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ABSOLUTE CONTINUITY OF SYMMETRIC DIFFUSIONS[1]

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Let X and Y be symmetric diffusion processes with a common state space, and let P^m (resp. Q^μ) be the law of X (resp. Y) with its symmetry measure m (resp. μ) as initial distribution. We study the consequences of the absolute continuity condition $Q^\mu \ll_{\mathrm{loc}} P^m$. We show that under this condition there is a "smooth" version ρ of the Radon–Nikodym derivative $d\mu/dm$ such that $\frac{1}{2}[\log \rho(X_t) - \log \rho(X_0)] = M_t + N_t$, $t < \sigma$, where M is a continuous local martingale additive functional, N is a zero-energy continuous additive functional and σ is an explosion time. The Girsanov density $L_t := dQ^\mu|_{F_t}/dP^m|_{F_t}$ then admits the representation $L_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)1_{\{t<\sigma\}}$. The density ρ also serves to link the Dirichlet forms of X and Y in a simple way. Our identification of L relies on notions of *even* and *odd* for additive functionals. These notions complement Fukushima's decomposition and the forward–backward martingale decomposition of Lyons and Zheng.

1. Introduction. Let $X = (X_t, P^x)$ be a symmetric Markov diffusion process with state space E, symmetry measure m, infinitesimal generator A, and Dirichlet form $(\mathbf{D}, \mathbf{E})$. Thus

$$\mathbf{E}(u, v) = (u, -Av)_m, \qquad u \in \mathbf{D},\ v \in \mathbf{D}(A),$$

where $(\cdot,\cdot)_m$ denotes the natural inner product in $L^2(m)$. If the domain of A contains a dense subalgebra, then the bilinear form defined by $\gamma(u,v) := A(uv) - uAv - vAu$ extends by continuity to a bilinear mapping of $\mathbf{D}\times\mathbf{D}$ into $L^1(m)$. The form $\mathbf{E}$ then admits the representation

$$\mathbf{E}(u, v) = \tfrac{1}{2}\int_E \gamma(u, v)\, dm, \qquad u, v \in \mathbf{D}. \tag{1.1}$$

Given a function $\rho\colon E \to]0,\infty[$ that is locally an element of $\mathbf{D}$, define a measure $\mu := \rho\cdot m$ and a Dirichlet form on $L^2(\mu)$ by

$$\hat{\mathbf{E}}(u, v) := \tfrac{1}{2}\int_E \gamma(u, v)\rho\, dm = \tfrac{1}{2}\int_E \gamma(u, v)\, d\mu. \tag{1.2}$$

Since X is a diffusion, the mapping $v \mapsto \gamma(u,v)$ satisfies the derivation identity $\gamma(u, vw) = v\gamma(u,w) + w\gamma(u,v)$. Using this fact, a formal calculation shows that the infinitesimal generator associated with $\hat{\mathbf{E}}$ is given by the formula

$$\hat{A}\, u = Au + \gamma(l, u) =: Au + Bu, \tag{1.3}$$

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where $l := \frac{1}{2}\log\rho$. The operator B, being a derivation, represents a first-order term. Thus, $\hat{A}$ may be thought of as a drift perturbation of A.

Indeed, the symmetric diffusion $Y = (Y_t, Q^x)$ associated with $\hat{E}$ may be constructed from X by a Cameron–Martin–Girsanov transformation as follows. Since ρ is locally in D, so is l; consequently we have the Fukushima decomposition $l(X_t) - l(X_0) = M_t^l + N_t^l$, where both M^l and N^l are continuous additive functionals of X, M^l is a local martingale and N^l is locally of zero quadratic variation. The stochastic exponential of M^l,

$$L_t := \exp\left(M_t^l - \tfrac{1}{2}\langle M^l\rangle_t\right), \tag{1.4}$$

is a local martingale and a multiplicative functional of X. Define path-space laws Q^x by $Q^x|_{F_t} := L_t \cdot P^x|_{F_t}$, $t \geq 0$. The resulting diffusion is a symmetric Markov process, with symmetry measure μ. Using Itô's formula, it is easy to check (at least formally) that the infinitesimal generator of the diffusion corresponding to $(Q^x)_{x\in E}$ is $\hat{A}$ as defined in (1.3).

The above considerations can be made precise in a very general framework, as is shown in Sections 4 and 5 of this paper. However the main issue addressed is the converse problem.

Suppose we have symmetric Markov diffusion processes (X_t, P^x) and (Y_t, Q^x) with common state space E and respective symmetry measures m and μ. Suppose that the path-space law Q^μ is (locally) absolutely continuous with respect to P^m. What can one say about the form taken by the (local) Radon–Nikodym density process $L_t := dQ^\mu|_{F_t}/dP^m|_{F_t}$, and about the relationship between the associated Dirichlet forms? The short answer is that things must be as described in the first paragraph of this introduction. That is, if $Q^\mu \ll_{\mathrm{loc}} P^m$, then (i) $\mu \ll m$, (ii) the density $\rho := d\mu/dm$ is locally in the Dirichlet space of X, and (iii) L is given by (1.4), where M^l is the martingale part in the Fukushima decomposition of $l(X_t) - l(X_0)$, and $l = \frac{1}{2}\log\rho$ as before. Moreover, the Dirichlet forms E and $\hat{E}$ of X and Y are related as in (1.2). For precise statements, see Theorems 3.2 and 4.9.

The problem of deciding when the law of one symmetric diffusion is absolutely continuous with respect to that of another (and of describing the relationship between their infinitesimal generators or Dirichlet forms) has been studied by many authors. A complete discussion in the case of one-dimensional diffusions may be found in Orey's paper [36]. The multidimensional case (of a restricted form of our problem) has been treated by Fukushima [21], wherein X is taken to be Brownian motion. We must also mention the earlier work of Kolmogorov [28] and Nelson [35] concerning the case in which X is a Brownian motion in a finite-dimensional Riemannian manifold, and Y is a diffusion on the manifold with smooth generator. For an elegant account of this work, see [27], pages 274–282. For the sake of comparison, let us give a brief statement of Fukushima's result. The notation is as earlier, but now X is Brownian motion in $\mathbb{R}^d$ and Y is a symmetric diffusion in $\mathbb{R}^d$. Of course, m is Lebesgue measure in this context. The main result of [21] states that if $Q^x \sim_{\mathrm{loc}} P^x$ for all $x \in \mathbb{R}^d$ outside an X-polar set, and if X-polar sets are Y-polar (and vice versa), then (i) $\mu \sim m$, (ii) $\rho := d\mu/dm$ is locally in $H^1(\mathbb{R}^d)$, the space of square integrable functions with square-integrable distribution sense gradients, and (iii) the Radon–Nikodym derivative L has the representation (for X-quasi-every starting point x)

$$L_t = \exp\left(\int_0^t \nabla l(X_s) \cdot dX_s - \tfrac{1}{2}\int_0^t |\nabla l(X_s)|^2\, ds\right), \qquad t \geq 0, \tag{1.5}$$

where $l = \frac{1}{2} \log \rho$. Moreover, the generator of Y, when restricted to smooth functions of compact support, has the form

$$\hat{A} f = \tfrac{1}{2} \Delta f + \nabla l \cdot \nabla f. \tag{1.6}$$

An extension of this result, valid for X in a wide class of finite-dimensional diffusions, has been proved by Oshima [37].

Our main results extend the above-mentioned work, as well as more recent work of Oshima and Takeda [38] and Albeverio, Röckner and Zhang [4]. See also Eberle [13, 14] for work related to Sections 4 and 5 of this paper. Diffusion processes with generators of the type (1.6) have been the subject of much research; see [1], [33], [22], [23], [40], [41] and the references therein.

Most of the work on this problem of which we are aware has been confined to situations in which the state space is a vector space (perhaps infinite dimensional) and the infinitesimal generator of X is, loosely speaking, a diagonalizable elliptic differential operator. In contrast to this work, we make no special hypotheses on our processes (or their state spaces) beyond the assumption of path continuity.

A crucial ingredient in our arguments is the method of forward–backward martingales developed by Lyons and Zheng [31] and applied by Takeda [44, 45]. (See also [33, 35] for early forms of the technique and [30] for more recent work.) As a complement to this method, we introduce notions of even and odd additive functionals of a symmetric diffusion. In addition to providing the path to a crucial uniqueness result (described below), these notions shed new light on Fukushima's decomposition $u(X_t) - u(X_0) = M_t^u + N_t^u$ for elements u of the Dirichlet space of X. For a fuller development of the idea of parity for continuous additive functionals, the reader is directed to [17]. (After submitting this manuscript, we learned of [12] and [26] in which the notion of odd additive functional is used to provide remarkably simple proofs of invariance principles in the context of symmetric Markov processes.]

Since the proof of our main result, Theorem 3.2, is rather long, as an aid to the reader we provide here a summary of the argument. Suppose $Q^\mu \ll_{\text{loc}} P^m$. Then clearly $\mu \ll m$, and $Q^x \ll_{\text{loc}} P^x$ for μ-a.e. x. For simplicity, let us suppose that there is no exceptional set, so that $Q^x \ll_{\text{loc}} P^x$ for all x. We also assume in the rest of this paragraph that both X and Y have infinite lifetime. By a result of Kunita, there is a positive martingale L, which is a continuous multiplicative functional of X, such that $Q^x|_{F_t} = L_t \cdot P^x|_{F_t}$ for all $t \geq 0$ and all x. Provided L is strictly positive, general theory tells us that L has the form

$$L_t = \exp\left(K_t - \tfrac{1}{2}\langle K \rangle_t\right) \tag{1.7}$$

where K is a local martingale and a continuous additive functional of X, and $\langle K \rangle$ is the quadratic variation of K. The symmetry of X and of Y now imply that

$$l(X_t) - l(X_0) = \tfrac{1}{2}[K_t - K_t \circ r_t], \tag{1.8}$$

where $l = \frac{1}{2}\log(d\mu/dm)$ as before, and r_t is the operator of time reversal on the time interval $[0, t]$. Estimates based on (1.8) allow us to conclude that l is locally in the Dirichlet space of X. This fact in hand, a symmetry argument shows that

$$\tfrac{1}{2}[M_t^l - M_t^l \circ r_t] = \tfrac{1}{2}[K_t - K_t \circ r_t], \tag{1.9}$$

where M^l is the martingale part in Fukushima's decomposition of $l(X_t) - l(X_0)$. The quantity on the right-hand side of (1.9) is what we shall later refer to as the odd part of K, and similarly for the quantity on the left. Thus, (1.9) says that the local martingale $M^l - K$ is even (i.e., $M^l - K$ has vanishing odd part). But we are able to show that a local martingale which is also an even continuous additive functional must vanish. Thus, $K = M^l$, and so (1.7) is the desired representation of L. Note that with the identification of K and M^l, (1.8) becomes a special case of the Lyons–Zheng decomposition

$$u(X_t) - u(X_0) = \tfrac{1}{2}[M_t^u - M_t^u \circ r_t],$$

which is valid for any function u locally in the Dirichlet space of X. This decomposition will play an important role in our description of the Dirichlet space of Y in terms of that of X.

The rest of the paper is organized as follows: we introduce our basic hypotheses and prove several preliminary results in Section 2; Section 3 contains the statement and proof of the main result discussed above; in Section 4, we discuss briefly the relationship between the Dirichlet spaces of X and Y; in Section 5 we use the results of Sections 3 and 4 to find sufficient conditions for $Q^\mu \ll_{\text{loc}} P^m$.

Although the context of the paper is symmetric right Markov processes, we shall use a number of results which have been proved in the literature only for symmetric processes associated with regular Dirichlet forms. This usage is justified by recent work on quasi-regular Dirichlet forms [2, 32, 8]: any symmetric Borel right process is quasi-homeomorphic to a symmetric process associated with a regular Dirichlet form, and the quasi-homeomorphism can be used to transfer results known for regular Dirichlet forms to our context.

For the most part, our notation is standard. Background on Markov processes can be found in [5] and [42], whereas for details on symmetric Markov processes and Dirichlet spaces the reader can consult [20], [6] and [32]. Let us mention here a few specifics. If $(F, \mathcal{F})$ is a measurable space, then $p\mathcal{F}$ and $b\mathcal{F}$ denote the classes of positive and bounded real-valued $\mathcal{F}$-measurable functions from F to $\mathbb{R}$; these prefixed have the same meaning when attached to other function classes. If μ is a measure on $(F, \mathcal{F})$ and f: $F \to [0, \infty]$ is $\mathcal{F}$-measurable, then $\mu(f)$ denotes the integral $\int_F f\,d\mu$ while $f\mu$ denotes the measure whose density with respect to μ is f. The term "additive functional" should be interpreted in the sense of ([20], page 124); that is, as "additive functional with an exceptional set of starting points."

