alpha(1-\alpha)^{-1} \max_{s \leq x \leq t}(f(x) - f(s))$, $s \leq t \leq b$.

(2.3) If $[a,b] \subset [0,\infty)$ and if $g(x) \leq g^*(a)$, $x \in [a,b]$, and if $r = \inf\{t \geq a: g^\#(a) = g(t)\}$, then $g(t) - g(a) = f(t) - f(a)$, $a \leq t \leq b$, if $r \geq b$, while if $r < b$, $g(t) - g(a) = f(t) - f(a)$, $a \leq t \leq r$, and $g(t) - g(r) = f(t) - f(r) + \beta(1-\beta)^{-1} \min_{r \leq x \leq t}(f(x) - f(r))$, $r \leq t \leq b$.

To see (2.2), note that the only nontrivial part concerns the formula for $g(t) - g(s)$ when $s < b$. Now (1.1) gives

$$f(t) - f(s) = (g(t) - g(s)) - \alpha(g^*(t) - g^*(s)),$$

which equals $(1-\alpha)(g^*(t) - g^*(s))$ when $g(t) = g^*(t)$ and is smaller than $(1-\alpha)(g^*(t) - g^*(s))$ when $g(t) < g^*(t)$. Thus $g(t) = g^*(t)$ exactly for those t for which $f(t) = \max_{s \leq x \leq t} f(x)$. This verifies $g(t) - g(s) = f(t) - f(s) + \alpha(1-\alpha)^{-1} \max_{s \leq x \leq t}(f(x) - f(s))$ if $s \leq t \leq b$ and $g(t) = g^*(t)$. To verify it for other $t \in [s,b]$, let $t_0 = \sup\{x < t: g(x) = \max_{0 \leq y \leq x} g(y)\}$ and use its truth for t_0 and the fact that, by (1.1), $f(t) - f(t_0) = g(t) - g(t_0)$. The proof that (1.1) implies (2.3) is similar.

It is also true that (2.2) and (2.3) imply that g solves (1.1) for f, provided $g \in C_0[0,\infty)$. We just sketch this argument. To show (1.1) it suffices to prove (2.4) and (2.5):

(2.4) If $[a,b] \subset [0,\infty)$ and $g(x) \geq g^{\#}(a)$, $x \in [a,b]$, then $g(b) - g(a) = f(b) - f(a) + \alpha(g^*(b) - g^*(a))$.

(2.5) If $[a,b] \subset [0,\infty)$ and $g(x) \leq g^*(a)$, $x \in [a,b]$, then $g(b) - g(a) = f(b) - f(a) + \beta(g^{\#}(b) - g^{\#}(a))$.

That (2.4) and (2.5) imply (1.1) is not difficult: fix t, let $0 < \varepsilon < t$ and break $[\varepsilon, t]$ into disjoint intervals $[a,b]$ on which either $g(x) \geq g^{\#}(a)$ or $g(x) \leq g^*(a)$. Take the results of (2.4) and (2.5) on these intervals and add them. Then let $\varepsilon \to 0$. To show (2.2) implies (2.4), let s be as in (2.2) and observe the implication is trivial if $s \geq b$, while if $s < b$, let $\theta = \max\{t \in [s,b]: g^*(t) = g(t)\}$. Then $g(s) - g(a) = f(s) - f(a)$, $g(b) - g(\theta) = f(b) - f(\theta)$ and, recalling the discussion after (2.3), $g(\theta) - g(s) = g^*(b) - g^*(a) = (1-\alpha)^{-1}(f(\theta) - f(s))$. Adding these three expressions gives (2.4). The proof that (2.3) implies (2.5) is similar.

We prove part (a) of Lemma 2.3 first. Suppose $c > 0$. We construct g by putting $g(x) = cx/(1-\alpha)$, $0 \leq x \leq \delta$, and then using (2.2) and (2.3) as a recipe for constructing $g(t)$ for $t > \delta$. Since $g(\delta) = g^*(\delta) > g^{\#}(\delta)$, (2.2) dictates $g(t)$, $\delta \leq t \leq y$, where $y = \inf\{x > \delta: g(x) = g^{\#}(x)\}$. Then (2.3) dictates $g(t)$, $y \leq t \leq z = \inf\{x > y: g(x) = g^*(x)\}$ and so on. The $c < 0$ case is very similar. Part (b) of Lemma 2.3 is established in a similar way by first explicitly exhibiting a solution on $[0, \delta]$, which is 0 on $[0, \delta/2]$ and linear on $[\delta/2, \delta]$. □

COROLLARY 2.4. *There is at least one solution of* (1.1) *for every* f.

PROOF. Suppose, first, that there is a sequence $t_n \downarrow 0$ such that $f(t_n) \neq 0$. Let $f_n(t) = tf(t_n)/t_n$, $0 \leq t \leq t_n$, and $f_n(t) = f(t)$, $t \geq t_n$. Then Lemma 2.3 guarantees that (1.1) has a solution for f_n, and Corollary 2.2 gives a solution for f. If f is not the 0 function, but $f = 0$ on $[0,\delta]$ for some $\delta > 0$, let $\varepsilon = \sup\{s: f(t) = 0, 0 \leq t \leq s\}$, put $g = 0$ on $[0, \varepsilon]$ and for $t \geq \varepsilon$ mimic the argument above. Of course, if f is the 0 function, we may take $g = f$. □

The proof of Theorem 1.1 will be completed by proving three propositions, each of which treats some of the α, β not covered by the CPY results. Recall these results settled the issue for $|\rho| < 1$. Our propositions consider, respectively, $\rho = 1$, $\rho = -1$ and $|\rho| > 1$.

Our methods will also handle the parts of Theorem 1.1 proved in CPY.

LEMMA 2.5. *Let* g_1 *and* g_2 *be solutions of* (1.1) *for* f *and suppose* $t > 0$ *and* $f^*(t) > f^{\#}(t)$. *It cannot happen that both* $g_1(t) = g_1^*(t)$ *and* $g_2(t) = g_2^{\#}(t)$.

