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APPROXIMATION OF SUBADDITIVE FUNCTIONS AND CONVERGENCE RATES IN LIMITING-SHAPE RESULTS[1]

BY KENNETH S. ALEXANDER[2]

University of Southern California

For a nonnegative subadditive function h on $\mathbb{Z}^d$, with limiting approximation $g(x) = \lim_n h(nx)/n$, it is of interest to obtain bounds on the discrepancy between $g(x)$ and $h(x)$, typically of order $|x|^\nu$ with $\nu < 1$. For certain subadditive $h(x)$, particularly those which are expectations associated with optimal random paths from 0 to x, in a somewhat standardized way a more natural and seemingly weaker property can be established: every x is in a bounded multiple of the convex hull of the set of sites satisfying a similar bound. We show that this convex-hull property implies the desired bound for all x. Applications include rates of convergence in limiting-shape results for first-passage percolation (standard and oriented) and longest common subsequences and bounds on the error in the exponential-decay approximation to the off-axis connectivity function for subcritical Bernoulli bond percolation on the integer lattice.

1. Introduction. Suppose h is a nonnegative subadditive function on $\mathbb{Z}^d$:

$$h(x+y) \le h(x) + h(y), \qquad x, y \in \mathbb{Z}^d.$$

Then for fixed x, the sequence $\{h(nx),\ n \ge 0\}$ is subadditive, so by standard methods the limit

$$g(x) := \lim_n h(nx)/n \tag{1.1}$$

exists, and it is approached from above: for all $x \in \mathbb{Z}^d$,

$$g(x) \le h(x). \tag{1.2}$$

In fact, for $x \in \mathbb{Q}^d$, if we restrict n to those values for which $nx \in \mathbb{Z}^d$, then the limit in (1.1) exists, so g extends to $\mathbb{Q}^d$.

If desired, one can restrict the domains of h and g to $\mathbb{Z}_+^d$ and $\mathbb{Q}_+^d$, where $\mathbb{Z}_+ := \{i \in \mathbb{Z}: i \ge 0\}$ and $\mathbb{Q}_+ := \{q \in \mathbb{Q}: q \ge 0\}$. To avoid cumbersome wording and notation, we will present the main result for h with domain $\mathbb{Z}^d$, but it remains valid if $\mathbb{Z}^d$, $\mathbb{Q}^d$ and $\mathbb{R}^d$ are replaced throughout by $\mathbb{Z}_+^d$, $\mathbb{Q}_+^d$ and $\mathbb{R}_+^d$.

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Let $|\cdot|$ denote the Euclidean norm; we use $|\cdot|_p$ for other L^p norms. Elements of $\mathbb{Z}^d$ are called sites. By a nearest-neighbor bond we mean a pair $\langle x, y\rangle$ of sites with $|x - y| = 1$. The ith unit vector in $\mathbb{R}^d$ is denoted e_i. When we use the term "lattice path" we always implicitly mean a self-avoiding one.

It is of interest in a variety of contexts to bound the discrepancy between g and h. The following examples will be analyzed below in detail.

EXAMPLE 1.1. In *first-passage percolation*, a nonnegative passage time t_b is attached to each nearest-neighbor bond b; these passage times are i.i.d. The passage time $T(x, y)$ is the minimum total passage time among all lattice paths from site x to site y. Here we let $h(x) = ET(0, x)$. First-passage percolation is an example of a growth model—the object at time t is

$$B(t) := \left\{x + [-1/2, 1/2]^d : T(0, x) \le t\right\}.$$

This object has an asymptotic shape; Cox and Durrett (1981) showed that under mild hypotheses, $t^{-1}B(t)$ converges to a compact convex set B_0. One aspect of interest in the physics of growth models is boundary roughness—the order of magnitude of the discrepancy between $B(t)$ and tB_0. Setting

$$G(t) := \left\{x + [-1/2, 1/2]^d : ET(0, x) \le t\right\},$$

we see that this discrepancy really has two parts—the random part, between $B(t)$ and $G(t)$, and the nonrandom part, between $G(t)$ and tB_0, which can be expressed in terms of the discrepancy between g and h. From (1.2) one has $G(t) \subset tB_0$, meaning that the growth is no faster than its asymptotic rate, up to any finite t. Under hypotheses on the distribution of the times t_b, Kesten (1993) obtained an exponential bound for the random part of the discrepancy and bounded the nonrandom part by showing that for some C, for all x,

$$g(x) \le h(x) \le g(x) + C|x|^{1-1/(2d+4)}(\log|x|)^{1/(d+2)}.$$

The value $\mu = g(e_1)$ is called the time constant, the time per unit of growth along an axis. Here a sharper bound is available: Alexander (1993) showed that for some C, for all $n \ge 1$,

$$n\mu \le ET(0, ne_1) \le n\mu + Cn^{1/2}\log n,$$

or equivalently,

$$g(ne_1) \le h(ne_1) \le g(ne_1) + Cn^{1/2}\log n.$$

EXAMPLE 1.2. Oriented first-passage percolation is similar to first-passage percolation except that lattice paths are restricted to those in which each step is from x to $x + e_i$ for some x and i; we call such lattice paths oriented. The domain of h is $\mathbb{Z}_+^d$. Again $h(x) = ET(0, x)$.

EXAMPLE 1.3. In Bernoulli bond percolation at density $p \in [0, 1]$, each nearest-neighbor bond is independently occupied with probability p; otherwise it is vacant. Here we let $h(x) = h_p(x) = -\log P_p[0 \leftrightarrow x]$. This probability is called the connectivity function; $\leftrightarrow$ denotes connection by a path of occupied bonds and P_p denotes probability at density p. Alexander (1990) showed that in dimensions 2 and 3, for values of p below the critical point, for some $c = c(p, d)$ and $r = r(d)$, for all x with $|x| > 1$,

$$\exp(-\sigma(p, \theta)|x|) \geq P_p[0 \leftrightarrow x] \geq c|x|^{-r} \exp(-\sigma(p, \theta)|x|), \tag{1.3}$$

where $\theta = x/|x|$ and $\sigma(p, \theta) = g(x)/|x|$ (which depends on x only through θ) is the inverse correlation length in direction θ. In the present formulation this means that for some $C = C(p, d)$, for all x with $|x| > 1$,

$$g(x) \leq h(x) \leq g(x) + C \log |x|. \tag{1.4}$$

The result (1.3) is related to the more general phenomenon of power-law corrections to exponential decay of correlations and other analogs of the connectivity, in a wide variety of statistical mechanical models, as discussed by Ornstein and Zernike (1914). The heuristic is roughly as follows. Consider x on an axis and let H_x be the hyperplane through x perpendicular to the axis. For connection to H_x, one expects to need essentially no correction to exponential decay, that is,

$$P_p[0 \leftrightarrow H_x] = (1 + o(1))\exp(-\sigma(p, \theta)|x|) \quad \text{as } |x| \to \infty.$$

Therefore the correction factor $P_p[0 \leftrightarrow x]/\exp(-\sigma(p, \theta)|x|)$ can be interpreted as the probability that there is a connection to x, given there is a connection to H_x. Since connections to H_x are rare, given one exists, the path from 0 probably hits H_x at only a few nearby sites, so it makes loose sense to talk about "the location where the path hits H_x." If the transverse fluctuations in the path are Gaussian, with standard deviations of order $|x|^{1/2}$, then the probability of hitting x given one hits H_x is of order $|x|^{-(d-1)/2}$; this is therefore the desired correction factor. Such "Ornstein–Zernike behavior" for Bernoulli bond percolation was established by Campanino, Chayes and Chayes (1991), but only for x near a coordinate axis, where symmetry can be exploited. By contrast, (1.3) covers all x, but with r much larger than $(d - 1)/2$. Ornstein–Zernike behavior is expected to hold for a wide variety of systems in statistical mechanics, but there are only a few other rigorous results. Bricmont and Fröhlich (1985a, b) consider systems at extreme temperatures, and Chayes and Chayes (1986) consider self-avoiding random walk.

EXAMPLE 1.4. Let $X_1^{(i)}X_2^{(i)}\dots$, $i = 1, \dots, d$, be i.i.d. sequences of letters selected from a finite alphabet A. A sequence of letters which is a subsequence of $X_1^{(i)} \cdots X_{n_i}^{(i)}$ for every $i \leq d$ is called a common subsequence of these sequences; $L(n_1, \dots, n_d)$ denotes the length of a longest common subsequence (abbreviated LCS) of the sequences $X_1^{(1)} \cdots X_{n_1}^{(1)}$ through $X_1^{(d)} \cdots X_{n_d}^{(d)}$. Define $U(n_1, \dots, n_d) := n_1 + \cdots + n_d - dL(n_1, \dots, n_d)$. For general d, $U(n_1, \dots, n_d)$ represents the number of letters from the d sequences which are "unused" for a given choice of LCS. For $d = 2$, $U(n_1, n_2)$ is an edit distance between $X_1^{(1)} \cdots X_{n_1}^{(1)}$ and $X_1^{(2)} \cdots X_{n_2}^{(2)}$; it is the minimal number of deletions plus insertions needed to change either sequence to the other one. For the LCS problem we let $h(x) = EU(x)$, $x \in \mathbb{Z}_+^d$. This problem can be reformulated as an oriented first-passage percolation model with dependent passage times; see Arratia and Waterman (1994) and Section 4. For $d = 2$, sequences of equal length are considered in Alexander (1994), where it is shown that for $c := \lim_n EL(n,n)/n$ and for some C, for all $n \geq 1$,

$$cn \geq EL(n,n) \geq cn - C(n \log n)^{1/2}, \tag{1.5}$$

which is equivalent, with a different C, to

$$g(n,n) \leq h(n,n) \leq g(n,n) + C(n \log n)^{1/2}.$$

Let Φ denote the set of all positive nondecreasing functions on $(1,\infty)$. Motivated by the preceding examples, we make the following definition.