2. Preliminaries. Let $X = (\Omega, F, F_t, X_t, \theta_t, P^x)$ be a right Markov process. We shall say that X is a *diffusion* provided the following additional properties obtain:

(2.1a) The state space E of X is homeomorphic to a Borel subset of some compact metric space. We write $B(E)$ for the class of Borel subsets of E.

(2.1b) The transition semigroup of X, $(P_t)_{t \geq 0}$, is Borel: $P_t f$ is $B(E)$-measurable for each bounded $B(E)$-measurable function f: $E \to \mathbb{R}$. The resolvent family of X, $U^\alpha := \int_0^\infty e^{-\alpha t} P_t \, dt$, $\alpha > 0$, then has the same measurability property.

(2.1c) The filtration $F_t^\circ := \sigma\{X_s;\ 0 \leq s \leq t\}$, $t \geq 0$, is quasi-left-continuous (up to null sets) and the lifetime of X, denoted ζ, is a *predictable* stopping time.

(2.1d) $t \mapsto X_t$ is continuous on $[0, \zeta[$ a.s. P^x for all $x \in E$.

In particular, X is a strong Markov process, and by (44.5) and (47.10) in [42], any (F_{t+}°)-stopping time is predictable. The quasi-left-continuity portion of condition (2.1)(c) is imposed only to simplify the exposition; it is a consequence of the symmetry hypothesis discussed below, at least after the deletion of an exception set of starting points (see [16], (5.2)).

Without loss of generality, we take the sample space Ω to be the space of paths ω from $[0, \infty[$ to $E \cup \{\Delta\}$ that are E-valued and continuous on $[0, \zeta(\omega)[$ and that hold the value $\Delta \notin E$ after time $\zeta(\omega)$. As usual, any function f defined on E is automatically extended to the cemetery state Δ by the convention $f(\Delta) = 0$. For technical reasons we work mainly over the filtration $(F_{t+})_{t \geq 0}$, where F_t denotes the universal completion of F_t°. Thus $F \in F_t$ if and only if F is in the P-completion of F_t° for all probability measures P on (Ω, F_t°). It is easy to check that if P and Q are probability measures on (Ω, F_∞°), then $Q|_{F_t} \ll P|_{F_t}$ if and only if $Q|_{F_t^\circ} \ll P|_{F_t^\circ}$. In view of known perfection theorems, all multiplicative (and additive) functionals encountered in the sequel are assumed to be adapted to the filtration (F_{t+}); see the Appendix of [24] for a detailed discussion of this point.

Various localization arguments occurring in the sequel require the following notion of local martingale. If $S \leq \zeta$ is a stopping time, then we say that an adapted process $M = (M_t)_{t \geq 0}$ is a *local martingale on* $[0, S[$ provided there is an increasing sequence (T_n) of stopping times, with $T_n \uparrow S$ as $n \to \infty$ a.s. P^x for all $x \in E$, such that $t \mapsto M_{t \wedge T_n}$ is a P^x uniformly integrable martingale for each $x \in E$. Since X is a diffusion, any such local martingale has continuous paths on $[0, S[$, almost surely. (See (47.6) and (51.24) in [42].)

We say that the diffusion X is *symmetric* (with symmetry measure m) provided there is a σ-finite measure m defined on $(E, B(E))$ such that

$$(f, P_t g)_m = (P_t f, g)_m \qquad \forall\, f, g \in L^2(m), \tag{2.2}$$

where $(u,v)_m := \int_E uv\,dm$ is the natural inner product in $L^2(m)$. A Borel set $B \subset E$ is (X,m)-*polar* provided $P^m(T_B < \infty) = 0$. Here, $T_B := \inf\{t > 0\colon X_t \in B\}$ is the hitting time of B. A property $P(x)$ which holds for each x outside an (X,m)-polar set is said to hold (X,m)-*quasi everywhere* (q.e.). It is well known that the symmetry of X implies that every X-semipolar set is (X,m)-polar. In particular, this means that every predictable increasing additive functional (AF) of X is continuous on $[0,\zeta[$, a.s. P^x for (X,m)-q.e. $x \in E$ (see [25], (16.21)).

Consider now a second symmetric E-valued diffusion $Y = (\Omega, F, F_t, Y_t, \theta_t, Q^x)$. The semigroup, resolvent and symmetry measure of Y are denoted (Q_t), (V^α), and μ, respectively. Notice that Y is realized on the same sample space as X, and that $X_t(\omega) = Y_t(\omega) = \omega(t)$. The processes are distinguished by their respective laws P^x and Q^x, but we use Y_t for emphasis when working with Q^x.

DEFINITION 2.3. Given two σ-finite measures P and Q on (Ω, F), we say that Q is locally absolutely continuous with respect to P (and write $Q \ll_{\text{loc}} P$) provided the restriction of Q to $F_t \cap \{t < \zeta\}$ is absolutely continuous with respect to the restriction of P to $F_t \cap \{t < \zeta\}$, for all $t \geq 0$.

The following preliminary result extends [4], Proposition 1.2.

PROPOSITION 2.4. *Let $X = (X_t, P^x)$ and $Y = (Y_t, Q^x)$ be E-valued symmetric diffusions, with symmetry measures m and μ, respectively. Then the following statements are equivalent*:

(a) $Q^\mu \ll_{\text{loc}} P^m$,
(b) $\mu \ll m$ *and* $Q^x \ll_{\text{loc}} P^x$ *for* μ-*a.e.* $x \in E$,
(c) $\mu \ll m$ *and* $Q^x \ll_{\text{loc}} P^x$ *for* (Y,μ)-*q.e.* $x \in E$,
(d) μ *charges no* (X,m)-*polar set and* $Q^x \ll_{\text{loc}} P^x$ *for* μ-*a.e.* $x \in E$.

The implications (c) ⇒ (b) ⇒ (a) are trivial. To prove (a) ⇒ (d) ⇒ (c) we need the following form of the Lebesgue decomposition, due to Kunita [29]. Recall from ([42], Section 54) that a multiplicative functional (MF) is a positive, right-continuous, (F_{t+})-adapted process $(L_t)_{t>0}$ such that $L_{T+s} = L_T L_s \circ \theta_T$ a.s. on $\{T < \infty\}$ for each $s \geq 0$ and for each (F_{t+})-stopping time T. [We do not assume that $L_t \leq 1$, but L will always satisfy $P^x(L_t) \leq 1$.] An (F_{t+})-stopping time T is a terminal time provided $t \mapsto 1_{\{t<T\}}$ is a MF; that is, provided $T = S + T \circ \theta_S$ a.s. P^x on $\{S < T\}$, for each stopping time S and each $x \in E$. It is easy to check that the terminal time τ of Lemma 2.5 is a terminal time for Y as well as for X.

LEMMA 2.5. *There is a MF $L = (L_t)$ of X, and a terminal time* (*of X and of Y*) *$\tau \leq \zeta$ such that for each (F_{t+})-stopping time T*,

$$Q^x(F; T < \zeta) = \int F \cdot L_T\, dP^x + Q^x(F; \tau \leq T < \zeta) \qquad \forall\, F \in pF_{T+}. \tag{2.6}$$

Moreover, L vanishes on $[\tau,\infty[$ *and*

$$P^x(\tau<\zeta)=0 \qquad \forall\ x\in E.$$

In particular, for fixed $x\in E$,

$$Q^x \ll_{\rm loc} P^x \quad\Leftrightarrow\quad Q^x(\tau<\zeta)=0.$$

REMARK 2.7. Clearly L is a supermartingale. In fact, L is a P^x-local martingale on $[0,S[$, where $S:=\inf\{t>0\colon\ L_t=0\}$. To see this, let $\{T_n\}$ announce τ under Q^x. [The predictability of τ follows from the remark below (2.1).] By (2.6),

$$P^x\big(F\cdot L_{T\wedge L_n}\big)=Q^x(F;T\wedge T_n<\zeta)=Q^x(F) \qquad \forall\, F\in pF_{(T\wedge T_n)+},$$

where T is any (F_{t+})-stopping. This ensures that $(L_{t\wedge T_n})_{t\geq 0}$ is a uniformly integrable martingale under P^x. Moreover, writing T_∞ for $\lim_n T_n$, we have

$$0=Q^x(T_\infty<\tau)=P^x\big(L_{T_\infty};T_\infty<\tau\big)=P^x\big(L_{T_\infty};T_\infty<\infty\big),$$

so that $T_\infty\geq S$ a.s. P^x.

PROOF OF PROPOSITION 2.4. (a) $\Rightarrow$ (d). Assume that $Q^\mu\ll_{\rm loc}P^m$. Let $B\in B(E)$ be an (X,m)-polar set. Then $P^m(T_B<\zeta)=0$, hence $Q^\mu(T_B<\zeta)=0$ as well. Consequently B is (Y,μ)-polar, and a fortiori μ-null. In view of Lemma 2.5, to prove the second assertion of (d) we must show that $Q^\mu(\tau<\zeta)=0$. But this is an immediate consequence of $Q^\mu\ll_{\rm loc}P^m$ because $P^x(\tau<\zeta)=0$ for all $x\in E$.

(d) $\Rightarrow$ (c). Assume that (d) holds. Fix $B\in B(E)$ such that $m(B)=0$. Then the X-finely open set $\{U^\alpha 1_B>0\}$ is m-null, hence (X,m)-polar. Thus $\mu U^\alpha 1_B=0$. But (d) implies that $V^\alpha(x,\cdot)\ll U^\alpha(x,\cdot)$ for μ-a.e. $x\in E$ and each $\alpha>0$. Therefore $\mu V^\alpha 1_B=0$, and consequently $\mu(B)=\lim_{\alpha\to\infty}\alpha\mu V^\alpha 1_B=0$. We have therefore shown that $\mu\ll m$. As for the second part of (c), the function $\varphi\colon\ x\mapsto Q^x\ (\tau<\zeta)$ is strongly supermedian with respect to Y, and $\mu(\varphi>0)=0$ by (d). Thus, if T is a Y-stopping time then $Q^\mu(\varphi(Y_T))\leq\mu(\varphi)=0$. Since $\{\varphi=0\}$ is a Borel set (see the proof of Proposition 2.10), the section theorem ([10], IV.84) allows us to conclude that $Q^\mu(T_{\{\varphi>0\}}<\infty)=0$. This means that $Q^x(\tau<\zeta)=0$ for (Y,μ)-q.e. x. $\square$

The following result is a by-product of the proof just given. Following ([32], III.2.1, IV.4.5) we say that an increasing sequence $\{K_n\}$ of compact subsets of E is an (X,m)-nest provided $P^x(\lim_n\tau(K_n)<\zeta)=0$ for (X,m)-q.e. $x\in E$, where $\tau(B):=\inf\{t\colon\ X_t\notin B\}$ denotes the first exit time from B.

COROLLARY 2.8. *If $Q^\mu\ll_{\rm loc}P^m$, then every (X,m)-polar Borel set in E is (Y,μ)-polar. Moreover, every (X,m)-nest of compacts is a (Y,μ)-nest.*

For the proof, the first assertion follows as in the proof of (2.4)(a) $\Rightarrow$ (d). The second assertion follows immediately from local absolute continuity.

The conditions listed in Proposition 2.4 are equivalent to absolute continuity on the germ σ-field F_{0+}. Actually, Proposition 2.4 is valid for symmetric processes with jumps, but the next proposition depends crucially on path continuity (cf. [29], Theorem 6.2). Although this proposition could be deduced from the work of Dawson [9], we give a short direct argument.

Note that because of Blumenthal's 0–1 law, $Q^x|_{F_{0+}} \ll P^x|_{F_{0+}}$ if and only if $Q^x|_{F_{0+}} = P^x|_{F_{0+}}$. Recall that $A \subset E$ is *X-absorbing* provided $P^x(T_{A^c} < \zeta) = 0$ for all $x \in A$.

PROPOSITION 2.9. *In the context of Proposition* 2.4, $Q^\mu \ll_{\text{loc}} P^m$ *if and only if* $Q^x|_{F_{0+}} \ll P^x|_{F_{0+}}$ *for* (Y, μ)*-q.e.* $x \in E$.

PROOF. The "only if" part of the assertion follows immediately from Proposition 2.4. Conversely, assume that $Q^x|_{F_{0+}} \ll P^x|_{F_{0+}}$ for (Y, μ)-q.e. $x \in E$. By (6.12) in [25], there is a (Y, μ)-polar set N, which is Y-absorbing, such that $Q^x|_{F_{0+}} \ll P^x|_{F_{0+}}$ for all $x \in E \setminus N$. With τ as in Lemma 2.5, we have $\{\tau = 0\} \in F_{0+}$, hence $0 = P^x(\tau = 0) = Q^x(\tau = 0)$ for all $x \in E \setminus N$. Thus τ is a thin (predictable) terminal time for Y restricted to $E \setminus N$. The μ-symmetry of Y then implies $Q^\mu(\tau < \zeta) = 0$, by (16.21) in [25]. $\square$

We end this section with a complement to Kunita's lemma, Lemma 2.5.

