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PACKING AND COVERING INDICES FOR A GENERAL LÉVY PROCESS

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There has been substantial interest in the indices $0 \le \beta'' \le \beta' \le \beta \le 2$, defined by Blumenthal and Getoor, determined by a general Lévy process in $\mathbf{R}^d$. Pruitt defined an index γ which determines the covering dimension and Taylor showed that an index γ', first considered by Hendricks, determines the packing dimension for the trajectory. In the present paper we prove that

$$\frac{\beta}{2} \le \gamma' \le \min(\beta, d),$$

and give examples to show that the whole range is attainable. However, we cannot completely determine the set of values of (γ, γ', β) which can be attained as indices of some Lévy process.

1. Introduction. Let X_t be a Lévy process taking values in $\mathbf{R}^d$. The question of interest here is the nature of the random trajectory of the process. Blumenthal and Getoor [1] introduced an upper index β and two lower indices β'' and β' and obtained certain properties of the sample paths of X_t in terms of these indices. They also showed that

$$0 \le \beta'' \le \beta' \le \beta \le 2.$$

Pruitt [8] showed that the Hausdorff dimension of the trajectory is γ a.s. where

$$\gamma = \sup\left\{\alpha \ge 0 \colon \limsup_{a \to 0} a^{-\alpha} \int_0^1 P\big[|X_t| \le a\big]\, dt < \infty\right\}, \tag{1.1}$$

while Taylor [15] showed that the packing dimension (defined in [12]) of the trajectory is γ' a.s. where

$$\gamma' = \sup\left\{\alpha \ge 0 \colon \liminf_{a \to 0} a^{-\alpha} \int_0^1 P\big[|X_t| \le a\big]\, dt < \infty\right\}. \tag{1.2}$$

The index γ' was first considered by Hendricks [4].

More precise information than just these fractal indices is already known for special processes. We will not refer here to the well-known results about Hausdorff measure functions, but we will refer to the newer results for

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ϕ-packing measures given in: [15] for Brownian motion for $d \geq 3$; [6] for planar Brownian motion; [14] for strictly stable processes; [12] for asymmetric Cauchy processes and the graph of any stable process; and [2] for arbitrary subordinators.

Our object in the present paper is to obtain more information about the possible values of the indices γ and γ'. We will show that given any pair δ_1 and δ_2 satisfying $0 \leq \delta_1 \leq \delta_2 \leq 2$, we can define a Lévy process whose trajectory has Hausdorff dimension δ_1 and packing dimension δ_2. In Section 4 we will show that γ' is related to the index β by

$$\frac{\beta}{2} \leq \gamma' \leq \min(\beta, d), \tag{1.3}$$

and give examples in Section 5 which show that the entire range in (1.3) is attainable. We have not solved the more difficult problem concerning the exact set of possible values of (γ, γ', β) in $\mathbf{R}^3$. Looking at the three indices simultaneously may introduce new restrictions.

Let $S(a)$ be the first passage time out of the ball of radius a for X_t and $T(a,1)$ the sojourn time in the ball of radius a up to time 1. That is,

$$S(a) = \inf\{t > 0: |X_t| > a\}, \qquad T(a,1) = \int_0^1 1\{|X_t| \leq a\}\, dt. \tag{1.4}$$

We will show in Section 3 that if T_1 and T_2 are independent copies of T, then

$$\liminf_{a \to 0} a^{-\alpha}\left[T_1(a,1) + T_2(a,1)\right] = 0 \quad \text{a.s. if } \alpha < \gamma',$$
$$\lim_{a \to 0} a^{-\alpha}\left[T_1(a,1) + T_2(a,1)\right] = \infty \quad \text{a.s. if } \alpha > \gamma'.$$

One would expect the simpler version of these statements to be true where we simply look at the lim inf of $a^{-\alpha}T(a,1)$. This will lead us below to the statement of a conjecture and an open problem concerning the connection between the lower growth conditions satisfied by a process and the sum of two independent copies of that process.

We will start with some preliminaries. We follow the customary practice of letting c, k denote finite positive constants whose value is unimportant and may change from line to line.

2. Preliminaries. The definition and properties of Hausdorff measure are well known; see, for example, Rogers [13]. The packing measure was defined more recently in [15], so we recall two versions of it. The measure functions ϕ under consideration map $[0,1] \to [0,1]$, are increasing, continuous with $\phi(0) = 0$ and satisfy a regularity condition: there is a constant $c > 0$ such that

$$\phi(2x) \leq c\phi(x), \qquad 0 \leq x \leq \tfrac{1}{2}.$$

For any collection $\mathscr{C}$ of bounded subsets of $\mathbf{R}^d$, let

$$\phi(\mathscr{C}) = \sum_{E \in \mathscr{C}} \phi(\operatorname{Diam} E),$$
$$\|\mathscr{C}\| = \sup\{\operatorname{diam} E: E \in \mathscr{C}\}.$$