PROOF. First note that $g_1^{\#}(t) < g_1^*(t)$, because, since f is not identically zero on $[0, t]$, neither is g_1. Thus if $g_1(t) = g_1^*(t)$, there is $0 < s < t$ such that $g_1(s) < g_1^*(t)$ and $g_1(r) \neq g_1^{\#}(r)$, $s \leq r \leq t$. Then (2.2) implies $f(t) = \max_{s \leq r \leq t} f(r) > f(s)$. Similarly, if $g_2(t) = g_2^{\#}(t)$, there is $0 < y < t$ such that $f(y) > \min_{y \leq r \leq t} f(r) = f(t)$. $\square$

LEMMA 2.6. *Let $0 < p < \infty$. Let a_k, $0 \leq k \leq n$, and b_k, $0 \leq k \leq n$, be sequences of numbers such that $b_{k+1} - b_k = -p(a_{k+1} - a_k)$, $n \geq 0$. Then $pa_k + b_k = pa_0 + b_0$, $0 \leq k \leq n$.*

The proof of Lemma 2.6 is immediate.

PROPOSITION 2.7. *If $0 < \alpha < 1$ and $\beta = 1 - \alpha$, then there do not exist two different solutions of* (1.1) *for any f.*

PROOF. Think of f as fixed. We assume that $f^*(t) - f^{\#}(t) > 0$, $t > 0$. Only minor alterations in our proof are required if this does not hold. Let g_1 and g_2 be solutions of (1.1) for f. We will prove

$$g_1(b) - g_2(b) \leq g_1(a) - g_2(a), \qquad 0 < a < b. \tag{2.6}$$

Upon letting a go to zero, (2.6) gives $g_1(b) \leq g_2(b)$ and, of course, switching the roles of g_1 and g_2, we get $g_1(b) \geq g_2(b)$, verifying the proposition.

For $t \geq 0$, define

$$\Delta(t) = g_1(t) - g_2(t), \qquad {}^*\Delta(t) = g_1^*(t) - g_2^*(t),$$
$$P^+(t) = \left[\left(g_2^*(t) - g_2(t)\right) - \left(g_1^*(t) - g_1(t)\right)\right] = \Delta(t) - {}^*\Delta(t)$$

and

$$P^-(t) = \left[\left(g_1(t) - g_1^{\#}(t)\right) - \left(g_2(t) - g_2^{\#}(t)\right)\right].$$

We say that an interval $I = [c, d] \subset [0, \infty)$ is *positive* if $g_i(t) > g_i^{\#}(c)$, $c < t < d$, $i = 1, 2$, and we say that I is *negative* if $g_i(t) < g_i^*(c)$, $c < t < d$, $i = 1, 2$. Equation (2.2) implies

$$\begin{aligned} g_i^*(d) - g_i^*(c) = (1-\alpha)^{-1} & \\ \times\left[f^*([c, d]) - f(c) - \left(g_i^*(c) - g_i(c)\right)\right]^+, & \\ i = 1, 2, [c, d] \text{ positive.} & \end{aligned} \tag{2.7}$$

To see this, note that $g_i^*(d) - g_i^*(c) = g_i^*(d) - g_i^*(t)$, where $t = \inf\{s \leq d\colon g_i(s) = g_i^*(c)\}$. Then, recalling the argument in the paragraph after (2.3), (2.2) gives $g_i(s) = g_i^*(s)$ if and only if $f(s) = f^*[c, s]$, if $t \leq s \leq d$, and (2.7) follows. Equation (2.7) implies that ${}^*\Delta(d) - {}^*\Delta(c)$ lies between 0 and $(1-\alpha)^{-1}P^+(c)$, inclusive. Now (2.2) gives

$$\begin{aligned} g_i(d) - g_i(c) = f(d) - f(c) + \alpha\left(g_i^*(d) - g_i^*(c)\right), & \\ i = 1, 2, [c, d] \text{ positive.} & \end{aligned} \tag{2.8}$$

Subtracting the $i = 2$ version of (2.8) from the $i = 1$ version gives

$$\Delta(d) - \Delta(c) = \alpha({}^*\Delta(d) - {}^*\Delta(c)), \qquad [c, d] \text{ positive,} \tag{2.9}$$

which in turn gives

$$P^+(d) - P^+(c) = (\alpha - 1)({}^*\Delta(d) - {}^*\Delta(c)), \qquad [c, d] \text{ positive.} \tag{2.10}$$

Also, (2.9), (2.10) and the fact that neither $g_1^\#$ nor $g_2^\#$ changes on a positive interval yield

$$\begin{aligned} P^-(d) - P^-(c) &= \Delta(d) - \Delta(c) \\ &= \frac{\alpha}{\alpha - 1}(P^+(d) - P^+(c)), \qquad [c, d] \text{ positive.} \end{aligned} \tag{2.11}$$

The sentence before (2.8), together with (2.10), shows it cannot happen that both $P^+(c) \geq 0$ and $P^+(d) < 0$ or that both $P^+(c) \leq 0$ and $P^+(d) > 0$. Furthermore,

$$|P^+(d)| \leq |P^+(c)|, \qquad [c, d] \text{ positive.} \tag{2.12}$$

A mirror set of equalities and inequalities holds for negative intervals. In particular, we have, recalling $\beta/(\beta - 1 = \alpha - 1/\alpha)$,

$$\begin{aligned} P^+(d) - P^+(c) &= \Delta(d) - \Delta(c) \\ &= \frac{\alpha - 1}{\alpha}(P^-(d) - P^-(c)), \qquad [c, d] \text{ negative.} \end{aligned} \tag{2.13}$$

Also we have

$$|P^-(d)| \leq |P^-(c)|, \qquad [c, d] \text{ negative,} \tag{2.14}$$

and it cannot happen that both $P^-(c) \geq 0$ and $P^-(d) < 0$ or that both $P^-(c) \leq 0$ and $P^-(d) > 0$ if $[c, d]$ is negative.

If $0 < r < s$, we say $r = a_0 < a_1 < a_2 < \cdots < a_n = s$ is a positive–negative decomposition of $[r, s]$ if each interval $[a_i, a_{i+1}]$, $1 \leq i \leq n$, is either positive or negative, and we let $\|[r, s]\|$ be the fewest such intervals possible. The following construction, here called the canonical decomposition, not only shows each interval has a positive–negative decomposition, but constructs one which clearly has no more than $2\|[r, s]\|$ intervals. Call t positive if either $g_1^*(t) = g_1(t)$ or $g_2^*(t) = g_2(t)$, and call t negative if either $g_2^\#(t) = g_2(t)$ or $g^\#(t) = g_1(t)$. Take $a_0 = r$, $a_1 = \min(\inf\{t > a_0: t \text{ is positive or negative}\}, s)$ if $a_1 < s$; take $a_2 = \min(\inf\{t > a_1: t \text{ is negative}\}, s)$ if a_1 is positive and $a_2 = \min(\inf\{t > a_1: t \text{ is positive}\}, s)$ if a_1 is negative; if $a_2 < s$, let a_3 be the next negative or positive number, depending on whether a_2 is positive or negative and so on. This process eventually yields an a_i equal to s, since otherwise Lemma 2.5 would be contradicted because the limit of positive (negative) numbers is positive (negative). We also observe that if $[u, v] \subset [r, s]$, then the intersection of $[u, v]$ with the intervals in the canonical decomposition of $[r, s]$ gives a positive–negative decomposition of $[u, v]$ with at most $2\|[u, v]\|$ intervals in it.