PROPOSITION 2.10. *Define* $E_a = \{x \in E: Q^x \ll_{\text{loc}} P^x\}$. *Then* E_a *is a Borel set which is absorbing for* Y *and finely open for* X.

PROOF. That E_a is a Borel set follows easily from the fact that each of the σ-algebras F_t° is countably generated. Also, E_a is the set where the Y strongly supermedian function $x \mapsto Q^x(\tau < \zeta)$ vanishes; this implies that E_a is Y-absorbing, as in the proof of Proposition 2.4. Finally, let $T = T_{E_a^c}$ and suppose that $x \in E_a$. Then $1 = Q^x(\Omega) = P^x(L_0)$, so $L_0 = 1$ a.s.-P^x. Using the fact that E_a is Y-absorbing, we have $0 = Q^x(T = 0) = P^x(L_0; T = 0)$, hence $P^x(T > 0) = 1$. It follows that x is in the X-fine interior of E_a, and since $x \in E_a$ was arbitrary, E_a is finely open for X. $\square$

3. Representation of dQ^x/dP^x. Let (X, m, P^x) be a symmetric diffusion as discussed in Section 2. Recall that the Dirichlet space of X is the inner product space $(D(X), E)$ defined by

$$D(X) = \left\{u \in L^2(m) : \sup_{t>0} \frac{1}{t}(u, u - P_t u)_m < \infty\right\};$$

$$E(u, v) = \lim_{t \to 0} \frac{1}{t}(u, v - P_t v)_m, \qquad u, v \in D(X).$$

Endowed with the inner product $E_1(u, v) := E(u, v) + (u, v)_m$, $D(X)$ is a Hilbert space. Each element $u \in D(X)$ admits an m-modification $\tilde{u}$ (a quasi-continuous version of u) such that $t \mapsto \tilde{u}(X_t)$ is continuous on $[0, \infty[$, a.s. P^m.

We then have Fukushima's decomposition [20], Theorem 5.2.2:

$$(3.1)\quad \tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \qquad \forall\ t \ge 0 \quad \text{a.s. } P^x \text{ for q.e. } x \in E,$$

where M^u and N^u are continuous additive functionals (CAF's) of X, M^u is a martingale such that $\sup_{t>0} t^{-1} P^m([M_t^u]^2) < \infty$, and $\lim_{t\to 0} t^{-1} P^m([N_t^u]^2) = 0$. This decomposition is unique, and we refer to M^u and N^u as the martingale and zero energy parts, respectively, of the CAF $A_t^u := \tilde{u}(X_t) - \tilde{u}(X_0)$.

Recall that $\tau(G)$ denotes the first exit time from $G \subset E$; that is, $\tau(G) := T_{G^c} = \inf\{t > 0\colon X_t \notin G\}$. A function $u\colon E \to \mathbb{R}$ is locally in $D(X)$ [notation: $u \in D_{\mathrm{loc}}(X)$] provided there is an increasing sequence $\{G_n\}$ of finely open sets such that $\tau(G_n) \uparrow \zeta$ a.s. P^m, and a sequence $\{u_n\}$ of elements of $D(X)$ such that $u = u_n$ a.e.-m on G_n for each $n \in \mathbb{N}$. [This is not the standard definition for $D_{\mathrm{loc}}(X)$, but it is convenient for our purposes.] Each $u \in D_{\mathrm{loc}}(X)$ admits a quasi-continuous modification $\tilde{u}$ such that (3.1) holds for $t \in [0, \zeta[$, where now M^u and N^u are locally in the classes described in the last paragraph. More precisely, M^u is a local martingale (on $[0, \zeta[$) CAF such that $\sup_{t>0} t^{-1} P^m([M^u_{t\wedge\tau(G_n)}]^2) < \infty$ for each n, and N^u is a CAF such that $\lim_{t\to 0} t^{-1} P^m([N^u_{t\wedge\tau(G_n)}]^2) = 0$ for each n. (See [20], pages 160–161; [21], Appendix; and [39].)

In what follows the term (Y, μ)-*inessential set* will refer to any (Y, μ)-polar subset of E whose complement is absorbing for Y. As noted earlier, from (6.12) in [25], we know that any (Y, μ)-polar set can be enclosed in a Borel (Y, μ)-inessential set.

We can now state the main result. When combined with the results of Sections 4 and 5, it generalizes [21], [37] and [4]. For a terminal time T, we use (X, T) to denote the subprocess derived by killing X at time T. This is a symmetric diffusion with state space $E_T := \{x \in E\colon P^x(T > 0) = 1\}$ and symmetry measure $m|_{E_T}$. In case $T = \tau(G)$, where G is finely open, the Dirichlet form of $(X, \tau(G))$ can be identified with the restriction of E to $D(X, \tau(G)) := \{u \in D(X)\colon \tilde{u} = 0,\ (X, m)\text{-q.e. on } G^c\}$ (cf. [20], Theorem 4.4.2 and [43], Theorem 7.3).

THEOREM 3.2. *Suppose that $Q^\mu \ll_{\mathrm{loc}} P^m$. Then $\mu \ll m$ and there is a (Y, μ)-inessential Borel set $N \subset E$ which is X-finely closed, and a version ρ of the Radon–Nikodym derivative $d\mu/dm$ such that $0 < \rho(x) < \infty$ for all $x \in E \setminus N$, and*:

(a) *$t \mapsto \rho(X_t)$ is continuous on $[0, T_N[$ a.s. P^x for all $x \notin N$; in particular $\rho|_{E\setminus N}$ is (X, T_N)-finely continuous*;

(b) *$l := \frac{1}{2}\log \rho|_{E\setminus N} \in D_{\mathrm{loc}}(X, T_N)$, and we have the Fukushima decomposition*

$$l(X_t) - l(X_0) = M_t^l + N_t^l, \qquad 0 \le t < T_N,\ a.s.\ P^x, \forall\ x \notin N,$$

where M^l is a CAF of (X, T_N) and a local martingale on $[0, T_N[$, and N^l is a CAF of (X, T_N) locally of zero energy;

(c) *For each* $x \in E \setminus N$, $Q^x \ll_{\text{loc}} P^x$ *and the Radon–Nikodym derivative*

$$L_t = dQ^x|_{F_{t+} \cap \{t < \zeta\}} / dP^x|_{F_{t+} \cap \{t < \zeta\}}$$

admits the representation

$$L_t := \exp\left(M_t^l - \tfrac{1}{2}\langle M^l \rangle_t\right) 1_{\{0 \le t < T_N\}} \qquad \forall\ t \ge 0,\ a.s.\ P^x.$$

REMARK 3.3. (a) Using Theorem 3.2(b) and the fact that $0 < \rho < \infty$ on $E \setminus N$, it is easy to show that $\psi := \rho^{1/2} \in D_{\text{loc}}(X, T_N)$.

(b) Let P_N^x denote the law of X started at x and killed at T_N. The strict positivity of L up to time T_N implies that $P_N^x \sim_{\text{loc}} Q^x$ for each $x \notin N$. In particular, $\int_{E \setminus N} P_N^x(\cdot) m(dx) \sim_{\text{loc}} Q^\mu$ since $m|_{E \setminus N} \sim \mu$. Another important consequence of $P_N^x \sim_{\text{loc}} Q^x$ is the fact that the fine topologies of X and Y coincide on $E \setminus N$.

(c) With ψ as in the first remark, one can also express L as

$$L_t = \frac{\psi(X_t)}{\psi(X_0)} \exp(-N_t) 1_{\{t < T_N\}},$$

where $N_t = N_t^l + \frac{1}{2}\langle M^l \rangle_t$ (cf. (2.2) in [38]). Note that $N_t = \int_0^t \psi(X_s)^{-1}\, dN_s^\psi$, the stochastic integral being that introduced by Nakao [34].

For the rest of this section, we fix X and Y such that $Q^\mu \ll_{\text{loc}} P^m$. We will prove Theorem 3.2 by building up the (Y, μ)-inessential set N in several steps. The inessential set produced in a given step will be deleted from the state space in subsequent steps, with no change in notation. For example, Step 1 yields an inessential set N_1; in Step 2, the symbol Y really refers to "Y restricted to $E \setminus N_1$" while X refers to "X killed at the first hitting time of N_1." We leave it to the reader to check that local absolute continuity is preserved by this deletion procedure.

Step 1. By Propositions 2.4 and 2.10, the Borel set N_1, consisting of those points $x \in E$ for which the relation $Q^x \ll_{\text{loc}} P^x$ fails, is (Y, μ)-inessential and X-finely closed. After restricting Y to $E \setminus N_1$ and killing X on its first exit from $E \setminus N_1$, we can, in subsequent steps, assume that $Q^x \ll_{\text{loc}} P^x$ for all x.

Step 2. Let $L = (L_t)$ be the multiplicative functional relating Q^x and P^x as in Lemma 2.5.

LEMMA 3.4. *Let* $S := \inf\{t > 0: L_t = 0\}$. *Then* $Q^x(S < \zeta) = 0$ *for all* $x \in E$, *while* $\{x \in E: P^x(S < \zeta) > 0\}$ *is an* (X, m)*-inessential set.*

PROOF. For any $t > 0$,

$$Q^x(S \le t < \zeta) = P^x(L_t;\ S \le t < \zeta) = 0,$$

since L, being a supermartingale, vanishes after time S. This proves the first assertion. As for the second, S is a terminal time, so $x \mapsto P^x(S < \zeta)$ is a strongly supermedian function of X. Thus, as in the proof of Proposition 2.4, it suffices to prove that $P^m(S < \zeta) = 0$. But $P^x(S > 0) = 1$ for all x, since $P^x(L_0 = 1) = 1$ for all x. Thus, S is a thin (predictable) terminal time of X. The symmetry of X now implies that $P^m(S < \zeta) = 0$ (see [25], (16.21)). $\square$

The set $\{x \in E: P^x(S < \zeta) > 0\}$ is (X, m)-inessential. Using (6.12) in [25], we can find a Borel (X, m)-inessential set $N_2 \supset \{x \in E: P^x(S < \zeta) > 0\}$, and then N_2 is (Y, μ)-inessential by Corollary 2.8 and the fact that $Q^x \ll_{\text{loc}} P^x$ for all x. By deleting N_2 from E, we shall assume in subsequent steps that $(Q^x + P^x)(S < \zeta) = 0$ for all x. In other words, L is strictly positive on $[0, \zeta[$ almost surely for X and for Y.

Step 3. We shall now establish the regularity of an appropriately chosen version of $d\mu/dm$. By Remark 2.7 and Step 2, L is a local martingale on $[0, \zeta[$. Consequently, $t \mapsto L_t$ is continuous on $[0, \zeta[$. Lemma 3.6 is a simple consequence of the following fact ([46], Theorem 2.1): given a path ω with $\zeta(\omega) > t$, we define the reversed (at time t) path $r_t\omega$ by

$$r_t(\omega)(u) = \begin{cases} \omega(t-u), & 0 \le u \le t, \\ \omega(0), & u > t. \end{cases}$$

We then have

$$P^m(F \circ r_t; t < \zeta) = P^m(F; t < \zeta) \qquad \forall\, F \in pF_t,\ t > 0. \tag{3.5}$$

This is a sophisticated (but useful) way to express the symmetry property (2.2). Of course, a similar result holds for Q^μ.