DEFINITION. For $\nu \geq 0$ and $\varphi \in \Phi$ we say that the subadditive function h satisfies the general approximation property (or GAP) with exponent ν and correction factor φ if there exist $M > 1$ and $C > 0$ such that for all $x \in \mathbb{Z}^d$ with $|x| \geq M$,

$$g(x) \leq h(x) \leq g(x) + C|x|^{\nu}\varphi(|x|).$$

When we want to specify the relevant constants, we say h satisfies GAP(ν, φ, M, C).

The following is similar to Proposition 3.2 of Alexander, Chayes and Chayes (1990).

LEMMA 1.5. *Suppose h is subadditive on $\mathbb{Z}^d$ and the corresponding limit $g(x)$ is finite for all $x \in \mathbb{Q}^d$. Then g extends to a function on $\mathbb{R}^d$ which is continuous, convex, and positive-homogeneous of order* 1. *In particular, if h is symmetric, then g is a norm.*

PROOF. Clearly g is subadditive. Directly from the definition we have

$$g(\lambda x) = \lambda g(x) \quad \text{for } x \in \mathbb{Q}^d \text{ and } \lambda \in \mathbb{Q}_+. \tag{1.6}$$

Therefore

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) \qquad \text{for } x \in \mathbb{Q}^d \text{ and } \lambda \in \mathbb{Q} \cap [0,1]. \tag{1.7}$$

Hence

$$g(x) \leq |x_1| \max(g(e_1), g(-e_1)) + \cdots + |x_d| \max(g(e_d), g(-e_d)) \qquad \text{for } x \in \mathbb{Q}^d,$$

so g is bounded on $\mathbb{Q}^d \cap R$ for bounded R. This and subadditivity show that g is uniformly continuous on $\mathbb{Q}^d$. Thus g extends by continuity, and the rationals can be replaced by the reals in (1.6) and (1.7). □

If g is symmetric with respect to interchanges of coordinates and reflections across all coordinate hyperplanes, it is easy to show using subadditivity that

$$|x|_\infty \le g(x)/g(e_1) \le |x|_1 \quad \text{for all } x \ne 0. \tag{1.8}$$

Let $B_0 := \{x: g(x) \le 1\}$. For $x \in \mathbb{R}^d$ let H_x denote a hyperplane tangent to $\partial g(x) B_0$ at x; note that if ∂B_0 is not smooth, there is not necessarily a unique choice of H_x. Let H_x^0 denote the hyperplane through 0 parallel to H_x. There is a unique linear functional g_x on $\mathbb{R}^d$ satisfying

$$g_x(y) = 0 \quad \text{for all } y \in H_x^0, \qquad g_x(x) = g(x).$$

Note that for $y \in \mathbb{R}^d$, $g_x(y)$ is the g-length of a projection of y onto the line through 0 and x. By convexity of g we have $g_x \le g$, and if g is symmetric,

$$|g_x(y)| \le g(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{1.9}$$

The value $g_x(y)$ may be thought of as the amount of progress (measured in the norm g) toward x made by a vector increment of y; for fixed x, $h(y) - g_x(y)$ is a measure of the error or inefficiency associated with such an increment, as we will now illustrate. For $x \in \mathbb{R}^d$ we define a set of vector increments for which this "error" is of order at most roughly $|x|^\nu$:

$$Q_x(\nu, \varphi, C, K) := \left\{ y \in \mathbb{Z}^d : |y| \le K|x|,\ g_x(y) \le g(x),\ h(y) \le g_x(y) + C|x|^\nu \varphi(|x|) \right\}.$$

Since h is subadditive and g_x is linear, one way to establish GAP with exponent ν and correction factor φ is to show that for some m, K and C, every $x \in \mathbb{Z}^d$ can be expressed as

$$x = \sum_{i=1}^m y_i \quad \text{with } y_i \in Q_x(\nu, \varphi, C, K); \tag{1.10}$$

specifically this establishes GAP($\nu, \varphi, 1, Cm$). Thus (1.10) says that x can be built from a bounded number of "good" increments. Unfortunately the sufficient condition (1.10) does not seem to be natural, in that we have no canonical procedure for trying to verify it in the examples of interest. We next define a related condition which is much less clearly sufficient for GAP, but which we will see is quite natural.

DEFINITION. We say that the subadditive function h satisfies the convex-hull approximation property (or CHAP) with exponent ν and correction factor φ, if there exist $M > 0$, $C > 0$, $K > 0$, $a > 1$ and $\varphi \in \Phi$ such that

$$\begin{aligned} &x/\alpha \in \mathrm{Co}\big(Q_x(\nu, \varphi, C, K)\big) \\ &\text{for some } \alpha \in [1, a], \text{ for all } x \in \mathbb{Q}^d \text{ with } |x| \ge M, \end{aligned} \tag{1.11}$$

where $\mathrm{Co}(\cdot)$ denotes the convex hull. When we want to specify the relevant constants, we say h satisfies CHAP(ν, φ, M, C, K, a).

In d dimensions, $z \in \mathrm{Co}(A)$ for some A implies that z is in the convex hull of some subset of A consisting of at most $d+1$ points, so (1.11) is equivalent to

$$x = \sum_{i=1}^{d+1} \alpha_i y_i \quad \text{with } \alpha_i \geq 0, \ \sum_{i=1}^{d+1} \alpha_i \in [1, a] \text{ and}$$
$$y_i \in Q_x(\nu, \varphi, C, K) \text{ for all } x \in \mathbb{R}^d \text{ with } |x| \geq M.$$

The following result illustrates why CHAP is natural. Roughly, it says that in expressing x/α as a convex combination, one can take the coefficient of each $y \in Q_x$ to be proportional to the number of times the increment y occurs when a path from 0 to nx is cut up into increments with each increment in Q_x.

LEMMA 1.6. *Let h be a nonnegative subadditive function on $\mathbb{Z}^d$ and let $\nu \geq 0$, $\varphi \in \Phi$, $M > 1$, $C > 0$, $K > 0$ and $a > 1$. Suppose that for each $x \in \mathbb{Q}^d$ with $|x| \geq M$, there exist $n \geq 1$, a lattice path γ from 0 to nx and a sequence of sites $0 = v_0, v_1, \dots, v_m = nx$ in γ such that $m \leq an$ and $v_i - v_{i-1} \in Q_x(\nu, \varphi, C, K)$ for all $1 \leq i \leq m$. Then h satisfies CHAP$(\nu, \varphi, M, C, K, a)$.*

PROOF. Given $y \in Q = Q_x(\nu, \varphi, C, K)$, $n \geq 1$ as in the lemma statement, and a corresponding path γ, let $\beta_n(y, \gamma)$ be the number of indices i such that $v_i - v_{i-1} = y$. Then

$$nx = \sum_{y \in Q} \beta_n(y, \gamma) y, \tag{1.12}$$

and

$$\sum_{y \in Q} \beta_n(y, \gamma) \leq an,$$

so

$$\alpha := \sum_{y \in Q} n^{-1} \beta_n(y, \gamma) \leq a,$$

and $x/\alpha \in \mathrm{Co}(Q)$. Applying g_x to both sides of (1.12) yields

$$ng(x) = \sum_{y \in Q} \beta_n(y, \gamma) g_x(y) \leq g(x) \sum_{y \in Q} \beta_n(y, \gamma),$$

from which we obtain $\alpha \geq 1$, and the lemma follows. □

REMARK 1.7. Let us call the $m+1$ sites in Lemma 1.6 marked sites. If m is unrestricted, it is easy to find inductively a sequence of marked sites for any path γ from 0 to nx—one can start at $v_0 = 0$, and given v_i, let v'_{i+1} be the first site (if any) in γ such that $v'_{i+1} - v_i \notin Q_x(\nu, \varphi, C, K)$; then let v_{i+1} be the last site in γ before v'_{i+1} if v'_{i+1} exists; otherwise let $v_{i+1} = nx$ and end the construction. We call the sequence of marked sites, obtained from a self-avoiding path γ in this way, the $Q_x(\nu, \varphi, C, K)$-skeleton of γ. The difficulty is that the $Q_x(\nu, \varphi, C, K)$-skeleton of a typical γ may have far more than $an + 1$ marked sites. Roughly, one would like many of the marks to be made at a distance at least of order $|x|$ beyond the previous mark, to keep the number of marks of order n. One method of controlling the number of marked sites is summarized following the proof of Theorem 3.1.

In Alexander (1990), GAP$(0, \log(\cdot), M, C)$ is established for the connectivity function of Bernoulli bond percolation in dimensions $d = 2$ and 3, by first proving CHAP$(0, \log(\cdot), M, C, 2d, 3)$ by roughly the method of Lemma 1.6 and Remark 1.7. The existence of an appropriate γ is established by showing that, conditionally on there being a path of occupied bonds from 0 to nx, the probability that it and its x-skeleton fail the conditions in Lemma 1.6 is less than 1. That is, we show that a certain nonrandom structure exists by showing that an event built on this structure has positive probability, so is nonempty. We will see that this method has analogs for other models; see the remarks following the proof of Theorem 3.1.

We say that h has sublinear growth if $h(x) \le r|x|$ for all x for some $r > 0$. The following is our main result; the proof is in Section 2.

THEOREM 1.8. *Suppose h is a nonnegative subadditive function on $\mathbb{Z}^d$ which has sublinear growth. If h satisfies CHAP with exponent ν and correction factor φ for some $\nu > 0$, then h satisfies GAP with exponent ν and correction factor φ.*

When h satisfies CHAP with exponent $\nu = 0$, Theorem 1.8 yields only that h satisfies GAP with arbitrarily small positive exponent. In particular, it gives a far from optimal result for the connectivity function of Bernoulli bond percolation (Example 1.3). The proof of GAP from CHAP, for that connectivity function, in Alexander (1990) is completely different from that of Theorem 1.8; it allows $\nu = 0$ but makes use of a purely geometric lemma about splicing segments of curves which is only known for dimension $d = 2$ and 3. For general h satisfying CHAP with exponent 0, the next result shows we can obtain GAP but at the expense of a log factor in the correction factor φ. The result is not optimal, because the extra log factor is unnecessary in Example 1.3. The proof is in Section 5.

