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# A CENTRAL LIMIT THEOREM FOR A ONE-DIMENSIONAL POLYMER MEASURE

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Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk on the integers having bounded steps. The self-repellent (resp., self-avoiding) walk is a sequence of transformed path measures which discourage (resp., forbid) self-intersections. This is used as a model for polymers. Previously, we proved a law of large numbers; that is, we showed the convergence of $|S_n|/n$ toward a positive number $\Theta$ under the polymer measure. The present paper proves a classical central limit theorem for the self-repellent and self-avoiding walks; that is, we prove the asymptotic normality of $(S_n - \Theta n)/\sqrt{n}$ for large $n$. The proof refines and continues results and techniques developed previously.

## 1. Introduction.

1.1. *Polymer measures.* Random walks are sometimes used as a stochastic model for the random spread of polymer chains which consist of a huge number of relatively small groups of atoms (so-called monomers) joined together by chemical bonds. Depending on the chemical properties, there are some favorite angles and spatial orientations which occur with certain probabilities. Throughout this paper, $(S_n)_{n\in\mathbb{N}_0}$ denotes a random walk on the $d$-dimensional lattice $\mathbb{Z}^d$ having i.i.d. steps and starting at 0. Its distribution is denoted by $P$, the corresponding expectation by $E$.

As a model for a polymer, the free random walk has the important disadvantage that self-intersections may occur, that is, pairs of time points $n < m$ satisfying $S_n = S_m$. Rather than the free walk, one usually considers a transformation of the walk which has few self-intersections or even none at all. Introducing a parameter $\alpha \in (0, 1]$, the *strength of self-repellence*, one defines a path measure $P_\alpha^n$ by

$$\frac{dP_\alpha^n}{dP} = \frac{(1-\alpha)^{X_n}}{E((1-\alpha)^{X_n})}, \qquad n \in \mathbb{N}, \tag{1.1}$$

where

$$X_n := \#\{(i,j) \in \{0,\dots,n\}^2\colon\ i < j \text{ and } S_i = S_j\} \tag{1.2}$$

denotes the number of self-intersections until time $n$. The so-called *polymer measure* $P_\alpha^n$ weights the path's probability with the factor $1-\alpha$ for every self-intersection until time $n$. For $\alpha \in (0,1)$, we call the distribution of the walk

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under $P_\alpha^n$ *self-repellent*, while the one under $P_1^n = P(\cdot \mid X_n = 0)$ is called *self-avoiding*. In both cases, the polymer measure is highly non-Markovian since the interaction involves the whole path until time $n$. Being transformed measures, the measures $P_\alpha^n$ do not even form a consistent family.

The large-$n$ behavior of $S_n$ under $P_\alpha^n$ is of particular interest. One expects that the self-repellence has the effect of spreading out the path more than the free random walk, and one expects that this effect becomes small in high dimensions. The standard reference about self-avoiding walks in any dimension is Madras and Slade (1993), which contains not only a detailed introduction both from a heuristic and a mathematically rigorous point of view but also presents many (surveys on) proofs and simulation algorithms. Usually, one takes a simple $d$-dimensional random walk as a basis, but even in this case there are no rigorous results for the most interesting dimensions $d = 2$ and 3 and very few for $d = 4$. In dimension $d \geq 5$, Brydges and Spencer (1985) prove that a certain scaling limit of $S_n$ is Gaussian. They use the technique of the lace expansion, which seems to admit good results only in high dimensions.

From now on, we restrict our attention to the dimension $d = 1$ for the remainder of the paper. It is the aim of this paper to prove a central limit theorem for the self-repellent and the self-avoiding walk for a certain class of one-dimensional random walks. Let us first mention some earlier works on this subject.

Aldous (1986) considers a random walk whose steps are uniformly distributed on $\{-r, \dots, r\}$ for some positive integer $r$ and proves that the sequence of successive self-intersection times of the free walk has a limit law as $r \to \infty$. Bolthausen (1990) imposes no boundedness condition on the step distribution, but only the existence of an exponential moment and a symmetry condition. Using large-deviation analysis, he shows, for small repulsion parameter $\alpha$, that $S_n/n$ is bounded and bounded away from 0 under $P_\alpha^n$ as $n \to \infty$. In the symmetric nearest-neighbor case, Greven and den Hollander (1993) prove a law of large numbers: they show the convergence of $|S_n|/n$ under $P_\alpha^n$ toward some number $\Theta(\alpha) \in (0,1)$, the so-called *effective drift* of the self-repellent walk. The main tool is a Ray–Knight-type description of the local times of a simple random walk. A quite explicit characterization of the drift $\Theta(\alpha)$ is obtained in that paper, and on this basis van der Hofstad and den Hollander (1995) derive the existence of $\lim_{\alpha \downarrow 0} \Theta(\alpha)/(-\log(1-\alpha))^{1/3}$ and a characterization in terms of a certain second-order differential operator. In König (1993, 1994) the law of large numbers is extended to a class of random walks whose steps may have larger size than 1, but are assumed to be bounded. The next subsection explains the two latter papers more closely since they serve as a starting point for the present paper.

1.2. *Statement of the results.* For the remainder of this paper we assume that the random walk $(S_n)_{n \in \mathbb{N}_0}$ makes only steps of sizes $\pm 1, \dots, \pm r$ with positive probability, with $r$ a positive integer. Recall that we assume that $S_0 = 0$ and that the steps are i.i.d. We impose a symmetry condition and introduce a drift parameter $h \in [0, \infty)$. More precisely, we assume that the steps are given by

$$S_{n+1} - S_n = k \in \{\pm 1, \dots, \pm r\} \quad \text{with probability } p_{|k|}\, e^{hk}/Z_h \tag{1.3}$$

for $n \in \mathbb{N}_0$, where $Z_h = \sum_{|k|=1}^{r} p_{|k|}\, e^{hk}$ is a normalization constant and $p_1, \dots, p_r$ are positive numbers. For convenience, we assume that $Z_0 = 1$. The associated path measure and expectation are denoted by $P_h$ and $E_h$, respectively. Under $P_h$ the steps have mean $(d/dh)\log Z_h$, which is positive if and only if $h$ is. We are mainly interested in the symmetric case $h = 0$, but our strategy a priori works for a walk with positive drift only. In order to be able to lead the symmetric case back to the positive-drift case, we inserted the drift parameter in such a way in (1.3) that the probability of a finite path depends on this parameter through the length and the endpoint only. The polymer measure defined in (1.1) (with $P = P_h$ and $E = E_h$) is denoted by $P^n_{h,\alpha}$ and the corresponding expectation by $E^n_{h,\alpha}$.

In this paper we prove a central limit theorem for the endpoint of the path under $P^n_{h,\alpha}$. The main result appears in Theorem 1.10 below.

Our starting point is the following law of large numbers.

THEOREM 1.4 [König (1993, 1994)]. *For every $h \geq 0$ and $\alpha \in (0,1]$, there is a number $\Theta(h,\alpha) \in (0,r]$ (called the effective drift of the walk) such that*

$$P^n_{h,\alpha}\left(\frac{S_n}{n}\right)^{-1} \Rightarrow_w \begin{cases} \delta_{\Theta(h,\alpha)}, & \text{if } h > 0,\\ (\delta_{\Theta(0,\alpha)} + \delta_{-\Theta(0,\alpha)})/2, & \text{if } h = 0, \end{cases} \tag{1.5}$$

*as $n \to \infty$.*

(We denote the distribution of a random variable $Y$ under a measure $\mu$ by $\mu Y^{-1}$.) In order to derive Theorem 1.4, we showed that the function

$$\Phi_{h,\alpha}(\theta) := \lim_{n\to\infty} \frac{1}{n} \log P^n_{h,\alpha}(S_n = \lfloor \theta n \rfloor), \qquad \theta \in [0,r] \tag{1.6}$$

($\lfloor x \rfloor$ denotes the largest integer not exceeding $x$), is well defined and possesses a strict maximum in $\theta = \Theta(h,\alpha)$. In particular, this implies that $P^n_{h,\alpha}(|S_n - \Theta(h,\alpha)n| \geq \varepsilon n)$ decays exponentially fast toward 0 as $n \to \infty$, for every positive $\varepsilon$, and this implies Theorem 1.4. As a pre-step to this result, we used large-deviation analysis to analyze the function

$$\tilde{\Phi}_{h,\alpha}(\theta) := \lim_{n\to\infty} \frac{1}{n} \log P^n_{h,\alpha}(0 < S_1, \dots, S_{n-1} < S_n = \lfloor \theta n \rfloor), \qquad \theta \in (0,r). \tag{1.7}$$

It turned out that $\tilde{\Phi}_{h,\alpha}$ is real-analytic in $(0,r)$ [resp., in $(1,r)$ in the case $\alpha = 1$], strictly concave in that interval and possesses a strict maximum point in $\Theta(h,\alpha)$. The latter property originally defined this number. Furthermore, we showed that $\tilde{\Phi}_{h,\alpha}$ possesses a strictly negative curvature in $\Theta(h,\alpha)$. Comparing this to König (1993) and König (1994), note that $\Phi_{h,\alpha}$ is identical to $J_{h,\beta} - J_{h,\beta}(\Theta(h,\,\alpha))$, where $\beta = -\log(1-\alpha)/2$, in the self-repellent case in König (1994) and to $J_h - J_h(\Theta(h))$ in König (1993) for $\alpha = 1$; the number

$\Theta(h)$ in König (1993) is called $\Theta(h,1)$ in the present paper. The analogous statements are valid for the tilded functions.

What has not been proved in König (1993) nor in König (1994) is the fact that $\Phi_{h,\alpha}$ is even identical to $\tilde{\Phi}_{h,\alpha}$ in a neighborhood of $\Theta(h,\alpha)$, and this can be proved by strengthening the techniques in Sections 4 of those papers. So, in particular, the rate function $\Phi_{h,\alpha}$, which governs the large deviations for the self-repellent (resp., self-avoiding) walk, possesses a strictly negative curvature in its maximum point $\Theta(h,\alpha)$.