Let $0 < \varepsilon < a$. We prove

$$\Delta(b) - \Delta(a) \leq C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|)\|[a, b]\|. \tag{2.15}$$

Before proving (2.13), we note that both $P^+(\varepsilon) \to 0$ and $P^-(\varepsilon) \to 0$ as $\varepsilon \to 0$, so that (2.15) implies (2.6).

To prove (2.15), first consider the case where both $P^+(\varepsilon) \geq 0$ and $P^-(\varepsilon) \geq 0$. Let $\varepsilon = s_0 < s_1 < \cdots s_n = b$, where $[s_i, s_{i+1}]$, $0 \leq i < n$, are all the intervals which arise by intersecting the intervals in the canonical decomposition of $[\varepsilon, b]$ with both $[\varepsilon, a]$ and $[a, b]$. Then one of the s_i is a; designate it by s_m. The two sequences $P^+(s_i)$, $0 \leq i \leq n$, and $P^-(s_i)$, $0 \leq i \leq n$, are nonnegative by the sentences just before (2.12) and after (2.14), and by (2.11) and (2.13) they satisfy the conditions of Lemma 2.6, with $p = \alpha/(1-\alpha)$, $a_k = P^+(s_k)$ and $b_k = P^-(s_k)$. Thus both $P^+(s_i)$ and $P^-(s_i)$ are no larger than $C_\alpha(P^+(s_0) + P^-(s_0)) = C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|)$ and we have, using (2.11) and (2.13),

$$\begin{aligned} |\Delta(b) - \Delta(a)| &\leq \sum_{k=m}^{n-1} |\Delta(s_{k+1}) - \Delta(s_k)| \\ &\leq \sum_{k=m}^{n-1} C_\alpha(|P^+(s_k)| + |P^-(s_k)|) \\ &\leq C_\alpha(n-m)(|P^+(\varepsilon)| + |P^-(\varepsilon)|) \\ &\leq C_\alpha\|[a, b]\|(|P^+(\varepsilon)| + |P^-(\varepsilon)|). \end{aligned} \tag{2.16}$$

If $P^+(\varepsilon) \leq 0$ and $P^-(\varepsilon) \leq 0$, then $P^+(s_i) \leq 0$ and $P^-(s_i) \leq 0$, $1 \leq i \leq n$, and so (2.11) and (2.13) imply that $\Delta(s_{k+1}) - \Delta(s_k) \leq 0$, implying $\Delta(b) - \Delta(a) \leq 0$. Alternatively, we could mimic the argument just given, to bound $|\Delta(b) - \Delta(a)|$.

Finally, if one of $P^+(\varepsilon)$, $P^-(\varepsilon)$ is positive and one is negative, (2.11)–(2.14), together with the comments before (2.12) and after (2.14), imply that if $m = \inf\{k\colon P^+(s_k)$ and $P^-(s_k)$ have the same sign$\}$, then $|P^+(s_{i+1})| \leq |P^+(s_i)|$ and $|P^-(s_{i+1})| \leq |P^-(s_i)|$, $0 \leq i < m-1$. Now if $m = \infty$, $|\Delta(s_{k+1}) - \Delta(s_k)| \leq C_\alpha(|P^+(s_k)| + |P^-(s_k)|) \leq C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|)$, and an analysis very similar to (2.16) gives (2.15). In addition, if $m < \infty$, (2.11)–(2.14) imply that

$$\begin{aligned} |P^+(s_m)| + |P^-(s_m)| &\leq C_\alpha(|P^+(s_{m-1})| + |P^-(s_{m-1})|) \\ &\leq C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|). \end{aligned}$$

Furthermore, $|P^+(s_{m+k})| + |P^-(s_{m+k})| \leq C_\alpha(|P^+(s_m)| + |P^-(s_m)|)$, $k > 0$, by the argument that led to the statement just before (2.16). Thus, once again, an analysis similar to (2.16) gives (2.15). □

LEMMA 2.8. *Let $0 < p < \infty$ and suppose $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$ are real numbers which satisfy the following condition. For each k, $0 \leq k < n$, either all of $a_{k+1} - a_k = p(b_{k+1} - b_k)$, $|a_{k+1}| \leq |a_k|$ and $a_{k+1}a_k \geq 0$ or all of $a_{k+1} - a_k = -p(b_{k+1} - b_k)$, $|b_{k+1}| \leq |b_k|$ and $b_{k+1}b_k \geq 0$ hold. Then*

$$|a_k| + p|b_k| \leq |a_0| + p|b_0|, \qquad 1 \leq k \leq n.$$

The proof, by induction, of this lemma is immediate.

PROPOSITION 2.9. *If* $\rho = -1$, *there do not exist two different solutions of* (1.1) *for any* f.

PROOF. Suppose, with no loss of generality, that $\alpha > 0$, so $\beta < 0$. Let g_1 and g_2 be two solutions for f. Define $\Delta(t)$, $P^+(t)$, $P^-(t)$ and positive and negative intervals symbolically exactly as they were defined in the proof of Lemma 2.8 and define $Q^-(t) = -P^-(t)$. All the equations, inequalities and discussion appearing between (2.8) and (2.12) inclusive still hold. In addition, (2.11) gives

$$\begin{aligned} -[Q^-(d) - Q^-(c)] &= \Delta(d) - \Delta(c) \\ &= \frac{\alpha}{\alpha - 1}(P^+(d) - P^+(c)), \qquad [c, d] \text{ positive.} \end{aligned} \tag{2.17}$$

We also have, by reasoning very similar to that which led to (2.11),

$$\begin{aligned} P^+(d) - P^+(c) &= \Delta(d) - \Delta(c) \\ &= \frac{\beta}{1 - \beta}(Q^-(d) - Q^-(c)), \qquad [c, d] \text{ negative.} \end{aligned} \tag{2.18}$$

Mirroring the comments before (2.12), if $[c, d]$ is negative, it cannot happen that both $Q^-(c) \geq 0$ and $Q^-(d) < 0$ or that both $Q^-(c) \leq 0$ and $Q^-(d) > 0$, and $|Q^-(d)| \leq |Q^-(c)|$.