THEOREM 1.9. *Suppose h is a nonnegative subadditive function on $\mathbb{Z}^d$ which has sublinear growth. If h satisfies CHAP with exponent 0 and correction factor φ, then h satisfies GAP with exponent 0 and correction factor $\varphi(\cdot)\log(\cdot)$.*

2. Proof of the main result. Observe that if h satisfies CHAP$(\nu, \varphi, M, C, K, a)$, then CHAP remains satisfied if M and C are replaced by larger constants. If h has sublinear growth, then clearly h satisfies GAP with exponent 1. This will allow us to prove Theorem 1.8 by iterating the following result.

PROPOSITION 2.1. *Suppose* $C > 1$, $K > 1$, $M > 1$, $a > 1$, $\nu \in (0,1)$, $\varphi \in \Phi$ *and* h *is a nonnegative subadditive function on* $\mathbb{Z}^d$ *with* $h(x) \le r|x|$ *for all* x *for some* $r > 1$. *There exists a constant* $\tilde{C}_0(\nu, \varphi, C, K, a, r, d, M)$ *such that if* $\tilde{C} \ge \tilde{C}_0$, $\beta \in (\nu, 1]$, *and* h *satisfies both* $GAP(\beta, \varphi, M, \tilde{C})$ *and* $CHAP(\nu, \varphi, M, C, K, a)$, *then* h *satisfies* $GAP(\beta', \varphi, M, \tilde{C})$ *where* $\beta' := \beta/(1 + \beta - \nu) < \beta$.

Loosely, this proposition says that CHAP enables one to reduce the exponent β in $\mathrm{GAP}(\beta, \varphi, M, \tilde{C})$ "for free," that is, without having to increase M or $\tilde{C}$. It is essential that $\tilde{C}_0$ does not depend on β, so that this reduction can be iterated.

PROOF. We begin with a sketch of the proof, at least for $|x|$ large relative to M (Case 1). Let q be large, but much smaller than $|x|$. Applying CHAP to x/q we express x/q as a convex combination (up to a bounded constant) of $d+1$ "good" increments y_{qi}—see (2.3). We then decompose x into an linear combination x^* of the y_{qi}'s, with nonnegative integer coefficients, plus a remainder $x - x^*$; see Figure 1. Because the coefficients are nonnegative integers, subadditivity bounds the error $h(x^*) - g_x(x^*)$; see (2.6). Then GAP is used to bound $h(x - x^*) - g_x(x - x^*)$; see (2.8). We then optimize over q to obtain the result.

Now to the details. Throughout this proof, $c_0, c_1, \ldots$ denote constants which may depend on ν, C, K, a, d and/or r, but not on M or β.

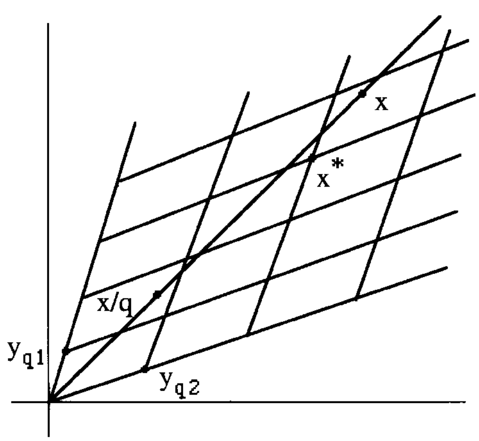

FIG. 1. *The situation with* $y_{q3} = 0$ *is shown*; x^* *is a corner of the cell containing* x *in the grid generated by* y_{q1} *and* y_{q2}.

Define

$$c_0 := C(d+1),$$
$$c_1 := Ca,$$
(2.1) $$c_2 := K(d+1)$$
$$c_3 := (1-\nu)(c_1+c_0)c_2^{2-2\nu}/\nu$$
$$c_4 := 2/(1-\nu)(c_1+c_0).$$

Since multiplying φ by a positive constant and dividing C and $\tilde{C}$ by the same constant does not alter the result, let us assume for now that

(2.2) $$\varphi(M) \geq 1.$$

Fix $M > 1$ and x with $|x| \geq M$. We consider two cases.

Case 1. $|x| \geq c_2 c_4 \tilde{C} M^{1+\beta-\nu}$. Let $q \in [c_2, |x|/M] \cap \mathbb{Q}$; later we will optimize over q. The optimal q will be of the order of a small power of $|x|$. Since $|x|/q \geq M$, we can apply CHAP to x/q to obtain

(2.3) $$x/q = \sum_{i=1}^{d+1} \alpha_{qi} y_{qi} \quad \text{with } \alpha_{qi} \geq 0,$$
$$\sum_{i=1}^{d+1} \alpha_{qi} \in [1, a] \text{ and } y_{qi} \in Q_{x/q}(\nu, \varphi, C, K).$$

Note that

(2.4) $$g_{x/q} = g_x.$$

Let

$$x^* := \sum_{i=1}^{d+1} \lfloor q\alpha_{qi} \rfloor y_{qi}.$$

where $\lfloor\cdot\rfloor$ denotes the integer part, and

$$\gamma_{qi} := q\alpha_{qi} - \lfloor q\alpha_{qi} \rfloor \in [0, 1),$$

so that

(2.5) $$x - x^* = \sum_{i=1}^{d+1} \gamma_{qi} y_{qi}.$$

Figure 1 illustrates $d = 2$; we have shown the case $y_{q3} = 0$, which makes the picture more clear but not alter the basic idea. By subadditivity, (2.3) and (2.4),

(2.6) $$h(x^*) \leq \sum_{i=1}^{d+1} \lfloor q\alpha_{qi} \rfloor h(y_{qi})$$
$$\leq \sum_{i=1}^{d+1} \lfloor q\alpha_{qi} \rfloor \left[g_x(y_{qi}) + C|x/q|^{\nu} \varphi(|x/q|) \right]$$
$$\leq g_x(x^*) + c_1 q^{1-\nu} |x|^{\nu} \varphi(|x|).$$

To bound $h(x - x^*)$ we consider two subcases.

Case 1a. $|x - x^*| \geq M$. Observe that

$$g(y_{qi}) - g_x(y_{qi}) \leq h(y_{qi}) - g_x(y_{qi}) \leq Cq^{-\nu}|x|^\nu\varphi(|x|/q)$$

so that

$$\sum_{i=1}^{d+1} \gamma_{qi}\big[g(y_{qi}) - g_x(y_{qi})\big] \leq c_0 q^{-\nu}|x|^\nu\varphi(|x|). \tag{2.7}$$

Now by (2.5),

$$|x - x^*| \leq \sum_{i=1}^{d+1} |y_{qi}| \leq c_2|x|/q.$$

so applying GAP to $x - x^*$ and using linearity of g_x and subadditivity of g, we obtain

$$\begin{aligned} h(x - x^*) &\leq g(x - x^*) + \tilde{C}|x - x^*|^\beta\varphi(|x - x^*|) \\ &\leq g_x(x - x^*) + \sum_{i=1}^{d+1} \gamma_{qi}\big[g(y_{qi}) - g_x(y_{qi})\big] \\ &\quad + \tilde{C}c_2^\beta q^{-\beta}|x|^\beta\varphi(c_2|x|/q). \end{aligned} \tag{2.8}$$

Therefore using (2.6), (2.8) and (2.7),

$$\begin{aligned} h(x) &\leq h(x^*) + h(x - x^*) \\ &\leq g(x) + c_1 q^{1-\nu}|x|^\nu\varphi(|x|) + c_0 q^{-\nu}|x|^\nu\varphi(|x|) + \tilde{C}c_2^\beta q^{-\beta}|x|^\beta\varphi(|x|) \\ &\leq g(x) + (c_1 + c_0)q^{1-\nu}|x|^\nu\varphi(|x|) + \tilde{C}c_2^\beta q^{-\beta}|x|^\beta\varphi(|x|). \end{aligned}$$

Case 1b. $|x - x^*| < M$. Observe that

$$0 \leq h(y_{qi}) \leq g_x(y_{qi}) + Cq^{-\nu}|x|^\nu\varphi(|x|/q)$$

so that, letting $I(q) := \{i \leq d + 1 \colon g_x(y_{qi}) < 0\}$, we have

$$\begin{aligned} g_x(x - x^*) &= \sum_{i=1}^{d+1} \gamma_{qi} g_x(y_{qi}) \\ &\geq -\sum_{i \in I(q)} |g_x(y_{qi})| \\ &\geq -c_0 q^{-\nu}|x|^\nu\varphi(|x|). \end{aligned}$$

Therefore

$$\begin{aligned} h(x - x^*) &\leq rM \\ &\leq g_x(x - x^*) + c_0 q^{-\nu}|x|^\nu\varphi(|x|) + rM, \end{aligned}$$

which with (2.6) yields

$$\begin{aligned} h(x) &\leq h(x^*) + h(x - x^*) \\ &\leq g(x) + (c_1 + c_0)q^{1-\nu}|x|^\nu\varphi(|x|) + rM. \end{aligned}$$

Thus in both Case 1a and Case 1b we have

$$h(x) \le g(x) + (c_1 + c_0) q^{1-\nu} |x|^{\nu} \varphi(|x|) + \tilde{C} c_2^{\beta} q^{-\beta} |x|^{\beta} \varphi(|x|) + rM\varphi(|x|). \tag{2.9}$$

Since the inequality (2.9) is valid for all rational q in $[c_2, |x|/M]$, it is also valid for all real q in the same interval. To minimize the right side of (2.9) over q we set

$$q := \left(c_4 \beta \tilde{C} c_2^{\beta}\right)^{1/(1+\beta-\nu)} |x|^{(\beta-\nu)/(1+\beta-\nu)}.$$

Let us verify that $q \in [c_2, |x|/M]$. Assume $\tilde{C} \ge c_3$. Since $c_2 > 1$ and $1 \ge \beta > \nu$, it follows that

$$c_4 \beta \tilde{C} c_2^{\beta} > c_4 \nu \tilde{C} c_2^{\nu} \ge c_2^{2-\nu} \ge c_2^{1+\beta-\nu}. \tag{2.10}$$

Since $|x| > 1$, it follows that $q > c_2$. By the assumption of Case 1,

$$\begin{aligned} |x| &= |x|^{1/(1+\beta-\nu)} |x|^{(\beta-\nu)/(1+\beta-\nu)} \\ &\ge \left(c_2 c_4 \tilde{C}\right)^{1/(1+\beta-\nu)} M |x|^{(\beta-\nu)/(1+\beta-\nu)} \ge qM, \end{aligned} \tag{2.11}$$

and the desired conclusion $q \in [c_2, |x|/M]$ follows.