LEMMA 3.6. *Fix* $0 < s \le t$. *Then*

$$\rho(X_0) L_s L_{t-s} \circ r_t = \rho(X_s) L_t \circ r_t \quad a.s.\ P^m \text{ on } \{t < \zeta\}. \tag{3.7}$$

PROOF. We first consider the special case $s = t$:

$$\rho(X_0) L_t = \rho(X_t) L_t \circ r_t \quad \text{a.s. } P^m \text{ on } \{t < \zeta\}. \tag{3.8}$$

Fix $t > 0$, and $F \in pF_t$. Then using (3.5) (first for Q^μ, then for P^m),

$$\begin{aligned} P^m(\rho(X_0) L_t F; t < \zeta) &= Q^\mu(F; t < \zeta) = Q^\mu(F \circ r_t; t < \zeta) \\ &= P^m(\rho(X_0) L_t F \circ r_t; t < \zeta) \\ &= P^m(\rho(X_t) L_t \circ r_t F; t < \zeta). \end{aligned}$$

Varying F, we see that (3.8) holds. Now notice that if $0 < s \le t$, then $r_s\omega(u) = \theta_{t-s} r_t \omega(u)$ for $0 \le u \le s$. Using this, the multiplicative property of L and (3.8), we compute, for $0 < s \le t$,

$$\begin{aligned} \rho(X_0) L_s L_{t-s} \circ r_t &= \rho(X_s) L_s \circ r_s L_{t-s} \circ r_t \\ &= \rho(X_s) L_s \circ \theta_{t-s} \circ r_t L_{t-s} \circ r_t \\ &= \rho(X_s)(L_s \circ \theta_{t-s} L_{t-s}) \circ r_t \\ &= \rho(X_s) L_t \circ r_t, \end{aligned}$$

a.s. P^m on $\{t < \zeta\}$. $\square$

Lemma 3.9. *There is a version of the Radon–Nikodym density* $\rho = d\mu/dm$ *such that* $t \mapsto \rho(X_t)$ *is continuous on* $[0, \zeta[$ *a.s.* P^m. *With this choice of* ρ, *the sets* $\{\rho = 0\}$ *and* $\{\rho > 0\}$ *are invariant in the following strong sense*:

$$P^m\big(\rho(X_0) = 0;\ \rho(X_t) > 0 \text{ for some } 0 < t < \zeta\big) = 0; \tag{3.10}$$

$$P^m\big(\rho(X_0) > 0;\ \rho(X_t) = 0 \text{ for some } 0 < t < \zeta\big) = 0. \tag{3.11}$$

Proof. First note that because of Lemma 3.4 and (3.5),

$$\inf_{0 \le s \le t} L_s L_{t-s} \circ r_t > 0$$

a.s. P^m on $\{t < \zeta\}$, for all $t > 0$. Also, $s \mapsto L_s$ is continuous on $[0, t]$ a.s. P^m on $\{t < \zeta\}$, as is $L_{t-s} \circ r_t$ because of (3.5). It now follows from (3.7) and Fubini's theorem that for P^m-a.e. $\omega \in \Omega$,

$$\rho(X_s(\omega)) = Z_s(\omega) \quad \text{for a.e. } s \in [0, \zeta(\omega)[,$$

for some process Z whose paths are P^m-a.s. continuous on $[0, \zeta[$. The method of essential limits (see, e.g., [19], Section 2) now yields the existence of an m-version of ρ with the stated continuity property. With the continuity of $\rho(X_t)$ in hand, (3.10) and (3.11) follow easily from (3.7). □

By Lemma 3.9 and (6.12) in [25], there is an (X, m)-inessential Borel set N_3 such that ρ is finite-valued and finely continuous on $E \setminus N_3$. Using (3.11) one can now find a (Y, μ)-polar set $N_4 \supset N_3 \cup \{\rho = 0\}$ whose complement is X-absorbing. Since $Q^x \ll_{\text{loc}} P^x$ for all x, N_4 is (Y, μ)-inessential. By deleting N_4 from E we can assume in the sequel that ρ is finite, strictly positive and finely continuous on E.

Step 4. The MF L is strictly positive on $[0, \zeta[$ and is a continuous local martingale on $[0, \zeta[$. Therefore, L admits an exponential representation:

$$L_t = \exp\big(K_t - \tfrac{1}{2}\langle K\rangle_t\big) 1_{\{t < \zeta\}}, \tag{3.12}$$

where $K_t := \int_0^t L_s^{-1}\, dL_s$ is a CAF of X and a local martingale on $[0, \zeta[$, and $\langle K\rangle$ denotes the quadratic variation process of K. In particular, $\langle K\rangle$ is an increasing CAF of X, finite on $[0, \zeta[$. By a result of Walsh ([46], Section 4),

$$\langle K\rangle_s \circ r_t = \langle K\rangle_t - \langle K\rangle_{t-s} \qquad \forall\, s \in [0, t], \text{ a.s. } P^m \text{ on } \{t < \zeta\}. \tag{3.13}$$

Combining these results with (3.7), we arrive at the following proposition.

Proposition 3.14. *Define* $l := \frac{1}{2}\log \rho$. *Then for fixed* $t > 0$,

$$l(X_s) - l(X_0) = \tfrac{1}{2}\big[K_s + K_{t-s} \circ r_t - K_t \circ r_t\big] \qquad \forall\, s \in [0, t],$$

a.s. P^m *on* $\{t < \zeta\}$.

We are now going to show that $l \in D_{\text{loc}}(X)$. For the proof of the next lemma, and for later reference, recall that if A is an increasing CAF of X, the the *Revuz measure* ν_A of A is the measure on E defined by

$$\nu_A(f) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t f(X_s)\, dA_s, \qquad f \in p\,B(E).$$

The measure ν_A is σ-finite and charges no (X, m)-polar set.

LEMMA 3.15. *Let* $f: E \to \mathbb{R}$ *be a Borel function such that* $t \mapsto f(X_t)$ *is continuous on* $[0, \zeta[$ *a.s.* P^m. *Suppose there is a CAF* M *of* X *that is a local martingale on* $[0, \zeta[$ *such that for each* $t > 0$,

$$f(X_t) - f(X_0) = \tfrac{1}{2}[M_t - M_t \circ r_t] \qquad a.s.\ P^m \text{ on } \{t < \zeta\}. \tag{3.16}$$

Then $f \in D_{\text{loc}}(X)$.

REMARK 3.17. It turns out that the local martingale M in (3.16) is M^f, the martingale part of $f(X_\cdot) - f(X_0)$ in Fukushima's decomposition. See Lemma 3.21.

PROOF. The quadratic variation process $\langle M \rangle$ is an increasing CAF of X. Choose a strictly positive Borel function g on E such that $g \le 1$ everywhere and $m(g) < \infty$. Define $\varphi: E \to \mathbb{R}$ by

$$\varphi(x) = P^x \int_0^\infty \exp(-t - \langle M \rangle_t) g(X_t)\, dt, \qquad x \in E.$$

Then $0 < \varphi \le U^1 g \le 1$, φ is finely continuous and by a standard computation

$$\begin{aligned} U^1_{\langle M \rangle} \varphi(x) &:= P^x \int_0^\infty \exp(-t) \varphi(X_t)\, d\langle M \rangle_t \\ &= U^1 g(x) - \varphi(x) \le U^1 g(x) \qquad \forall\, x \in E. \end{aligned}$$

Clearly $U^1 g$ is an element of $D(X)$, hence so is its 1-excessive minorant $U^1_{\langle M \rangle}\varphi$, as is their difference φ. Let ν denote the Revuz measure of $\langle M \rangle$ (relative to X and m), and let $\{g_n\} \subset bp\, B(E)$ be such that $U^1 g_n \uparrow 1$ [e.g., $g_n = n(1 - (n-1)U^n 1)$]. Then by the Revuz formula ([32], IV(4.7)) and (2.2),

$$\left(\varphi, U^1 g_n\right)_\nu = \left(U^1_{\langle M \rangle}\varphi, g_n\right)_m \le \left(U^1 g, g_n\right)_m = \left(g, U^1 g_n\right)_m \le m(g).$$

Thus,

$$\nu(\varphi) = \lim_{n \to \infty} \nu\left(\varphi U^1 g_n\right) \le m(g) < \infty.$$

Define $G_n := \{x \in E: |f(x)| \le n,\ \varphi(x) > 1/n\}$, and note that $\nu(G_n) \le n \cdot \nu(\varphi) < \infty$, while $m(G_n) \le n \cdot m(\varphi) \le n \cdot m(U^1 g) \le n \cdot m(g) < \infty$. Now $\varphi \in D(X)$, so that $\varphi(X_t) \to 0$ as $t \uparrow \zeta$, a.s. P^m on $\{\zeta < \infty\}$. It follows that $\tau(G_n) \to \zeta$ as $n \to \infty$. Define $F_n := \{x \in G_n: U^{G_n} g > 1/n\}$ and $H_n := \{x \in F_n: U^{F_n} g > 1/n\}$. Here, for example, U^{G_n} denotes the 0-potential operator for X killed at time $\tau(G_n)$. Clearly each H_n is finely open and of finite m-measure, and by [39], page 325, $\tau(H_n) \to \zeta$ as $n \to \infty$. We are going to produce functions $f_n \in D(X)$ such that $f = f_n$ on H_n for each $n \in \mathbb{N}$. This will prove that $f \in D_{\text{loc}}(X)$.

Consider the condenser potentials

$$w_n(x) := P^x\left(T_{H_n} < \tau(F_n)\right), \qquad x \in E.$$

Clearly $0 \le w_n \le 1$, $w_n \equiv 1$ on H_n, and w_n is excessive for $(X, \tau(F_n))$. Arguing as in [39], pages 322–324, one can show that $w_n \in D(X, \tau(F_n))$; in particular,

$w_n = 0$, (X, m)-q.e. on F_n^c. Following the argument given in [39], we now show that $f_n := fw_n \in D(X, \tau(G_n)) \subset D(X)$. For this, let P_t^n denote the transition semigroup of $(X, \tau(G_n))$. Since $|f_n| \le n w_n$ on G_n,

$$\begin{aligned}
&\limsup_{t\to 0} \frac{1}{t}\int_{G_n} f_n(x)\left[f_n(x) - P_t^n f_n(x)\right] m(dx) \\
&\quad\le \limsup_{t\to 0} \frac{1}{2t}\int_{G_n}\int_{G_n}\left[f_n(y) - f_n(x)\right]^2 m(dx)\, P_t^n(x, dy) \\
&\qquad + \limsup_{t\to 0} \frac{1}{t}\int_{G_n} f_n(x)^2\left(1 - P_t^n 1(x)\right) m(dx) \\
&\quad\le \limsup_{t\to 0} \frac{1}{t}\int_{G_n}\int_{G_n}\left[f(y) - f(x)\right]^2 m(dx)\, P_t^n(x, dy) \\
&\qquad + \limsup_{t\to 0} \frac{n^2}{t}\int_{G_n}\int_{G_n}\left[w_n(y) - w_n(x)\right]^2 m(dx)\, P_t^n(x, dy) \\
&\qquad + \limsup_{t\to 0} \frac{2n^2}{t}\int_{G_n} w_n(x)^2\left(1 - P_t^n 1(x)\right) m(dx) \\
&\quad\le \limsup_{t\to 0} \frac{1}{t}\int_{G_n}\int_{G_n}\left[f(y) - f(x)\right]^2 m(dx)\, P_t^n(x, dy) \\
&\qquad + \limsup_{t\to 0} \frac{2n^2}{t}\int_{G_n} w_n(x)\left[w_n(x) - P_t^n w_n(x)\right] m(dx) \\
&\quad= \limsup_{t\to 0}\left[I_1(t) + I_2(t)\right].
\end{aligned}$$

Now $\limsup_{t\to 0} I_2(t) < \infty$, since $w_n \in D(X, \tau(F_n)) \subset D(X, \tau(G_n))$. On the other hand, by Proposition 3.14,

$$|f(X_t) - f(X_0)|^2 \le \left[M_t^* + M_t^* \circ r_t\right]^2 \le 2\left[(M_t^*)^2 + (M_t^* \circ r_t)^2\right],$$

on $\{t < \zeta\}$, where $M_t^* := \sup_{0\le s\le t}|M_s|$. Using this observation, Doob's inequality and (3.5) [applied with X replaced by $(X, \tau(G_n))$], we may estimate

$$\begin{aligned}
I_1(t) &= \frac{1}{t}\int_{G_n} P^x\left(\left[f(X_t) - f(X_0)\right]^2;\ t < \tau(G_n)\right) m(dx) \\
&\le \frac{2}{t}\int_{G_n} P^x\left(\left[M_t^*\right]^2;\ t < \tau(G_n)\right) m(dx) \\
&\quad + \frac{2}{t}\int_{G_n} P^x\left(\left[M_t^* \circ r_t\right]^2;\ t < \tau(G_n)\right) m(dx)
\end{aligned}$$

$$= \frac{4}{t}\int_{G_n} P^x\Big([M_t^*]^2;\, t < \tau(G_n)\Big)\, m(dx)$$
$$\le \frac{4}{t}\int_{G_n} P^x\Big(\big[M_{t\wedge\tau(G_n)}^*\big]^2\Big)\, m(dx)$$
$$\le \frac{16}{t}\int_{G_n} P^x\big(\langle M\rangle_{t\wedge\tau(G_n)}\big)\, m(dx)$$
$$\le \frac{16}{t}\int_{G_n} P^x\left(\int_0^t 1_{G_n}(X_s)\, d\langle M\rangle_s\right) m(dx)$$
$$\le 16\nu(G_n) < \infty.$$

Thus $\limsup_{t\to 0} I_1(t) < \infty$ as well. It follows that $f_n \in D(X, \tau(G_n)) \subset D(X)$. Since $f = fw_n = f_n$ on H_n, $f \in D_{\mathrm{loc}}(X)$ as claimed. □

Step 5. Our final task in finishing the proof of Theorem 3.2 is to identify the local martingale K appearing in (3.12). By Proposition 3.14 and Lemma 3.15, $l \in D_{\mathrm{loc}}(X)$, so we can apply Fukushima's decomposition: there is a CAF M^l which is a local martingale on $[0, \zeta[$, and a CAF locally of zero energy N^l such that

(3.18) $$l(X_t) - l(X_0) = M_t^l + N_t^l \qquad \forall\ t \in [0, \zeta[, \quad \text{a.s. } P^m.$$

Now by a slight extension of the forward–backward martingale decomposition of Lyons and Zheng ([31], 1.5), we have

(3.19) $$l(X_t) - l(X_0) = \tfrac{1}{2}\big[M_t^l - M_t^l \circ r_t\big] \quad \text{a.s. } P^m \text{ on } \{t < \zeta\},$$

for all $t > 0$. Formula (3.19) follows from Lemma 3.21. Combining Proposition 3.14 and (3.19) we find that

(3.20) $$M_t^l - M_t^l \circ r_t = K_t - K_t \circ r_t \quad \text{a.s. } P^m \text{ on } \{t < \zeta\}, \forall\ t > 0.$$

As we shall see, (3.20) implies that $K = M^l$, which will establish the desired representation for L.

