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ε-CLOSE MEASURES PRODUCING NONISOMORPHIC FILTRATIONS[1]

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A consequence of the preceding two papers is this. Let $\{\mathscr{A}_t: 0 \le t < \infty\}$ be the filtration of a stochastic process on $(\Omega, \mathscr{A}, P)$. Under a mild assumption on the process, there exist, for any $\varepsilon > 0$, uncountably many probability measures Q_α with $(1-\varepsilon)P \le Q_\alpha \le (1+\varepsilon)P$ so that no two of the filtrations $(\Omega, (\mathscr{A}_t)_{0 \le t}, Q_\alpha)$ and $(\Omega, (\mathscr{A}_t)_{0 \le t}, Q_\beta)$, $\alpha \ne \beta$, can be generated by equivalent stochastic processes.

1. Statement of results. This paper makes two easy but striking deductions from the preceding two papers: "Decreasing sequences of σ-fields and a measure change for Brownian motion" by Dubins, Feldman, Smorodinsky and Tsirelson, hereafter referred to as [I]; and "Decreasing sequences of σ-fields and a measure change for Brownian motion. II" by Feldman and Tsirelson, hereafter referred to as [II].

The main result of [I] was the construction, for Brownian motion B on $[0, \infty)$, of a new probability measure Q equivalent to the old one P in the sense of absolute continuity, but for which the filtration of B equipped with the new measure is not Brownian; that is, it is not the filtration of any Brownian motion with respect to Q. This may be rephrased by saying that there is no measure space isomorphism carrying P to Q and carrying the filtration to itself. Then in [II] it was shown that Q may even be chosen so that $(1-\varepsilon)P \le Q \le (1+\varepsilon)P$.

In this paper a stronger version of this phenomenon is shown to hold for all "sufficiently rich" stochastic processes. The result is obtained by applying two "bootstrapping" procedures to the example constructed in [I] and improved in [II]. Specifically, it will be shown that there exist *uncountably many* different probability measures equivalent to P for which *no two* of the corresponding filtrations are isomorphic, and which all lie between $(1-\varepsilon)P$ and $(1+\varepsilon)P$ (this is what we mean by ε-close). Furthermore (and this is the surprising part), the phenomenon really has nothing to do with Brownian motion: it holds for *any* sufficiently rich stochastic process.

THEOREM 1. *Let A_t, $0 \le t < \infty$, be a stochastic process on $(\Omega, \mathscr{A}, P)$. Let $\mathscr{A}_t$ be the complete σ-field generated by $\{A_s: 0 \le s < t\}$, and $\mathbf{A} = (\mathscr{A}_t: 0 \le t < \infty)$.*

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Suppose that there exist $t_1 > t_2 > \cdots \to 0$ *with the following properties*:

(i) $\bigcap_n \mathscr{A}_{t_n}$ *is trivial, that is, consists of sets of measure* 0 *or* 1.
(ii) $P|\mathscr{A}_{t_n}$ *conditioned by* $\mathscr{A}_{t_{n+1}}$ *is almost everywhere nonatomic.*

Then for any $\varepsilon > 0$ *one may construct for each countable ordinal* α *a probability measure* Q_α *with* $Q_1 = P$, $(1-\varepsilon)P \le Q_\alpha \le (1+\varepsilon)P$ *for all* α, *and such that no two of the filtrations* $(\Omega, \mathbf{A}, Q_\alpha)$ *are isomorphic. Thus no two of them may be generated by equivalent stochastic processes.*

The assumption of the existence of a sequence with the stated properties is a very mild one; typically, any sequence $t_1 > t_2 > \cdots \to 0$ will do.

As in [I], the theorem is proved by first obtaining a parallel result for the analogous but simpler case of "reverse filtrations"; see [I], Section 2, (2.1) to (2.4), for background. However, we repeat the basic definitions.

DEFINITION. Let $(X, \mathscr{F}, \mu)$ be a Lebesgue probability space, and $\mathscr{F} = \mathscr{F}_0 \supset \mathscr{F}_1 \supset \mathscr{F}_2 \cdots$ a sequence of (complete) σ-fields with $\bigcap_n \mathscr{F}_n$ trivial. The sequence $\mathbf{F} = (\mathscr{F}_n)_0^\infty$ is called a *reverse filtration* on $(X, \mathscr{F}, \mu)$. It is called *nonatomic* if for every nonnegative integer n almost every measure obtained by conditioning $\mu|\mathscr{F}_n$ on $\mathscr{F}_{n+1}$ is nonatomic.