We next bound the second and third terms on the right-hand side of (2.9). Let $c_5 := 1/c_4 \nu (c_1 + c_0)$ and $c_6 := (1 + c_5)(c_1 + c_0)$. Assume $\tilde{C} \ge c_7 := (2c_6(1 + c_2 c_4))^{1/\nu}$ and $\tilde{C} > 1$; then since $\beta > \nu$,

$$\tilde{C}^{\beta/(1+\beta-\nu)} \ge \tilde{C}^{\nu} \ge 2c_6 (c_2 c_4)^{(1-\nu)/(1+\beta-\nu)}$$

and so

$$\tilde{C} \ge 2c_6 \left(c_2 c_4 \tilde{C}\right)^{(1-\nu)/(1+\beta-\nu)}. \tag{2.12}$$

We have also

$$q^{1+\beta-\nu} = c_4 \beta \tilde{C} c_2^{\beta} |x|^{\beta-\nu} \ge \left(\tilde{C} c_2^{\beta} / c_5 (c_1 + c_0)\right) |x|^{\beta-\nu}$$

so that, rearranging,

$$\tilde{C} c_2^{\beta} q^{-\beta} |x|^{\beta} \le c_5 (c_1 + c_0) q^{1-\nu} |x|^{\nu}.$$

Using this, $\beta \le 1$ and (2.12), we obtain

$$\begin{aligned} &(c_1 + c_0) q^{1-\nu} |x|^{\nu} + \tilde{C} c_2^{\beta} q^{-\beta} |x|^{\beta} \\ &\quad \le c_6 q^{1-\nu} |x|^{\nu} \\ &\quad \le c_6 \left(c_2 c_4 \tilde{C}\right)^{(1-\nu)/(1+\beta-\nu)} |x|^{\beta/(1+\beta-\nu)} \\ &\quad \le \tilde{C} |x|^{\beta/(1+\beta-\nu)}/2. \end{aligned} \tag{2.13}$$

Next we bound the last term on the right-hand side of (2.9). Let $c_8 := (2r)^{(2-\nu)/(3-\nu)} (c_2 c_4)^{-1/(3-\nu)}$ and assume that $\tilde{C} \ge c_8 M^{(1-\nu)(2-\nu)/\nu(3-\nu)}$, $\tilde{C} \ge$

$1/c_2c_4$ and $\tilde{C} \geq 2r$. Then

$$\left(c_2c_4\tilde{C}\right)^{1/(1+\beta-\nu)}\left(\tilde{C}/2r\right)^{1/\beta} \geq \left(c_2c_4\tilde{C}\right)^{1/(2-\nu)}\tilde{C}/2r$$
$$= (c_2c_4)^{1/(2-\nu)}\tilde{C}^{(3-\nu)/(2-\nu)}/2r$$
$$\geq M^{(1-\nu)/\nu}$$
$$\geq M^{(1-\beta)/\beta},$$

where the second inequality uses the first lower bound assumed for $\tilde{C}$. Rearranging gives

$$\left(c_2c_4\tilde{C}\right)^{1/(1+\beta-\nu)}M \geq \left(2rM/\tilde{C}\right)^{1/\beta}.$$

which using the assumption of Case 1 yields

$$|x| \geq c_2c_4\tilde{C}M^{1+\beta-\nu} \geq \left(2rM/\tilde{C}\right)^{(1+\beta-\nu)/\beta}.$$

Another rearrangement then shows that

$$rM \leq \tilde{C}|x|^{\beta/(1+\beta-\nu)}/2. \tag{2.14}$$

Combining (2.9), (2.13) and (2.14) we obtain

$$h(x) \leq g(x) + \tilde{C}|x|^{\beta/(1+\beta-\nu)}\varphi(|x|). \tag{2.15}$$

Case 2. $M \leq |x| < c_2c_4\tilde{C}M^{1+\beta-\nu}$. Let $c_9 := r^{1/\nu}(1 + (c_2c_4)^{(1-\nu)/\nu})$ and assume $\tilde{C} \geq c_9M^{(1-\nu)/\nu}$. Then since $\beta > \nu$, this assumption yields

$$\tilde{C} \geq r^{(1+\beta-\nu)/\beta}(c_2c_4)^{(1-\nu)/\beta}M^{(1-\nu)(1+\beta-\nu)/\beta},$$

which is equivalent to

$$\tilde{C}^{\beta/(1+\beta-\nu)} \geq r(c_2c_4)^{(1-\nu)/(1+\beta-\nu)}M^{(1-\nu)}$$

and thus also to

$$\tilde{C} \geq r\left(c_2c_4\tilde{C}\right)^{(1-\nu)/(1+\beta-\nu)}M^{(1-\nu)}.$$

Using this and the assumption of Case 2 we obtain

$$\begin{aligned} h(x) &\leq r|x| \\ &= r|x|^{(1-\nu)/(1+\beta-\nu)}|x|^{\beta/(1+\beta-\nu)} \\ &\leq r\left(c_2c_4\tilde{C}\right)^{(1-\nu)/(1+\beta-\nu)}M^{(1-\nu)}|x|^{\beta/(1+\beta-\nu)} \\ &\leq \tilde{C}|x|^{\beta/(1+\beta-\nu)} \\ &\leq g(x) + \tilde{C}|x|^{\beta/(1+\beta-\nu)}\varphi(|x|). \end{aligned} \tag{2.16}$$

This and (2.15) prove the proposition under (2.2), with

$$\tilde{C}_0 := \max\left(1, 1/c_3, 1/c_2c_4, c_7, 2r, c_8M^{(1-\nu)(2-\nu)/\nu(3-\nu)}, c_9M^{(1-\nu)/\nu}\right).$$

If (2.2) does not hold, then the result is obtained by applying the result under (2.2) to $\varphi(\cdot)/\varphi(M)$, $\varphi(M)C$ and $\varphi(M)\tilde{C}$ in place of φ, C and $\tilde{C}$, respectively.

□

PROOF OF THEOREM 1.8. Suppose h satisfies CHAP$(\nu, \varphi, M, C, K, a)$. We may assume that $C > 1$, $K > 1$, $\nu \le 1$, $r > 1$ and $M > 1$. Let $\tilde{C} := \max(r/\varphi(M), \tilde{C}_0(\nu, \varphi, C, K, a, r, d, M))$. Since $h(x) \le r|x|$, h satisfies GAP$(1, \varphi, M, r/\varphi(M))$ and hence also GAP$(1, \varphi, M, \tilde{C})$. For $\beta \in [\nu, 1]$ set $f(\beta) := \beta/(1 + \beta - \nu)$; then $f(\beta) < \beta$ for $\beta \in (\nu, 1]$ and the unique fixed point of f is at ν. Set $\beta_0 := 1$ and $\beta_{n+1} := f(\beta_n)$, so $\beta_n \to \nu$. We have that h satisfies GAP$(\beta_0, \varphi, M, \tilde{C})$, and by Proposition 2.1 if h satisfies GAP$(\beta_n, \varphi, M, \tilde{C})$ then h satisfies GAP$(\beta_{n+1}, \varphi, M, \tilde{C})$. Therefore, taking the limit, h satisfies GAP$(\nu, \varphi, M, \tilde{C})$. □

3. First-passage percolation. In this section we will apply Theorem 1.8 to Example 1.1 on first-passage percolation. Recall that there is a bond between each nearest-neighbor pair of sites in $\mathbb{Z}^d$, that is, each pair $\langle x, y\rangle$ with $|x - y| = 1$. Attached to each bond b is a random passage time $t_b \ge 0$; these passage times are i.i.d. with d.f. F. For a lattice path γ, the passage time $T(\gamma)$ is the sum of the passage times of the bonds comprising γ. Then

$$T(x, y) := \inf\{T(\gamma) : \gamma \text{ a lattice path from } x \text{ to } y\}, \qquad x, y \in \mathbb{Z}^d.$$

To visualize this, one may imagine injecting liquid at the origin at time 0, and assume that the passage time t_b of each bond b tells how long the liquid takes to pass through that bond. For each time $t \ge 0$ there is then a set of sites which are wet at time t; centering a unit cube at each of these sites determines the wet region

$$B(t) := \left\{x + [-1/2, 1/2]^d : T(0, x) \le t\right\}.$$

Richard (1973) and Cox and Durrett (1981) showed that under mild conditions there is a nonrandom compact convex symmetric set $B_0 \subset \mathbb{R}^d$ such that $t^{-1}B(t)$ converges to B_0, in the sense that for every $\varepsilon > 0$, with probability 1, for sufficiently large t,

$$(1 - \varepsilon) B_0 \subset t^{-1} B(t) \subset (1 + \varepsilon) B_0. \tag{3.1}$$

Here we investigate the speed of this convergence, or more specifically, the nonrandom part of the discrepancy between $t^{-1}B(t)$ and B_0. Throughout this section, $C_1, C_2, \ldots$ will denote constants which depend only on F and/or d. Kesten (1993) has shown that under a moment condition on F there are constants $C_i = C_i(F, d)$ such that with probability 1, for sufficiently large t,

$$\begin{aligned} &\left(1 - C_1 t^{-1/(2d+4)} (\log t)^{1/(d+2)}\right) B_0 \subset t^{-1} B(t) \\ &\qquad\qquad \subset \left(1 + C_2 t^{-1/2} \log t\right) B_0. \end{aligned} \tag{3.2}$$

Note that the inner bound for $B(t)$ in (3.2) is weaker than the outer bound. This is related to subadditivity of the function

$$h(x) := ET(0, x),$$

which is a consequence of the inequality

$$T(x, y) \le T(x, z) + T(z, y) \quad \text{for all } x, y, z \in \mathbb{Z}^d.$$

Specifically, subadditivity yields the one-sided bound (1.2) for $ET(0, x)$, making it easier to bound passage times from below than to bound them from above.