Given a CAF A of X, we define the even and odd parts of A by the formulas

$$A_t^{\mathrm{even}} = [A_t + A_t \circ r_t]/2, \qquad A_t^{\mathrm{odd}} = [A_t - A_t \circ r_t]/2, \qquad t < \zeta.$$

Both A^{even} and A^{odd} are additive functionals, and $A = A^{\mathrm{even}} + A^{\mathrm{odd}}$ on $[0, \zeta[$. Let us say that A is even (resp. odd) provided $A_t = A_t^{\mathrm{even}}$ (resp. $A_t = A_t^{\mathrm{odd}}$) a.s. P^m on $\{t < \zeta\}$ for each $t > 0$. In view of the lemma to follow, if $u \in D_{\mathrm{loc}}(X)$, then the even part of M^u is $-N^u$ while the odd part is $\tilde{u}(X) - \tilde{u}(X_0)$.

LEMMA 3.21. *If $u \in D_{\mathrm{loc}}(X)$, then the zero-energy part N^u is an even CAF.*

PROOF. Suppose first that $u \in D(X)$. Then by the proof of Theorem 5.2.2 in [20], there is a sequence of Borel functions g_n such that $N_t^{(n)} := \int_0^t g_n(X_s)\, ds$ converges to N_t^u uniformly on compact t-intervals, a.s. P^m. It is easy to check that each $N^{(n)}$ is even, hence so is the limit N^u because of (3.5).

Now assume $u \in D_{\mathrm{loc}}(X)$. Then there is an increasing sequence $\{G_n\}$ of finely open sets with $\tau(G_n) \to \zeta$ a.s. P^m, and a sequence $\{u_n\} \subset D(X)$ such that $u = u_n$ m-a.e. on G_n for all n. Since X is a diffusion, we have by [20], Lemma 5.4.6 that $N^u = N^{u_n}$ on $[0, \tau(G_n)[$ a.s. P^m. But N^{u_n} is even, so

$$(3.22) \qquad N_t^u \circ r_t = N_t^{u_n} \circ r_t = N_t^{u_n} = N_t^u \quad \text{a.s. } P^m \text{ on } \{X_0 \in G_n,\, t < \tau(G_n)\}.$$

We have used here the fact that $P^m(r_t^{-1}[X_0 \in G_n,\ t < \tau(G_n)]\Delta[X_0 \in G_n, t < \tau(G_n)]) = 0$ because of (3.5). Since $\tau(G_n) \to \zeta$ a.s. P^m, we can let $n \to \infty$ in (3.22) to conclude that N^u is even. □

PROPOSITION 3.23. *Let A and B be CAF's of X and semimartingales on $[0, \zeta[$. Suppose that A is even and B is odd. Then the covariation process $\langle A, B \rangle$ vanishes identically on $[0, \zeta[$.*

PROOF. Fix $t > 0$ and choose a sequence of partitions $\pi_n = \{0 = t(1, n) < t(2, n) < \cdots < t(k(n), n) = t\}$ with mesh sizes tending to zero, such that

$$(3.24) \qquad \langle A, B \rangle_t = \lim_n \sum_{i=1}^{k(n)-1} \left[A_{t(i+1,n)} - A_{t(i,n)}\right]\left[B_{t(i+1,n)} - B_{t(i,n)}\right] \qquad \text{a.s. } P^m \text{ on } \{t < \zeta\}.$$

In view of VIII.20 in [11], we can (and do) assume that each π_n is symmetric with respect to $[0, t]$, in the sense that for each i there exists i^* such that $t - t(i, n) = t(i^*, n)$. By the result of Walsh mentioned earlier, $\langle A, B \rangle_t \circ r_t = \langle A, B \rangle_t$ a.s. P^m on $\{t < \zeta\}$. (Any CAF of finite variation can be expressed as the difference of increasing CAF's.) But since A is even and $r_t\omega(u) = r_s \circ \theta_{t-s}\omega(u)$ for $u \in [0, s]$, $A_s \circ r_t = A_t - A_{t-s}$ if $0 < s < t$; since B is odd, $B_s \circ r_t = B_{t-s} - B_t$. Thus,

$$\begin{aligned}
&\left(\left[A_{t(i+1,n)} - A_{t(i,n)}\right]\left[B_{t(i+1,n)} - B_{t(i,n)}\right]\right) \circ r_t \\
&\qquad = -\left[A_{t-t(i,n)} - A_{t-t(i+1,n)}\right]\left[B_{t-t(i,n)} - B_{t-t(i+1,n)}\right],
\end{aligned}$$

so (3.24) and the symmetry of each π_n imply that $\langle A, B \rangle_t = \langle A, B \rangle_t \circ r_t = -\langle A, B \rangle_t$ a.s. P^m on $\{t < \zeta\}$. Thus $\langle A, B \rangle_t = 0$ a.s. P^m on $\{t < \zeta\}$. Varying t and invoking the continuity of $\langle A, B \rangle$, we conclude that $\langle A, B \rangle$ vanishes on $[0, \zeta[$ a.s. P^m. □

COROLLARY 3.25. *If M is an even CAF of X and a local martingale on $[0, \zeta[$, then $M \equiv 0$.*

PROOF. Let $(A, D(A))$ denote the $L^2(m)$ infinitesimal generator of (P_t). Given $u \in D(A) \subset D(X)$, let $M^u := \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t Au(X_s)\,ds$ be the martingale part of $A^u := \tilde{u}(X_\cdot) - \tilde{u}(X_0)$. Then $M^u = A^u - N^u$, where A^u is odd and $N_t^u := -\int_0^t Au(X_s)\,ds$ is even. By Proposition 3.23,

$$0 = \langle M, A^u \rangle = \langle M, M^u \rangle + \langle M, N^u \rangle = \langle M, M^u \rangle,$$

because N^u is of locally finite variation. Thus, $\langle M, M^u\rangle = 0$ on $[0,\zeta[$, for all $u \in D(A)$. Let us take $u = U^\alpha f$, where f is a bounded positive element of $L^1(m)$. Then $Au = \alpha u - f$, and using Itô's formula one can verify that

$$Z_t := \int_0^t e^{-\alpha s}\, dM_s^u = e^{-\alpha t}\tilde{u}(X_t) - \tilde{u}(X_0) + \int_0^t e^{-\alpha s} f(X_s)\, ds.$$

Clearly Z is a P^x-martingale for all x, and $\langle M, Z\rangle_t = \int_0^t e^{-\alpha s}\langle M, M^u\rangle_s = 0$. Consider now the local martingale multiplicative functional

$$J_t := \exp\big(M_t - \tfrac{1}{2}\langle M\rangle_t\big)1_{\{t<\zeta\}},$$

and the associated path-space law P_J^x whose Radon–Nikodym density process with respect to P^x is J. By Girsanov's theorem ([11], VII.49), Z is a P_J^x-local martingale on $[0,\zeta[$. In fact, Z is a true P_J^x-martingale, being uniformly bounded on compact time intervals. Consequently, $P_J^x(Z_t) = 0$, whence (letting $t \to \infty$),

$$P_J^x\left(\int_0^\infty e^{-\alpha s} f(X_s)\, ds\right) = U^\alpha f(x), \qquad \text{q.e. } x \in E,$$

for all $\alpha > 0$ and all f as specified. From this and separability considerations, it follows that the L^2-resolvent associated with $\{P_J^x\colon x \in E\}$ coincides with $(U^\alpha)_{\alpha>0}$. Thus, $P_J^x = P^x$ for q.e. $x \in E$, hence $J \equiv 1$ on $[0,\zeta[$ a.s. P^x for q.e. x. Therefore $M \equiv \frac{1}{2}\langle M\rangle$, and so $M \equiv 0$ a.s. P^x for q.e. x, since a continuous local martingale of finite variation is constant. □

In view of (3.20), $M^l - K$ is an even CAF and a local martingale. By Corollary 3.25, $M^l = K$. This finishes the proof of Theorem 3.2.

REMARK 3.26. Using the main result of [39], one can strengthen Lemma 3.21 as follows: any CAF of X locally of zero energy is even. It therefore follows from Corollary 3.25 that in the class of CAF's decomposable as the sum of a local martingale CAF and a CAF locally of zero energy, the class of even CAF's coincides with the class of CAF's locally of zero energy. The situation for odd martingale CAF's is more involved. Consider, for example, the case in which X is Brownian motion in $\mathbb{R}^d$. For a smooth vector field F, the odd part of the stochastic integral $Z_t := \int_0^t F(X_s)\cdot dX_s$ is $Z_t + \frac{1}{2}\int_0^t \operatorname{div} F(X_s)\, ds$, which is precisely the Stratonovich integral of $F(X_\cdot)$ with respect to X. In particular, Z is an odd martingale CAF of X if and only if $\operatorname{div} F = 0$. In general, the odd part of a local martingale CAF can be computed using the stochastic integral of Nakao [34]. These matters are developed in more detail in [17].

4. Absolute continuity and Dirichlet forms. Let (X, m, P^x) and (Y, μ, Q^x) be symmetric diffusions such that $Q^\mu \ll_{\mathrm{loc}} P^m$. In this section we shall study the Dirichlet form of Y as it relates to that of X. The results presented here extend work found in [23], [33], [37], [38], [4] and [13].

To simplify the exposition, we assume that the (Y, μ)-inessential set N of Theorem 3.2 has been deleted from E (and X killed at T_N). Thus, the density $\rho = d\mu/dm$ has the properties listed in Theorem 3.2 on all of E. We shall write ψ for $\rho^{1/2}$. As remarked in Section 3, $\psi \in D_{\mathrm{loc}}(X)$. Having deleted N, we have $Q^x \sim_{\mathrm{loc}} P^x$ for all x. In particular, the fine topologies of X and Y coincide, and every (X, m)-nest of compacts is a (Y, μ)-nest, and vice versa. Consequently, the (X, m)-quasi-continuous version of a function is (Y, μ)-quasi-continuous, and vice versa.

Given $u \in D_{\mathrm{loc}}(X)$, let M^u be the (local) martingale part of $\tilde{u}(X_t) - \tilde{u}(X_0)$. The quadratic variation process $\langle M^u \rangle$ is an increasing CAF of X, finite on $[0,\zeta[$ in general, and finite on $[0,\infty[$ if $u \in D(X)$. We write $\Gamma(u)$ for the associated Revuz measure. By [20], Theorem 5.2.2,

$$E(u, u) = \tfrac{1}{2}\Gamma(u)(E) \qquad \forall\, u \in D(X). \tag{4.1}$$

Since X is a diffusion, the form E is (strongly) local ([20], Section 4.5). For our purposes, this property is best expressed as follows: if $u \in D_{\mathrm{loc}}(X)$ and f: $\mathbb{R} \to \mathbb{R}$ is C^1 on an open set containing the range of u, then $F \circ u \in D_{\mathrm{loc}}(X)$ and

$$M_t^{F \circ u} = \int_0^t F'(\tilde{u}(X_s))\, dM_s^u, \qquad 0 \le t < \zeta, \text{ a.s. } P^m. \tag{4.2}$$

(See V.1 in [32], and I.5 in [6], for discussions of this and equivalent conditions.)