The rest of the proof of Proposition 2.9 closely models the proof of Proposition 2.7. We fix $[a, b]$ and again make the additional assumption that $f^*(t) - f^\#(t) > 0$, $t > 0$. Let $0 < \varepsilon < a < b$ and let $\varepsilon = s_0 < s_1 < \cdots < s_m = b$ be constructed exactly as they were in the proof of Proposition 2.8. Let $a_i = P^+(s_i)$ and $b_i = Q^-(s_i)$, and $p = (\alpha - 1)/\alpha = -\beta/(\beta - 1)$. If $[s_k, s_{k+1}]$ is negative, the comments after (2.14) imply that either $b_k \geq b_{k+1} \geq 0$ or $b_k \leq b_{k+1} \leq 0$, and (2.17), (2.12) and the comments after (2.14) show that if $[s_k, s_{k+1}]$ is positive, either $a_k \geq a_{k+1} \geq 0$ or $a_k \leq a_{k+1} \leq 0$. Together with (2.17) and (2.18) this shows Lemma 2.8 applies. The remainder of the argument is virtually identical to the proof of Proposition 2.7 and is omitted. □

The following proposition provides the rest of the proof of Theorem 1.1.

PROPOSITION 2.10. *If* $|\rho| > 1$, *there is a function* $f_{\alpha,\beta} = f$ *for which* (1.1) *has at least two solutions.*

PROOF. We prove the case $\rho < -1$ and then briefly discuss the case $\rho > 1$. We assume, without loss of generality, that $\alpha > 0$. If f is a piecewise linear function on $[0, t]$, it is easy to see there is a unique piecewise linear solution g of (1.1) "on $[0, t]$." These solutions may be found explicitly as in the proof of Lemma 2.3.

We now construct two functions f_1 and f_2 on $[0, \infty)$ by recursively defining them on successively larger intervals. We define $P^+(t)$, $Q^-(t)$ and $\Delta(t)$ symbolically exactly as they were defined in the proof of Proposition 2.8, where g_1 and g_2 are the piecewise linear solutions of (1.1) on these intervals for f_1 and f_2, respectively. Put $f_1(t) = 0$, $0 \le t \le 1$, and $f_2(t) = \gamma t$, $0 \le t \le 1/2$, $f_2(t) = (\gamma/2) - \delta(t - (1/2))$, $1/2 \le t \le 1$, where $\gamma, \delta > 0$ are chosen so that $g_2(1) = 0$, $g_2^{\#}(1) = 0$ and $g_2^*(1) = 1$. Of course $g_1(t) = 0$, $0 \le t \le 1$, so $P^+(1) = 1$ and $Q^-(1) = 0$. We now define $h(t) := f_1(t) - f_1(1) = f_2(t) - f_2(1)$, thereby defining f_1 and f_2 on the rest of $[0, \infty]$.

Of course $h(1) = 0$ and we put $h'(t) = 1$, $1 < t < t_1$, where $t_1 = \inf\{s: P^+(s) = 0\}$. Note that since h is increasing on $[1, t]$, this interval must be positive and so (2.17) gives $P^+(t_1) = 0$ and $Q^-(t_1) = -\alpha/(1-\alpha)$. It is worth noting that since $P^+(1) > 0$, $g_2^*(1) - g_2(1) > g_1^*(1) - g_1(1)$ (of course, we knew this anyhow) and thus the increments of both g_1 and g_2 after 1 equal those of h until $g_1 = g_1^*$, after which g_1 increases at a faster rate than g_2 until $g_2 = g_2^*$, which occurs at t_1.

Next define $h'(t) = -1$ on $t_1 < t < t_2$, where $t_2 = \inf\{t \ge t_1: Q^-(t) = 0\}$. Then $P^+(t_2) = (-\alpha/(1-\alpha))(-\beta/(1-\beta)) = \rho$ using (2.18). Then define $h(t) = -1$, $t_2 < t < t_3$, where $t_3 = \inf\{t > t_2: P^+(t_3) = 0\}$ and so on. We have $P^+(t_{2n}) = \rho^n$, $Q^-(t_{2n}) = 0$ and $P^+(t_{2n+1}) = 0$, $Q^-(t_{2n+1}) = (-\alpha/(1-\alpha)) \times P^+(t_{2n})$, $n \ge 0$.

We will show

$$(2.19) \qquad c|\rho|^n < t_{2n} < C|\rho|^n, \qquad n \ge 0.$$

To prove the left-hand side of (2.19), we first note that

$$(2.20) \qquad |P^+(s) - P^+(t)| \le C|s - t|, \qquad 1 \le s < t,$$

since, roughly, none of g_1, g_2, g_1^* or g_2^* changes on $[1, \infty)$ at a rate faster than an absolute constant C, since $|h'| = 1$ for all but a discrete set of points. For example, if k is even and $t_k \le s < t \le t_{k+1}$, $h(t) - h(s) = t - s$, and so (2.2) gives $0 \le g_1(t) - g_1(s) \le C(t-s)$ and now (2.4) gives $g_1^*(t) - g_1^*(s) \le C(t-s)$. Thus $t_{2n} - t_{2n-1} \ge c|P^+(t_{2n}) - P^+(t_{2n-1})| = c|\rho^n|(t-s)$ and now (2.4) gives $g_1^*(t) - g_1^*(s) \le C(t-s)$. Thus $t_{2n} - t_{2n-1} \ge c|P^+(t_{2n}) - P^+(t_{2n-1})| = c|\rho^n|$.