THEOREM 2. *Given a nonatomic reverse filtration* $(X, \mathbf{F}, \mu)$, *one may construct for each countable ordinal* α *a probability measure* ν_α *with* $\nu_1 = \mu$, $(1-\varepsilon)\mu \le \nu_\alpha \le (1+\varepsilon)\mu$ *for all* α, *and such that no two of the inverse filtrations* $(X, \mathbf{F}, \nu_\alpha)$ *are isomorphic.*

It should be noted that uncountable families of mutually nonisomorphic reverse filtrations were already constructed in Stepin (1971) and Vershik (1971). But these had *atomic* conditional measures and, more important, were not obtained from a single reverse filtration via absolutely continuous changes of measure.

Here is how to deduce Theorem 1 from Theorem 2. Given $(\Omega, \mathbf{A}, P)$ and $t_1 > t_2 > \cdots$ as in Theorem 1, let $\mathscr{F}_0 = \mathscr{A}$ and $\mathscr{F}_n = \mathscr{A}_{t_n}$ for $n \ge 1$. There is no loss of generality in assuming that P conditioned by $\mathscr{A}_{t_1}$ is almost everywhere nonatomic, since this can be achieved by just shifting the subscripts of the t_n by 1. Then the reverse filtration $(\Omega, \mathbf{F}, P)$ satisfies the assumptions of Theorem 2 with $X = \Omega$ and $\mu = P$. Letting Q_α be the ν_α of Theorem 2, we see that the reverse filtrations $(\Omega, \mathbf{F}, Q_\alpha)$ are nonisomorphic, and therefore also the filtrations $(\Omega, \mathbf{A}, Q_\alpha)$, since any isomorphism of $(\Omega, \mathbf{A}, Q_\alpha)$ with $(\Omega, \mathbf{A}, Q_\beta)$ induces an isomorphism of $(\Omega, \mathbf{F}, Q_\alpha)$ with $(\Omega, \mathbf{F}, Q_\beta)$.

2. Reverse filtrations and standardness. Recall from [I] that a *standard* reverse filtration is one which is isomorphic to $(X, \mathbf{F}, \lambda)$, where $X = [0,1]^{\mathbb{N}}$ ($\mathbb{N}$ being the positive integers), λ is the product of the Lebesgue measures on the coordinate factors and $\mathscr{F}_n$ is the completed σ-field generated by all coordinates greater than n.

Let S be an infinite set of positive integers. Write it in ascending order as $s_1, s_2, \dots,$ and set $s_0 = 0$. If Φ is the reverse filtration $(X, \mathbf{F}, \mu)$, then $\Phi|S$ will denote the reverse filtration $(X, \mathbf{F}|S, \mu)$, where $\mathbf{F}|S$ is the sequence $(\mathscr{F}_{s_0}, \mathscr{F}_{s_1}, \dots)$. It is easy to see that if $S \supset T$ and $\Phi|T$ is standard, then so is $\Phi|S$; in fact, in the nonatomic case, if $T - S$ is finite and $\Phi|T$ is standard, then so is $\Phi|S$. (As a matter of notational convenience, we have written $T - S$ rather than $T \cap S^c$.) This may be shown by routine Lebesgue space constructions.

We will need the following consequence of Corollary 2 of [II].

THEOREM 3. *Let $(X, \mathbf{F}, \mu)$ be a standard nonatomic reverse filtration. Then for any $\varepsilon > 0$ there is a measure ν such that $(1 - \varepsilon)\mu \leq \nu \leq (1 + \varepsilon)\mu$ and $(X, \mathbf{F}, \nu)$ is not standard.*

We will also need the following statement, which is the nonatomic version of the main result of Vershik (1968), and may be proved in much the same way; see Ganikhodzhaev and Vinokurov (1978).

THEOREM 4. *For each nonatomic reverse filtration Φ, there is an infinite set S of positive integers for which $\Phi|S$ is standard.*

The collection of all infinite sets S of positive integers for which $\Phi|S$ is standard is what is called in Vershik (1970) the *fundamental invariant* of Φ. It is clearly an isomorphism invariant.

3. Proof of Theorem 2.

We will need two lemmas.