The Dirichlet space of Y is denoted $(D(Y), \hat{E})$. If $u \in D_{\mathrm{loc}}(Y)$, then $\hat{\Gamma}(u)$ denotes the Revuz measure of $\langle \hat{M}^u \rangle$, where $\hat{M}^u$ is the martingale part of $\tilde{u}(Y_t) - \tilde{u}(Y_0)$. Of course, the Dirichlet form of Y is also local.

Here is the key result of this section.

THEOREM 4.3. *Let G be a finely open subset of E on which ρ is bounded away from zero and infinity. Furthermore, assume $\Gamma(\psi)(G) < \infty$. Then $D(Y, \tau(G)) = D(X, \tau(G))$, and if $u \in D(X, \tau(G))$ then $\hat{\Gamma}(u) = \rho\Gamma(u)$.*

Before proceeding to the proof of Theorem 4.3, we record two lemmas.

LEMMA 4.4. *Let B be an increasing CAF of X which is finite on $[0, \zeta[$. Let ν (resp. $\hat{\nu}$) be the Revuz measure of B computed with respect to (X, m) [resp. (Y,μ)]. Then $\hat{\nu} = \rho\nu$.*

PROOF. If B had the special form $B_t = \int_0^t b(X_s)\, ds$, then the conclusion would follow immediately. For in this case we would have $B_t = \int_0^t b(Y_s)\, ds$ a.s. Q^μ, and then $\nu = bm$ and $\hat{\nu} = b\mu$, so that $\hat{\nu} = b\mu = b\rho m = \rho b m = \rho\nu$. Our argument consists of showing how to reduce the general case to this special situation through time change.

Define a strictly increasing CAF H by $H_t = B_t + t \wedge \zeta$. The Revuz measure of H computed relative to (X, m) [resp. relative to (Y, μ)] is $\overline{m} = \nu + m$ (resp. $\overline{\mu} = \hat{\nu} + \mu$). Now consider the time changed processes $\overline{X}_t = X_{H^{-1}(t)}$ and

$\overline{Y}_t = Y_{H^{-1}(t)}$. These are symmetric diffusions, with symmetry measures $\overline{m}$ and $\overline{\mu}$, respectively. (See [20], Section 5.5, and [15].) The additive functional B time changes to $\overline{B}_t = B_{H^{-1}(t)}$, which by (6.2) in [18] has Revuz measure ν relative to $(\overline{X}, \overline{m})$ [resp. $\hat{\nu}$ relative to $(\overline{Y}, \overline{\mu})$]. But now $d\overline{B}_t \le dt$ since $dB_t \le dH_t$. By Motoo's theorem ([42], (66.2)), there is a bounded positive Borel function b such that $\overline{B}_t = \int_0^t b(\overline{X}_s)\,ds$ for all $t \ge 0$, a.s. P^m. By the discussion in the first paragraph of the proof, $\hat{\nu} = \overline{\rho}\nu$, where $\overline{\rho}$ is the precise version of $d\overline{\mu}/d\overline{m}$ resulting from an application of Theorem 3.2 to $\overline{X}$ and $\overline{Y}$. [To see that $\overline{Q}^x \ll_{\mathrm{loc}} \overline{P}^x$ for all x, use Lemma 2.5 with $T = H^{-1}(t)$, $t > 0$.] We are now going to show that $\hat{\rho} = \rho$, (X, m)-q.e. Since neither ν nor $\hat{\nu}$ charges (X, m)-polars, the equality $\hat{\nu} = \overline{\rho}\nu$ will then imply $\hat{\nu} = \rho\nu$, as desired.

To see that $\overline{\rho} = \rho$, (X, m)-q.e., note that from

$$\mu + \hat{\nu} = \overline{\mu} = \overline{\rho}\,\overline{m} = \overline{\rho}(m + \nu) = \overline{\rho}m + \hat{\nu}$$

we can deduce that $\mu = \overline{\rho}m$. But $\mu = \rho m$, so $\overline{\rho} = \rho$ a.e. m. Since both $\overline{\rho}$ and ρ are finely continuous (and the fine topology is invariant under strictly increasing time change) we have $\overline{\rho} = \rho$ outside an (X, m)-polar set. □

LEMMA 4.5. *Let G be a finely open subset of E such that $\Gamma(\psi)(G) < \infty$. If $u \in b\,D(X) \cap L^2(\mu)$, $\hat{\Gamma}(u)(E) < \infty$ and $\tilde{u} = 0$, (Y, μ)-q.e. on G^c, then $u \in D(Y)$.*

PROOF. Using Girsanov's theorem and Lemma 3.15, one can show that $D(X) \subset D_{\mathrm{loc}}(Y)$; see the discussion around (4.6) for more detail on this point. In particular, the hypothesis "$\hat{\Gamma}(u)(E) < \infty$" is meaningful. Fix u as in the statement of the lemma. Then $u \in D_{\mathrm{loc}}(Y) \cap L^2(\mu)$ and $\hat{\Gamma}(u)(E) < \infty$, so that u is an element of the reflected Dirichlet space associated with Y ([43], Section 14). By [43], Theorem 14.4, or [7], Theorem 3.3 (applied to the 1-subprocess of Y), $u = u_0 + h$ where $u_0 \in D(Y)$, $h \in D_{\mathrm{loc}}(Y)$ is 1-harmonic, and $\hat{E}_1(u, u) = \hat{E}_1(u_0, u_0) + \hat{E}_1(h, h)$. Thus $\lim_{t\uparrow\zeta} \tilde{u}_0(Y_t) = 0$ a.s. Q^μ on $\{\zeta < \infty\}$, while $H_t := e^{-t}h(Y_t)$ is a continuous local martingale on $[0, \zeta[$ such that $Q^\mu(\langle H\rangle_t) \le 2t\hat{E}_1(h, h) < \infty$ for all $t > 0$. It follows that $\langle H\rangle_t < \infty$ for all $t > 0$, a.s. Q^μ, and in particular that $\lim_{t\uparrow\zeta} h(Y_t) = e^\zeta \lim_{t\to\zeta} H_t$ exists a.s. Q^μ on $\{\zeta < \infty\}$. Thus, $\lim_{t\uparrow\zeta} \tilde{u}(Y_t)$ exists a.s. Q^μ on $\{\zeta < \infty\}$. Consequently $h = 0$ [in which case $u = u_0 \in D(Y)$] if and only if

$$\lim_{t\uparrow\zeta} \tilde{u}(Y_t) = 0 \quad \text{a.s. } Q^\mu \text{ on } \{\zeta < \infty\},$$

by [43], Theorem 14.5.

Observe that because of the local property (4.2),

$$\langle M^l\rangle_t = \int_0^t \psi(X_s)^{-2}\,d\langle M^\psi\rangle_s$$

for all $t > 0$ a.s. P^m, since $l = \log \psi$. But $Q^\mu \ll_{\mathrm{loc}} P^m$, so $\langle M^l \rangle_t = \int_0^t \psi(Y_s)^{-2} \, d\langle M^\psi \rangle_s$ for all $t > 0$ a.s. Q^μ as well. Thus,

$$Q^\mu \int_0^t 1_G(Y_s) \, d\langle M^l \rangle_s = Q^\mu \int_0^t 1_G(Y_s) \psi(Y_s)^{-2} \, d\langle M^\psi \rangle_s$$

$$\leq t \int_G \psi(x)^{-2} \rho(x) \Gamma(\psi)(dx) = t\Gamma(\psi)(G) < \infty,$$

because by Lemma (4.4) the Revuz measure of $\langle M^\psi \rangle$, when computed relative to (Y, μ), is $\rho\Gamma(\psi)$. Thus, $\int_0^t 1_G(Y_s) d\langle M^l \rangle_s < \infty$ a.s. Q^μ for all $t > 0$. Fix $f \in bL^1(m)$ with $f > 0$ and set $v := U^1 f$. Then v is a strictly positive element of $D(X)$. Consequently, if $H_n := \{|\tilde{u}| + \tilde{v} > 1/n\}$ then the sequence of exit times $\tau(H_n)$, $n \geq 1$, announces ζ with respect to P^m. An argument in [33], page 15, now shows that Q^μ-a.s. on $\{\zeta < \infty\}$, either $\tau(H_n) < \zeta$ for all n or $\langle M^l \rangle_t \to \infty$ as $t \uparrow \zeta$. In the first case, $\lim_{t \uparrow \zeta} [|\tilde{u}(Y_t)| + \tilde{v}(Y_t)] = 0$. On the other hand, the finiteness of $\int_0^t 1_G(Y_s) d\langle M^l \rangle_s$ forces $\liminf_{t \uparrow \zeta} 1_G(Y_t) = 0$ a.s. Q^μ on $\{\zeta < \infty, \langle M^l \rangle_{\zeta-} = \infty\}$. But $\tilde{u}$ vanishes (Y, μ)-q.e. on G^c, so $\lim_{t \uparrow \zeta} |\tilde{u}(Y_t)| = 0$ on $\{\zeta < \infty, \langle M^l \rangle_{\zeta-} = \infty\}$ as well. $\square$

Proof of Theorem 4.3. Fix $u \in b\,D(X, \tau(G))$ and let M^u be the associated martingale CAF. Recall from Section 3 that the Radon–Nikodym density for the law of Y relative to that of X is $L_t = \exp(M_t^l - \frac{1}{2}\langle M^l \rangle_t) 1_{\{t < \zeta\}}$. Thus, by Girsanov's theorem ([11], VII.49, VIII.20), $M := M^u - \langle M^u, M^l \rangle$ is a local martingale CAF of Y on $[0, \zeta[$. Moreover, $\langle M \rangle = \langle M^u \rangle$ a.s. Q^μ. By Lemma 4.4, the Revuz measure of $\langle M \rangle$ [viewed as an increasing CAF of (Y, μ)] is $\rho\Gamma(u)$. By Lemma 3.21,

$$\tilde{u}(Y_t) - \tilde{u}(Y_0) = \tfrac{1}{2}[M_t - M_t \circ r_t] \quad \text{a.s. } Q^\mu \text{ on } \{t < \zeta\}, \tag{4.6}$$

whence $u \in D_{\mathrm{loc}}(Y)$ because of Lemma 3.15. In view of Lemma 3.21, $M = \hat{M}^u$, the martingale part of $\tilde{u}(Y_t) - \tilde{u}(Y_0)$. Now Lemma 4.4 implies that $\hat{\Gamma}(u)$, the Revuz measure of $\langle \hat{M}^u \rangle$, is just $\rho\Gamma(u)$. In particular,

$$Q^\mu\Big([\tilde{u}(Y_t) - \tilde{u}(Y_0)]^2; t < \zeta\Big) \leq CQ^\mu\big(\langle \hat{M}^u \rangle_t\big) \leq Ct \int_E \rho \, d\Gamma(u). \tag{4.7}$$

We claim that $\int_E \rho \, d\Gamma(u) < \infty$. Since ρ is bounded above on G (by b, say), this will follow once we show that $\Gamma(u)$ is carried by the X-fine closure $\overline{G}^f$ of G. Indeed, ρ is finely continuous, hence bounded above by b on $\overline{G}^f$, so we will have

$$\int_E \rho \, d\Gamma(u) = \int_{\overline{G}^f} \rho \, d\Gamma(u) \leq b \int_E d\Gamma(u) = 2bE(u, u) < \infty. \tag{4.8}$$

To see that $\Gamma(u)$ is carried by $\overline{G}^f$, note that since $u \in D(X, \tau(G))$, the martingale M^u is constant on the excursions of X from $\overline{G}^f$. But a continuous martingale has the same intervals of constancy as its quadratic variation process. Thus, $\int_0^t 1_{E \setminus \overline{G}^f}(X_s) d\langle M^u \rangle_s = 0$ for all $t > 0$, a.s. P^m. Since $\Gamma(u)$ is the Revuz measure of $\langle M^u \rangle$, the assertion follows. Lemma 4.5 now tells us that $u \in D(Y)$ since it is clear from the hypotheses that $u \in L^2(\mu)$. We have shown that $b\,D(X, \tau(G)) \subset D(Y, \tau(G))$ and that $\hat{\Gamma}(u) = \rho\Gamma(u)$ for all $u \in b\,D(X, \tau(G))$. The boundedness assumption is easily removed by a truncation argument, and the reverse inclusion follows upon switching the roles of X and Y—notice that $\psi^{-1} \in D_{\mathrm{loc}}(Y)$ and [by (4.2)]

$$\int_G d\hat{\Gamma}(\psi^{-1}) = \int_G \rho^{-1}\, d\Gamma(\psi) < \infty. \qquad \square$$

THEOREM 4.9. *Let $\{G_n\} \subset E$ be an increasing sequence of finely open sets such that* (i) $E \setminus \bigcup_n G_n$ *is (X, m)-polar,* (ii) $\Gamma(\psi)(G_n) < \infty$ *for all $n \in \mathbb{N}$ and* (iii) *there are constants $0 < C_n < \infty$ such that $C_n^{-1} \le \rho \le C_n$ on G_n, $\forall\, n$. Then $\bigcup_n D(X, \tau(G_n))$ is $\hat{E}_1$-dense in $D(Y)$, and*

$$(4.10) \qquad \hat{E}(u, u) = \tfrac{1}{2}\int_E \rho(x)\Gamma(u)\,(dx) \qquad \forall\, u \in \bigcup_n D(X, \tau(G_n)).$$

REMARK. The existence of such a sequence $\{G_n\}$ follows easily since ψ is strictly positive, finite-valued, finely continuous and an element of $D_{\mathrm{loc}}(X)$.