For a fixed subset E of $\mathbf{R}^d$, $\mathscr{C}_E$ denotes the family of balls $B_r(x)$ of radius $r > 0$ and center $x \in E$, and Γ_E^{**} the family of semidyadic cubes whose central cubes intersect E. To make this precise, $C \in \Gamma_E^{**}$ has a projection on the ith axis

$$\mathrm{proj}_i C\left[k_i 2^{-n-1}, (k_i + 2)2^{-n-1}\right],$$

with $k_i \in \mathbf{Z}$, and there is an $x \in E$ such that the complement of C is at distance 2^{-n-2} from the (unique) dyadic cube of side 2^{-n-2} which contains x. We define

$$\phi - P(E) = \lim_{\varepsilon \to 0} \sup\{\phi(\mathscr{R}): \|\mathscr{R}\| < \varepsilon, \text{disjoint } \mathscr{R} \subset \mathscr{C}_E\},$$

$$\phi - P^{**}(E) = \lim_{\varepsilon \to 0} \sup\{\phi(\mathscr{R}): \|\mathscr{R}\| < \varepsilon, \text{disjoint } \mathscr{R} \subset \Gamma_E^{**}\}.$$

These two functions are defined on all subsets of $\mathbf{R}^d$. Their properties are explored in [12], but we note here that they are premeasures and there are positive finite constants c_1, c_2 such that, for all $E \subset \mathbf{R}^d$,

$$c_1 \phi - P^{**}(E) \le \phi - P(E) \le c_2 \phi - P^{**}(E).$$

The final step is to generate outer measures

$$\phi - p(E) = \inf\{\Sigma_i \phi - P(E_i): E \subset \bigcup E_i\},$$

$$\phi - p^{**}(E) = \inf\{\Sigma_i \phi - P^{**}(E_i): E \subset \bigcup E_i\}.$$

We call $\phi - p$ the ϕ-packing measure and use $\phi - p^{**}$ as a computational aid, since both measures have the same class of sets having finite positive measure. Both measures are Borel regular and have good topological properties. The following density theorem, proved in [15], is a key tool.

THEOREM 1. *Suppose μ is a finite Borel measure on $\mathbf{R}^d$ and ϕ is a measure function as described above. Then there is a constant $\lambda > 0$ such that, for all $E \subset \mathbf{R}^d$,*

$$\lambda \mu(E) \inf_{x \in E} \Lambda(x) \le \phi - p(E) \le \|\mu\| \sup_{x \in E} \Lambda(x),$$

where

$$\Lambda(x) = \limsup_{r \to 0} \frac{\phi(2r)}{\mu(B_r(x))}.$$

In the present paper, we are interested only in the fractal indices determined by the Hausdorff and packing measure for the functions $\phi(s) = s^\alpha$, $\alpha > 0$. For any set $E \subset \mathbf{R}^d$, define

$$\dim E = \inf\{\alpha > 0: s^\alpha - m(E) = 0\},$$

$$\operatorname{Dim} E = \inf\{\alpha > 0: s^\alpha - p(E) = 0\},$$

called the Hausdorff and packing dimensions of E. Since

$$\phi - m(E) \le \phi - p(E) \quad \text{for all } E,$$

we clearly have

$$0 \leq \dim E \leq \operatorname{Dim} E \leq d,$$

and, given α, β satisfying $0 \leq \alpha \leq \beta \leq d$, it is not difficult to construct a deterministic set $E \subset \mathbf{R}^d$ for which $\dim E = \alpha$, $\operatorname{Dim} E = \beta$.

A Lévy process is one with stationary independent increments, taking values in $\mathbf{R}^d$, and characteristic function

$$E \exp\{i(u, X_t)\} = \exp\{-t\psi(u)\},$$

where

$$\psi(u) = i(b, u) + \int \left(1 - e^{i(u,x)} + \frac{i(u,x)}{1 + |x|^2}\right) \nu(dx),$$

with $b \in \mathbf{R}^d$ and ν a Borel measure on $\mathbf{R}^d$ satisfying

$$\int \frac{|x|^2}{1 + |x|^2} \nu(dx) < \infty.$$

It is also customary to include a Gaussian part, but since its behavior is well known we will omit this component in order to simplify the formulas. We will assume that $X_0 = 0$, and that we are dealing with a version which has almost all sample functions right continuous and having left limits.

We define, for $x > 0$,

$$G(x) = \nu\{y : |y| > x\}, \qquad K(x) = x^{-2} \int_{|y| \leq x} |y|^2 \nu(dy), \tag{2.1}$$

$$M(x) = x^{-1} \left| b + \int_{|y| \leq x} \frac{y|y|^2}{1 + |y|^2} \nu(dy) - \int_{|y| > x} \frac{y}{1 + |y|^2} \nu(dy) \right|, \tag{2.2}$$

$$Q(x) = G(x) + K(x), \qquad h(x) = Q(x) + M(x). \tag{2.3}$$

The function h is fairly well behaved; in particular, for $C > 1$,

$$\frac{1}{2C^2} \leq \frac{h(Ca)}{h(a)} \leq 2. \tag{2.4}$$

Furthermore, if we let $M_t = \sup_{0 \leq s \leq t} |X_t|$, then there exists $C > 0$ such that

$$P\{M_t \geq a\} \leq Cth(a), \qquad P\{M_t \leq a\} \leq \frac{C}{(th(a))^2}. \tag{2.5}$$

(See (3.2) and the remark on page 951 of [10].) These tail estimates for M_t lead immediately to similar estimates for the first passage time $S(a)$, and one easily obtains (see Theorem 1 in [10])

$$\begin{aligned} ES(a) &\approx \{h(a)\}^{-1} \approx E\{S(a) \wedge 1\} \\ &\approx E\big\{S(a) \wedge \{h(a)\}^{-1}\big\}, \qquad 0 < a \leq 1. \end{aligned} \tag{2.6}$$

(The symbol $\approx$ here means that the ratio of the two sides is bounded above and below by finite, positive constants.)