To remedy this asymmetry, one must bound $ET(0, x) - g(x)$ for the function $g(x)$ of (1.1). Kesten (1993) shows that

$$ET(0, x) - g(x) \le C_3 |x|^{1 - 1/(2d+4)} (\log |x|)^{1/(d+2)},$$

which is GAP with exponent $1 - 1/(2d + 4)$. Our main effort will be to improve this exponent to $1/2$, or more precisely, to show that

$$ET(0, x) - g(x) \le C_4 |x|^{1/2} \log |x|, \tag{3.3}$$

then use this to obtain an almost sure bound improving (3.2). For x on an axis, (3.3) was proved in Alexander (1993). But the reflection argument therein uses the symmetry of the lattice about the hyperplane perpendicular to x, which does not exist for off-axis x. Therefore we will use Lemma 1.6 and Remark 1.7 to help establish CHAP, then apply Theorem 1.8.

In view of Lemma 1.5, the limiting set can be described by

$$B_0 = \{x \in \mathbb{R}^d : g(x) \le 1\}.$$

An almost sure analog of (1.1) is the fact that

$$\lim_n T(0, nx)/n = g(x) \quad \text{a.s.} \tag{3.4}$$

provided the passage times have a finite first moment. This follows from the subadditive ergodic theorem of Kingman (1968).

Let $p_c(\mathbb{Z}^d)$ denote the critical probability for Bernoulli bond percolation on $\mathbb{Z}^d$; this means the probability that the origin is part of an infinite connected set of occupied bonds is positive at densities $p > p_c(\mathbb{Z}^d)$ and zero at densities $p < p_c(\mathbb{Z}^d)$. Kesten (1993) showed that if

$$F(0) < p_c(\mathbb{Z}^d) \tag{3.5}$$

and

$$\int e^{\lambda x}\, dF(x) < \infty \quad \text{for some } \lambda > 0, \tag{3.6}$$

then

$$P\left[|T(0, x) - ET(0, x)| \ge u|x|^{1/2}\right] \le C_5 e^{-C_6 u} \quad \text{for } u \le C_7 |x|. \tag{3.7}$$

The condition (3.5) is natural because it is equivalent to the positivity of the time constant μ and to the compactness of B_0; see Kesten (1986).

Here now is the main theorem, to be proved later in this section.

THEOREM 3.1. *Under the hypotheses* (3.5) *and* (3.6), *for some constants* $C_i(F, d)$, *with probability* 1, *for all sufficiently large* t,

$$\left(1 - C_8 t^{-1/2} \log t\right) B_0 \subset t^{-1} B(t) \subset \left(1 + C_9 t^{-1/2} \log t\right) B_0.$$

In the Eden model, examined by Eden (1961) and Richardson (1973), one begins with the cube $[-1/2, 1/2]^d$ and adds one cube at a time, chosen from among the unselected cubes which are adjacent to the set of cubes selected so far, with probability proportional to the number of adjacent already selected cubes. All cubes are of form $x + [1/2, 1/2]^d$ with $x \in \mathbb{Z}^d$. Except for a time change, this is identical to the growth of $B(t)$ when the passage times are exponential. Richardson (1973) proved an asymptotic shape result like (3.1) for the Eden model, and Theorem 3.1 for exponential passage times gives a rate of convergence in Richardson's result.

The $t^{-1/2} \log t$ rate in Theorem 3.1 may well not be optimal. Simulations [see Kesten (1993) for references and comments] suggest that $t^{-2/3}$ may be the right rate. In our method, increasing the exponent beyond 1/2 in Theorem 3.1 would require a similar increase in (3.7).

For $x, y \in \mathbb{Z}^d$, a route from x to y is a lattice path γ of minimal total passage time, that is, $T(\gamma) = T(x, y)$. Routes always exist; see Smythe and Wierman (1977).

As we have described, the main ingredient in proving Theorem 3.1 will be the establishment of GAP with exponent 1/2, which we now state as a separate theorem.

THEOREM 3.2. *Under the hypotheses* (3.5) *and* (3.6), *for some constant* $C_4(F, d)$, *for all* $x \in \mathbb{Z}^d$ *with* $|x| > 1$,

$$g(x) \le ET(0, x) \le g(x) + C_4 |x|^{1/2} \log|x|. \tag{3.8}$$

Before proving this we need some preliminary definitions and results. Define

$$s_x(y) := ET(0, y) - g_x(y), \qquad y \in \mathbb{Z}^d.$$

By (1.2) and (1.9), s_x is nonnegative. From subadditivity of h and linearity of g_x we obtain

$$s_x(y + z) \le s_x(y) + s_x(z) \quad \text{for all } y, z \in \mathbb{Z}^d.$$

As in the discussion preceding (1.10), we can view $s_x(y)$ as a measure of the inefficiency of an increment of y when trying to reach x. With C_6 from (3.7), let $C_{10} := 32d(8d)^{1/2}/C_6$, $\varphi(t) := \log t$,

$$\begin{aligned} Q_x &:= Q_x(1/2, \varphi, C_{10}, 2d + 1), \\ G_x &:= \{y \in \mathbb{Z}^d : g_x(y) > g(x)\}, \\ \Delta_x &:= \{y \in Q_x : y \text{ adjacent to } \mathbb{Z}^d \setminus Q_x,\ y \text{ not adjacent to } G_x\}, \\ D_x &:= \{y \in Q_x : y \text{ adjacent to } G_x\}. \end{aligned}$$

The next lemma, which is analogous to Lemma 2.2 of Alexander (1990), summarizes some basic properties of the quantities we have defined. Let m_F denote the mean of the distribution F.

LEMMA 3.3. *Assume that conditions* (3.5) *and* (3.6) *hold. There exists a constant* C_{11} *such that if* $|x| \geq C_{11}$ *then the following hold.*

(i) *If* $y \in Q_x$, *then* $g(y) \leq 2g(x)$ *and* $|y| \leq 2d|x|$.
(ii) *If* $y \in \Delta_x$, *then* $s_x(y) \geq C_{10}|x|^{1/2}(\log|x|)/2$.
(iii) *If* $y \in D_x$, *then* $g_x(y) \geq 5g(x)/6$.

PROOF. (i) Suppose $g(y) > 2g(x)$ and $g_x(y) \leq g(x)$. Then using (1.2) and (1.9),

$$2g(x) < g(y) \leq ET(0, y) = g_x(y) + s_x(y) \leq g(x) + s_x(y),$$

so from (1.8), $s_x(y) \geq g(x) > C_{10}|x|^{1/2}\log|x|$, provided $|x| \geq C_{11}$. Thus $y \notin Q_x$ and the first conclusion in (i) follows. The second conclusion then follows from (1.8).

(ii) Note that $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap Q_x^c \cap G_x^c$ and $i \leq d$. From (i) we have $|y| \leq 2d|x|$, so $|z| < (2d+1)|x|$, provided $|x| > 1$. Since $z \notin Q_x$ we must then have $s_x(z) > C_{10}|x|^{1/2}\log|x|$, while using (1.9),

$$m_F \geq ET(0, \pm e_i) = s_x(\pm e_i) + g_x(\pm e_i) \geq s_x(\pm e_i) - \mu.$$

Consequently

$$s_x(y) \geq s_x(z) - s_x(\pm e_i) \geq C_{10}|x|^{1/2}\log|x| - m_F - \mu \geq C_{10}|x|^{1/2}(\log|x|)/2.$$

(iii) As in (ii) we have $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap G_x$ and $i \leq d$. Therefore using (1.9),

$$g_x(y) \geq g_x(z) - g_x(\pm e_i) \geq g(x) - \mu \geq 5g(x)/6. \qquad \square$$

The notion of the Q_x-skeleton of a lattice path from 0 to nx was defined in Remark 1.7. Given such a skeleton $(v_0, \dots, v_m)$, abbreviated (v_i), of some lattice path, we divide the corresponding indices into two classes, corresponding to "short" and "long" increments:

$$S((v_i)) := \{i: 0 \leq i < m-1,\ v_{i+1} - v_i \in \Delta_x\},$$
$$L((v_i)) := \{i: 0 \leq i < m-1,\ v_{i+1} - v_i \in D_x\}.$$

Note that the final index m is in neither class, and by Lemma 3.3(ii),

$$j \in S((v_i)) \quad \text{implies} \quad s_x(v_{j+1} - v_j) > C_{10}|x|^{1/2}(\log|x|)/2. \tag{3.9}$$

The next result is analogous to Lemma 2.3 of Alexander (1990), though the proof is significantly different. Lemma 1.6 shows that it is the main ingredient in establishing CHAP.