LEMMA 5. *Given $\varepsilon > 0$ and an infinite set S of positive integers and a nonatomic reverse filtration $(X, \mathbf{F}, \mu)$, there is a probability measure μ' satisfying $(1 - \varepsilon)\mu \leq \mu' \leq (1 + \varepsilon)\mu$ and for which $(X, \mathbf{F}, \mu')|S$ is not standard.*

PROOF. If $(X, \mathbf{F}, \mu)|S$ is already nonstandard, use μ; otherwise apply Theorem 3 to the standard reverse filtration $(X, \mathbf{F}, \mu)|S$. □

LEMMA 6. *Let $S_1, S_2, \dots$ be a sequence of infinite sets of positive integers with each $S_{n+1} - S_n$ finite. Then there is an infinite set S of positive integers with each $S - S_n$ finite.*

PROOF. By removing finitely many elements from each S_n, we may suppose $S_{n+1} \subset S_n$ for all n. Then a diagonal argument gives the result. □

Now the proof of Theorem 2 will be completed. Let $\Phi = (X, \mathbf{F}, \mu)$ be a given nonatomic inverse filtration. We will construct for each countable ordinal α

an infinite set S_α of positive integers and a probability measure $\nu_\alpha \sim \mu$ so that, denoting $(X, \mathbf{F}, \nu_\alpha)$ by Φ_α, we have:

1. $(1-\varepsilon)\mu \le \nu_\alpha \le (1+\varepsilon)\mu$.
2. $\alpha < \alpha' \Rightarrow S_{\alpha'} - S_\alpha$ is finite.
3. $\Phi_\alpha | S_\alpha$ is nonstandard.
4. $\alpha < \alpha' \Rightarrow \Phi_\alpha | S_{\alpha'}$ is standard.

It will then be clear that Φ_α is not isomorphic to $\Phi_{\alpha'}$ if $\alpha \ne \alpha'$, since their fundamental invariants will be different. The construction will of course be made by a transfinite induction.

To begin, let $S_1 = N$ and choose $\nu_1 \sim \mu$ so that $(1-\varepsilon)\mu \le \nu_1 \le (1+\varepsilon)\mu$ and $\Phi_1 = (X, \mathbf{F}, \nu_1)$ is not standard. Inductively, given a countable ordinal $\bar{\alpha}$, suppose S_α and ν_α have been defined for all $\alpha < \bar{\alpha}$ and satisfy conditions 1–4. We need to extend the definition to $\alpha = \bar{\alpha}$, that is, to replace $\bar{\alpha}$ by $\bar{\alpha} + 1$, with the conditions still satisfied.

(a) If $\bar{\alpha}$ is a successor ordinal, that is, $\bar{\alpha} = \alpha_0 + 1$, then use Theorem 4 to choose an infinite set T of positive integers such that $(\Phi_{\alpha_0} | S_{\alpha_0}) | T$ is standard. Enumerate S_{α_0} as $s_1 < s_2 < \cdots$ and let $S_{\bar{\alpha}} = \{s_t : t \in T\} \subset S_{\alpha_0}$. Apply Lemma 5 by choosing $\nu_{\bar{\alpha}}$ with $(1-\varepsilon)\mu \le \nu_{\bar{\alpha}} \le (1+\varepsilon)\nu$ and $(X, \mathbf{F}, \nu_{\bar{\alpha}}) | S_{\bar{\alpha}}$ not standard. This extends the induction one step, as required.

(b) If $\bar{\alpha}$ is a limit ordinal, then there exist $\alpha_1 < \alpha_2 < \cdots \uparrow \bar{\alpha}$. Use Lemma 6 to choose S so $S - S_n$ is finite for each n, and call this set $S_{\bar{\alpha}}$. If $\alpha < \bar{\alpha}$, then $\alpha < \alpha_n$ for some n, so $\Phi_\alpha | S_{\alpha_n}$ is standard for all n, so $\Phi_\alpha | S_{\bar{\alpha}}$ is also standard. Now apply Lemma 5 to produce a measure ν such that $(1-\varepsilon)\mu \le \nu_{\bar{\alpha}} \le (1+\varepsilon)\mu$ and $(X, \mathbf{F}, \nu_{\bar{\alpha}}) | S_{\bar{\alpha}}$ is nonstandard. Again, the induction has been advanced one step, and we are done. □

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