The right-hand side of (2.19) follows from

$$(2.21) \qquad t_{k+1} - t_k < C|\rho|^{k/2}.$$

To prove (2.21), we first prove

$$(2.22) \qquad \big(g_1^*(t_k) - g_1^{\#}(t_k)\big) + \big(g_2^*(t_k) - g_2^{\#}(t_k)\big) \le C|\rho|^{k/2}, \qquad k \ge 0.$$

Suppose first that j is even and $j/2$ is an even integer. Let $y = y_j = \inf\{t > t_j: g_1(t) = g_1^*(t)\}$. Then $g_1(s) = g_1^*(s)$, $y \le s \le t_{j+1}$ and $t_{j+1} = \inf\{t > y: g_2(s) = g_2^*(s)\}$. Thus $|\rho|^{j/2} = P^+(t_j) - P^+(t_{j+1}) = P^+(y) - P^+(t_{j+1})$, which in turn equals $t_{j+1} - y$, since $g_2(t_{j+1}) - g_2(y) = h(t_{j+1}) - h(y) = t_{j+1} - y$.

Now (2.2), (2.4) and (2.20) yield

$$g_1^*(t_{j+1}) - g_1^*(t_j) = g_1^*(t_{j+1}) - g_1^*(y) \le C(t_{j+1} - y) = C\big(P^+(y) - P^+(t_{j+1})\big) = C|\rho|^{j/2}. \tag{2.23}$$

Since $g_1^\#(t_{j+1}) = g_1^\#(t_j)$, $g_2^\#(t_{j+1}) = g_2^\#(t_j)$ and $g_2^*(t_{j+1}) = g_2^*(t_j)$, this gives

$$\big(g_1^*(t_{j+1}) - g_1^*(t_j)\big) + \big(g_2^*(t_{j+1}) - g_2^*(t_j)\big) + \big(g_1^\#(t_j) - g_1^\#(t_{j+1})\big) + \big(g_2^\#(t_j) - g_2^\#(t_{j+1})\big) \le C|\rho|^{j/2}. \tag{2.24}$$

The proof of (2.24) for j odd and for j even when $j/2$ is not an integer is similar, and adding these inequalities for $j = 0$ to $k - 1$ gives an inequality which immediately implies (2.22). To derive (2.21) from (2.22), let k and $k/2$ be even, as the argument for other k is very similar, and let $y = y_k$ be as defined just after (2.22). Then $t_{k+1} - t_k = (t_{k+1} - y) + (y - t_k)$. Now

$$y - t_k = g_1^*(t_k) - g_1(t_k) \le g_1^*(t_k) - g_1^\#(t_k) \le C|\rho|^{k/2}$$

using (2.22), and (2.23) gives $t_{k+1} - y \le C|\rho|^{k/2}$.

Finally we note

$$|\Delta(t_{2n+1}) - \Delta(t_{2n})| \ge C|\rho|^n, \qquad n \ge 1, \tag{2.25}$$

which follows from (2.11), so that

$$\sup_{0 \le s \le t_{2n+1}} |g_1(s) - g_2(s)| \ge C|\rho|^n, \qquad n \ge 1. \tag{2.26}$$

Now define f_n^1 and f_n^2 by $f_n^1(t) = n^{-1}f_1(nt)$ and $f_n^2(t) = n^{-1}f_2(nt)$. Their solutions for (1.1) equal $n^{-1}g_1(nt)$ and $n^{-1}g_2(nt)$, respectively, which we designate g_1^n and g_2^n. Pick a subsequence $n(m)$, $n \ge 1$, of the integers, such that $f_1^{n(m)}$, $f_2^{n(m)}$, $g_1^{n(m)}$, $g_2^{n(m)}$ converge uniformly on compact subintervals of $[0, \infty)$. This is possible since $\{f_n^1,\ n \ge 1\}$ and $\{f_n^2,\ n \ge 1\}$ are both absolutely continuous and bounded by their explicit construction and thus so are $\{g_n^1,$ $n \ge 1\}$ and $\{g_n^2,\ n \ge 1\}$ by Lemma 2.1.

Now $f_{n(m)}^1$ and $f_{n(m)}^2$ clearly converge to the same function, again by their constructions. Call this function f. Corollary 2.2 guarantees that the limits of $g_{n(m)}^1$ and $g_{n(m)}^2$, call them g_1 and g_2, are both solutions of (1.1) for f. Finally, (2.26) and (2.15) guarantee that g_1 and g_2 cannot be the same function.

If $\rho > 1$, a very similar argument can be made: again define two functions f_1 and f_2 so that they have the same differences on $[1, \infty)$ and have $P^+(1) = 1$ and $P^-(1) = 0$, and define t_i, $i \ge 1$, and the differences of f_1 and f_2 so that both $P^+(t_i)$ and $P^-(t_i)$ oscillate between 0 and numbers which have geometrically increasing absolute value, while the absolute values of the t_i are controlled, using (2.11) and (2.13); then proceed as before. □

We just note that in the $\rho > 1$ we can explicitly construct a function f for which (1.1) has one nonnegative (and nonzero) and one nonpositive solution.

This function depends on α and β and switches slopes between $+1$ and -1. If $\alpha = \beta$, the lengths of the intervals where the function is linear increase geometrically, approaching infinity, and decrease geometrically, approaching zero. The basic approach is the same as above: alter the function slightly two ways on $[0, \varepsilon_n]$, show that one solution never goes below its minimum on $[0, \varepsilon_n]$ and the other never goes above its maximum on$[0, \varepsilon_n]$ and then take subsequential limits as $\varepsilon_n \to 0$.

When $|\rho| = 1$, if we try to mimic the examples above, we get unbounded differences between g_1 and g_2, but not the geometrical increase of (2.26).

3. Proof of Theorem 1.2. The basis of our proof of Theorem 1.2 is the following formula of CPY. If g solves (1.1) for f, then

$$g^*(t) = \frac{1}{1-\alpha} \sup_{s \le t} \left(f(s) - \frac{\beta}{1-\beta} \sup_{u \le s} (-f(u) - \alpha g^*(u)) \right). \tag{3.1}$$

Let $\|h\|_T = \sup_{0 \le s \le T} |h(s)|$, $T > 0$. Throughout this section we assume $|\rho| < 1$.