PROPOSITION 3.4. *Assume that conditions* (3.5) *and* (3.6) *hold. There exists a constant* C_{12} *such that if* $|x| \geq C_{12}$ *then for sufficiently large n there exists a lattice path from* 0 *to* nx *with* Q_x*-skeleton of* $2n+1$ *or fewer vertices.*

PROOF. Let $(v_0, \dots, v_m)$ be a Q_x-skeleton of some lattice path and let

$$Y_i := ET(v_i, v_{i+1}) - T(v_i, v_{i+1}).$$

We need an exponential bound for Y_i. We have

$$Y_i \le ET(v_i, v_{i+1}) \le m_F |v_{i+1} - v_i|_1 \le m_F d^{1/2} |v_{i+1} - v_i|,$$

so there exists a constant C_{13} such that if $|x| \ge C_{13}$ and

$$u > (2d)^{1/2} C_7 |v_{i+1} - v_i|$$

(with C_7 from (3.7)) then

$$P\big[Y_i > u|x|^{1/2}\big] = 0.$$

By Lemma 3.3(i),

$$|v_{i+1} - v_i| \le 2d|x|,$$

so for $0 \le u \le (2d)^{1/2} C_7 |v_{i+1} - v_i|$, by (3.7),

$$P\big[Y_i > u|x|^{1/2}\big] \le P\big[Y_i > (2d)^{-1/2} u |v_{i+1} - v_i|^{1/2}\big] \le C_5 \exp\big(-(2d)^{-1/2} C_6 u\big).$$

It follows that, for $\beta < (2d)^{-1/2} C_6$,

$$E \exp\big(\beta Y_i / |x|^{1/2}\big) \le C_5 (2d)^{-1/2} C_6 \big/ \big((2d)^{-1/2} C_6 - \beta\big).$$

In particular, for $\beta_0 := (8d)^{-1/2} C_6$,

$$E \exp\big(\beta_0 Y_i / |x|^{1/2}\big) \le 2C_5. \tag{3.10}$$

Let $Y_0', \dots, Y_{m-1}'$ be independent r.v.'s with Y_i' having the distribution of Y_i. Let $T(0, w; (v_j))$ be the minimum passage time among all lattice paths from 0 to site w with Q_x-skeleton (v_j). By (4.13) of Kesten (1986), or Theorem 2.3 of Alexander (1993), for all $t \ge 0$,

$$P\left(\sum_{i=0}^{m-1} Y_i' \ge t\right) \ge P\left(\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, v_m; (v_j)\big) \ge t\right).$$

Therefore for $C_{14} := 2d(8d)^{1/2}/C_6$, by (3.10),

$$\begin{aligned} &P\left[\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, v_m; (v_j)\big) > C_{14} m |x|^{1/2} \log|x|\right] \\ &\quad \le P\left[\exp\left(\beta_0 \sum_{i=0}^{m-1} Y_i' / |x|^{1/2}\right) > \exp\big(\beta_0 C_{14} m \log|x|\big)\right] \\ &\quad \le \big(2C_5 \exp(-\beta_0 C_{14} \log|x|)\big)^m. \end{aligned} \tag{3.11}$$

By Lemma 3.3(i), for some constant C_{15} there are at most $(C_{15}|x|^d)^m$ Q_x-skeletons with $m+1$ vertices. Since $\beta_0 C_{14} > d$, with (3.11) this shows that for some constants C_{16} and C_{17}, for all $m \ge 1$ and all $|x| \ge C_{17}$,

$$P\Bigg[\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, v_m; (v_j)\big) > C_{14} m |x|^{1/2} \log|x| \text{ for some } Q_x\text{-skeleton with } m+1 \text{ vertices}\Bigg] \le \exp\big(-C_{16} m \log|x|\big).$$

This in turn yields that for some constant C_{18}, for all $|x| \geq C_{18}$,

$$P\Bigg[\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, v_m; (v_j)\big) > C_{14}\, m|x|^{1/2} \log|x|$$

(3.12)
$$\text{for some } m \geq 1 \text{ and some } Q_x\text{-skeleton with } m+1 \text{ vertices}\Bigg]$$

$$\leq 2\exp\big(-C_{16} \log|x|\big).$$

Now let $\omega = \{t_b \colon b \text{ a nearest-neighbor bond in } \mathbb{Z}^d\}$ be a fixed configuration of passage times (to be further specified later) and let $(v_0, \dots, v_m)$ be the Q_x-skeleton of a route in ω from 0 to nx. Then since $v_{i+1} - v_i \in Q_x$,

$$mg(x) \geq \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx) = ng(x)$$

so

(3.13)
$$n \leq m.$$

By (3.4),

$$P\big[T(0, nx) \leq ng(x) + n\big] \to 1 \quad \text{as } n \to \infty,$$

so by (3.12) if n is large we can choose ω so that

(3.14)
$$T\big(0, nx; (v_j)\big) = T(0, nx) \leq ng(x) + n$$

and

$$\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, nx; (v_j)\big) \leq C_{14}\, m|x|^{1/2} \log|x|.$$

Then for some constant C_{19}, if $|x| \geq C_{19}$ then by (3.13) and (3.14),

(3.15)
$$\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) \leq ng(x) + n + C_{14}\, m|x|^{1/2} \log|x|$$

$$\leq ng(x) + 2C_{14}\, m|x|^{1/2} \log|x|.$$

But by (3.9),

$$\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) = \sum_{i=0}^{m-1} \big(g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)\big)$$

$$\geq g_x(nx) + |S((v_i))| C_{10} |x|^{1/2} (\log|x|)/2,$$

which with (3.15) yields

(3.16)
$$|S((v_i))| \leq 4C_{14}\, m / C_{10} = m/4.$$

At the same time, using Lemma 3.3(iii),

$$\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) = \sum_{i=0}^{m-1} \big(g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)\big)$$

$$\geq 5|L((v_i))| g(x)/6.$$

With (3.15) and (1.8) this yields that there is a constant C_{20} such that, provided $|x| \geq C_{20}$,

$$|L((v_i))| \leq 6n/5 + 2C_{14}m(d/|x|)^{1/2}\log|x| \leq 6n/5 + m/8.$$

This and (3.16) give

$$m = |L((v_i))| + |S((v_i))| + 1 \leq 6n/5 + 3m/8 + 1,$$

which, for n large, implies $m \leq 2n$, which proves the proposition. □

PROOF OF THEOREM 3.2. By Proposition 3.4 and Lemma 1.6, h satisfies CHAP$(1/2, \varphi, C_{12}, C_{10}, 2d+1, 2)$. By Theorem 1.8, it follows that for some C_4 and M, (3.8) is valid for all $x \in \mathbb{Z}^d$ with $|x| \geq M$. By increasing C_4 if necessary, we may assume $M = 1$. □

It is perhaps worth summarizing how we found a lattice path as in the statement of Proposition 3.4. Let us call a lattice path γ from 0 to nx *fast* if $T(\gamma) \leq ng(x) + n$ [see (3.14)] and call the Q_x-skeleton of a path from 0 to nx *good* if it consists of $2n+1$ or fewer vertices, and *bad* otherwise. Note that "fast" is a property depending on the random configuration of passage times, but "good" and "bad" are deterministic properties. To show that good Q_x-skeletons exist, we show roughly that the probability that there exists a fast lattice path which follows a bad Q_x-skeleton is strictly less than the probability that there exists a fast lattice path. This is essentially the same method that was used in Alexander (1990) to establish CHAP for the connectivity function of Bernoulli bond percolation, on the way to the result (1.3) of Example 1.3, provided we replace "fast" with "consisting entirely of occupied bonds." We will see in the next section that this method also works for longest common subsequences and for oriented first-passage percolation. Analogs of the method may well work in other problems in which $h(x)$ involves an expected value or probability associated with an optimal path of some kind from 0 to x.

PROOF OF THEOREM 3.1. The outer bound for $B(t)$ is part of the result (3.2) from Kesten (1993), but we include a short proof here for convenience. Suppose $c, t > 0$ and

$$\text{there exists } x \in B(t) \cap \mathbb{Z}^d \quad \text{with } x \notin \left(t + ct^{1/2}\log t\right)B_0. \tag{3.17}$$

Then $T(0,x) \leq t$, but

$$ET(0,x) \geq g(x) > t + ct^{1/2}\log t,$$

so

$$ET(0,x) - T(0,x) \geq g(x) - t \geq ct^{1/2}\log t. \tag{3.18}$$

If $g(x) \leq 2t$ then by (1.8) $|x|/2d \leq t$ so (3.18) yields

$$ET(0,x) - T(0,x) \geq (2d)^{-1}c|x|^{1/2}\log|x|.$$

If $g(x) > 2t$ then by the first inequality in (3.18), provided $|x|$ is large,

$$ET(0,x) - T(0,x) \geq g(x)/2 \geq (2d)^{-1}c|x|^{1/2}\log|x|.$$

Thus if (3.17) occurs for arbitrarily large t then

$$ET(0,x) - T(0,x) \ge (2d)^{-1} c|x|^{1/2} \log|x| \quad \text{for infinitely many } x.$$

However, for $|x|$ large, by (3.7),

$$P\Big[ET(0,x) - T(0,x) \ge (2d)^{-1} c|x|^{1/2} \log|x| \Big]$$
$$\le C_5 \exp\big(-C_6 (2d)^{-1} c \log|x| \big),$$

which is summable over x if we choose

$$c := C_{21} := 2d(d+1)/C_6.$$

Thus with probability 1, for sufficiently large t,

$$B(t) \subset \big(t + C_{21} t^{1/2} \log t\big) B_0. \tag{3.19}$$

Turning to the inner bound for $B(t)$, suppose $c > 0$ and suppose that for arbitrarily large t there exists $x \in \mathbb{Z}^d$ with $x \in (t - ct^{1/2} \log t) B_0$ but $x \notin B(t)$. Then

$$|x|/d \le t, \qquad T(0,x) > t \quad \text{and} \quad g(x) \le t - ct^{1/2} \log t$$

and for large t we must have $|x| > 1$. Hence by Theorem 3.2,

$$ET(0,x) \le t - ct^{1/2} \log t + C_4 |x|^{1/2} \log|x|$$
$$\le t - (c/d - C_4)|x|^{1/2} \log|x|,$$

so that

$$T(0,x) - ET(0,x) \ge (c/d - C_4)|x|^{1/2} \log|x|.$$

Since $x \notin B(t)$ and t is arbitrarily large, this must be true for infinitely many x. But as in the proof of (3.19), with probability 1 this does not happen infinitely often if c is chosen large enough. □