PROOF. Both (4.10) and the inclusion $\bigcup_n D(X, \tau(G_n)) \subset D(Y)$ follow from Theorem 4.3. From the (X, m)-polarity of $E \setminus \bigcup_n G_n$, we deduce that $P^m(\lim_n \tau(G_n) < \zeta) = 0$. Using [3], Theorem 2.2, we can choose compacts $K_n \subset G_n$, $n \in \mathbb{N}$, such that $P^m(f(X_0)[\exp(-\tau(K_n)) - \exp(-\tau(G_n))]) < 1/n$, where $f > 0$ is a fixed bounded element of $L^1(m)$. Replacing K_n by $K_1 \cup K_2 \cup \cdots \cup K_n$, we can (and do) assume that $\{K_n\}$ is an increasing sequence. It then follows easily that $P^m(\lim_n \tau(K_n) < \zeta) = 0$. Thus, $\{K_n\}$ is an (X, m)-nest. Because of Corollary 2.8, $\{K_n\}$ is also a (Y, μ)-nest. The characterization ([32], III.2.1, IV.4.5) of nests now implies that $\bigcup_n D(Y, \tau(K_n))$ is $\hat{E}_1$-dense in $D(Y)$; a fortiori, $\bigcup_n D(Y, \tau(G_n))$ is $\hat{E}_1$-dense in $D(Y)$. $\square$

A variant of the above argument yields the following result, which will be useful in the next section.

THEOREM 4.11. $D_{\mathrm{loc}}(Y) = D_{\mathrm{loc}}(X)$, *and* $\hat{\Gamma}(u) = \rho\Gamma(u)$ *for all* $u \in D_{\mathrm{loc}}(X)$.

COROLLARY 4.12. *Suppose that $Q^\mu(\zeta < \infty) = 0$. Then $D(Y) = \{u \in D_{\mathrm{loc}}(X)\colon \int \rho\, d\Gamma(u) < \infty\} \cap L^2(\mu)$ and $\hat{E}(u, u) = \frac{1}{2}\int_E \rho\, d\Gamma(u)$ for all $u \in D(Y)$.*

PROOF. Since the lifetime of Y is assumed to be infinite, an element u of $L^2(\mu)$ lies in $D(Y)$ if and only if $\lim_{t\downarrow 0} t^{-1} Q^\mu([u(Y_t) - u(Y_0)]^2)$ is finite. In view of the estimate (4.7), this follows for $u \in D_{\mathrm{loc}}(X)$ provided $\int_E \rho\, d\Gamma(u) < \infty$. The assertion thus follows from Theorem 4.11, as does the evaluation of $\hat{E}(u, u)$. $\square$

A simple sufficient condition ensuring $Q^{\mu}(\zeta < \infty) = 0$ is $\psi \in D(X)$; if X has infinite lifetime, this condition can be weakened to $\Gamma(\psi)(E) < \infty$. See [33] and also [38], Section 4.

5. A converse to Theorem 3.2. One can use the results of Sections 3 and 4, in combination with Girsanov's theorem, to provide sufficient conditions for the relation $Q^{\mu} \ll_{\mathrm{loc}} P^{m}$. We describe two such results in this section. The first of these, Theorem 5.2, when combined with Theorems 3.2 and 4.11, yields an abstract form of [21], Theorem 2, and [37]. The second, Theorem 5.19, extends work found in [4].

Let (X, m, P^{x}) and (Y, μ, Q^{x}) be symmetric diffusions with respective Dirichlet spaces $(D(X), E)$ and $(D(Y), \hat{E})$. Throughout this section we shall impose the following supplementary condition:

(5.1) $\mu \ll m$; $\psi := \sqrt{d\mu/dm} \in D(X)$; there is a vector space $C \subset D(X) \cap D(Y)$ which is E_1-dense in $D(X)$ and $\hat{E}_1$-dense in $D(Y)$ such that $\hat{\Gamma}(u) = \tilde{\psi}^2 \Gamma(u)$ for all $u \in C$.

Here, $\tilde{\psi}$ denotes an (X, m)-quasi-continuous version of ψ. In the context of Theorem 2 of [21], an appropriate choice for C is $C_0^{\infty}(\mathbb{R}^d)$, the space of smooth functions $f: \mathbb{R}^d \to \mathbb{R}$ with compact support. In the infinite-dimensional context studied in [4], one can take C to be FC_b^{∞}, as defined in that paper. Note that because of Theorem 4.9, the conditions listed in (5.1), with the exception of $\psi \in D(X)$, are consequences of the relation $Q^{\mu} \ll_{\mathrm{loc}} P^{m}$.

THEOREM 5.2. *In addition to condition* (5.1), *assume that*

(5.3) $Q^{\mu}(\zeta < \infty) = 0$, *and every* (X, m)-*nest of compacts is a* (Y, μ)-*nest*.

Under these conditions $Q^{\mu} \ll_{\mathrm{loc}} P^{m}$.

REMARK 5.4. Condition (5.3) is easy to verify if ψ is bounded; see, for example, [13].

As preparation for the proof of Theorem 5.2, we require the following lemma. Given $u, v \in D(X)$, let M^{u} and M^{v} be the corresponding martingale parts. The covariation process $\langle M^{u}, M^{v} \rangle$ is a CAF of X of locally finite variation. Let $\Gamma(u, v)$ denote the corresponding (signed) Revuz measure. Notice that by polarizing (4.1) we obtain

$$E(u, v) = \tfrac{1}{2}\Gamma(u, v)(E). \tag{5.5}$$

See I.4 in [6] for a version of the following result under more restrictive conditions; see also [34], Lemma 2.3. If ν is a signed measure, the associated total variation measure is denoted $|\nu|$.

LEMMA 5.6. *If u and v are elements of $D(X)$, then*

$$|\Gamma(u)-\Gamma(v)|(E) \le E(u-v,u-v)^{1/2}\left[E(u,u)^{1/2}+E(v,v)^{1/2}\right]. \tag{5.7}$$

In particular, if $u_n \to u$ in E_1-norm, then $\Gamma(u_n)$ converges to $\Gamma(u)$ in total variation norm.

PROOF. By the Kunita–Watanabe inequality ([11], VII.54), if $f \in bp\,B(E)$,

$$\left[\int_0^t f(X_s)|d\langle M^u, M^v\rangle|\right]^2 \le \left[\int_0^t f(X_s)\,d\langle M^u\rangle\right]\left[\int_0^t f(X_s)\,d\langle M^v\rangle\right], \tag{5.8}$$

a.s. P^m for all $t>0$. Consequently, $\Gamma(u,v)$ being the Revuz measure of $\langle M^u, M^v\rangle$, we have

$$\begin{aligned}|\Gamma(u,v)|(f) &= \lim_{t\to 0} t^{-1}P^m\int_0^t f(X_s)|d\langle M^u, M^v\rangle|\\ &\le \lim_{t\to 0} t^{-1}P^m\left(\left[\int_0^t f(X_s)\,d\langle M^u\rangle\right]^{1/2}\left[\int_0^t f(X_s)\,d\langle M^v\rangle\right]^{1/2}\right)\\ &\le \Gamma(u)(f)^{1/2}\Gamma(v)(f)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}\Gamma(u-v)(f) &= \Gamma(u)(f) - 2\Gamma(u,v)(f) + \Gamma(v)(f)\\ &\ge \Gamma(u)(f) - 2\Gamma(u)(f)^{1/2}\Gamma(v)(f)^{1/2} + \Gamma(v)(f)\\ &= \left(\Gamma(u)(f)^{1/2} - \Gamma(v)(f)^{1/2}\right)^2,\end{aligned}$$

and so

$$\begin{aligned}&|\Gamma(u)(f) - \Gamma(v)(f)|\\ &\quad = \left|\Gamma(u)(f)^{1/2} - \Gamma(v)(f)^{1/2}\right|\left(\Gamma(u)(f)^{1/2} + \Gamma(v)(f)^{1/2}\right)\\ &\quad \le \Gamma(u-v)(f)^{1/2}\left(\Gamma(u)(f)^{1/2} + \Gamma(v)(f)^{1/2}\right),\end{aligned} \tag{5.9}$$

which implies (5.7). □

PROOF OF THEOREM 5.2. Put $l := \log\tilde{\psi}$ and let $S := \inf\{t>0: \tilde{\psi}(X_t) = 0\}$. Then $l \in D_{\mathrm{loc}}(X,S)$, and we have the Fukushima decomposition

$$l(X_t) - l(X_0) = M_t^l + N_t^l, \qquad 0 \le t < S \quad \text{a.s. } P^x\ \forall\, x \in E \setminus N_0,$$

where N_0 is an (X,m)-inessential subset of E, outside of which l is finely continuous (with values in $[-\infty,\infty[$). The process

$$L_t := \exp\left(M_t^l - \tfrac{1}{2}\langle M^l\rangle_t\right)1_{\{t<S\}}$$

is a local martingale on $[0,S[$ and a multiplicative functional of X. In particular, L is a supermartingale. Let $(\overline{Y}, \overline{Q}^x)$ be the right process whose path-space probabilities are determined by

$$\overline{Q}^x(F; t<\zeta) := P^x(FL_t; t<S), \qquad F \in pF_t,\ t>0.$$

See, for example, [42], Section 62. To ensure that $\overline{Q}^x(\overline{Y}_0 = x)$ we agree that $\overline{Q}^x$ is the point mass at the path constantly equal to x whenever $x \in N_1 := N_0 \cup B$, where $B := \{x \in E \setminus N_0 : P^x(S = 0) = 1\}$. With this specification we have $\overline{Q}^x \ll_{\mathrm{loc}} P^x$ for all $x \in E \setminus N_1$. Now N_0 is m-null, hence μ-null, while $B \subset \{\tilde{\psi} = 0\}$. Since μ is carried by $\{\tilde{\psi} > 0\}$, we have $\mu(N_1) = 0$. Consequently $\overline{Q}^\mu \ll_{\mathrm{loc}} P^m$. Writing $\rho := \psi^2$ and using (3.19),

$$[\rho(X_0) L_t] \circ r_t = \rho(X_t) L_t \circ r_t = \rho(X_0) L_t \quad \text{on } \{t < S\} \text{ a.s. } P^m.$$

From this and (3.5) [applied to (X, S)] it follows that $\overline{Y}$ is a μ-symmetric diffusion. Indeed, writing $(\overline{Q}_t)$ for the transition semigroup of $\overline{Y}$,

$$\begin{aligned}
(f, \overline{Q}_t g)_\mu &= P^m(f(X_0)\rho(X_0) g(X_t) L_t; t < S) \\
&= P^m([f(X_0)\rho(X_0) g(X_t) L_t] \circ r_t; t < S) \\
&= P^m(f(X_t)\rho(X_0) g(X_0) L_t; t < S) \\
&= (\overline{Q}_t f, g)_\mu.
\end{aligned}$$

Let $(D(\overline{Y}), \overline{E})$ denote the Dirichlet space of $\overline{Y}$. For the moment, suppose that $\tilde{\psi}$ is everywhere strictly positive. A careful reading of the proof of Theorem 3.2 reveals that in this case we can take the exceptional set N to be (X, m)-inessential. Thus, by Theorems 3.2 and 4.11, we have $D_{\mathrm{loc}}(\overline{Y}) = D_{\mathrm{loc}}(X)$ and $\overline{\Gamma}(u) = \tilde{\psi}^2 \Gamma(u)$ for $u \in D_{\mathrm{loc}}(\overline{Y})$. But as noted at the end of Section 4, $\psi \in D(X)$ implies that $\overline{Y}$ has infinite lifetime, which means that $D(\overline{Y}) = \{u \in D_{\mathrm{loc}}(\overline{Y}) : \overline{\Gamma}(u)(E) < \infty\} \cap L^2(\mu)$, by Corollary 4.12. It now follows from (5.1) that $C \subset D(\overline{Y})$, and $\hat{\Gamma}(u) = \overline{\Gamma}(u) = \tilde{\psi}^2 \Gamma(u)$ for all $u \in C$.