PROPOSITION 3.1. *Let $|\rho| < 1$. Then if g_1 and g_2 are solutions of* (1.1) *for f_1 and f_2, respectively, we have*

$$\|g_1 - g_2\|_T \le C \|f_1 - f_2\|_T, \qquad T > 0. \tag{3.2}$$

PROOF. Subtracting the version of (3.1) for f_2 from that for f_1,

$$\begin{aligned} &|g_1^*(T) - g_2^*(T)| \\ &\quad \le \frac{1}{1-\alpha} \Biggl[\sup_{s \le T} |f_1(s) - f_2(s)| \\ &\qquad\qquad + \frac{|\beta|}{1-\beta} \sup_{s \le T} \Bigl| \sup_{u \le s} (-f_1(u) - \alpha g_1^*(u)) \\ &\qquad\qquad\qquad - \sup_{u \le s} (-f_2(u) - \alpha g_2^*(u)) \Bigr| \Biggr] \\ &\quad \le \frac{1}{1-\alpha} \Biggl[\|f_1 - f_2\|_T + \frac{|\beta|}{1-\beta} \Bigl(\sup_{s \le T} \Bigl| \sup_{u \le s} |f_1(u) - f_2(u)| \\ &\qquad\qquad\qquad + |\alpha| \sup_{u \le s} |g_1^*(u) - g_2^*(u)| \Bigr| \Bigr) \Biggr] \\ &\quad \le \frac{1}{1-\alpha} \|f_1 - f_2\|_T + \frac{|\beta|}{1-\beta} \|f_1 - f_2\|_T + |\rho| \, \|g_1^* - g_2^*\|_T. \end{aligned} \tag{3.3}$$

Upon noticing that $|g_1^*(T) - g_2^*(T)|$ may be replaced in (3.3) by any of $|g_1^*(t) - g_2^*(t)|$, $0 < t < T$, since the right-hand side of (3.3) is increasing in T, (3.3) yields

$$\|g_1^* - g_2^*\|_T (1 - |\rho|) \le \frac{1 - \beta + |\beta| - \alpha|\beta|}{(1-\alpha)(1-\beta)} \|f_1 - f_2\|_T, \tag{3.4}$$

so

$$|g_1^*(t) - g_2^*(t)| \le C\|f_1 - f_2\|_T, \qquad 0 \le t \le T. \tag{3.5}$$

Similarly we have

$$|g_1^\#(t) - g_2^\#(t)| \le C\|f_1 - f_2\|_T, \qquad 0 \le t \le T.$$

We claim that the truth of (3.4) and (3.5) for all f_1 and f_2 implies the apparently stronger inequality (3.2).

We show this by showing that if $\|f_1 - f_2\|_T > 0$ and

$$4 < K = K_{f_1, f_2, T} = \|g_1 - g_2\|_T / \|f_1 - f_2\|_T, \tag{3.6}$$

then there are functions $\tilde{f}_1$ and $\tilde{f}_2$, with solutions $\tilde{g}_1$ and $\tilde{g}_2$, respectively, and $S > 0$, such that $\|\tilde{f}_1 - \tilde{f}_2\|_S = \|f_1 - f_2\|_T$ and either $|\tilde{g}_1^*(S) - \tilde{g}_2^*(S)| > (K/2)\|\tilde{f}_1 - \tilde{f}_2\|_S$ or $|\tilde{g}_1^\#(S) - \tilde{g}_2^\#(S)| > (K/2)\|\tilde{f}_1 - \tilde{f}_2\|_S$.

Suppose, first, that $\alpha > 0$ and $\beta > 0$, and suppose without loss of generality that $|g_1(T) - g_2(T)| = \|g_1 - g_2\|_T$ and that $g_1(T) > g_2(T)$. Let $w = \sup\{x \le T\colon (g_1(T) - g_2(T)) - (g_1(x) - g_2(x)) > (f_1(T) - f_2(T)) - (f_1(x) - f_2(x))\}$.

Note that 0 is in the set we are taking the supremum of, since $K > 2$. Now either $g_1(w) = g_1^*(w)$ or $g_2(w) = g_2^\#(w)$, since otherwise $(g_1(w) - g_2(w)) - (g_1(w - \varepsilon) - g_2(w - \varepsilon)) \le (f_1(w) - f_1(w - \varepsilon)) - (f_2(w) - f_2(w - \varepsilon))$ for small enough $\varepsilon > 0$, using (2.2) and (2.3). Suppose $g_1(w) = g_1^*(w)$. Define $\tilde{f}_1$ and $\tilde{f}_2$ by $\tilde{f}_1(t) = f_1(t)$, $t \le w$, $\tilde{f}_1(t) - \tilde{f}_1(w) = (t - w)$, $t > w$, and $\tilde{f}_2(t) = f_2(t)$, $t \le w$, $\tilde{f}_2(t) - \tilde{f}_2(w) = (t - w)$, $t > w$. Let $\gamma = \inf\{t \ge w\colon \tilde{g}_2(t) = \tilde{g}_2^*(w)\}$. Now $\tilde{g}_1(s) = \tilde{g}_1^*(s)$, $w \le s \le \gamma$, and since $\alpha > 0$, $\tilde{g}_1(s) - \tilde{g}_2(s)$ is increasing on (w, γ) and so

$$\begin{aligned}\tilde{g}_1^*(\gamma) - \tilde{g}_2^*(\gamma) &= \tilde{g}_1(\gamma) - \tilde{g}_2(\gamma)\\ &\ge \tilde{g}_1(w) - \tilde{g}_2(w) = g_1(w) - g_2(w).\end{aligned}$$

However,

$$\begin{aligned}g_1(w) - g_2(w) &\ge (g_1(T) - g_2(T)) - |(f_1(T) - f_1(w)) - (f_2(T) - f_2(w))|\\ &\ge (g_1(T) - g_2(T)) - 2\|f_1 - f_2\|_T\\ &> \frac{1}{2}(g_1(T) - g_2(T)) \quad [\text{by } (3.6)]\\ &= \frac{K}{2}\|f_1 - f_2\|_T.\end{aligned}$$

Finally, note $\|\tilde{f}_1 - \tilde{f}_2\|_\gamma = \|\tilde{f}_1 - \tilde{f}_2\|_w = \|f_1 - f_2\|_w \le \|f_1 - f_2\|_T$ and so we get $\tilde{g}_1^*(\gamma) - \tilde{g}_2^*(\gamma) \ge (K/2)\|\tilde{f}_1 - \tilde{f}_2\|_\gamma$, which verifies the sentence containing (3.6).