There is a heuristic calculation which supports the conjecture that $(S_n - \Theta(h,\alpha)n)/\sqrt{n}$ may be asymptotically centered Gaussian with variance

$$\sigma^2_{h,\alpha} := -\frac{1}{\tilde{\Phi}''_{h,\alpha}(\Theta(h,\alpha))} \in (0,\infty). \tag{1.8}$$

Use the approximation $P^n(S_n \approx \theta n) = \exp(n\Phi(\theta) + o(n))$ for $\theta$ close to $\Theta$ (we suppress $h$ and $\alpha$ from the notation) and expand $\Phi$ in a Taylor series around $\Theta$ up to second order. Then we see, for $c < C$ (up to a factor of $e^{o(n)}$),

$$\begin{aligned} P^n\left(c \le \frac{S_n - \Theta n}{\sqrt{n}} \le C\right) &\approx \int_c^C P^n\left(S_n \approx n\left(\Theta + \frac{x}{\sqrt{n}}\right)\right) dx \\ &\approx \int_c^C \exp\left(n\Phi\left(\Theta + \frac{x}{\sqrt{n}}\right)\right) dx \approx \int_c^C \exp\left(\frac{x^2}{2}\Phi''(\Theta)\right) dx. \end{aligned} \tag{1.9}$$

The present paper proves the conjecture. Write $\mathscr{N}(\sigma^2)$ for the Gaussian distribution with mean 0 and variance $\sigma^2 \in (0,\infty)$. Then our main result is stated as follows.

THEOREM 1.10. *For any $r \in \mathbb{N}$, $\alpha \in (0,1]$ and $h \ge 0$ (with the exception of the trivial case $r = 1 = \alpha$), the distribution of $(S_n - \Theta(h,\alpha)n)/\sqrt{n}$ under $P^n_{h,\alpha}(\cdot \mid S_n > 0)$ converges weakly to $\mathscr{N}(\sigma^2_{h,\alpha})$ as $n \to \infty$. For positive $h$, the measure $P^n_{h,\alpha}(\cdot \mid S_n > 0)$ in this statement may be replaced by $P^n_{h,\alpha}$.*

The continuous analog of this theorem (*i.e.*, a central limit theorem for a transformed Brownian path measure that suppresses self-intersections, the so-called *Edwards measure*) is proved in van der Hofstad, den Hollander and König (1995).

1.3. *Outline of the proof.* The Gaussian behavior of $(S_n - \Theta(h,\alpha)n)/\sqrt{n}$ under $P^n_{h,\alpha}(\cdot \mid S_n > 0)$ exclusively stems from the self-intersections the path $(S_0,\dots,S_n)$ has in the spatial area $\{S_0,\dots,S_n\}$. A much finer knowledge about the influence of the number of self-intersections which occur outside of this area is needed than for working on an exponential scale as is done in König (1993, 1994). Their influence will eventually turn out to be convergent such that their contribution cancels in the definition of the polymer measure [recall (1.1)] in the limit as $n \to \infty$.

In Section 2 we derive a precise formula for the expectation of $(1-\alpha)^{X_n}$ on $\{S_n = a+1\}$ for fixed $n$ and $a \in \mathbb{N}_0$ in terms of a certain Markov chain.

This chain is introduced in König (1993) and also used in König (1994) and admits a Ray–Knight-type description of the local times of the walk. By using some independence properties of this chain, we separate the influence of the self-intersection numbers in $-\mathbb{N}_0$, $\{1, \dots, S_n - 1\}$ and $\{S_n, S_n + 1, \dots\}$ from each other. This chain exists in the case of positive drift only, and this is the reason why we inserted a drift parameter in (1.3).

The further course of the proof relies on a certain characterization of the function $\tilde{\Phi}_{h,\alpha}$ which is derived in König (1993) [resp., König (1994)]. These results (which are listed at the beginning of Section 3) enable us to construct a certain *ergodic* Markov chain (a modification of the one above) which is highly adapted to the path's best strategy to minimize its number of self-intersections. Section 3 shows how to let this new chain work for us.

In Section 4 we may apply a standard central limit theorem for stationary ergodic Markov chains to our problem since the modified chain turns out to be highly mixing; in fact, its return times possess exponential moments. Section 4 finishes the proof of Theorem 1.10 for the self-repellent walk. In particular, we show at the end of the section how to lead the symmetric case back to the positive-drift case.

Sections 2 through 4 treat the self-repellent case and Section 5 the self-avoiding case. In fact, the latter case is completely analogous and even technically easier, so in Section 5 we will only point out the differences to the self-repellent case.

**2. Local times.** This and the following two sections are devoted to the proof of the following result.

PROPOSITION 2.1. *For every $\alpha \in (0,1)$ and $h \geq 0$ there exist $b = b(h,\alpha) \in \mathbb{R}$ and $S = S(h,\alpha) \in (0,\infty)$ such that, for all $C \in \overline{\mathbb{R}}$,*

$$\lim_{n\to\infty} e^{bn} E_h\big((1-\alpha)^{X_n} 1_{0<S_n<\Theta(h,\alpha)n + C\sqrt{n}}\big) = S\mathscr{N}(\sigma^2_{h,\alpha})((-\infty, C]). \tag{2.2}$$

Theorem 1.10 for the self-repellent case $\alpha \in (0,1)$ follows from this proposition [divide the l.h.s. of (2.2) by the same expression with $C = \infty$ and recall (1.1)].

The present section develops, for every $a \in \mathbb{N}_0$, a representation of the expected value of $(1-\alpha)^{X_n} 1_{S_n = a+1}$ in terms of a certain Markov chain, more exactly, as a weighted sum of expected values over this chain. This chain is used to describe the number of self-intersections which the path $(S_0, \dots, S_n)$ possesses in the spatial area $\{1, \dots, a\}$. A certain independence property of this chain enables us to separate their influence from that of the remaining self-intersections, that is, from those which occur in $-\mathbb{N}_0$ and $\{a+1, a+2, \dots\}$.

Let $\alpha \in (0,1)$ and $h > 0$ be fixed. We regard $P_h$ as a probability measure on

$$\begin{aligned}\Omega := \big\{(\omega_n)_{n\in\mathbb{N}_0} \in \mathbb{Z}^{\mathbb{N}_0}\colon\ \omega_0 = 0,\ |\omega_{n+1} - \omega_n| \in \{1, \dots, r\} \text{ for every } n \in \mathbb{N}_0 \\ \text{and } \lim_{n\to\infty} \omega_n = +\infty\big\}.\end{aligned} \tag{2.3}$$

2.1. *Ray–Knight-type description.* The starting observation is that the random variable under interest, $(1-\alpha)^{X_n}$, can be expressed in terms of the $n$-step local times

$$\ell_n(x) := \#\{i \in \{0,1,\ldots,n\}\colon\ S_i = x\}, \qquad x \in \mathbb{Z},\ n \in \mathbb{N}_0. \tag{2.4}$$

Introduce the abbreviation

$$\beta := -\frac{1}{2}\log(1-\alpha) \in (0,\infty). \tag{2.5}$$

Since $\#\{(i,j) \in \{0,\ldots,n\}^2 \mid S_i = S_j\} = \sum_{x\in\mathbb{Z}} \ell_n(x)^2$, one easily calculates that

$$(1-\alpha)^{X_n} = \exp(\beta(n+1))\exp\biggl(-\beta\sum_{x\in\mathbb{Z}}\ell_n(x)^2\biggr). \tag{2.6}$$

Since the factor $\exp(\beta(n+1))$ cancels in the definition of the transformed measure $P^n_{h,\alpha}$ [see (1.1)], we concentrate on the analysis of

$$E_h\biggl(\exp\biggl[-\beta\sum_x \ell_n(x)^2\biggr]1_{S_n=a+1}\biggr).$$

We are going to use a certain Markov chain which is introduced in König (1993) and serves for a Ray–Knight-type description of the local times. For $j$, $k \in \{1,\ldots,r\}$ and $x \in \mathbb{N}_0$, let

$$\begin{aligned}\eta_{j,k}(x) := \#\{(m,n) \in \mathbb{N}_0^2 \mid m < n,\ S_{m-1} \le x,\ S_m = x+j,\\ S_{m+1},\ldots,S_{n-1} > x,\ S_n = x+1-k\}\end{aligned} \tag{2.7}$$

be the number of excursions beyond $x$, starting at $x+j$ and ending at $x+1-k$. Furthermore, let

$$\tau(x) := \max\{n \in \mathbb{N} \mid S_{n-1} \le x < S_n\} \tag{2.8}$$

be the time of the last jump beyond $x$ and let

$$q(x) := S_{\tau(x)} - x \tag{2.9}$$

register the walker's position at this time. Lemma 2.2 in König (1993) states that

$$\Lambda(x) := \bigl((\eta_{j,k}(x))_{j,k\in\{1,\ldots,r\}}, q(x)\bigr) \tag{2.10}$$

defines a homogeneous Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ on

$$E := \mathbb{N}_0^{r\times r} \times \{1,\ldots,r\} \tag{2.11}$$

under $P_h$. Its transition probability function is denoted by $Q_h\colon E\times E \to [0,1]$. Note that $Q_h$ is irreducible and aperiodic as can be shown by adapting the proof of Lemma 2.6 in König (1993); in fact, every component of $Q_h^{3r}$ is positive. The infinite-time local times

$$\ell(x) := \lim_{n\to\infty} \ell_n(x) = \#\{k \in \mathbb{N}_0 \mid S_k = x\}, \qquad x \in \mathbb{Z}, \tag{2.12}$$

are a two-block functional of this Markov chain since we have

$$\ell(x) = g(\Lambda(x-1), \Lambda(x)), \qquad x \in \mathbb{N}, \tag{2.13}$$

where

$$g(\Lambda_1, \Lambda_2) := \sum_{k=1}^{r} \left(\eta_{1,k}^{(1)} + \eta_{k,1}^{(2)}\right) + 1_{q^{(1)}=1} \tag{2.14}$$

for elements $\Lambda_i = ((\eta_{j,k}^{(i)})_{j,k}, q^{(i)})$ of $E$, $i = 1, 2$. In the proof of Lemma 4.1 below we will need the following necessary condition for the positivity of $Q_h(\cdot,\cdot)$:

$$Q_h(\Lambda_1, \Lambda_2) > 0 \Rightarrow \sum_{j=1}^{r} \eta_{j,k+1}^{(2)} \le \sum_{j=1}^{r} \eta_{j,k}^{(1)} \quad \text{for every } k = 1, \dots, r-1. \tag{2.15}$$

The proof of Lemma 2.2 in König (1993) actually proved more than the Markov property and its homogeneity.