To lift the strict positivity condition imposed on $\tilde{\psi}$ in the last paragraph, we use a truncation argument. Given $n \in \mathbb{N}$, let $\psi_n = (1/n) \vee \psi$. Then $\psi_n \in D_{\mathrm{loc}}(X)$, $\Gamma(\psi_n)(E) \le \Gamma(\psi)(E) < \infty$ and $\tilde{\psi}_n = \tilde{\psi} \vee (1/n)$. Let $L^{(n)}$ and $\overline{Y}^{(n)}$ be the analogs of L and $\overline{Y}$, with ψ_n substituted for ψ. By the foregoing discussion, we have $C \subset D(\overline{Y}^{(n)})$, and (with the obvious notation),

$$\text{(5.10)} \quad \overline{\Gamma}(u)^{(n)}(B \cap H_n) = \int_{B \cap H_n} \tilde{\psi}_n^2(x) \Gamma(u)(dx) = \hat{\Gamma}(u)(B \cap H_n)$$

for all $u \in C$ and all $B \in B(E)$, where $H_n := \{\tilde{\psi} > 1/n\}$. It is easy to check that $L^{(n)}$ and L coincide on $[0, \tau(H_n)[$, from which it follows immediately that $\overline{Y}$ and $\overline{Y}^{(n)}$ agree in law up to time $\tau(H_n)$. In particular, $\overline{\Gamma}^{(n)}(u)$ and $\overline{\Gamma}(u)$ agree on subsets of H_n whenever $u \in C$. Using these observations and a localization argument, we obtain, for $u \in C$,

$$\tilde{u}(\overline{Y}_t) - \overline{u}(\overline{Y}_0) = \overline{M}_t^u + \overline{N}_t^u, \qquad 0 \le t < \sigma \text{ a.s. } \overline{Q}^\mu,$$

where $\sigma := \lim_n \tau(H_n)$, $\tilde{u}$ is an X-quasi-continuous m-version of u, $\overline{M}^u$ is a local martingale (on $[0, \sigma[$) CAF of $(\overline{Y}, \sigma)$, and $\overline{N}^u$ is a CAF of $(\overline{Y}, \sigma)$ locally of zero energy. By Theorem 3.2, $\{\tilde{\psi} = 0\}$ is $(\overline{Y}, \mu)$-inessential, and so $\sigma = \zeta$ a.s. $\overline{Q}^\mu$. In view of 5.10, the Revuz measure of $\langle \overline{M}^u \rangle$ is $\overline{\Gamma}(u) = \hat{\Gamma}(u)$ on $\{\tilde{\psi} > 0\}$.

This identity extends to all of E because both $\overline{\Gamma}(u)$ and $\hat{\Gamma}(u)$ vanish on subsets of $\{\tilde{\psi} = 0\}$. Thus, if $u \in C$, then $\overline{\Gamma}(u)(E) = \hat{\Gamma}(u)(E) = 2\hat{E}(u, u) < \infty$, and then Lemma 3.15 implies that $u \in D_{\text{loc}}(\overline{Y})$. Because $\overline{Y}$ has infinite lifetime, we conclude that if $u \in C$ then $u \in D(\overline{Y})$, and clearly $\hat{E}(u, u) = \overline{E}(u, u)$.

We have shown that $C \subset D(\overline{Y})$ and that $\hat{E} = \overline{E}$ on C. Using condition (5.3) we will now show that $D(\overline{Y}) \subset D(Y)$. This will allow us to conclude that $(D(Y), \hat{E}) = (D(\overline{Y}), \overline{E})$, so that (Y, Q^μ) and $(\overline{Y}, \overline{Q}^\mu)$ are equivalent processes, and $Q^\mu = \overline{Q}^\mu \lll_{\text{loc}} P^m$.

Fix $u \in D(\overline{Y})$. Then $u \in D_{\text{loc}}(X, T_{N_1})$, by Theorem 4.11, where N_1 is as described in the first paragraph of this proof. Thus, there is a sequence $\{u_n\} \subset D(X, T_{N_1}) \subset D(X)$ and a sequence $\{G_n\}$ of finely open subset of $E \setminus N_1$ with $\tau(G_n) \to T_{N_1}$ as $n \to \infty$, such that $u = u_n$, m-a.e. on G_n for each n. Substituting $\{\rho \le n\} \cap G_n$ for G_n, we can assume that ρ is bounded above by n on G_n. Now using (5.1), for each n there is a sequence $\{w_{nk}\}_{k \in \mathbb{N}} \subset C$ with $w_{nk} \to u_n$ in E_1-norm. Passing to a subsequence if necessary, we can assume that $w_{nk} \to u_n$, m-a.e. Since $C \subset D(Y)$, we have

$$\tilde{w}_{nk}(Y_t) - \tilde{w}_{nk}(Y_0) = \tfrac{1}{2}\left[M_t^{nk} - M_t^{nk} \circ r_t\right] \quad \text{on } \{t < \zeta\} \text{ a.s. } Q^\mu, \forall\, t > 0,$$

where M^{nk} is the martingale part of $\tilde{w}_{nk}(Y_t) - \tilde{w}_{nk}(Y_0)$. This and the proof of Lemma 3.15 lead to the estimate

$$Q^\mu\Big(\left[\tilde{w}_{nk}(Y_t) - \tilde{w}_{nk}(Y_0)\right]^2;\, t < \tau(G_n)\Big) \le tC\int_{G_n} \rho(x)\Gamma(w_{nk})(dx). \tag{5.11}$$

Now by Lemma 5.6, $\Gamma(w_{nk}) \to \Gamma(u_n)$ in total variation norm, since $w_{nk} \to u_n$ in E_1-norm. But the integrand $1_{G_n}\rho$ is bounded, so the right side of (5.11) converges to $tC\int_{G_n} \rho\, d\Gamma(u_n)$ as $k \to \infty$. Using Fatou's lemma we can pass to the limit in (5.11) to obtain

$$Q^\mu\Big(\left[u_n(Y_t) - u_n(Y_0)\right]^2;\, t < \tau(G_n)\Big) \le tC\int_{G_n} \rho\, d\Gamma(u_n). \tag{5.12}$$

But $u = u_n$ on G_n, so the local nature of X means that the measures $\Gamma(u)$ and $\Gamma(u_n)$ coincide on subsets of G_n. [Indeed, the martingale $M^u - M^{u_n} = M^{u - u_n}$ is constant on the excursions of X into G_n, so $\int_0^t 1_{G_n}(X_s)d\langle M^u - M^{u_n}\rangle_s = 0$ for all $t > 0$, a.s. P^m. This means that $\Gamma(u - u_n)(G_n) = 0$, whence the claim, because of (5.9).] Thus, we can replace u_n by u on both sides of (5.12) and let $n \to \infty$. There results

$$Q^\mu\Big(\left[u(Y_t) - u(Y_0)\right]^2;\, t < \lim_n \tau(G_n)\Big) \le tC\int_E \rho\, d\Gamma(u). \tag{5.13}$$

But $\lim_n \tau(G_n) = \zeta$ a.s. Q^μ. Indeed, since $\mu(\tilde{\psi} = 0) = 0$ and $t \mapsto \tilde{\psi}(Y_t)$ is right continuous a.s. Q^μ, we have $\sigma := \inf\{t > 0\colon \tilde{\psi}(Y_t) = 0\} > 0$ a.s. Q^μ. Therefore, by a now familiar application of (16.21) in [25], we have $Q^\mu(\sigma < \zeta) = 0$. Also, recalling the definition of N_1, we see that $\{\tilde{\psi} > 0\} \setminus \bigcup_n G_n$ is (Y, μ)-polar because of (5.3) [(X, m)-nests are (Y, μ)-nests]. Thus, $\lim_n \tau(G_n) = \sigma = \zeta$ a.s.

Q^{μ}. Since the right side of (5.13) is finite and Y has infinite lifetime, we must have $u \in D(Y)$. This proves $D(\overline{Y}) \subset D(Y)$. □

By imposing additional hypotheses on the core C, we can replace condition (5.3) by a Markovian uniqueness hypothesis of the type studied in [40] and [41]. [I am grateful to A. Eberle and M. Röckner (personal communication) for pointing out the inadequacy of an earlier form of the theorem to follow.] For example, let the situation be as for Theorem 5.2 and assume in addition that

(5.14) $\quad C$ is an algebra of bounded functions, and $C \subset D(A^X) \cap D(A^Y)$,

(5.15) $\quad \gamma(u,v) := A^X(uv) - uA^Xv - vA^Xu \in L^{\infty}(m), \qquad \forall\, u, v \in C,$

where A^X (resp. A^Y) is the L^2-infinitesimal generator associated with X (resp. Y). These conditions are satisfied in the setting of [4], [40] and [41]. Because of (5.14) and (5.15), if $u \in C$ then the measure $\Gamma(u)$ is absolutely continuous with respect to m. Indeed, for $u, v \in C$ we have

$$\Gamma(u,v) = \gamma(u,v) \cdot m, \qquad u, v \in C. \tag{5.16}$$

See, for example, ([6], I.4). Because of the Kunita–Watanabe inequality, we have $\Gamma(u,v) \ll m$ for $u \in C$, $v \in D(X)$; we shall write $\gamma(u,v)$ for the Radon–Nikodym derivative $d\Gamma(u,v)/dm$ even in this case. Furthermore, we assume that the operator A^{ψ} defined by

$$A^{\psi}u := A^Xu + \psi^{-1}\gamma(\psi, u), \qquad u \in C, \tag{5.17}$$

maps C into $L^2(\mu)$. Then A^Y restricted to C coincides with A^{ψ}. Indeed, if ψ is bounded above then $\psi^2 \in D(X)$ and for $u, v \in C$,

$$\begin{aligned}(A^{\psi}u, v)_{\mu} &= (A^Xu, \psi^2 v)_m + (\gamma(\psi,u), \psi v)_m \\ &= -E(u, \psi^2 v) + (\gamma(\psi,u), \psi v)_m \\ &= -\tfrac{1}{2}\big(\gamma(u, \psi^2 v), 1\big)_m + (\gamma(\psi,u), \psi v)_m \\ &= -\tfrac{1}{2}\big(\gamma(u,v), \psi^2\big)_m = -\hat{E}(u,v) = (A^Yu, v)_{\mu},\end{aligned} \tag{5.18}$$

since $\gamma(u, \psi^2 v) = 2v\psi\gamma(u,\psi) + \psi^2\gamma(u,v)$ as a consequence of (4.2). A simple truncation argument shows that the equality of the extreme terms in (5.18) persists even for unbounded ψ. The assumed density of C (5.1) now implies that $A^Yu = A^{\psi}u$, a.e. m, for all $u \in C$, as asserted.

We can now state a sharpened form of Theorem 5.2, which generalized [4], Theorem 1.3. For discussion (in specific situations) of the Markovian uniqueness hypothesized below, see [40], [41] and the references therein.

THEOREM 5.19. *In addition to* (5.1), (5.14) *and* (5.15), *assume that* $(D(Y), \hat{E})$ *is the unique Dirichlet form on* $L^2(\mu)$ *for which the associated infinitesimal generator is an extension of* A^{ψ} $(D(A^{\psi}) = C)$. *Under these conditions,* $Q^{\mu} \ll_{\text{loc}} P^m$.

PROOF. Arguing as in the proof of Theorem 5.2 (and using the notation established there), we need only show that $C \subset D(A^{\overline{Y}})$ and $A^{\overline{Y}}u = A^{\psi}u$ for $u \in C$, where $(D(A^{\overline{Y}}), A^{\overline{Y}})$ denotes the infinitesimal generator of $\overline{Y}$. But if $u \in C \subset D(A^X)$, then by Girsanov's theorem the process

$$M_t := u(\overline{Y}_t) - u(\overline{Y}_0) - \int_0^t A^{\psi}u(\overline{Y}_s)\, ds, \qquad t \geq 0, \tag{5.20}$$

is a $\overline{Q}$-local martingale. Moreover, because of (5.14) and (5.15), the quadratic variation of M is given by $\int_0^t \gamma(u,u)(\overline{Y}_s)\, ds$ for which we have the estimate

$$\overline{Q}^{\mu}\int_0^t \gamma(u,u)(\overline{Y}_s)\, ds \leq t\mu(\gamma(u,u)) = t\hat{E}(u,u) < \infty.$$

Doob's inequality now implies that M is a $\overline{Q}$-martingale. Taking expectations in (5.20), we see that $t^{-1}(\overline{Q}_t u - u) = t^{-1}\int_0^t \overline{Q}_s A^{\psi}u\, ds \to A^{\psi}u$ in $L^2(\mu)$ as $t \to 0$. Thus, $u \in D(A^{\overline{Y}})$ and $A^{\overline{Y}}u = A^{\psi}u$. □

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