We now recall the definitions given by Blumenthal and Getoor [1] of the indices that will be relevant here:

$$\beta = \inf\{ \alpha > 0 : r^{\alpha} G(r) \to 0 \text{ as } r \to 0\}, \tag{2.7}$$

$$\beta' = \sup\left\{ \alpha > 0 : \int |x|^{\alpha - d} \frac{1 - \exp[-\operatorname{Re} \psi(x)]}{\operatorname{Re} \psi(x)} \, dx < \infty \right\}. \tag{2.8}$$

When $d = 1$ and the process is increasing, that is, a subordinator, it is customary to use the Laplace transform instead of the characteristic function:

$$E \exp(-uX_t) = \exp(-tg(u)),$$

where

$$g(u) = \int_0^{\infty} (1 - e^{-ux}) \nu(dx). \tag{2.9}$$

For a subordinator, it is clear that

$$\beta = \inf\{ \alpha > 0 : u^{-\alpha} g(u) \to \infty \text{ as } u \to \infty\}. \tag{2.10}$$

For subordinators, Blumenthal and Getoor considered, in addition to β', the lower index

$$\sigma = \sup\{ \alpha > 0 : u^{-\alpha} g(u) \to \infty \text{ as } u \to \infty\}. \tag{2.11}$$

We recall that for any subordinator

$$0 \le \beta' \le \sigma \le \beta \le 1.$$

The indices γ, γ' defined in (1.1) and (1.2) which give the Hausdorff and packing dimensions, respectively, of $X[0, 1]$ satisfy the inequalities

$$0 \le \beta' \wedge d \le \gamma \le \gamma' \le \beta \wedge d.$$

In case X is a subordinator $\gamma = \sigma$, while if X is a symmetric process $\gamma = \beta' \wedge d$.

Hawkes and Pruitt ([3], Theorem 3.1) obtain a uniform upper bound for the Hausdorff dimension of $X(E)$ in the following form.

THEOREM 2. *Let X_t be a Lévy process with upper index β. Then*

$$P\{\dim X(E) \le \beta \dim E \textit{ for all } E \subset \mathbf{R}^+\} = 1.$$

For the purpose of constructing examples, we note the following theorem which is a corollary of more precise results in Perkins and Taylor [7].

THEOREM 3. *If Y_t is any strictly stable process of index α in $\mathbf{R}^d$, $\alpha \le d$, then, with probability* 1,

$$\dim Y(E) = \alpha \dim E$$

and

$$\operatorname{Dim} Y(E) = \alpha \operatorname{Dim} E,$$

uniformly for all Borel E.

The first result of this type was obtained for planar Brownian motion by Kaufman [5].

3. Implications of the packing dimension result. In order to explore its meaning and to state two natural problems, we will repeat the proof that $\operatorname{Dim} X[0,1] = \gamma'$ a.s. It is relatively easy given the preliminary results that have been mentioned. Let $T(a, 1)$ denote the sojourn time in the ball $B_a(0)$ up to time $t = 1$ [see (1.4)]. Then

$$ET(a,1) = \int_0^1 P\{|X_t| \le a\}\, dt.$$

As usual, we consider the occupation time measure for the trajectory given by

$$\mu_\omega(A) = |\{t \in (0,1): X_t \in A\}|,$$

where $|\cdot|$ denotes the Lebesgue measure. Note that, for $0 < t < 1$,

$$\begin{aligned}\mu_\omega(B_a(X_t)) &= \int_0^1 1_{B_a(X_t)}(X_s)\, ds \le \int_{t-1}^{t+1} 1_{B_a(X_t)}(X_s)\, ds \\ &= \int_{-1}^0 1_{B_a(X_t)}(X_{t+s})\, ds + \int_0^1 1_{B_a(X_t)}(X_{t+s})\, ds \\ &= T_1(a) + T_2(a),\end{aligned} \tag{3.1}$$

say. By the stationary and independent increment properties of Lévy processes, $T_1(a)$ and $T_2(a)$ are independent and both have the same distribution as $T(a, 1)$. Thus

$$E\mu_\omega(B_a(X_t)) \le 2ET(a,1).$$

By the definition of γ' [see (1.2)], if $0 < \alpha < \gamma'$, $\liminf a^{-\alpha} ET(a, 1) = 0$, so that by Fatou's lemma

$$\liminf a^{-\alpha}\mu_\omega(B_a(X_t)) = 0 \quad \text{a.s.}$$

To this point, t has been fixed, but now Fubini gives

$$\left|\{t \in (0,1): \liminf a^{-\alpha}\mu_\omega(B_a(X_t)) = 0 \text{ a.s.}\}\right| = 1.$$