LEMMA 2.16. *Under $P_h(\cdot \mid \Lambda(0))$, the Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ is independent of the $\sigma$-field $\sigma(\ell(x)\colon x \in -\mathbb{N}_0)$. For every $a \in \mathbb{N}$, the $\sigma$-fields $\sigma(\Lambda(0), \dots, \Lambda(a-1))$ and $\sigma(\ell_{\tau(a)}(x)\colon x > a) \vee \sigma(S_{\tau(a)+1}, \dots, S_{\tau(a+1)})$ are independent under $P_h(\cdot \mid \Lambda(a))$.*

PROOF. Fix $x \in \mathbb{N}_0$. We say that an *excursion beyond* $x$ is the path segment between an upcrossing and the following downcrossing of the line between $x$ and $x+1$. The proof of Lemma 2.2 in König (1993) states and proves the independence of the $\sigma$-field generated by the excursions beyond $x$ [including the tail $(S_{\tau(x)}, \dots, S_{\tau(x+1)})$] and the $\sigma$-field generated by the path segments between these excursions (including the first part between time 0 and the first jump beyond $x$) under $P_h(\cdot \mid \Lambda(x))$ for $x \in \mathbb{N}$. This directly implies our second assertion. It is easy to see that the proof literally applies for $x = 0$, too, and this implies the first assertion. □

2.2. *Application to the polymer measure.* We shall now express $E_h(\exp[-\beta \sum_x \ell_n(x)^2] 1_{S_n=a+1})$ in terms of the Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ introduced in the preceding subsection. Define two functionals of this chain for $a \in \mathbb{N}$ by

$$Y_a := \sum_{x=1}^{a} g(\Lambda(x-1), \Lambda(x)) \quad \text{and} \quad V_a := \sum_{x=1}^{a} g(\Lambda(x-1), \Lambda(x))^2. \tag{2.17}$$

Abbreviate $P_{h,\Lambda_0} = P_h(\cdot\,|\Lambda(0) = \Lambda_0)$ and let $E_{h,\Lambda_0}$ denote the corresponding expectation. The goal of the present subsection is to prove the following lemma.

LEMMA 2.18. *There is a function $K\colon \mathbb{N}_0^2 \times E^2 \to [0,\infty)$ such that, for every $a, n \in \mathbb{N}_0$,*

$$E_h\biggl(\exp\biggl(-\beta\sum_x \ell_n(x)^2\biggr)1_{S_n=a+1}\biggr) \tag{2.19}$$
$$= \sum_{n_1,n_2=0}^{\infty}\sum_{\Lambda_0,\Lambda\in E} K(n_1,n_2,\Lambda_0,\Lambda)$$
$$\times E_{h,\Lambda_0}(\exp(-\beta V_a)1_{Y_a=n-n_1-n_2}1_{\Lambda(a)=\Lambda}).$$

REMARK 2.20. The constant $K(n_1,n_2,\Lambda_0,\Lambda)$ summarizes the contributions to the random variable $\exp(-\beta\sum_x \ell_n(x)^2)$ that come from the boundary pieces, that is, from the parts of the path in $-\mathbb{N}_0$ and in $\{a+1, a+2, \ldots\}$, and those that come from paths that spend $n_1$ time units in $-\mathbb{N}_0$, $n_2$ units in $\{a+1, a+2, \ldots\}$ and have as many excursions beyond 0 (resp. $a$) as is given by the entries in $\Lambda_0$ (resp., $\Lambda$).

PROOF OF LEMMA 2.18. We will treat the path classes $\{S_{n-1} < a+1 = S_n\}$ and $\{S_{n-1} > a+1 = S_n\}$ separately in slightly different manners. The idea is to sum over the amounts of time the path spends below 0 and above $a$, and over the numbers of the path's excursions beyond 0 (resp. $a$). As a preliminary step, observe that [writing $\Lambda = ((\eta_{j,k})_{j,k}, q)$ in the sequel]

$$\bigl\{S_{n-1} < a+1 = S_n \le S_{n+m} \text{ for every } m \in \mathbb{N}\bigr\}$$
$$= \{\tau(a) = n, q(a) = 1\}$$
$$= \bigcup_{n_1,n_2=0}^{\infty}\bigcup_{\Lambda_0,\Lambda\in E\colon q=1}\biggl\{\sum_{x\le 0}\ell(x) = 1+n_1,\ \Lambda(0) = \Lambda_0, \tag{2.21}$$
$$\sum_{x=1}^{a} g(\Lambda(x-1),\ \Lambda(x)) = n - n_1 - n_2,$$
$$\Lambda(a) = \Lambda,\ \sum_{x>a}\ell_{\tau(a)}(x) = n_2\biggr\}.$$

Of course, this union is disjoint. Put $\pi_1 := P_h(S_m \ge 0 \text{ for every } m \in \mathbb{N}) \in (0,1)$. Then Lemma 2.16 implies that

$$E_h\biggl(\exp\biggl(-\beta\sum_x \ell_n(x)^2\biggr)1_{S_{n-1}<a+1=S_n}\biggr) \tag{2.22}$$
$$= \sum_{n_1,n_2=0}^{\infty}\sum_{\Lambda_0,\Lambda\in E\colon q=1} K_-(n_1,\Lambda_0)K_+^{\uparrow}(n_2,\Lambda)$$
$$\times E_{h,\Lambda_0}(\exp(-\beta V_a)1_{Y_a=n-n_1-n_2}1_{\Lambda(a)=\Lambda}),$$

where the constants are given by

$$
\begin{aligned}
K_-(n_1,\Lambda_0) &:= E_h\biggl(\exp\Bigl(-\beta\sum_{x\le 0}\ell(x)^2\Bigr)1_{\sum_{x\le 0}\ell(x)=1+n_1}1_{\Lambda(0)=\Lambda_0}\biggr),\\
K_+^{\uparrow}(n_2,\Lambda) &:= \frac{1}{\pi_1}E_h\biggl(\exp\Bigl(-\beta\sum_{x>a}\ell_{\tau(a)}(x)^2\Bigr)1_{\sum_{x>a}\ell_{\tau(a)}(x)=n_2}\mid\Lambda(a)=\Lambda\biggr)
\end{aligned}\tag{2.23}
$$

if $q=1$ and $n_2\in\mathbb{N}$. The latter constant is in fact independent of $a$ since the underlying random walk is homogeneous and since $\ell_{\tau(a)}(x)$, for $x>a$, does not depend on the order of the excursions. For formal reasons, define $K_+(n_2,\Lambda)=0$ if $q\neq 1$ or $n_2=0$.

The paths in $\{S_{n-1}>a+1=S_n\}$ are treated in an analogous way, but we have to introduce the time $\tilde\tau(a+1):=\max\{n\in\mathbb{N}\colon S_{n-1}>a+1\ge S_n\}$ of the last downcrossing of the line between $a+1$ and $a+2$, and this on the set $\{\sum_{j,k=1}^r\eta_{j,k}(a+1)\ge 1\}$. We consider the representation

$$
\begin{aligned}
&\{S_{n-1}>a+1=S_n<S_{n+m}\text{ for every }m\in\mathbb{N}\}\\
&\quad=\{\tau(a)<n=\tilde\tau(a+1)=\tau(a+1)-1\}\\
&\quad=\bigcup_{n_1,n_2=0}^{\infty}\ \bigcup_{\Lambda_0,\Lambda\in E}\biggl\{\sum_{x\le 0}\ell(x)=1+n_1,\ \Lambda(0)=\Lambda_0,\ Y_a=n-n_1-n_2,\\
&\qquad\qquad\Lambda(a)=\Lambda,\sum_{x>a}\ell_{\tilde\tau(a+1)}(x)=n_2,\ \tau(a)<\tau(a+1)-1\biggr\}.
\end{aligned}\tag{2.24}
$$

With the help of Lemma 2.16, we obtain the formula

$$
\begin{aligned}
&E_h\biggl(\exp\Bigl(-\beta\sum_x\ell_n(x)^2\Bigr)1_{S_{n-1}>a+1=S_n}\biggr)\\
&\quad=\sum_{n_1,n_2=1}^{\infty}\ \sum_{\Lambda_0,\Lambda\in E}K_-(n_1,\Lambda_0)K_+^{\downarrow}(n_2,\Lambda)\\
&\qquad\qquad\times E_{h,\Lambda_0}\bigl(\exp(-\beta V_a)1_{Y_a=n-n_1-n_2}1_{\Lambda(a)=\Lambda}\bigr),
\end{aligned}\tag{2.25}
$$

where the new constant is given by

$$
\begin{aligned}
K_+^{\downarrow}(n_2,\Lambda) := \frac{1}{\pi_2}E_h\biggl(&\exp\Bigl(-\beta\sum_{x>a}\ell_{\tilde\tau(a+1)}(x)^2\Bigr)1_{\sum_{x>a}\ell_{\tilde\tau(a+1)}(x)=n_2}\\
&\times 1_{\tau(a)<\tau(a+1)-1}\mid\Lambda(a)=\Lambda\biggr)
\end{aligned}\tag{2.26}
$$

for $n_2\in\mathbb{N}$ where $\pi_2:=P_h(S_m>0\text{ for every }m\in\mathbb{N})$. For the same reason as for the constant $K_+^{\uparrow}$, the constant $K_+^{\downarrow}$ does not depend on $a$. Let $K_+^{\downarrow}(0,\Lambda):=0$. Combining formulas (2.22) and (2.25), we obtain Lemma 2.18 with the constant $K$ identified as

$$
K(n_1,n_2,\Lambda_0,\Lambda)=K_-(n_1,\Lambda_0)\bigl(K_+^{\uparrow}+K_+^{\downarrow}\bigr)(n_2,\Lambda).\tag{2.27}
$$

□

**3. Passing to an ergodic Markov chain.** In this section, we shall construct a transformation of the Markov chain introduced in Section 2.1. This will turn the expectation of $e^{-\beta V_a}$ into a probability with respect to this chain. The transformed chain will turn out to have strong recurrence properties, and it will play a key role in the proof of Proposition 2.1. Informally and intuitively, this chain describes the path's best (random) strategy to minimize its self-intersection number.

Along the way, we shall be able to use some results from König (1994) which will be presented now. Some general theoretical background of the following is provided by Seneta (1981), pages 200–207, while the concrete assertions are proved in Section 3 of König (1994). Keep $\alpha \in (0,1)$ fixed and assume that $h > 0$. Unlike König (1994), we will not indicate the dependence of the following objects on $\beta = -\log(1-\alpha)/2$ in the notation.

3.1. *A transformed Markov chain.* Recall (2.11) and (2.14) and introduce an infinite matrix

$$\begin{aligned} A_{h,b} &= \big(A_{h,b}(\Lambda, \Lambda')\big)_{\Lambda, \Lambda' \in E} \\ &:= \big(Q_h(\Lambda, \Lambda') \exp[-\beta g(\Lambda, \Lambda')^2 + b g(\Lambda, \Lambda')]\big)_{\Lambda, \Lambda' \in E}, \qquad b \in \mathbb{R}, \end{aligned} \tag{3.1}$$

which is nonnegative, irreducible and aperiodic. Define its convergence parameter by

$$\lambda_h(b) := \lim_{n\to\infty} \big(A^n_{h,b}(\Lambda, \Lambda')\big)^{1/n}, \qquad b \in \mathbb{R}, \tag{3.2}$$

and note that this latter definition does not depend on $\Lambda, \Lambda' \in E$. The function $\lambda_h$ is real-analytic, strictly increasing and strictly log-convex.