4. Longest common subsequences and oriented first-passage percolation. In this section we apply Theorem 1.4 to Examples 1.2 and 1.4 on oriented first-passage percolation and the LCS problem. As observed by several authors [see, e.g., Arratia and Waterman (1994)], the LCS problem can be reformulated as a dependent version of oriented first-passage percolation, as follows. We use the notation of Example 1.4. Let $e^* := (1, \dots, 1) \in \mathbb{Z}^d$. In the positive quadrant of the integer lattice with all nearest-neighbor bonds, we add a diagonal bond b_x from $x - e^*$ to x for each x with strictly positive coordinates. We define the passage time of each vertical and horizontal bond to be 1, and define the passage time of b_x to be 0 if $X^{(1)}_{x_1} = \cdots = X^{(d)}_{x_d}$, and ∞ otherwise. For purposes of the LCS problem, we extend the definition of oriented lattice paths from Example 1.2 by allowing them to include diagonal bonds b_x traversed from $x - e^*$ to x. We then have $T(0,x) = U(x)$. This enables us to use $h(x) = ET(0,x)$, and to define $B(t)$ as in Section 3, for both independent oriented first-passage percolation and the LCS problem. Again, $B_0 := \{x \in \mathbb{R}^d\colon\ g(x) \le 1\}$. Note that for both models, if $0 =$

$v_0, v_1, \dots, v_m = nx$ are a sequence of sites (in order) in an oriented lattice path from 0 to nx, then the passage times $T(v_i, v_{i+1})$, $i = 0, \dots, m-1$, are independent.

For independent oriented first-passage percolation, we replace the hypothesis (3.5) with

$$F(0) < p_{c,\mathrm{or}}(\mathbb{Z}_+^d). \tag{4.1}$$

where $p_{c,\mathrm{or}}(\mathbb{Z}_+^d)$ denotes the critical probability for the existence of infinite oriented lattice paths consisting of occupied bonds. Given $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, let $\bar{z} := (z_1 + \cdots + z_d)/d$. The natural time constant is $\mu_{\mathrm{or}} := g(e^*)$. The proof of Kesten (1986) that (3.5) is equivalent to positivity of the time constant $\mu = g(e_1)$, and the proof in Kesten (1993) of (3.7) then carry over essentially without change to their oriented analogs. Inequalities (1.8) and (1.9) are no longer necessarily valid, but we still have $g_x \le g$. Since g is convex and symmetric about the diagonal through 0 and e^*, we have $g(z) \ge g(\bar{z}e^*) = g(e^*)|z|_1/d$. From subadditivity and symmetry we have $g(z) \le g(e_1)|z|_1$. Therefore we can replace Lemma 3.3(i) with the statement that if $y \in Q_x$ then $g(y) \le 2g(x)$ and $|y| \le 2d^{3/2}g(e_1)|x|/g(e^*)$. The rest of Lemma 3.3 carries over provided that $2d + 1$ is replaced with $2d^{3/2}g(e_1)/g(e^*) + 1$ in the definition of Q_x preceding that lemma, and then the proofs of Theorems 3.1 and 3.2 carry over. Thus we obtain the following.

THEOREM 4.1. *For independent oriented first-passage percolation in d dimensions, under the hypotheses* (4.1) *and* (3.6), *for some constants $C_i(F, d)$, for all $x \in \mathbb{Z}^d$ with $|x| > 1$,*

$$g(x) \le ET(0, x) \le g(x) + C_{22}|x|^{1/2}\log|x|.$$

Further, with probability 1, *for all sufficiently large t,*

$$\left(1 - C_{23}t^{-1/2}\log t\right)B_0 \subset t^{-1}B(t) \subset \left(1 + C_{24}t^{-1/2}\log t\right)B_0.$$

Turning to longest common subsequences, we observe first that there is no analog of (3.6). The proof of Proposition 5.8 of Kesten (1986) shows that the natural time constant $\mu_{\mathrm{LCS}} := g(e^*)$ is positive provided the letter distribution is nondegenerate, so there is no need for an analog of (4.1). Finiteness of μ_{LCS} means that for d letter sequences each of length n, the LCS length as a fraction of n is asymptotically strictly less than 1. Further, as in Lemma 2.3 of Alexander (1994), by an inequality of Azuma (1967), (3.7) can be replaced by

$$P\left[|U(x) - EU(x)| < \lambda\right] \le 2\exp\left(-\lambda^2/2|x|_1\right)$$

for all $\lambda \ge 0$. But in fact the use of (3.7) in the proof of the analog of Proposition 3.4 can be avoided altogether, as Azuma's inequality yields directly the following replacement for (3.11): for all $\lambda > 0$,

$$P\left[\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\left(0, v_m; (v_j)\right) > \lambda\right] \le \exp\left(-\lambda^2/2|v_m|_1\right).$$

In particular, when the final site $v_m = nx$ and $m \geq n$, for all $c > 0$,

$$P\left[\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T\big(0, v_m; (v_j)\big) > cm(|x|\log|x|)^{1/2}\right]$$
$$\leq \exp\big(-c^2 d^{1/2} m \log|x|\big).$$

This yields an analog of (3.12), with the following changes: $\log|x|$ is replaced by $(\log|x|)^{1/2}$, including in the definition of Q_x; $m \geq 1$ is replaced by $m \geq n$, and the restriction that $v_m = nx$ is added. The rest of the proof of Proposition 3.4 remains similar, with $\log|x|$ replaced by $(\log|x|)^{1/2}$. We then obtain the following analog of Theorems 3.1, 3.2 and 4.1; rather than formulate it entirely in terms of $g(x)$, it is more natural for the LCS problem to use the function

$$\psi(x) := \lim_n EL(nx)/n = (|x|_1 - g(x))/d.$$

THEOREM 4.2. *Let $X_1^{(i)} X_2^{(i)}, \ldots,$ $i = 1, \ldots, d$, be d i.i.d. sequences of letters selected from a finite alphabet A, with each letter having the nondegenerate distribution α. There exist constants $C_i(\alpha, d)$ as follows. For all $x \in \mathbb{Z}^d$ with $|x| > 1$,*

$$\psi(x) \geq EL(x) \geq \psi(x) - C_{25}(|x|\log|x|)^{1/2}. \tag{4.2}$$

With probability 1, uniformly in x as $|x| \to \infty$,

$$|L(x) - \psi(x)| = O\Big((|x|\log|x|)^{1/2}\Big).$$

Finally, with probability 1, for all sufficiently large t,

$$\big(1 - C_{26} t^{-1/2} \log t\big) B_0 \subset t^{-1} B(t) \subset \big(1 + C_{27} t^{-1/2} \log t\big) B_0. \tag{4.3}$$

Taking $d = 2$ and $x = (n, n)$ in (4.2) reproduces the result (1.5), though unlike the proof in Alexander (1994), the present proof does not readily yield an explicit value for C_{25}. For the coin-tossing case of alphabet $A = \{0, 1\}$ and $\alpha(0) = \alpha(1) = 1/2$, the value of $\psi(e^*)$, particularly for $d = 2$, has been an object of some study; see Dančík and Paterson (1995), Deken (1979) and Sankoff and Kruskal (1983). For $d = 2$, $\psi(e^*)$ is the c of (1.5), and simulations performed by Eggert and Waterman [see Alexander (1994)] suggest that its value is near 0.81.

There is no shape result for $L(x)$ analogous to the result (4.3) for $U(x)$, because $\{x \in \mathbb{Z}_+^d\colon L(x) \leq t\}$ includes the unbounded set $\{x \in \mathbb{Z}_+^d\colon x_i \leq t$ for some $i \leq d\}$ for all $t \geq 0$.

5. The case of $\nu = 0$. We will prove Theorem 1.9. We begin with an analog of Proposition 2.1. Note that here, however, the reduction in the exponent β is not quite "for free," as the analog of the $\tilde{C}$ of Proposition 2.1 is z/β, which unlike $\tilde{C}$ does increase as β decreases.

PROPOSITION 5.1. *Suppose $C > 1$, $K > 1$, $M > 1$, $a > 1$, $\varphi \in \Phi$ and h is a nonnegative subadditive function on $\mathbb{Z}^d$ with $h(x) \le r|x|$ for all x for some $r > 1$. There exists a constant $z_0(\varphi, C, K, a, r, d, M)$ such that if $z \ge z_0$, $\beta \in (0, 1]$, and h satisfies both $GAP(\beta, \varphi, M, z/\beta)$ and $CHAP(0, \varphi, M, C, K, a)$, then h satisfies $GAP(\beta', \varphi, M, z/\beta')$ where $\beta' := \beta/(1+\beta) < \beta$.*

PROOF. Let c_0, c_1 and c_2 be as in the proof of Proposition 2.1, assume (2.1) holds, and let $c_4' := 1/(c_1 + c_0)$.

Fix $M > 1$ and x with $|x| \ge M$. Again we consider two cases.