Next, an application of Theorem 1 gives $s^\alpha - p(X[0,1]) = +\infty$ a.s. Allowing α to increase to γ' through a countable set then shows that

$$\operatorname{Dim} X[0,1] \ge \gamma' \quad \text{a.s.} \tag{3.2}$$

In the other direction, we start with $\gamma' < \delta < \alpha$. Then

$$a^{-\delta} ET(a,1) \to +\infty$$

as $a \to 0$. Semidyadic cubes of side 2^{-k} cover $\mathbf{R}^d$ 2^d times. Using Lemma 5.1 of [11], if N_k is the number of such cubes hit by $X[0, 1]$, then

$$EN_k = o(2^{k\delta}) \quad \text{as } k \to \infty. \tag{3.3}$$

Even if we do not require the cubes to have a point of $X[0,1]$ in their central area and if we also forget the requirement that they be disjoint, packing by cubes of side at most 2^{-m} leads to the estimate

$$Es^{\alpha} - P^{**}(X[0,1]) \le C \sum_{k=m}^{\infty} 2^{-k\alpha} EN_k \le \sum_{k=m}^{\infty} 2^{-k(\alpha-\delta)} \to 0 \quad \text{as } m \to \infty;$$

we have used (3.3) at the last step. Hence $s^{\alpha} - P^{**}(X[0,1]) = 0$ a.s., which in turn implies that $s^{\alpha} - P(X[0,1]) = 0$ a.s. by the comparison mentioned above. Allowing α to decrease to γ' through a countable set then gives

$$\operatorname{Dim} X[0,1] \le \gamma' \quad \text{a.s.}$$

With (3.2), this completes the proof of the packing dimension result.

The definition of γ' involves the lower growth rate of $ET(a,1)$ as $a \to 1$. We can deduce, from what we have done, an almost sure local growth rate for $T(a,1)$, as follows.

THEOREM 4. *If X_t is a Lévy process, $T_1(a,1)$ and $T_2(a,1)$ are independent copies of $T(a,1)$, the corresponding sojourn time process defined by* (1.4), *and γ' is the index defined in* (1.2), *then, with probability* 1:

(i) *for $\alpha < \gamma'$, $\liminf a^{-\alpha}(T_1(a,1) + T_2(a,1)) = 0$;*
(ii) *for $\alpha > \gamma'$, $a^{-\alpha}(T_1(a,1) + T_2(a,1)) \to \infty$.*

PROOF. (i) follows from Fatou's lemma as in the proof of the first part of the packing dimension result above. If (ii) fails and $\gamma' < \delta < \alpha$, then with positive probability $\liminf a^{-\delta}(T_1(a,1) + T_2(a,1)) = 0$. By using the estimate in (3.1) and Fubini, we see that

$$\left|\left\{t \in (0,1) \colon P\left\{\liminf a^{-\delta}\mu_{\omega}(B_a(X_t)) = 0\right\} > 0\right\}\right| > 0.$$

By Theorem 1, it would then follow that $s^{\delta} - p(X[0,1]) = +\infty$ with positive probability, contradicting the above packing dimension result. □

COROLLARY. *For any Lévy process,*

$$\gamma' = \inf\left\{\alpha \ge 0 \colon a^{-\alpha}\left(T_1(a,1) + T_2(a,1)\right) \to \infty \textit{ as } a \to 0 \textit{ a.s.}\right\}.$$

PROBLEM A. It is true that

$$\gamma' = \inf\{\alpha \ge 0 \colon a^{-\alpha} T(a,1) \to \infty \text{ as } a \to 0 \text{ a.s.}\}?$$

We believe this has to be true and even state a much stronger conjecture.

CONJECTURE. *If ϕ is a monotone function such that*

$$\liminf_{a \to 0} \frac{T(a,1)}{\phi(a)} \le C \quad a.s.$$

and h is a monotone function such that

$$\int_{0^+} s^{-1} h(s)\, ds < \infty,$$

then

$$\liminf_{a \to 0} \frac{h(a)(T_1(a,1) + T_2(a,1))}{\phi(a)} = 0 \quad a.s.$$

Existing results about particular processes show that the lower growth rate of $T_1(a,1) + T_2(a,1)$ may differ from that of $T(a,1)$ by a factor of $|\log a|(\log|\log a|)^{1/2}$, but no examples are known where the ratio is as great as $|\log a|^{1+\varepsilon}$.

PROBLEM B. For a monotone stochastic process $Z(a)$, what conditions are sufficient to ensure that $\delta_1 = \delta_2$, where

$$\delta_1 = \inf\{\alpha \geq 0: a^{-\alpha} Z(a) \to \infty \text{ as } a \to 0\ a.s.\},$$
$$\delta_2 = \inf\{\alpha \geq 0: a^{-\alpha}(Z_1(a) + Z_2(a)) \to \infty \text{ as } a \to 0\ a.s.\}$$

and Z_1, Z_2 are independent copies of Z?

We note that a solution to Problem B could provide an affirmative answer to Problem A.

4. Inequalities relating γ' and β. We start with some lemmas that give information about the growth of the functions G, K and M defined in (2.1) and (2.2). Similar results were obtained in [9] for Q, but these were easier since Q is continuous. The function M is more complicated, and G and K depend only on the Lévy measure of the complements of balls centered at the origin. Recall the definitions:

$$G(x) = \nu\{y: |y| > x\}, \qquad K(x) = x^{-2} \int_{|y| \leq x} |y|^2 \nu(dy),$$
$$M(x) = x^{-1} \left| b + \int_{|y| \leq x} \frac{y|y|^2}{1 + |y|^2} \nu(dy) - \int_{|y| > x} \frac{y}{1 + |y|^2} \nu(dy) \right|,$$
$$Q(x) = G(x) + K(x), \qquad h(x) = Q(x) + M(x).$$

LEMMA 4.1. *For* $0 < x < y$,

$$|yM(y) - xM(x)| \leq \min\{y(G(x) - G(y)),\, x^{-1}(y^2 K(y) - x^2 K(x))\}.$$

PROOF. Applying the inequality $||u| - |v|| \le |u - v|$ yields

$$|yM(y) - xM(x)| \le \left| \int_E \frac{z|z|^2}{1+|z|^2} \nu(dz) + \int_E \frac{z}{1+|z|^2} \nu(dz) \right|$$
$$= \left| \int_E z \nu(dz) \right| \le \int_E |z| \nu(dz),$$

where $E = \{z\colon x < |z| \le y\}$. The first inequality is now clear, and the second follows on replacing the integrand $|z|$ by $x^{-1}|z|^2$. □

LEMMA 4.2. *If either*

$$\lim_{x\to 0} \frac{K(x)}{M(x)} = 0 \quad \textit{or} \quad \lim_{x\to 0} \frac{G(x)}{M(x)} = 0,$$

then $xM(x)$ is slowly varying at 0.