The number $\lambda_h(b)$ is even an eigenvalue of the matrix $A_{h,b}$ with corresponding positive right and left eigenvectors $\tau^r_b$, $\tau^l_b \in (0,\infty)^E$ satisfying $\langle \tau^l_b, \tau^r_b \rangle = 1$. Thus, the stochastic matrix

$$Q_{h,b} := \left( \frac{A_{h,b}(\Lambda, \Lambda')}{\lambda_h(b)} \, \frac{\tau^r_b(\Lambda')}{\tau^r_b(\Lambda)} \right)_{\Lambda, \Lambda' \in E}, \qquad b \in \mathbb{R}, \tag{3.3}$$

is the transition matrix of a positive recurrent Markov chain [which will be denoted by $(\Lambda(x))_{x\in\mathbb{N}_0}$, too] on $E$. We denote the invariant distribution of this chain by

$$\mu_b := (\tau^l_b(\Lambda)\tau^r_b(\Lambda))_{\Lambda\in E}. \tag{3.4}$$

The distribution of the stationary Markov chain with initial distribution $\mu_b$ and transition kernel $Q_{h,b}$ is denoted by $P^{\beta,b}_h$ and we write

$$P^{\beta,b}_{h,\Lambda_0} := P^{\beta,b}_h(\,\cdot \mid \Lambda(0) = \Lambda_0).$$

We shall write $E^{\beta,b}_{h,\Lambda_0}$ for expectation with respect to $P^{\beta,b}_{h,\Lambda_0}$ and $E^{\beta,b}_h$ for expectation with respect to $P^{\beta,b}_h$ and so on.

We intend to multiply (2.19) by $e^{nb}$ and to switch from the underlying distribution $P_h$ to $P_h^{\beta,b}$. The following lemma formulates the relation between the free and the transformed Markov chain [recall (2.17)].

LEMMA 3.5. *For every $n, n_1, n_2, a \in \mathbb{N}_0$, $b \in \mathbb{R}$ and $\Lambda_0, \Lambda \in E$, we have*

$$\begin{aligned} &e^{nb} E_{h,\Lambda_0}\big(e^{-\beta V_a} 1_{Y_a = n-n_1-n_2} 1_{\Lambda(a)=\Lambda}\big) \\ &\qquad = P_{h,\Lambda_0}^{\beta,b}(Y_a = n - n_1 - n_2,\ \Lambda(a) = \Lambda) e^{b(n_1+n_2)} \frac{\tau_b^r(\Lambda_0)}{\tau_b^r(\Lambda)} \lambda_h(b)^a. \end{aligned} \tag{3.6}$$

PROOF. Note that, on the set $A = \{Y_a = n - n_1 - n_2,\ \Lambda(0) = \Lambda_0,\ \Lambda(a) = \Lambda\}$, we have

$$\begin{aligned} nb - \beta V_a &= b(n_1 + n_2) - \beta V_a + bY_a \\ &= b(n_1 + n_2) + \log \prod_{x=1}^{a} \frac{Q_{h,b}(\Lambda(x-1), \Lambda(x))}{Q_h(\Lambda(x-1), \Lambda(x))} \\ &\qquad - \log \frac{\tau_b^r(\Lambda)}{\tau_b^r(\Lambda_0)} + a \log(\lambda_h(b)). \end{aligned} \tag{3.7}$$

Now take the exponentials of both sides, multiply them by the indicator on $A$ and take expectations w.r.t. $P_{h,\Lambda_0}$ to arrive at (3.6) [recall (3.3) and (3.1)]. □

3.2. *Properties of the transformed chain.* The variance $\sigma^2_{h,\alpha}$ defined in (1.8) and some more quantities related to the polymer measure can be completely characterized in terms of the convergence parameter $\lambda_h$ defined in (3.2) and hence in terms of properties of the transformed Markov chain introduced in the preceding subsection. We are going to list these connections. Still we assume that $h > 0$, suppress $\beta = -\log(1-\alpha)/2$ from the notation and write $\Theta_h = \Theta(h, \alpha)$.

Most of the work in König (1994) was spent on analyzing the function

$$\tilde{J}_h(\theta) := \lim_{n\to\infty} \frac{1}{n} \log E_h\bigg(\exp\bigg(-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\bigg) 1_{0<S_1,\ldots,S_{n-1}<S_n=\lfloor\theta n\rfloor}\bigg), \qquad \theta \in (0, r), \tag{3.8}$$

where it is to be understood that, in the nearest-neighbor case, the condition $S_n = \lfloor\theta n\rfloor$ should be replaced by $S_n \in \{\lfloor\theta n\rfloor, \lfloor\theta n\rfloor + 1\}$. Note that $\tilde{\Phi}_{h,\alpha} = \tilde{J}_h - \tilde{J}_h(\Theta_h)$ [recall (1.7)].

The function $\tilde{J}_h$ can be characterized in terms of $\lambda_h$ as follows. Recall that $\lambda_h$ is a strictly log-convex function. The inverse function of $\lambda_h/\lambda_h'$,

$$b_h(\theta) := \left(\frac{\lambda_h}{\lambda_h'}\right)^{-1}(\theta), \qquad \theta \in (0, r), \tag{3.9}$$

is real-analytic and strictly decreasing. Then we have the following identities [see Theorem 1.7 in König (1994) and note the typographical error in the first line there] for all $\theta \in (0, r)$:

$$\tilde{J}_h(\theta) = \theta \log \lambda_h(b_h(\theta)) - b_h(\theta), \tag{3.10a}$$

$$\tilde{J}'_h(\theta) = \log \lambda_h(b_h(\theta)), \tag{3.10b}$$

$$\tilde{J}''_h(\theta) = b'_h(\theta)/\theta < 0. \tag{3.10c}$$

Thus, in particular, $\tilde{J}_h$ is real-analytic and strictly concave. In König (1994) it is also proved that $\tilde{J}'_h$ has a 0 in $\Theta_h \in (0, r)$, and this property originally defined this number. Thus $\tilde{J}_h$ possesses a strict maximum in $\Theta_h$.

Formulas (3.10a)–(3.10c), in particular, imply that

$$b^*_h := b_h(\Theta_h) = -\lim_{n\to\infty} \frac{1}{n} \log E_h\left(\exp\left(-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\right)\right), \tag{3.11}$$

and

$$\lambda_h(b^*_h) = 1 \quad \text{and} \quad \lambda'_h(b^*_h) = \frac{1}{\Theta_h} \tag{3.12}$$

and

$$\sigma^2_h(\theta) := -\frac{1}{\tilde{J}''_h(\theta)} = \theta^3 \left(\frac{\lambda''_h}{\lambda_h}(b_h(\theta)) - \frac{1}{\theta^2}\right), \qquad \theta \in (0, r). \tag{3.13}$$

Thus, we have, in particular [recall (1.8)],

$$\sigma^2_{h,\alpha} = \sigma^2_h(\Theta(h, \alpha)) = \Theta^3_h \left(\lambda''_h(b^*_h) - \frac{1}{\Theta^2_h}\right). \tag{3.14}$$

We shall write $\tau^l = \tau^l_{b^*_h}$ and $\tau^r = \tau^r_{b^*_h}$ in the sequel.

From (3.7) in König (1994) we have

$$E^{\beta,b}_h(g(\Lambda(0), \Lambda(1))) = \frac{\lambda'_h(b)}{\lambda_h(b)}, \qquad b \in \mathbb{R}, \tag{3.15}$$

and, in particular, writing $P^\beta_h := P^{\beta, b^*_h}_h$,

$$E^\beta_h(g(\Lambda(0), \Lambda(1))) = \frac{1}{\Theta_h}. \tag{3.16}$$

It should be noted that the objects $Q_h$, $A_{h,b}$, $\lambda_h$, $b_h$, $\tau^l_b$, $\tau^r_b$, $\mu_b$ and $P^{\beta,b}_h$ exist for positive-drift parameter $h$ only, while $\tilde{J}_0$, $\Theta_0$, $\sigma^2_{0,\alpha}$ and $\tilde{\Phi}_{0,\alpha}$ are well defined as well.

3.3. *The boundary pieces.* In the next section we shall prove a central limit theorem for $(Y_a)_{a\in\mathbb{N}}$ under $P_h^{\beta,b}$ for various $b$. The following lemma will be important at the end of that section for proving that exclusively this Gaussian behavior in fact determines the behavior of the self-repellent walk. We show that the influence of the self-intersections in $-\mathbb{N}_0 \cup \{S_n, S_n+1, \dots\}$ is much smaller than the influence of the self-intersections in $\{1, \dots, S_n - 1\}$. For the definition of the constant $K$ appearing below, recall Lemma 2.18 and (2.27).

LEMMA 3.17. *There is a positive number $\varepsilon$, not depending on $h$, such that, for every $h > 0$ and $\theta \in (0, \Theta_h]$,*

$$\limsup_{n_1,n_2\to\infty} \frac{1}{n_1+n_2} \log \sum_{\Lambda_0,\Lambda\in E} K(n_1,n_2,\Lambda_0,\Lambda) \frac{\tau^r_{b_h(\theta)}(\Lambda_0)}{\tau^r_{b_h(\theta)}(\Lambda)} < -b_h(\theta) + r\log\lambda_h(b_h(\theta)) - \varepsilon. \tag{3.18}$$

PROOF. This proof is a refinement of the proof of Proposition 4.1 in König (1994). Roughly speaking, the l.h.s. of our asserted inequality is the exponential rate of the expected self-intersection punishment for paths which stay for $n_1$ (resp. $n_2$) time units below 0 (resp. above) $a$, starting and ending approximately at the same site, weighted with the eigenvector term. Our proof will be in the spirit of estimating the l.h.s. above by $\tilde{J}_h(\theta)$. [Note that, with some more effort, one could show that $\tilde{J}_h(\theta) - \varepsilon$ is indeed an upper bound, too.]

We will show only that $\limsup_{n\to\infty}(1/n)\log\sum_{\Lambda_0} K_-(n,\Lambda_0)\tau^r_{b_h(\theta)}(\Lambda_0)$ is smaller than the r.h.s. of (3.18) for some appropriate positive $\varepsilon$ since the proof for the analogous property of $K_+^{\downarrow}$ and $K_+^{\uparrow}$ is similar. Recall (2.3).