The proof if one or both of α, β is not positive is very similar, in fact, somewhat easier: if both α and β are not positive, and w is defined as above, either $g_1(w) = g_1^{\#}(w)$ or $g_2(w) = g_2^*(w)$. In the first case, $|g_1^{\#}(w) - g^{\#}(w)| \geq (K/2)\|f_1 - f_2\|$; in the second case, $|g_1^*(w) - g^*(w)| \geq (K/2)\|f_1 - f_2\|_w$. $\square$

We use $\Rightarrow$ to indicate convergence in distribution of processes and retain the convention extending discrete time processes to and identifying them with continuous time processes, mentioned before the statement of Theorem 1.2. For a process $\mathbf{Z}$, we let $\mathbf{Z}^n$ be the process $n^{-1/2}Z_{nt}$, $t \geq 0$. We let $\mathbf{R}$ be fair random walk, started at 0, let $\mathbf{B}$ and $\mathbf{Y}$ be as in (1.2) and let $\mathbf{X}$ be as in the statement of Theorem 2.1. It is classical that $\mathbf{R}^n \Rightarrow \mathbf{B}$. The continuous mapping theorem [see Pollard (1984), page 70] and Lemma 3.1 now give that if $\mathbf{S}$ solves (1.1) for $\mathbf{R}$, then $\mathbf{S}^n \Rightarrow \mathbf{Y}$. If $\mathbf{S}$ had the distribution of $\mathbf{X}$, this would verify Theorem 1.2, but it does not. To circumvent this problem we find a process $\mathbf{U}$ such that $\mathbf{U}^n \Rightarrow \mathbf{B}$ and such that the solution of (1.1) for $\mathbf{U}$ has exactly the distribution of $\mathbf{X}$.

Process $\mathbf{U}$ is constructed from $\mathbf{R}$. We describe its construction and properties for α, β both nonpositive. The other cases are very similar. We let A_i, $i \geq 1$, be iid indicator variables with $P(A_i = 1) = -\alpha/(2-\alpha)$ and let B_i, $i \geq 1$, be indicator variables independent of the A_i with $P(B_i = 1) = -\beta/(2-\beta)$. Let $M_0 = R_0$ and $M_1 = R_1$, and if $i \geq 1$, put $M_{i+1} - M_i = R_{i+1} - R_i$ if either $M_i^{\#} < M_i < M_i^*$, or $M_i = M_i^{\#}$ and $R_{i+1} - R_i = +1$, or $M_i = M_i^*$ and $R_{i+1} - R_i = -1$. Define $M_{i+1} - M_i = R_{i+1} - R_i - 2A_{J(i)}$ if $M_i = M_i^*$ and $R_{i+1} - R_i = 1$, where $J(i)$ is the number of k, $1 \leq k \leq i$, such that $M_k = M_k^*$ and $R_{k+1} - R_k = 1$. Define $M_{i+1} - M_i = R_{i+1} - R_i + 2B_{\Theta(i)}$ if $M_i^{\#} = M_i$ and $R_{i+1} - R_i = -1$, where $\Theta(i)$ is the number of those k, $1 \leq k \leq i$, such that $M_k = M_k^{\#}$ and $R_{i+1} - R_i = -1$. Then $\mathbf{M}$ has exactly the distribution of $\mathbf{X}$. We define the process $\mathbf{U}$ as follows: $U_{n+1} - U_n = M_{n+1} - M_n$ except on $\{M_{n+1} - M_n = 1, M_n = M_n^*\}$, where we define $U_{n+1} - U_n = (1-\alpha)$, and on $\{M_{n+1} - M_n = -1, M_n = M_n^{\#}\}$, where we define $U_{n+1} - U_n = -1 + \beta$. Then $\mathbf{M}$ is the solution of (1.1) for $\mathbf{U}$. Furthermore, we have

$$
\begin{aligned}
U_{n+1} - U_n &= R_{n+1} - R_n \quad \text{if } M_n \neq M_n^* \text{ or } M_n^{\#} \text{ or } n = 0,\\
U_{n+1} - U_n - (R_{n+1} - R_n) &= [(1-\alpha) - 1] I(R_{n+1} - R_n = 1, A_{J(n)} = 0)\\
&\quad - 2I(R_{n+1} - R_n = 1, A_{J(n)} = 1)\\
&:= \Delta_n^+ \quad \text{if } M_n = M_n^*, n > 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
U_{n+1} - U_n - (R_{n+1} - R_n) &= [(-1+\beta) + 1] I(R_{n+1} - R_n = -1, B_{\Theta(n)} = 0)\\
&\quad + 2I(R_{n+1} - R_n = -1, B_{\Theta(n)} = 1)\\
&:= \Delta_n^- \quad \text{if } M_n = M_n^{\#}, n > 0.
\end{aligned}
$$

Thus $U_n - R_n = \sum_{k=0}^n \Delta^+(k) + \sum_{k=0}^n \Delta^-(k)$. It is easily checked that $\Delta^+(k)$, $k > 0$, and $\Delta^-(k)$, $k > 0$, are both martingale difference sequences, that $|\Delta^+(k)| \le C_\alpha$ and $|\Delta^-(k)| \le C_\beta$, that $\Delta^+(k) = 0$ except on $\{M_k = M_k^*\}$ and that $\Delta^-(k) = 0$ except on $\{M_k = M_k^\#\}$.

LEMMA 3.2. *If* $\mathbf{X}$ *is as in the statement of Theorem* 1.2, *then*

$$n^{-1} \sum_{k=1}^{n} I\big(X_k = X_k^* \textit{ or } X_k^\#\big) \to 0 \quad \textit{in probability}.$$

PROOF. Fix $M > 1 > 0$. Let $\tau_1 = \inf\{k\colon X_k^* - X_k^\# = M\}$. Clearly $\tau_1 < \infty$ a.s. Let

$$\tau_{2k} = \inf\big\{i \ge \tau_{2k-1}\colon X_i \in \big(X_i^\#, X_i^*\big)\big\}, \qquad k \ge 1,$$
$$\tau_{2k+1} = \inf\big\{i \ge \tau_{2k}\colon X_i = X_i^* \text{ or } X_i^\#\big\}, \qquad k \ge 1.$$

Let $\mathscr{A}_k = \sigma(X_i, i \le k)$.