PROOF. Letting $y = Cx$ in Lemma 4.1 yields

$$\left| \frac{CxM(Cx)}{xM(x)} - 1 \right| \le C \frac{G(x)}{M(x)}, \qquad \left| 1 - \frac{xM(x)}{CxM(Cx)} \right| \le C \frac{K(Cx)}{M(Cx)}.$$

Letting $x \to 0$ completes the proof. □

LEMMA 4.3. *If $0 < \alpha < 1$ and $G(z) < \alpha^{-1}(1-\alpha)M(z)$ for $x < z < y$, then $z^{\alpha}(G(z) + M(z))$ is decreasing on $[x, y]$. If $1 < \alpha < 2$ and $K(z) < (2-\alpha)^{-1}(\alpha - 1)M(z)$ for $x < z < y$, then $z^{\alpha}(K(z) + M(z))$ is increasing on $[x, y]$.*

REMARK. The stronger result that if $G(z) \le \alpha^{-1}(1-\alpha)M(z)$ on (x, y), then $z^{\alpha}(G(z) + M(z))$ is nonincreasing on $[x, y]$ follows by a perturbation argument. This is comparable to the results for Q in [9].

PROOF. First note that G, K, M are all right continuous and have left limits. Letting $x \nearrow y$ in Lemma 4.1, we have

$$|M(y) - M(y^-)| \le \min(G(y^-) - G(y), K(y) - K(y^-));$$

the two terms on the right are equal. Thus, at any discontinuity $G + M$ can only jump down while $K + M$ can only jump up. Now, let

$$u = \sup\{v \ge x\colon z^{\alpha}(G(z) + M(z)) \text{ is decreasing on } [x, v]\}.$$

We will show $u \ge y$. First $z^{\alpha}(G(z) + M(z))$ decreases on $[x, u]$ if $u > x$ since this function can only jump down. Next, if $u < j$, since G and M are right continuous, you may choose $v > u$ such that

$$\xi = \sup_{u \le w \le v} \frac{G(w)}{M(w)} < \frac{1-\alpha}{\alpha}.$$

Then, for $u \le w < z \le v$, we have, by Lemma 4.1,

$$zM(z) + zG(z) \le wM(w) + zG(w),$$

so that, for $0 < \alpha < 1$,

$$z^{\alpha}(M(z) + G(z)) \le wz^{\alpha-1}M(w) + z^{\alpha}G(w). \tag{4.1}$$

The derivative with respect to z of the function on the right is

$$z^{\alpha-1}\big[(\alpha-1)wz^{-1}M(w) + \alpha G(w)\big] < 0,$$

provided that

$$wz^{-1} > \frac{\alpha G(w)}{(1-\alpha)M(w)}. \tag{4.2}$$

Thus, if we also require $z < (1-\alpha)(\alpha\xi)^{-1}u$, we have

$$wz^{-1} \ge ux^{-1} > (1-\alpha)^{-1}\alpha\xi,$$

so that (4.2) holds. Then, by (4.1),

$$z^{\alpha}(M(z) + G(z)) < w^{\alpha}(M(w) + G(w)),$$

so that $z^{\alpha}(M(z) + G(z))$ is decreasing on $[x, v \wedge (1-\alpha)(\alpha\xi)^{-1}u]$ which strictly contains $[x, u]$. This is impossible so we must have $u \ge y$. The other statement in the lemma is proved in the same way. $\square$

Now we are ready to prove the inequalities for γ'.

THEOREM 5. *For any Lévy process in* $\mathbf{R}^d$, *we have*

$$\gamma \wedge \frac{\beta}{2} \le \gamma' \le \beta \wedge d,$$

where γ, γ' *are defined in* (1.1) *and* (1.2) *and* β *is the upper index of Blumenthal and Getoor defined in* (2.7).

PROOF. The inequality $\gamma \le \gamma'$ is immediate from the definitions. Since $\beta \le 2$, we only need to prove $\gamma' \le d$ when $d = 1$. If not, choose $a_k \to 0$ so that $a_k^{-1}ET(a_k, 1) \to 0$, and partition $[-1, 1]$ into intervals of length a_k. Let T_j be the time spent in the jth interval before time 1. By starting over when we hit this interval we see that $ET_j \le ET(a_k, 1)$. Thus

$$ET(1,1) \le \left(2a_k^{-1} + 1\right)ET(a_k, 1) \to 0,$$

so $ET(1,1) = 0$, a contradiction. It remains to prove the inequalities involving β. We will use the definition

$$\beta = \inf\Big\{\alpha \ge 0\colon \lim_{a\to 0} a^{\alpha}h(a) = 0\Big\}.$$

This is equivalent to (2.7)—see page 954 of [10]. (If there is a Brownian component, $\beta = 2$.) By using the inequalities (3.2) of [10] (see Theorem 1 of [10] for a similar argument), it is easy to see that

$$ES(a) \approx (h(a))^{-1} \approx E(S(a) \wedge 1).$$

Then the inequality $E(S(a) \wedge 1) \le ET(a,1)$ leads to $\gamma' \le \beta$. [If there is a Brownian component, then $E(S(a) \wedge 1) \approx a^2$, so $\gamma' \le 2 = \beta$.] The final inequality $\beta/2 \le \gamma'$ requires more work, and we must consider three cases. If $\beta = 0$, there is nothing to prove.