*Step* 1. There are $c, \varepsilon > 0$ and a sequence $\gamma_n = e^{o(n)}$ such that, for $h > 0$, $\Lambda_0 \in E$ and $n \in \mathbb{N}_0$,

$$K_-(n,\Lambda_0) \le \gamma_n E_h\biggl(\exp\biggl(-\beta\sum_{x\le 0}\ell_n(x)^2\biggr)1_{A_n(\Lambda_0,c,\varepsilon)}\biggr), \tag{3.19}$$

where

$$\begin{aligned} A_n(\Lambda_0,c,\varepsilon) := \Omega \cap \biggl\{&\Lambda(0) = \Lambda_0,\ \sum_{x\le 0}\ell(x) = n,\ \exists\, t^*\colon\ S_1,\dots,S_{t^*} \le 0,\\ &S_{t^*} = \min_{t\le\tau(0)} S_t \le -cn \text{ and } U_{t^*} \ge \frac{\varepsilon n}{2\beta}\biggr\} \end{aligned} \tag{3.20}$$

and

$$U_{t^*} = \sum_{x\le 0}\ell_{t^*}(x)(\ell_{\tau(0)}(x) - \ell_{t^*}(x)). \tag{3.21}$$

PROOF. On the set $\{\sum_{x\le 0}\ell(x) = n, \min_{t\le\tau(0)} S_t > -cn\}$, the random variable $\exp[-\beta\sum_{x\le 0}\ell(x)^2]$ is not larger than $e^{-\beta n/c}$ whose exponential rate decays to $-\infty$ as $c \downarrow 0$. Thus, for small $c > 0$, the exponential rate of $K_-(n,\Lambda_0)$

is not changed if we insert the indicator on $\{\min_{t\le\tau(0)} S_t \le -cn\}$ in the expectation on the r.h.s. in the first line of (2.23).

Now cut out a path segment $(S_{t_1},\dots,S_{t_2})$ that satisfies $0\le t_1<t_2\le\tau(0)$ and $S_{t_1}+r\ge 0\ge S_{t_1},\dots,S_{t_2-1}\le 0<S_{t_2}$ and $S_{t^*}=\min_{t\le\tau(0)} S_t\le -cn$ for some $t^*\in\{t_1,\dots,t_2\}$, put it in the beginning of the path and insert contingent joining steps such that a path in

$$\begin{aligned}B_n(\Lambda_0,c):=\Omega\cap\Big\{&\Lambda(0)=\Lambda_0,\ \sum_{x\le 0}\ell(x)=n,\\ &\exists\, t^*\colon\ S_1,\dots,S_{t^*}\le 0,\ S_{t^*}=\min_{t\le\tau(0)} S_t\le -cn\Big\}\end{aligned}\tag{3.22}$$

is obtained. This map changes the variable $\exp[-\beta\sum_{x\le 0}\ell(x)^2]$ by a factor of maximal size $e^{o(n)}$, and the number of pre-images under the map is also smaller than $e^{o(n)}$, uniformly in $\Lambda_0\in E$. So far we have seen that (3.19) is true with $A_n(\Lambda_0,c,\varepsilon)$ replaced by $B_n(\Lambda_0,c)$.

The random variable $U_{t^*}$ is equal to the number of intersections the path segment $(S_0,\dots,S_{t^*})$ has with $(S_{t^*},\dots,S_{\tau_0})$. In the proof of Proposition 4.1 in König (1994) between formulas (4.4) and (4.6), it is made precise and it is proved that $U_{t^*}$ behaves linearly in $n$ on $B_n(\Lambda_0,c)$. More precisely, it is shown there that the exponential rate of

$$E_h\bigg(\exp\bigg(-\beta\sum_{x\le 0}\ell(x)^2\bigg)1_{B_n(\Lambda_0,c)}1_{U_{t^*}=xn+o(n)}\bigg)$$

is strictly maximal in some positive $x$. This corresponds to the intuition that, in order to minimize the self-intersection number of a path which stays altogether $n$ time units in $-\mathbb{N}_0$, it is not the best strategy to let the two segments $(S_0,\dots,S_{t^*})$ and $(S_{t^*},\dots,S_{\tau(0)})$ avoid each other but rather assume they have some intersections with each other whose number is linear in $n$.

Thus, the rate of $K_-(n,\Lambda_0)$ is not changed if the indicator on $\{U_{t^*}\ge\varepsilon n/2\beta\}$ for some small $\varepsilon>0$ [in addition to that on $B_n(\Lambda_0,c)$] is inserted, which proves Step 1.

*Step* 2. With $A_n$, $\varepsilon$, $U_{t^*}$ and $c$ as in Step 1 and $V_a$ and $Y_a$ as in (2.17), we have

$$\begin{aligned}&E_h\bigg(\exp\bigg(-\beta\sum_{x\le 0}\ell(x)^2\bigg)1_{A_n(\Lambda_0,c,\varepsilon)}\bigg)\\ &\qquad\le\frac{\exp(-\varepsilon n)}{p_r}\sum_{a=\lfloor 2cn\rfloor}^{rn}E_h(\exp(-\beta V_a)1_{Y_a=n}1_{\Lambda(0)=\Lambda^*}1_{\Lambda(a)=\Lambda_0}),\end{aligned}\tag{3.23}$$

where $\Lambda^*=((0),r)$ and $p_r$ is given in (1.3).

PROOF. Define a map

$$\Gamma_n\colon\ A_n(\Lambda_0,c,\varepsilon)\to\Omega\cap\bigcup_{a=\lfloor 2cn\rfloor}^{rn}\bigg\{\Lambda(0)=\Lambda^*,\ \Lambda(a)=\Lambda_0,\sum_{x=1}^{a}\ell(x)=n\bigg\}$$

by the following procedure: reflect the path segment $(S_0, \dots, S_{t^*})$ [with the smallest $t^*$ as in (3.20)] around $S_{t^*}$, add a single step of size $r$ upwards in the beginning and lift the whole path by $a := -2S_{t^*} + r$ sites.

The map $\Gamma_n$ is injective. Under $\Gamma_n$ the path's probability is multiplied by the probability for one step upwards, $p_r$ [see (1.3)], and is additionally increased for $h > 0$ [observe from (1.3) that a horizontal reflection does not change the probability under $P_0$]. Furthermore, $\Gamma_n$ carries the variable $\exp[-\beta \sum_{x \le 0} \ell(x)^2]$ into $\exp[-\beta \sum_{x=1}^{a} \ell(x)^2] \exp(-2\beta U_{t^*})$ [see (3.21)] which is not larger than $\exp(-\varepsilon n) \exp[-\beta \sum_{x=1}^{a} \ell(x)^2]$. Now insert the Markov chain $(\Lambda(x))_{x \in \mathbb{N}_0}$ and use (2.17) to prove Step 2.

PROOF OF LEMMA 3.17. Use Steps 1 and 2 and note that $\lambda_h(b_h(\theta)) \ge \lambda_h(b_h(\Theta_h)) = 1$ and apply (3.6) to $b = b_h(\theta)$ to obtain

$$\begin{aligned}
&e^{o(n)+\varepsilon n} e^{b_h(\theta) n} \sum_{\Lambda_0 \in E} K_-(n, \Lambda_0) \tau^r_{b_h(\theta)}(\Lambda_0) \\
&\quad \le \sum_{\Lambda_0 \in E} \sum_{a = \lfloor 2cn \rfloor}^{rn} e^{b_h(\theta) n} E_{h, \Lambda^*}\big(e^{-\beta V_a} 1_{Y_a = n} 1_{\Lambda(a) = \Lambda_0}\big) \tau^r_{b_h(\theta)}(\Lambda_0) \\
&\quad \le \sum_{a \in \mathbb{N}} P^{\beta, b_h(\theta)}_{h, \Lambda^*}(Y_a = n) \tau^r_{b_h(\theta)}(\Lambda^*) \lambda_h(b_h(\theta))^{rn} \\
&\quad \le r \tau^r_{b_h(\theta)}(\Lambda^*) \lambda_h(b_h(\theta))^{rn}.
\end{aligned}$$

In the last step, we used the fact that $P_h(Y_a = n = Y_{a+l}) = 0$ if $l > r$ which is a by-product of the proof of Lemma 4.13 below and is due to the simple fact that the walker must hit at least one of any $r$ subsequent sites. The estimation above implies our assertion. □

**4. The central limit theorem.** Recall that $P^{\beta, b}_h$ is the distribution of a positive recurrent Markov chain $(\Lambda(x))_x$ on $E$ with transition kernel $Q_{h,b}$ defined by (2.9) and invariant starting distribution $\mu_b$ [recall (3.4)]. This section completes the proof of Proposition 2.1 for the self-repellent case $\alpha \in (0,1)$. As a pre-step, we will prove a central limit theorem for $Y_a = \sum_{x=1}^{a} g(\Lambda(x-1), \Lambda(x))$ under $P^{\beta, b}_{h, \Lambda_0} = P^{\beta, b}_h(\,\cdot \mid \Lambda(0) = \Lambda_0)$. This is essentially done by applying a standard central limit theorem for ergodic Markov chains; however, the justification of this application will require some effort. Roughly speaking, the main obstacle is to show that the return times possess exponential moments under $P^{\beta, b}_h$. Further on, we keep $\beta = -\log(1-\alpha)/2 \in (0, \infty)$ fixed and assume the drift parameter $h$ to be positive. Recall (3.2) and (3.9).

LEMMA 4.1. *Fix $\theta \in (0, r)$ and $\Lambda_0 \in E$. As $a \to \infty$, the distribution of $(Y_a/a - 1/\theta)\sqrt{a}$ under $P^{\beta, b_h(\theta)}_{h, \Lambda_0}$ tends weakly toward $\mathscr{N}((\lambda''_h/\lambda_h)(b_h(\theta)) - 1/\theta^2)$.*

PROOF. Let $T_0 := 0$ and, recursively, $T_{i+1} := \inf\{n > T_i : \Lambda(n) = \Lambda_0\}$ for $i \in \mathbb{N}_0$ denote the subsequent hitting times of $\Lambda_0$. By a suitable reduction of the underlying probability space, we may and will assume that $T_i$ is finite for every $i \in \mathbb{N}_0$. Note that $(T_{i+1}-T_i)_{i\in\mathbb{N}_0}$ is an i.i.d. sequence with $E^{\beta,b_h(\theta)}_{h,\Lambda_0}(T_{i+1}-T_i) = (\mu_{b_h(\theta)}(\Lambda_0))^{-1}$. The random variables [recall (2.14)]

$$U_i := \sum_{x=T_{i-1}+1}^{T_i} g(\Lambda(x-1),\Lambda(x)) = Y_{T_i} - Y_{T_{i-1}}, \qquad i \in \mathbb{N}, \tag{4.2}$$

are i.i.d. under $P^{\beta,b_h(\theta)}_{h,\Lambda_0}$, and they are integrable with

$$E^{\beta,b_h(\theta)}_{h,\Lambda_0}(U_i) = \frac{1}{\theta\mu_{b_h(\theta)}(\Lambda_0)} \tag{4.3}$$

[see Chung (1967), Chapter 1, Theorem 14.5 and (14.16); here we use (3.15), the stationarity and the fact that the function $g(\cdot,\cdot)$ is the sum of two functions of its arguments].