Now if $i \ge 1$, the conditional distribution of $\tau_{2i} - \tau_{2i-1}$ given $\mathscr{A}_{\tau_{2i-1}}$ is the geometric distribution with parameter $1/(2-\alpha)$ on $\{X_{\tau_{2i-1}} = X^*_{\tau_{2i-1}}\}$ and it is the geometric distribution with parameter $1/(2-\beta)$ on $\{X_{\tau_{2i-1}} = X^\#_{\tau_{2i-1}}\}$. The conditional distribution of $\{\tau_{2i+1} - \tau_{2i}\}$ given $\mathscr{A}_{2i}$ is stochastically no smaller than the distribution of the time it takes fair random walk, started at 1, to exit from $(0, M)$. Especially if E_M is the expected time it takes random walk started at 1 to exit $(0, M)$, we have

$$\limsup_{n\to\infty} \sum_{k=1}^{n} (\tau_{2k} - \tau_{2k-1}) \Big/ \sum_{k=1}^{n-1} (\tau_{2k+1} - \tau_{2k}) \le C/E_M,$$

where C is the maximum of the expectation of the two geometric variables mentioned above. Since the sum in the denominator is smaller than τ_{2n}, this implies

$$\limsup_{n\to\infty} \sum_{k=1}^{n} I\big(X_k = X_k^* \text{ or } X_k^\#\big) \Big/ n \le \limsup_{n\to\infty} \sum_{k=1}^{\tau_{2n}} I\big(X_k = X_k^* \text{ or } X_k^\#\big)/\tau_{2n}$$
$$= \limsup_{n\to\infty} \sum_{k=1}^{n} (\tau_{2k} - \tau_{2k-1})/\tau_{2n} \le C/E_M.$$

Since $\sup_M E_M = \infty$, this proves the lemma. □

Note that this lemma is equivalent to

(3.7) $$Q_n/n \to 0 \quad \text{in probability},$$

where $Q_n := \sum_{k=I}^n I(M_n = M_n^* \text{ on } M_n^\#)$ a.s., since $\mathbf{X}$ and $\mathbf{M}$ have the same distribution.

To complete the proof of Theorem 1.2, we prove the following lemma.

LEMMA 3.3. $\mathbf{U}^n \to \mathbf{B}$ *in distribution as* $n \to \infty$.

PROOF. The proof will be accomplished by showing that $\sup_{0 \le s \le t} |U^n(s) - R^n(s)| \to 0$ in probability for each fixed t. This follows from

$$\begin{aligned} E \max_{1 \le k \le n} (U(k) - R(k))^2/n &= E \max_{1 \le k \le n} \left(\sum_{i=1}^{k} \Delta^+(i) + \sum_{i=1}^{k} \Delta^-(i) \right)^2 \Big/ n \\ &\le 4E\left(\max_{1 \le k \le n} \sum_{i=1}^{k} \Delta^+(k) \right)^2 \Big/ n + 4E\left(\max_{1 \le k \le n} \sum_{i=1}^{k} \Delta^-(k) \right)^2 \Big/ n \\ &\le 16E\left(\sum_{i=1}^{n} \Delta^+(i) \right)^2 \Big/ n + 16E\left(\sum_{i=1}^{n} \Delta^-(i) \right)^2 \Big/ n \\ &= C_\alpha EQ_n/n + C_\beta EQ_n/n \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The last inequality by Doob's martingale maximal inequality [Doob (1951), page 317] applied to the martingales $\sum_{i=1}^{k} \Delta^+(i)$ and $\sum_{i=1}^{k} \Delta^-(i)$, and the convergence to zero is by (3.4). □

REFERENCES

BOLTHAUSEN, E. and SCHMOCK, U. (1994). On self-attracting one-dimensional random walks. Preprint.

CARMONA, P., PETIT, F. and YOR, M. (1993). Beta variables as times spent in $[0, \infty]$ by certain perturbed reflecting Brownian motions. *Bull. London Math Soc*. To appear.

CARMONA, P., PETIT, F. and YOR, M. (1994). *Probab. Theory Related Fields* **100** 1–29.

DAVIS, B. (1989). Loss of recurrence in reinforced random walk. In *Almost Everywhere Convergence: Proceedings of a Conference* (G. Edgar and L. Sucheston, eds.) 179–185. Academic Press, New York.

DAVIS, B. (1990). Reinforced random walk. *Probab. Theory Related Fields* **84** 203–229.

DIACONIS, P. (1988). Recent progress on de Finetti's notions of exchangeability. In *Bayesian Statistics* (J. Bernardo, M. H. DeGroot, D. V. Lindley and A. F. M. Smith, eds.) 111–125. Oxford Univ. Press.

DOOB, J. L. (1951). *Stochastic Processes*. Wiley, New York.

HARRISON, J. M. and SHEPP, L. A. (1981). On skew-Brownian motion. *Ann. Probab*. **9** 309–313.

LE GALL, J. F. (1986). L'équation stochastique $Y_t = B_t + \alpha M_t^Y + \beta I_t^Y$ comme limite des équations de Norris–Rogers–Williams. Unpublished notes.

LE GALL, J. F. AND YOR, M. (1992). Enlacement du mouvement brownien autour des courbes de l'espace. *Trans. Amer. Math. Soc*. **317** 687–722.

NESTER, D. (1994). A random walk with partial reflection or attraction at its extrema. Preprint.

OTHMER, H. and STEVENS, A. (1995). Aggregation, blow-up, and collapse: the ABC's of taxis in reinforced random walks. Preprint.

PEMANTLE, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab*. **16** 1229–1241.

PEMANTLE, R. (1992). Vertex-reinforced random walk. *Probab. Theory Related Fields* **92** 117–136.

PERMAN, M. (1995). An excusion approach to Ray–Knight theorems for perturbed reflecting Brownian motion. *Stochastic Process. Appl*. To appear.

POLLARD, D. (1982). *Convergence of Stochastic Processes*. Springer, New York.

SELLKE, T. (1994a). Recurrence of reinforced random walk on a ladder. *Probab. Theory Related Fields*. To appear.

SELLKE, T. (1994b). Reinforced random walk on the d-dimensional lattice. *Probab. Theory Related Fields*. To appear.

TÓTH, B. (1994). "True" self-avoiding walk with generalized bond repulsion on $\mathbb{Z}$. *J. Statist. Phys.* **77** 17–33.

TÓTH, B. (1995). The "true" self avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems. *Ann. Probab.* **23** 1523–1556.

TÓTH, B. (1996). Generalized Ray–Knight theory and limit theorems for self-interacting random walks on $\mathbb{Z}$. *Ann. Probab.* **24** 1324–1367.

WERNER, W. (1995). Some remarks on perturbed reflecting Brownian motion. *Seminaire de Probabilités XXIX. Lecture Notes in Math.* **1613** 37–43. Springer, Berlin.

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