$\beta < 1$. First note that there exists $\rho < 1$ such that

$$a^{\rho} M(a) \le a^{\rho} h(a) \to 0.$$

Thus $aM(a)$ is not slowly varying, and so

$$\limsup_{x \to 0} \frac{G(x)}{M(x)} > 0$$

by Lemma 4.2. Choose $\alpha \in (0, \beta)$. Then there exists $a_k \to 0$ such that

$$a_k^{\alpha} h(a_k) \ge 1.$$

Now choose $\eta < \alpha^{-1}(1-\alpha)$ and $\eta < \limsup G(x)/M(x)$ and define

$$b_k = \sup\{x < a_k : G(x) \ge \eta M(x)\}.$$

Then $b_k > 0$ and $b_k \to 0$. If $M(a_k) > K(a_k)$, then since $G(x) < \eta M(x)$ for $b_k < x < a_k$ we have, by Lemma 4.3,

$$b_k^{\alpha}(G(b_k) + M(b_k)) \ge a_k^{\alpha}(G(a_k) + M(a_k)) > \frac{a_k^{\alpha} h(a_k)}{2} \ge \frac{1}{2}.$$

Then, since $G + M$ can only jump down, we can find $c_k \le b_k$ such that

$$c_k^{\alpha} h(c_k) \ge c_k^{\alpha}(G(c_k) + M(c_k)) \ge \tfrac{1}{4},$$

and $G(c_k) \ge \eta M(c_k)$. Letting $d_k = c_k$ in this case or $d_k = a_k$ in the case $M(a_k) \le K(a_k)$ we have

$$d_k^{\alpha} h(d_k) \ge \tfrac{1}{4} \quad \text{and} \quad Q(d_k) \ge (\eta \wedge 1) M(d_k), \tag{4.3}$$

and $d_k \to 0$.

The rest of the proof will also be used in case $\beta > 1$. If

$$\lim_{x \to 0} \frac{K(x)}{G(x)} = 0,$$

then by Lemma 2.4 of [7] we have $a^{\alpha/2} Q(a) \nearrow$ for $a \le a_0$. (This lemma was proved for distribution functions instead of Lévy measures, but the proof applies in either case.) Then, by (4.3),

$$d_k^{-\alpha/2} \le 4 d_k^{\alpha/2} h(d_k) \le C d_k^{\alpha/2} Q(d_k) \le C a_0^{\alpha/2} Q(a_0)$$

for large k, a contradiction. Thus we may choose c so that

$$0 < c < \limsup_{x \to 0} \frac{K(x)}{G(x)} \quad \text{and} \quad \frac{2c}{1+c} < \alpha, \tag{4.4}$$

and define

$$e_k = \inf\{x \ge d_k : K(x) \ge c G(x)\}.$$

Note that $e_k \to 0$. Since $K(x) < cG(x)$ for $d_k < x < e_k$ we have $x^\lambda Q(x) \nearrow$ on $[d_k, e_k]$ by Lemma 2.4 of [9], where $\lambda = 2c(1+c)^{-1} < \alpha$. Thus $x^\alpha Q(x) \nearrow$ also, so, by (4.3),

$$e_k^\alpha Q(e_k) \geq d_k^\alpha Q(d_k) \geq C d_k^\alpha h(d_k) \geq C_1. \tag{4.5}$$

Since K and G are right continuous, $K(e_k) \geq cG(e_k)$ so that

$$(1 + c^{-1}) K(e_k) \geq Q(e_k). \tag{4.6}$$

Finally, by page 955 of [10], (4.6) and (4.5)

$$\begin{aligned} ET(e_k, 1) &= \int_0^1 P\{|x_t| \leq e_k\} \leq \int_0^t \frac{C}{\{tK(e_k)\}^{1/2}}\,dt = \frac{2C}{\{K(e_k)\}^{1/2}} \\ &\leq \frac{C_2}{\{Q(e_k)\}^{1/2}} \leq C_3 e_k^{\alpha/2}. \end{aligned}$$

Thus $\gamma' \geq \alpha/2$, which is sufficient since α is arbitrary in $(0, \beta)$.