The main difficulty in this proof is to show that the i.i.d. integrable and centered variables

$$Z_i := U_i - \frac{T_i - T_{i-1}}{\theta}, \qquad i \in \mathbb{N}, \tag{4.4}$$

are square integrable under $P^{\beta,b_h(\theta)}_{h,\Lambda_0}$ and to identify their variance, which we will do now.

Since $Q_{h,b}$ is, in particular, recurrent for real $b$ in a neighborhood of $b_h(\theta)$, we have

$$1 = \sum_{n=1}^{\infty} \sum_{\Lambda_1,\dots,\Lambda_{n-1}\in E\setminus\{\Lambda_0\}} \prod_{i=1}^{n} \frac{A_{h,b}(\Lambda_{i-1},\Lambda_i)}{\lambda_h(b)}, \qquad \Lambda_n := \Lambda_0, \tag{4.5}$$

for those $b$. We intend to differentiate both sides of this identity two times with respect to $b$, to interchange this differentiation with the sum over $n$ and to evaluate it at $b = b_h(\theta)$. For this purpose, we will first show that the series on the r.h.s. of (4.5) converges even uniformly in $b$ in a *complex* neighborhood of $b_h(\theta)$. More exactly, we will show the existence of some $\eta > 0$ such that

$$\sup_{b\in\mathbb{C}:\,|b-b_h(\theta)|\le\eta} \limsup_{n\to\infty} \frac{1}{n} \log\left| \sum_{\Lambda_1,\dots,\Lambda_{n-1}\in E\setminus\{\Lambda_0\}} \prod_{i=1}^{n} \frac{A_{h,b}(\Lambda_{i-1},\Lambda_i)}{\lambda_h(b)} \right| < 0. \tag{4.6}$$

This assertion implies, in particular, the existence of an exponential moment of $T_1$ under $P^{\beta,b}_{h,\Lambda_0}$, uniformly in $b \in [b_h(\theta)-\eta, b_h(\theta)+\eta]$. Recall that $Q_h^{3r}$ does not possess any zeros and define $E_R = \{0,\dots,R\}^{r\times r} \times \{1,\dots,r\}$ for $R \in \mathbb{N}$. Choose any $\eta > 0$ and observe that

$$c_R := \sup_{b\in[b_h(\theta)-\eta,\,b_h(\theta)+\eta]} \max_{\Lambda\in E_R} \sum_{\Lambda'\in E\setminus\{\Lambda_0\}} Q^{3r}_{h,b}(\Lambda,\Lambda') \tag{4.7}$$

is strictly smaller than 1 for every $R \in \mathbb{N}$. Now put

$$K_R := \sup_{b\in[b_h(\theta)-\eta,\,b_h(\theta)+\eta]} \sup_{x\ge R} \exp(-\beta x^2 + bx) \downarrow 0, \qquad R \uparrow \infty. \tag{4.8}$$

So we have $A^{3r}_{h,b}(\Lambda,\Lambda') \le Q^{3r}_h(\Lambda,\Lambda')K_0^{3r}$ for every $b \in [b_h(\theta)-\eta, b_h(\theta)+\eta]$ and $\Lambda,\Lambda' \in E$, and we even have $A^{3r}_{h,b}(\Lambda,\Lambda') \le Q^{3r}_h(\Lambda,\Lambda')K_0^{3r-1}K_R$ if additionally $\Lambda'$ is in $E\setminus E_R$. The latter statement follows from (2.15) since, for $e_0=\Lambda$, $e_1,\dots,e_{3r-1} \in E$ and $e_{3r}=\Lambda' \in E\setminus E_R$ satisfying $\prod_{i=1}^{3r} Q_h(e_{i-1},e_i) > 0$, there is an $i \in \{2r,\dots,3r\}$ such that $g(e_{i-1},e_i) \ge R$.

In order to derive (4.6), we first estimate, for complex $b$ with $|b-b_h(\theta)| \le \eta$, writing $\Re b$ for the real part of $b$,

$$\begin{aligned}\Bigg|\sum_{\Lambda_1,\dots,\Lambda_{n-1}\in E\setminus\{\Lambda_0\}} \prod_{i=1}^n \frac{A_{h,b}(\Lambda_{i-1},\Lambda_i)}{\lambda_h(b)}\Bigg| \\ \le |\lambda_h(b)|^{-n} \sum_{\Lambda_1,\dots,\Lambda_{\lfloor n/3r\rfloor}\in E\setminus\{\Lambda_0\}} \Bigg(\prod_{i=1}^{\lfloor n/3r\rfloor} A^{3r}_{h,\Re b}(\Lambda_{i-1},\Lambda_i)\Bigg) \\ \times A^{n-\lfloor n/3r\rfloor 3r}_{h,\Re b}(\Lambda_{\lfloor n/3r\rfloor},\Lambda_0)\end{aligned} \tag{4.9}$$

and split the sum on the r.h.s. into the sum over those $\Lambda_1,\dots,\Lambda_{\lfloor n/3r\rfloor} \in E\setminus\{\Lambda_0\}$ which satisfy $\#\{i\colon \Lambda_i \in E_R\} \ge n/6r$ and the sum over the remaining multi-indices. In the first sum, we replace $A_{h,\Re b}$ by $\lambda_h(\Re b)Q_{h,\Re b}$ on every occurrence in the product and estimate simply $Q^{n-\lfloor n/3r\rfloor 3r}_{h,\Re b}(\Lambda_{\lfloor n/3r\rfloor},\Lambda_0) \le 1$ and, recursively, for $i=\lfloor n/3r\rfloor-1, \lfloor n/3r\rfloor-2,\dots,1$,

$$\sum_{\Lambda_{i+1}\in E\setminus\{\Lambda_0\}} Q^{3r}_{h,\Re b}(\Lambda_i,\Lambda_{i+1}) \le c_R \tag{4.10}$$

if the $i$th summing index $\Lambda_i$ runs over $E_R\setminus\{\Lambda_0\}$ and simply $\le 1$ if it runs over $E\setminus\{\Lambda_0\}$. In the second sum, we use the two estimates below (4.8). Summarizing, we obtain the following upper bound for the r.h.s. of (4.9):

$$|\lambda_h(b)|^{-n}\Big(\lambda_h(\Re b)^n c_R^{\lfloor n/6r\rfloor} + K_0^n K_R^{\lfloor n/6r\rfloor}\Big), \qquad R \in \mathbb{N},\ |b-b_h(\theta)| \le \eta. \tag{4.11}$$

Now choose $R$ large and make $\eta$ smaller (if necessary) to arrive at (4.6).

According to Vitali's theorem, the series in (4.5) defines an analytic function in a complex neighborhood of $b_h(\theta)$ and may be differentiated termwise infinitely many times. Carrying out the first derivative and inserting $b=b_h(\theta)$, we get (4.3) back. The second termwise derivation yields at $b=b_h(\theta)$ the integrability of $Z_1^2$ and the formula

$$E^{\beta,b_h(\theta)}_{h,\Lambda_0}(Z_1^2) = \frac{(\lambda_h''/\lambda_h)(b_h(\theta)) - 1/\theta^2}{\mu_{b_h(\theta)}(\Lambda_0)}. \tag{4.12}$$

Now, since the function $g(\cdot,\cdot)$ is the sum of two functions of its arguments, we can apply Theorem 16.1 in Chapter 1 of Chung (1967), which yields the assertion of this lemma. □

In Lemma 4.16 below, we will consider the sum of (3.6) over $a$, and the next lemma states that, on the r.h.s., we have the sum of $P_h$-disjoint events.

LEMMA 4.13. *For every $n \in \mathbb{N}$ and any distinct natural numbers $a_1$ and $a_2$ and every $\Lambda \in E$, we have*

$$P_h(Y_{a_1} = n = Y_{a_2} \textit{ and } \Lambda(a_1) = \Lambda = \Lambda(a_2)) = 0. \tag{4.14}$$

PROOF. We may assume that $m := a_2 - a_1$ is positive, and we write $\Lambda = ((\eta_{j,k})_{j,k}, q)$. Assume that the assertion is not true. Then we know that the set

$$A := \{0 = \ell(a_1 + 1) = \cdots = \ell(a_2) \text{ and } \Lambda(a_1) = \Lambda = \Lambda(a_2)\} \tag{4.15}$$

has positive probability under $P_h$. If $\eta_{j,k} > 0$ for some $j$ and $k$, then $j$ must be larger than $m$ since otherwise an excursion beyond $a_1$ would produce a hit in $\{a_1+1, \dots, a_2\}$. So, on the set $A$, there exists an excursion beyond $a_1$, starting and ending with a jump over the sites $a_1 + 1, \dots, a_2$ each. This excursion is, in particular, an excursion beyond $a_2$ with parameters $j - m$ and $k + m$, so it follows that $\eta_{j-m,k+m} > 0$ and thus $j-m > m$. A repetition of these arguments leads to a contradiction. So we have $\Lambda = ((0), q)$, and the same arguments as above lead to a contradiction by considering the possible values of $q$. □

LEMMA 4.16. *For any $\theta \in (0, r)$, $\Lambda_0$, $\Lambda \in E$ and $c$, $C \in \overline{\mathbb{R}}$ satisfying $c < C$ and for every $n_1$, $n_2 \in \mathbb{N}_0$, we have*

$$\begin{aligned}\sum_{a=\lfloor\theta n + c\sqrt{n}\rfloor}^{\lfloor\theta n + C\sqrt{n}\rfloor} P_{h,\Lambda_0}^{\beta, b_h(\theta)}(Y_a = n - n_1 - n_2 \textit{ and } \Lambda(a) = \Lambda) \\ \to \theta\mu_{b_h(\theta)}(\Lambda)\mathscr{N}(\sigma_h^2(\theta))([c, C]) \quad \textit{as } n \to \infty,\end{aligned} \tag{4.17}$$

*where $\sigma_h^2(\theta)$ is defined in* (3.13).

PROOF. As will become clear from the course of the proof, it is sufficient to prove the assertion only for $n_1 = n_2 = 0$ and $C = +\infty$, which we shall assume henceforth.

Define by $T_0 := 0$ and $T_{i+1} := \inf\{n > T_i \colon \Lambda(n) = \Lambda\}$ for $i \in \mathbb{N}_0$ the subsequent hitting times of $\Lambda$. Then the l.h.s. of (4.17) is equal to the $P_{h,\Lambda_0}^{\beta, b_h(\theta)}$-probability of the set

$$\{Y_{\lfloor\theta n + c\sqrt{n}\rfloor} \le n\} \cap \{\text{there is a } k \in \mathbb{N} \text{ such that } Y_{T_k} = n\}. \tag{4.18}$$

First, we explain why the probability of the first of the two events tends to $\mathscr{N}(\sigma_h^2(\theta))([c, \infty))$ and the probability of the second to $\theta\mu_{b_h(\theta)}(\Lambda)$.