Case 1. $|x| \ge c_2^\beta c_4' z M^{1+\beta}$. Let

$$q := \left(c_2^\beta c_4' z\right)^{1/(1+\beta)} |x|^{\beta/(1+\beta)}.$$

Assume $z \ge c_2/c_4'$; then analogously to (2.11), using the assumption of Case 1, we obtain $|x| \ge qM$. Further, we have $q \ge c_2 \ge 1$. As in the proof of Proposition 2.1, we obtain the analog of (2.9) with $\nu = 0$:

$$h(x) \le g(x) + (c_1 + c_0) q \varphi(|x|) + z\beta^{-1} c_2^\beta q^{-\beta} |x|^\beta \varphi(|x|) + rM\varphi(|x|). \tag{5.1}$$

Now

$$(c_1 + c_0) q^{(1+\beta)} = c_2^\beta z |x|^\beta$$

so, analogously to the first line of (2.13),

$$(c_1 + c_0) q + z\beta^{-1} c_2^\beta q^{-\beta} |x|^\beta = \left(1 + \beta^{-1}\right)(c_1 + c_0) q. \tag{5.2}$$

Assume $z \ge c_2 e^2 / c_4'$; using $(1+\beta)^{(1+\beta)/\beta} \le e^2$, we obtain

$$(c_1 + c_0)^{1+\beta} (1+\beta)^{(1+\beta)} c_4' z c_2^\beta = (c_1 + c_0)^\beta (1+\beta)^{(1+\beta)} z c_2^\beta \le z^{1+\beta},$$

so, taking the $1/(1+\beta)$ power and then dividing by β,

$$(c_1 + c_0)\left(1 + \beta^{-1}\right)\left(c_2^\beta c_4' z\right)^{1/(1+\beta)} \le z\beta^{-1}.$$

Therefore, analogously to the latter part of (2.13),

$$\left(1 + \beta^{-1}\right)(c_1 + c_0) q \le z\beta^{-1} |x|^{\beta/(1+\beta)}. \tag{5.3}$$

Assume $z \ge rM$; then by (5.1), (5.2) and (5.3), analogously to (2.15),

$$\begin{aligned} h(x) &\le g(x) + z\beta^{-1}|x|^{\beta/(1+\beta)}\varphi(|x|) + z\varphi(|x|) \\ &\le g(x) + z(\beta')^{-1}|x|^{\beta'}\varphi(|x|). \end{aligned} \tag{5.4}$$

Case 2. $M \le |x| < c_2^\beta c_4' z M^{1+\beta}$. Assume $z \ge \exp(c_2 c_4' r^2 M^2 / e)$. Since $(\beta u)^{1/\beta} \le e^{u/e}$ for all β, $u > 0$, we then have

$$\beta c_2^\beta c_4' (rM)^{1+\beta} \le \beta c_2 c_4' r^2 M^2 \le z^\beta$$

so that

$$|x| < c_2^\beta c_4' z M^{1+\beta} \le (z/r)^{1+\beta}/\beta < (z/r\beta)^{1+\beta}$$

and then, analogously to (2.16),

$$\begin{aligned} h(x) &\le r|x| = r|x|^{1/(1+\beta)}|x|^{\beta/(1+\beta)} \\ &\le z\beta^{-1}|x|^{\beta/(1+\beta)} \le g(x) + z(\beta')^{-1}|x|^{\beta'}\varphi(|x|). \end{aligned}$$

This and (5.4) prove the proposition under (2.2), with

$$z_0 := \max\bigl(\exp\bigl(c_2 c_4' r^2 M^2/e\bigr), rM, c_2 e^2/c_4'\bigr).$$

The result without (2.2) then follows as in the proof of Proposition 2.1. □

PROOF OF THEOREM 1.9. Suppose h satisfies CHAP$(0, \varphi, M, C, K, a)$. In the notation of Proposition 5.1, we may assume that $C > 1$, $K > 1$, $r > 1$ and $M > 1$. Let $z := \max(r/\varphi(M), z_0(\varphi, C, K, a, r, d, M))$. Since $h(x) \le r|x|$, h satisfies GAP$(1, \varphi, M, r/\varphi(M))$ and hence also GAP$(1, \varphi, M, z)$. By Proposition 5.1, if h satisfies GAP$(1/n, \varphi, M, nz)$ for some n then h also satisfies GAP$(1/(n+1), \varphi, M, (n+1)z)$. Therefore h satisfies GAP$(1/n, \varphi, M, nz)$ for all n, that is,

$$g(x) \le h(x) \le g(x) + nz|x|^{1/n}\varphi(|x|)$$
$$\text{for all } x \in \mathbb{Z}^d \text{ with } |x| \ge M \text{ and all } n \ge 1.$$

Taking $n = 1 + \lfloor \log|x| \rfloor$ completes the proof. □

6. The connectivity function for bond percolation. In this section we will apply Theorem 1.9 to Example 1.3 on the connectivity function for Bernoulli bond percolation. Let $x \leftrightarrow y$ denote the event that site x is connected to site y by a path of occupied bonds, and fix $0 < p < p_c(\mathbb{Z}^d)$. Here $p_c(\mathbb{Z}^d)$ denotes the percolation critical point, above which there exists a.s. an infinite connected component of occupied bonds. From the Harris–FKG inequality [Harris (1960)] we have $P_p[0 \leftrightarrow x + y] \ge P_p[0 \leftrightarrow x]P_p[x \leftrightarrow x + y]$; from this and translation invariance it follows that the function $h(x) = -\log P_p[0 \leftrightarrow x]$ is subadditive. Recall that $\sigma(p, \theta)$ denotes the inverse correlation length in direction θ, which is strictly positive for all $p < p_c(\mathbb{Z}^d)$ [see Grimmett (1990), Theorem 5.78]; for $\theta = x/|x|$ we have $\sigma(p, \theta) = g(x)/|x|$, which depends on x only through θ. One would expect that something like (1.3), or equivalently (1.4), is true in all dimensions, and in fact Lemma 2.4 of Alexander (1990) shows that in all dimensions h satisfies CHAP with exponent 0 and correction factor $\varphi(t) = \log t$. [Strictly speaking, (1.11) is only established in Alexander (1990) with Q_x replaced by a set of sites adjacent to Q_x. This is because, in the notation of Remark 1.7, in Alexander (1990), v_{i+1} is defined to be v'_{i+1} rather than the last site in γ before v'_{i+1}. But since the values of h at adjacent sites do not differ by more than a factor of p, slight technical modifications to the proof in Alexander (1990) yield CHAP exactly in the form (1.11).] Theorem 1.9 only comes close to (1.3) and (1.4), yielding an extra factor of $\log|x|$ in the result, but it applies to all dimensions.

THEOREM 6.1. *For Bernoulli bond percolation on the d-dimensional integer lattice, for each $0 < p < p_c(\mathbb{Z}^d)$, uniformly in x as $|x| \to \infty$,*

$$-\sigma(p, \theta)|x| \geq \log P_p[0 \leftrightarrow x] \geq -\sigma(p, \theta)|x| - O\big((\log|x|)^2\big).$$

where $\theta = x/|x|$ and $\sigma(p, \theta)$ is the inverse correlation length in direction θ.

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REFERENCES

ALEXANDER, K. S. (1990). Lower bounds on the connectivity function in all directions for Bernoulli percolation in two and three dimensions. *Ann. Probab.* **18** 1547–1562.

ALEXANDER, K. S. (1993). A note on some rates of convergence in first-passage percolation. *Ann. Appl. Probab.* **3** 81–90.

ALEXANDER, K. S. (1994). The rate of convergence of the mean length of the longest common subsequence. *Ann. Appl. Probab.* **4** 1074–1082.

ALEXANDER, K. S., CHAYES, J. T. and CHAYES, L. (1990). The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation. *Comm. Math. Phys.* **131** 1–50.

ARRATIA, R. and WATERMAN, M. S. (1994). A phase transition for the score in matching random sequences allowing deletions. *Ann. Appl. Probab.* **4** 200–225.

AZUMA, K. (1967). Weighted sums of certain dependent random variables. *Tôhuku Math. J.* **19** 357–367.

BRICMONT, J. and FRÖHLICH, J. (1985a). Statistical mechanical methods in particle structure analysis of lattice field theories. I. General results. *Nuclear Phys. B* **251** 517–552.

BRICMONT, J. and FRÖHLICH, J. (1985b). Statistical mechanical methods in particle structure analysis of lattice field theories. II. Scalar and surface models. *Comm. Math. Phys.* **98** 553–578.

CAMPANINO, M., CHAYES, J. T. and CHAYES, L. (1991). Gaussian fluctuations of connectivities in the subcritical regime of percolation. *Probab. Theory Related Fields* **88** 269–341.

CHAYES, J. T. and CHAYES, L. (1986). Ornstein–Zernike behavior for self-avoiding walks at all noncritical temperatures. *Comm. Math. Phys.* **105** 221–238.

COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603.

DANČÍK, V. and PATERSON, M. (1994). Upper bound for the expected length of a longest common subsequence of two binary sequences. *Random Structures and Algorithms* **6** 449–458.

DEKEN, J. (1979). Some limit results for longest common subsequences. *Discrete Math.* **26** 17–31.

EDEN, M. (1961). A two-dimensional growth process. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **4** 223–239. Univ. California Press, Berkeley.

HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** 13–20.

GRIMMETT, G. (1989). *Percolation*. Springer, New York.

KESTEN, H. (1986). Aspects of first passage percolation. *Ecole d'Eté de Probabilités de Saint Flour XIV 1984. Lecture Notes in Math.* **1180** 125–264. Springer, New York.

KESTEN, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3** 296–338.

KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

ORNSTEIN, L. S. and ZERNIKE, F. (1914). Accidental deviations of density and opalescence at the critical point of a single substance. *Proceedings of the Section of Sciences* (*Academy of Sciences, Amsterdam*) **17** 793–806.

RICHARDSON, D. (1973). Random growth in a tesselation. *Proc. Cambridge Philos. Soc.* **74** 515–528.

SANKOFF, D. and KRUSKAL, J. B., eds. (1983). *Time Warps, String Edits, and Macromolecules: The Theory and Practice of Sequence Comparison*. Addison-Wesley, Reading, MA.

SMYTHE, R. and WIERMAN, J. C. (1977). *First Passage Percolation on the Square Lattice, Lecture Notes in Math.* **671**. Springer, Berlin.

DEPARTMENT OF MATHEMATICS DRB 155
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90089-1113
E-MAIL: alexandr@math.usc.edu