$\beta > 1$. First, there exists $\rho > 1$ such

$$\limsup_{a \to 0} a^\rho h(a) = \infty. \tag{4.7}$$

If $K(x)/M(x) \to 0$, then

$$x(K(x) + M(x)) \sim xM(x) \text{ and is slowly varying}$$

by Lemma 4.2. Then, by (4.7), $h(x) \sim G(x)$ as $x \to 0$. Take $\alpha \in (1, \beta)$ and find $a_k \to 0$, so that $a_k^\alpha h(a_k) \geq 1$. Since

$$Q(a_k) \geq G(a_k) \sim h(a_k) \geq M(a_k),$$

in this case we have (4.3) with $d_k = a_k$ and $\eta \wedge 1$ replaced by $\frac{1}{2}$. It remains to consider the case when

$$\limsup_{x \to 0} \frac{K(x)}{M(x)} > 0.$$

In this case choose

$$\eta < \limsup \frac{K(x)}{M(x)}$$

and $\eta < (\alpha - 1)(2 - \alpha)^{-1}$, and define

$$b_k = \inf\{x \geq a_k : K(x) \geq \eta M(x)\}.$$

Note that $b_k \to 0$. Since $K(x) < \eta M(x)$ for $a_k < x < b_k$ we have, using Lemma 4.3, if $G(a_k) \leq M(a_k)$, then

$$b_k^\alpha h(b_k) \geq b_k^\alpha (K(b_k) + M(b_k)) \geq a_k^\alpha (K(a_k) + M(a_k)) \geq \frac{a_k^\alpha h(a_k)}{2} \geq \frac{1}{2}.$$

In this case we take $d_k = b_k$, while if $G(a_k) > M(a_k)$ we take $d_k = a_k$, and we see that (4.3) holds in either case. The proof is complete in this case as in the final paragraph of the case $\beta < 1$.

$\beta = 1$. In this case, we use subordination to reduce it to the case when $\beta < 1$. Let $Y_t = X_{U_t}$ where U_t is a stable subordinator of index $\sigma < 1$. Then we have $\beta(Y) = \sigma \cdot \beta(X) = \sigma < 1$, so that $\gamma'(Y) \geq \sigma/2$ by the above argument. But $Y([0,1]) \subset X([0, U_1])$, so that, with probability 1,

$$\gamma'(X) = \operatorname{Dim} X[0, U_1] \geq \operatorname{Dim} Y[0,1] = \gamma'(Y) \geq \frac{\sigma}{2}.$$

Since σ is arbitrary, we must have $\gamma'(X) \geq \frac{1}{2}$. □

5. Examples. We give a variety of examples which show that the bounds in Theorem 5 are sharp. We start with a subordinator since these are easy to work with. We must have $\gamma = \sigma$ and $\gamma' = \beta$ for any subordinator. [See Theorems 1 and 3 of [8] and page 954 of [10], along with the observation that $S(a) \wedge 1 = T(a, 1)$ for a subordinator.] Then β and σ are relatively easy to compute.

EXAMPLE 1. Given any γ, γ' with $0 = \gamma \leq \gamma' \leq 1$, there is a corresponding subordinator. Take $\beta \in [0,1]$, let

$$x_k = \exp(-k^k), \qquad p_k = x_k^{-\beta},$$

and consider the Lévy measure which assigns mass p_k to the point x_k. [If $\beta = 1$, use $p_k = x_k^{-1} \exp(-k^{(k-1)/2})$.] Since $\sum x_k p_k < \infty$, this is permissible. By (2.7), it is clear that β is the upper index, and so $\gamma' = \beta$. Noting that, if $0 < \beta < 1$ and k is large,

$$g(u) = \sum_{j=1}^{\infty} \big(1 - \exp(-u x_j)\big) p_j \leq \sum_{j=1}^{k-1} p_j + \sum_{j=k}^{\infty} u x_j p_j \leq 2(p_{k-1} + u x_k p_k)$$

for all u, we see that if $\alpha > 0$ and $u_k = (x_{k-1})^{-1/\alpha}$, then

$$u_k^{-\alpha} g(u_k) \leq 2(x_{k-1})^{1-\beta} + 2(x_{k-1})^{1-1/\alpha} x_k^{1-\beta} \to 0.$$

If $\beta = 1$, then $x_k p_k$ is different, but the argument works in the same way. If $\beta = 0$, then the sum of the first $k-1$ p's is $k-1$, but this still will approach 0 when multiplied by $u_k^{-\alpha}$. Thus we have

$$\liminf_{u \to \infty} u^{-\alpha} g(u) = 0$$

for all $\alpha > 0$ and so $\gamma = \sigma = 0$.

To obtain a subordinator with indices $0 < \gamma < \gamma' \leq 1$, it is sufficient to add a continuous part to the Lévy measure, with density $x^{-\gamma-1}$. This will not change the upper index γ' but will increase the lower index to $\sigma = \gamma$.

EXAMPLE 2. Given $0 < \gamma \leq \gamma' < 1$ and $\beta/2 < \gamma' < \beta \wedge 1$, there is a corresponding symmetric process in $\mathbf{R}$. With x_k as above, we use a Lévy measure that has mass $x_k^{-\xi}$ at $\pm x_k$ and, in addition, has a density $|x|^{-\alpha-1}$ on the entire real line. The parameters are to satisfy

$$0 < \xi < 2 \quad \text{and} \quad 0 < \alpha < \xi \wedge 1.$$

Then it is clear from (2.7) that $\beta = \xi$, and one may show that $\gamma = \alpha$ quite easily but computing γ' is more difficult. To do this, we will use the form of the characteristic function of X_t given by

$$E \exp(iuX_t) = c^{-t\psi(u)},$$

where

$$\psi(u) = \int (1 - \cos ux)\nu(dx),$$

and ν is the Lévy measure. In this case, we see that $\psi(u) \geq cu^{\alpha}$ so that the characteristic function of X_t is integrable for all t. This means that X_t has a density $p(t, x)$, and we can use the inversion theorem for it in the following computation of the expected sojourn time:

$$\begin{aligned} ET(a,1) &= \int_0^1 P\{|X_t| \leq a\} = \int_0^1 \int_{-a}^{a} p(t,x)\,dx\,dt \\ &= (2\pi)^{-1} \int_0^1 \int_{-a}^{a} \int e^{-t\psi(u)} \cos ux\,du\,dx\,dt \\ &= \pi^{-1} \int \frac{\sin ua}{u} \frac{1}{\psi(u)} (1 - e^{-\psi(u)})\,du. \end{aligned}$$