1. Substitute $k_n = \lfloor\theta n + c\sqrt{n}\rfloor$, which means that $n = k_n(1/\theta + o(1)) - \sqrt{k_n}(c\theta^{-3/2} + o(1))$ as $n \to \infty$. Now use Lemma 4.1 and recall (3.13) to see that

$$\begin{aligned}\lim_{n\to\infty} P_{h,\Lambda_0}^{\beta, b_h(\theta)}(Y_{k_n} \le n) &= \lim_{n\to\infty} P_{h,\Lambda_0}^{\beta, b_h(\theta)}\left(Y_{k_n} - \frac{k_n}{\theta} \le \sqrt{k_n}c\theta^{-3/2}\right) \\ &= \mathscr{N}(\sigma_h^2(\theta)\theta^{-3})((-\infty, -c\theta^{-3/2}]) = \mathscr{N}(\sigma_h^2(\theta))([c, +\infty)).\end{aligned}$$

2. As we have noted in the beginning of the proof of Lemma 4.1, the variables $Y_{T_{i+1}} - Y_{T_i}$ with $i \in \mathbb{N}$ are i.i.d. under $P^{\beta, b_h(\theta)}_{h, \Lambda_0}$ and have expectation $(\theta \mu_{b_h(\theta)}(\Lambda))^{-1}$. Lemma 4.13 implies that they are positive, $P^{\beta, b_h(\theta)}_{h, \Lambda_0}$-a.s., which, in particular, implies that $\theta \mu_{b_h(\theta)}(\Lambda) \le 1$. Thus, an application of the renewal theorem [see Spitzer (1964), Chapter 2, Proposition 3] yields that the random walk $(Y_{T_k})_{k \in \mathbb{N}}$ hits the point $n$ with a probability that tends to $\theta \mu_{b_h(\theta)}(\Lambda)$ as $n \to \infty$.

So our task is to show the asymptotic independence of the two events in (4.18).

Fix some positive numbers $\delta$ and $\eta$ and abbreviate $c_n = \lfloor \theta n + c\sqrt{n} \rfloor$ and $d_n = \lfloor \theta n + (c - \eta\theta)\sqrt{n} \rfloor$. Observe first that

$$\lim_{n\to\infty} P^{\beta, b_h(\theta)}_{h, \Lambda_0}\left(\frac{\eta}{2}\sqrt{n} \le Y_{c_n} - Y_{d_n} \le 2\eta\sqrt{n}\right) = 1 \tag{4.19}$$

and choose a natural number $R$ so large that

$$\sup_{a \in \mathbb{N}} P^{\beta, b_h(\theta)}_{h, \Lambda_0}(\Lambda(a) \notin E_R) < \delta, \tag{4.20}$$

where $E_R := \{0, \dots, R\}^{r \times r} \times \{1, \dots, r\}$ as in the proof of Lemma 4.1. Then we have

$$\begin{aligned}
&\liminf_{n\to\infty} P^{\beta, b_h(\theta)}_{h, \Lambda_0}\big(Y_{c_n} \le n,\ \exists\, k \in \mathbb{N}\colon\ Y_{T_k} = n\big) \\
&\quad \ge \liminf_{n\to\infty} P^{\beta, b_h(\theta)}_{h, \Lambda_0}\big(Y_{d_n} \le n - 2\eta\sqrt{n},\ \exists\, k \in \mathbb{N}\colon\ Y_{T_k} = n\big) \\
&\quad \ge \liminf_{n\to\infty} \sum_{\tilde\Lambda \in E_R} \sum_{y=1}^{\lfloor n - 2\eta\sqrt{n} \rfloor} P^{\beta, b_h(\theta)}_{h, \Lambda_0}\big(Y_{d_n} = y,\ \Lambda(d_n) = \tilde\Lambda\big) \\
&\qquad\qquad \times P^{\beta, b_h(\theta)}_{h, \tilde\Lambda}\big(\exists\, k \in \mathbb{N}\colon\ Y_{T_k} = n - y\big) \\
&\quad \ge \lim_{n\to\infty} \Big(P^{\beta, b_h(\theta)}_{h, \Lambda_0}\big(Y_{d_n} < n - 2\eta\sqrt{n}\big) - \delta\Big)\, \theta \mu_{b_h(\theta)}(\Lambda) \\
&\quad = \big(\mathscr{N}(\sigma_h^2(\theta))\big([c - 3\eta\theta, \infty)\big) - \delta\big)\, \theta \mu_{b_h(\theta)}(\Lambda) \\
&\quad \to \mathscr{N}(\sigma_h^2(\theta))([c, \infty))\theta \mu_{b_h(\theta)}(\Lambda), \quad \text{as } \eta \downarrow 0,\ \delta \downarrow 0.
\end{aligned}$$

The derivation of the analogous upper bound is similar. □

We are now ready to prove Proposition 2.1.

PROOF OF PROPOSITION 2.1 IN THE CASE $\alpha \in (0, 1)$ AND $h > 0$. Fix $C \in \mathbb{R}$ and abbreviate $p = \mathscr{N}(\sigma^2_{h,\alpha})((-\infty, C])$. According to Lemma 3.17 (applied to $\theta = \Theta_h$), the series

$$\tilde{S} := \sum_{n_1, n_2=0}^{\infty} \exp[(n_1+n_2)b_h^*] \sum_{\Lambda_0, \Lambda \in E} K(n_1, n_2, \Lambda_0, \Lambda) \frac{\tau^r(\Lambda_0)}{\tau^r(\Lambda)} \mu_{b_h^*}(\Lambda) \tag{4.21}$$

converges. This yields, with the help of formulas (2.19) and (3.6) [recall (3.12)] and Lemma 4.16 (applied to $\theta = \Theta_h$, too), that

$$\lim_{n\to\infty} \exp(nb_h^*) E_h\biggl(\exp\biggl(-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\biggr) 1_{0\le S_n \le \Theta_h n + C\sqrt{n}}\biggr) = p\tilde{S}\Theta_h.$$

This proves Proposition 2.1 with $S = \tilde{S}\Theta_h e^{\beta}$ and $b = b_h^* + \beta$ [recall (2.6)]. □

For this proof of the central limit theorem, the finiteness of the infinite-time local times and the existence of the random variables $\Lambda(x)$ defined at the beginning of Section 2 are crucial. It should be mentioned that there is an analog for this chain in the symmetric nearest-neighbor case which is introduced in Knight (1963), Corollary 1.1. This chain enables us to carry out the proof in this case in a completely analogous way as shown above. In the symmetric case, I did not succeed in finding a promising analog for this chain in the general case of bounded steps. Our proof in this case works by inserting an artificial positive drift and using the nice fact that the drift dependence of the probability of a path of fixed length and fixed endpoint can easily be isolated.

PROOF OF PROPOSITION 2.1 IN THE CASE $\alpha \in (0,1)$ AND $h = 0$. We fix some positive $h$ (to be determined later). Our special choice of the walker's step distribution [see (1.3)] implies the nice relation [recall (3.8)]

$$\tilde{J}_h(\theta) = \tilde{J}_0(\theta) + h\theta + \log\frac{Z_0}{Z_h}, \qquad \theta \in (0, r). \tag{4.22}$$

Use (4.22) as well as (1.3) once more to calculate, for every $n$, $a \in \mathbb{N}_0$,

$$\begin{aligned} &\exp[-\tilde{J}_0(\Theta_0)n] E_0\biggl(\exp\biggl(-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\biggr) 1_{S_n=a+1}\biggr) \\ &\quad= \exp[-\tilde{J}_h(\Theta_0)n] E_h\biggl(\exp\biggl(-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\biggr) 1_{S_n=a+1}\biggr) (\exp(-h))^{a+1-n\Theta_0}. \end{aligned} \tag{4.23}$$

We intend to apply Lemma 4.1 to $\theta = \Theta_0$ and point out first that one can deduce from formulas (3.10a) and (3.10b) with the help of (4.22) that

$$b_h(\Theta_0) = \tilde{J}_h(\Theta_0) - h\Theta_0 \quad \text{and} \quad \lambda_h(b_h(\Theta_0)) = e^h. \tag{4.24}$$

Thus, an application of (3.6) to $\theta = \Theta_0$ leads to

$$\begin{aligned} &e^{-\tilde{J}_h(\Theta_0)n} E_{h,\Lambda_0}\bigl(e^{-\beta V_a} 1_{Y_a = n-n_1-n_2} 1_{\Lambda(a)=\Lambda}\bigr)(e^{-h})^{a+1-n\Theta_0} \\ &\quad= e^{b_h(\Theta_0)(n_1+n_2)} \frac{\tau^r_{b_h(\Theta_0)}(\Lambda_0)}{\tau^r_{b_h(\Theta_0)}(\Lambda)} P^{\beta, b_h(\Theta_0)}_{h,\Lambda_0}\bigl(Y_a = n - n_1 - n_2,\ \Lambda(a) = \Lambda\bigr) \end{aligned} \tag{4.25}$$

for every $a, n, n_1, n_2 \in \mathbb{N}_0$ and $\Lambda_0$, $\Lambda \in E$. We now choose $h$ smaller than $\varepsilon/r$, where $\varepsilon$ is the positive number whose existence is asserted in Lemma 3.17. Since $\Theta_0 \in (0, \Theta_h]$ [see the second equation in (4.24) and recall (3.12) and the fact that $\lambda \circ b_h$ is decreasing], the l.h.s. of (3.18) for $\theta = \Theta_0$ is still strictly smaller than $b_h(\Theta_0)$. Now an application of that lemma and Lemma 4.16 to $\theta = \Theta_0$ yields the central limit theorem for the zero-drift case in the same manner as in the positive-drift case. $\square$

**5. The self-avoiding case.** This section proves Theorem 1.10 in the self-avoiding case $\alpha = 1$ with $r \geq 2$, which is the nontrivial case. The proof proceeds in a way which is completely analogous to the preceding three sections and is even technically easier since the self-avoidance constraint allows us to restrict our attention to a certain *finite* subset of the infinite state space $E$ as we will see soon. Because of the large similarities to the proofs for the self-repellent (resp., self-avoiding) case, we will give a survey only, explaining the differences and pointing out the analogies. The reader should keep the fact in mind that

$$\lim_{\beta\to\infty} \exp\biggl(-\beta \sum_x \ell_n(x)^2\biggr) \exp[\beta(n+1)] = 1_{X_n=0}, \qquad n \in \mathbb{N}. \tag{5.1}$$

Like in the self-repellent case, it is sufficient to prove the following result.