To obtain estimates, we first consider ψ. If $(x_k)^{-1} < u < (x_{k+1})^{-1}$, then ψ is comparable to $\psi_1 + \psi_2 + \psi_3$ where

$$\psi_1(u) = |u|^{\alpha}, \qquad \psi_2(u) = (x_k)^{-\xi}(1 - \cos ux_k), \qquad \psi_3(u) = u^2 (x_{k+1})^{2-\xi}.$$

The last term comes from using the approximation $1 - \cos ux_j \approx (ux_j)^2$ for $j \geq k + 1$. The terms like ψ_2, but with $j < k$, are dominated by ψ_1. Then ψ is comparable to the maximum of these three terms. We obtain

$$\begin{aligned} &\psi(u) \approx \psi_1(u) + \psi_2(u), && (x_k)^{-1} < u < (x_k)^{-\xi/\alpha}, \\ &\psi(u) \approx \psi_1(u), && (x_k)^{-\xi/\alpha} < u < (x_{k+1})^{-(2-\xi)/(2-\alpha)}, \\ &\psi(u) \approx \psi_3(u), && (x_{k+1})^{-(2-\xi)/(2-\alpha)} < u < (x_{k+1})^{-1}. \end{aligned}$$

On the first of these intervals, both terms play a role due to the periodicity of ψ_2. We may now use these estimates to obtain good estimates of the expected sojourn time. One replaces $\sin ua$ by ua for $|ua| \leq 1$, and uses the fact that ψ is large when u is large. The main contribution comes from

$$a \int_0^{1/a} \frac{du}{\psi(u)}.$$

[To consider the intervals $(x_k)^{-1} < u < (x_k)^{-\xi/\alpha}$, one must break them up into smaller intervals of the form $(2\pi l/x_k, 2\pi(l+1)/x_k)$, and then subdivide these further to reflect which of ψ_1, ψ_2 is larger.] For the range of u when $|ua| > 1$, one bounds the sin term by 1 and shows that even the integral of the absolute value is smaller than the term above. However, in some cases the two terms are the same order of magnitude so that some care is required. We omit the details. To describe the result, let

$$a_k = x_k^{\xi/\alpha}, \qquad b_k = x_k^{(4-4\alpha+\xi\alpha)/(2-\alpha)^2}, \qquad c_k = x_k^{(2-\xi)/(2-\alpha)}.$$

Then

$$ET(a,1) \approx \begin{cases} a^{\alpha}, & c_{k+1} < a \le a_k, \\ a^{\alpha/2} x_k^{\xi/2}, & a_k < a \le b_k, \\ a x_k^{-(2-\xi)(1-\alpha)/(2-\alpha)}, & b_k < a \le c_k. \end{cases}$$

This gives $\gamma = \alpha$, as mentioned above, while

$$\gamma' = \frac{2\alpha + 2\xi - 2\alpha^2 - 2\xi\alpha + \xi\alpha^2}{4 - 4\alpha + \xi\alpha};$$

this power is achieved at $a = b_k$. γ' is an increasing function of both parameters. To obtain the first class of examples, fix $\alpha = \gamma$, and let ξ vary over $(\alpha, 2)$. Then γ' varies over the interval $(\alpha, 1)$. For the other class of examples, we fix $\xi = \beta$, and let α vary over the interval $(0, \xi \wedge 1)$. Then γ' varies over the interval $(\beta/2, \beta \wedge 1)$.

By an easy modification of this example, we can achieve $\gamma' = \beta/2$ at the cost of having $\gamma = 0$: it is sufficient to omit the component of the Lévy measure with density $|x|^{-\alpha-1}$. We remark that $\gamma' = \beta/2$ is also achieved by Brownian motion on $\mathbf{R}$; in that case

$$1 = \gamma = \gamma' = \frac{\beta}{2}.$$

EXAMPLE 3. To obtain values of γ or γ' larger than 1, we must use examples in $\mathbf{R}^2$. To obtain $1 < \gamma = \gamma' \le \beta$, we use a process with stable components, that is, we run independent stable processes of indices $1 \vee \alpha_2 \le \alpha_1$, $\alpha_1 > 1$, on the coordinate axes. Then by Lemma 5.1 of [11],

$$\beta = \alpha_1 \quad \text{and} \quad \gamma = \gamma' = 1 + \alpha_2\left(1 - \frac{1}{\alpha_1}\right).$$

Letting α_2 vary over $(0, \alpha_1]$ gives the desired range of γ'.

EXAMPLE 4. To obtain $0 \le \gamma < \gamma'$ when $\gamma' \ge 1$, we use planar Brownian motion subordinated by the subordinator of Example 1. Let B_s be planar Brownian motion and consider B_{X_t} with X_t as in the first example with indices $\gamma/2$ and $\gamma'/2$. By Theorem 3,

$$\gamma(B_{X_t}) = \dim(B_{X_t}[0,1]) = 2\dim(X_t[0,1]) = 2\left(\frac{\gamma}{2}\right) = \gamma,$$

with a similar argument for γ' using packing dimension.

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