PROPOSITION 5.2. *For every $h \geq 0$ there are constants $b = b(h,1) \in \mathbb{R}$ and $S = S(h,1) \in (0,\infty)$ such that, for all $C \in \overline{\mathbb{R}}$,*

$$\lim_{n\to\infty} e^{bn} P_h\left(0 = X_n,\ 0 < S_n < \Theta(h,1)n + C\sqrt{n}\right) = S\mathscr{N}(\sigma^2_{h,1})((-\infty, C]). \tag{5.3}$$

PROOF. Assume the drift parameter $h$ to be positive and recall Section 2.1. The starting observation is that, for a self-avoiding path, the Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ exclusively runs within the finite set

$$\begin{aligned} E^{\mathrm{sa}} := \{\Lambda_0 \in E\colon\ & \text{there are } \Lambda_{1-r}, \dots, \Lambda_{-1}, \Lambda_1, \dots, \Lambda_r \in E \\ & \text{such that } g(\Lambda_{i-1}, \Lambda_i) \in \{0,1\} \text{ and } Q_h(\Lambda_{i-1}, \Lambda_i) > 0 \\ & \text{for } i = 2-r, \dots, r\}. \end{aligned} \tag{5.4}$$

[The superscript "sa" reminds the reader of the self-avoidance in the sequel; the set $E^{\mathrm{sa}}$ is identical to $\tilde{E}$ in König (1993).] More exactly, we have $\Lambda(x) \in E^{\mathrm{sa}}$ on $\{X_{\tau(x+r)} = 0\}$ for $x \in \mathbb{N}_0$ [recall (1.2)]. Due to this fact, the proof which is given in Sections 2 through 4 applies to the self-avoiding situation when the infinite state space $E$ is replaced by the finite subset $E^{\mathrm{sa}}$ and the random variable $e^{-\beta\sum_x \ell_n(x)^2}$ by 1, roughly speaking. Let us explain some details.

Analogously to Lemma 2.18, one derives the representation

$$\begin{aligned} &P_h(X_n = 0,\ S_n = a+1) \\ &\quad = \sum_{n_1,n_2=0}^{\infty} \sum_{\Lambda_0,\Lambda\in E^{\mathrm{sa}}} K_-^{\mathrm{sa}}(n_1, \Lambda_0)\left(K_+^{\uparrow,\mathrm{sa}} + K_+^{\downarrow,\mathrm{sa}}\right)(n_2, \Lambda) \\ &\qquad\qquad \times P_{h,\Lambda_0}\bigl(Y_a = n - n_1 - n_2,\ \Lambda(a) = \Lambda, \\ &\qquad\qquad\qquad g(\Lambda(x-1), \Lambda(x)) \leq 1 \text{ for } x = 1, \dots, a\bigr), \end{aligned} \tag{5.5}$$

where (as in the preceding sections) $P_{h,\Lambda_0}$ is the distribution of a Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ on $E$ with transition kernel $Q_h$, starting in $\Lambda_0$, $Y_a$ is an abbreviation for $\sum_{x=1}^{a} g(\Lambda(x-1),\Lambda(x))$ and the constants are given by

$$
\begin{aligned}
K_-^{\mathrm{sa}}(n_1,\Lambda_0) &:= P_h\biggl(\sum_{x\le 0}\ell(x)=1+n_1, \ell(x)\le 1 \text{ for } x\in -\mathbb{N}_0, \Lambda(0)=\Lambda_0\biggr),\\
K_+^{\uparrow,\mathrm{sa}}(n_2,\Lambda) &:= \frac{1}{\pi_1}P_h\biggl(\sum_{x>a}\ell_{\tau(a)}(x)=n_2,\ \ell_{\tau(a)}(x)\le 1 \text{ for } x>a\ \bigg|\ \Lambda(a)=\Lambda\biggr)
\end{aligned}
\tag{5.6}
$$

if $q=1\le n_2$ and $q=0$ otherwise [we write $\Lambda=((\eta_{j,k})_{j,k},q)$], and

$$
\begin{aligned}
K_+^{\downarrow,\mathrm{sa}}(n_2,\Lambda) := \frac{1}{\pi_2}P_h\biggl(&\sum_{x>a}\ell_{\tilde\tau(a+1)}(x)=n_2,\ \ell_{\tilde\tau(a+1)}(x)\le 1 \text{ for } x>a,\\
&\tau(a)<\tau(a+1)-1\ \bigg|\ \Lambda(a)=\Lambda\biggr)
\end{aligned}
\tag{5.7}
$$

for $n_2\ge 1$ and $n=0$ otherwise. Like $K_+^{\uparrow}$ and $K_+^{\downarrow}$, the latter two constants do not depend on $a$.

The main part of König (1993) was devoted to the analysis of the function

$$
\tilde J_{h,\mathrm{sa}}(\theta) := \lim_{n\to\infty}\frac{1}{n}\log P_h\left(0=X_n,\ 0<S_1,\dots,S_{n-1}<S_n=\lfloor\theta n\rfloor\right), \qquad \theta\in(1,r),
\tag{5.8}
$$

and of its maximum point $\Theta_h^{\mathrm{sa}} := \Theta(h,1)$. We obtained a characterization which is very similar to that of $\tilde J_h$ which is given in Section 3.2. The finite matrix $A_{h,b}^{\mathrm{sa}}=(A_{h,b}^{\mathrm{sa}}(\Lambda,\Lambda'))_{\Lambda,\Lambda'\in E^{\mathrm{sa}}}$, for $b\in\mathbb{R}$, given by

$$
A_{h,b}^{\mathrm{sa}}(\Lambda,\Lambda') = \begin{cases} Q_h(\Lambda,\Lambda')e^{bg(\Lambda,\Lambda')}, & \text{if } g(\Lambda,\Lambda')\in\{0,1\},\\ 0, & \text{otherwise,}\end{cases}
\tag{5.9}
$$

is nonnegative, aperiodic and irreducible by Lemma 2.6 in König (1993), and it possesses the Frobenius eigenvalue

$$
\lambda_{h,\mathrm{sa}}(b) := \lim_{n\to\infty}\bigl((A_{h,b}^{\mathrm{sa}})^n(\Lambda,\Lambda')\bigr)^{1/n}.
\tag{5.10}
$$

This formula does not depend on $\Lambda,\Lambda'\in E^{\mathrm{sa}}$. The function $\lambda_{h,\mathrm{sa}}$ is real-analytic and strictly increasing and strictly log-convex. The inverse function

$$
b_{h,\mathrm{sa}}(\theta) := \left(\frac{\lambda_{h,\mathrm{sa}}}{\lambda'_{h,\mathrm{sa}}}\right)^{-1}(\theta), \qquad \theta\in(1,r),
\tag{5.11}
$$

is real-analytic and strictly decreasing.

Since $A^{\mathrm{sa}}_{h,b}$ has finitely many rows and columns, it clearly possesses right and left eigenvectors $\tau_b^{\mathrm{sa},r}$, $\tau_b^{\mathrm{sa},l} \in (0,\infty)^{E^{\mathrm{sa}}}$ corresponding to the eigenvalue $\lambda_{h,\mathrm{sa}}(b)$ and satisfying $\langle \tau_b^{\mathrm{sa},l}, \tau_b^{\mathrm{sa},r}\rangle = 1$. The stochastic matrix

$$Q^{\mathrm{sa}}_{h,b} := \left( \frac{A^{\mathrm{sa}}_{h,b}(\Lambda,\Lambda')}{\lambda_{h,\mathrm{sa}}(b)} \frac{\tau_b^{\mathrm{sa},r}(\Lambda')}{\tau_b^{\mathrm{sa},r}(\Lambda)} \right)_{\Lambda,\Lambda' \in E^{\mathrm{sa}}}, \qquad b \in \mathbb{R}, \tag{5.12}$$

possesses the vector $\mu_b^{\mathrm{sa}} := (\tau_b^{\mathrm{sa},l}(\Lambda)\tau_b^{\mathrm{sa},r}(\Lambda))_{\Lambda \in E^{\mathrm{sa}}}$ as its invariant distribution. We denote the distribution of the Markov chain $(\Lambda(x))_{x\in\mathbb{N}_0}$ on $E^{\mathrm{sa}}$ with transition matrix $Q^{\mathrm{sa}}_{h,b}$ and invariant starting measure $\mu_b^{\mathrm{sa}}$ by $P_h^{\infty,b}$ and write $P^{\infty,b}_{h,\Lambda_0} = P_h^{\infty,b}(\cdot \mid \Lambda(0) = \Lambda_0)$. We now switch in the r.h.s. of (5.5) to the distribution $P_h^{\infty,b}$ and note that a direct calculation which is very similar to (3.7) yields that $\exp[nb_{h,\mathrm{sa}}(\theta)]$ times the last line of (5.5) is equal to

$$\begin{aligned} &P^{\infty, b_{h,\mathrm{sa}}(\theta)}_{h,\Lambda_0}(Y_a = n - n_1 - n_2, \Lambda(a) = \Lambda) \\ &\quad \times \exp[b_{h,\mathrm{sa}}(\theta)(n_1+n_2)] \frac{\tau^{\mathrm{sa},r}_{b_{h,\mathrm{sa}}(\theta)}(\Lambda_0)}{\tau^{\mathrm{sa},r}_{b_{h,\mathrm{sa}}(\theta)}(\Lambda)} \lambda_{h,\mathrm{sa}}(b_{h,\mathrm{sa}}(\theta))^a \end{aligned} \tag{5.13}$$

for every $a, n, n_1, n_2 \in \mathbb{N}_0$ and $\Lambda_0, \Lambda \in E^{\mathrm{sa}}$, $\theta \in (1,r)$ and $h > 0$.

We now terminate our description of the proof's course by noting that Sections 3 and 4 apply almost literally after replacing the objects by the analogous ones for the self-avoiding case, that is, $P_h^{\beta,b}$ by $P_h^{\infty,b}$, $\Theta_h$ by $\Theta_h^{\mathrm{sa}}$, $K_-$, $K_+^{\uparrow}$ and $K_+^{\downarrow}$ by $K_{-,\mathrm{sa}}$, $K^{\uparrow}_{+,\mathrm{sa}}$ and $K^{\downarrow}_{+,\mathrm{sa}}$, $b_h$ by $b_{h,\mathrm{sa}}$, $Q_{h,b}$ by $Q^{\mathrm{sa}}_{h,b}$ and so on. Note also that the condition $\theta \in (0,r)$ must be replaced by $\theta \in (1,r)$ on every occurrence. For the proof, there are some facts and results from König (1993) needed which are completely analogous to those cited from König (1994) in the preceding sections and can be found on analogous places in König (1993). In fact, many details of the proof are considerably simpler in the self-avoiding case than in the self-repellent case, which is simply due to the finiteness of the state space $E^{\mathrm{sa}}$ and to the absence of a random variable in the exponential. □

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