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STRONG LAWS FOR LOCAL QUANTILE PROCESSES

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We show that increments of size h_n from the uniform quantile and uniform empirical processes in the neighborhood of a fixed point $t_0 \in (0,1)$ may have different rates of almost sure convergence to 0 in the range where $h_n \to 0$ and $nh_n/\log n \to \infty$. In particular, when $h_n = n^{-\lambda}$ with $0 < \lambda < 1$, we obtain that these rates are identical for $1/2 < \lambda < 1$, and distinct for $0 < \lambda < 1/2$. This phenomenon is shown to be a consequence of functional laws of the iterated logarithm for local quantile processes, which we describe in a more general setting. As a consequence of these results, we prove that, for any $\varepsilon > 0$, the best possible uniform almost sure rate of approximation of the uniform quantile process by a normed Kiefer process is not better than $O(n^{-1/4}(\log n)^{-\varepsilon})$.

1. Introduction and statement of main results. Denote by $\mathbb{U}_n(s) = n^{-1}\#\{U_i \le s\colon 1 \le i \le n\}$ for $-\infty < s < \infty$ the empirical distribution function, and by $\mathbb{V}_n(t) = \inf\{s \ge 0\colon \mathbb{U}_n(s) \ge t\}$ for $0 \le t \le 1$, $\mathbb{V}_n(t) = 0$ for $t < 0$, $\mathbb{V}_n(t) = \mathbb{V}_n(1)$ for $t > 1$, the empirical quantile function based upon the first $n \ge 1$ observations from an i.i.d. sequence $U_1, U_2, \dots$ of uniform $[0,1]$ random variables, Here, $\#A$ denotes the cardinality of A. We are concerned with the *local behavior* of the uniform quantile process $\beta_n(t) = n^{1/2}(\mathbb{V}_n(t) - t)$ and of the uniform empirical process $\alpha_n(t) = n^{1/2}(\mathbb{U}_n(t) - t)$ for $-\infty \le t \le \infty$, in a neighorhood of $t_0 \in [0,1)$. For $a \ge 0$ and $-\infty < s,\ t < \infty$, introduce the increment functions

$$\zeta_n(a, t; s) = \beta_n(t + as) - \beta_n(t) \quad \text{and} \quad \xi_n(a, t; s) = \alpha_n(t + as) - \alpha_n(t).$$

Because of the central role they play in nonparametric statistics, *local oscillations* of β_n and α_n have been very much investigated in the literature [refer to Csörgő and Révész (1981), Shorack and Wellner (1986), Csörgő and Horváth (1993)]. These are conveniently described through $\zeta_n(h_n, t; \cdot)$ and $\xi_n(h_n, t; \cdot)$, where $\{h_n\colon n \ge 1\}$ is a bounded sequence of positive constants satisfying conditions among (H.1–H.6) below. Set $\log_1 u = \log_+ u = \log(u \vee e)$ and $\log_p u = \log_+(\log_{p-1} u)$ for $p \ge 2$.

(H.1) (i) $h_n \downarrow 0$, (ii) $nh_n \uparrow$;
(H.2) $nh_n/\log_2 n \to \infty$;
(H.3) $nh_n/\log_+(1/h_n) \to \infty$;

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(H.4) $(\log(1/h_n))/\log_2 n \to c \in [0,\infty]$;
(H.5) $nh_n/\log_+(1/(h_n\sqrt{n})) \to \infty$;
(H.6) $(\log(1/(h_n\sqrt{n})))/\log_2 n \to d \in [-\infty,\infty]$.

Introduce the sequences of constants, depending upon h_n and $n \geq 1$,

$$
\begin{aligned}
a_n &= (2h_n \log_2 n)^{1/2},\\
b_n &= \Big(2h_n\big\{\log_+\big(1/\big(h_n\sqrt{n}\big)\big) + \log_2 n\big\}\Big)^{1/2},\\
c_n &= h_n + n^{-1/2}(\log_2 n)^{1/2},\\
d_n &= \big(2h_n\{\log_+(1/h_n) + \log_2 n\}\big)^{1/2}.
\end{aligned}
\tag{1.1}
$$

The following results are now well known. First, Kiefer (1972a), Mason (1988), Einmahl and Mason (1988) and Deheuvels and Mason (1990b) showed that, under (H.1) and (H.2),

$$
\begin{aligned}
&\limsup_{n\to\infty}\big\{\pm a_n^{-1}\zeta_n(h_n,0;1)\big\}\\
&\quad= \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\zeta_n(h_n,0;s)\Big\} = 1 \quad \text{a.s.},
\end{aligned}
\tag{1.2}
$$

and

$$
\begin{aligned}
&\limsup_{n\to\infty}\big\{\pm a_n^{-1}\xi_n(h_n,0;1)\big\}\\
&\quad= \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\xi_n(h_n,0;s)\Big\} = 1 \quad \text{a.s.}
\end{aligned}
\tag{1.3}
$$

Second, Mason (1984), Stute (1982), Mason, Shorack and Wellner (1983), Deheuvels and Mason (1992) and Deheuvels (1992), showed that, under (H.1)–(H.4),

$$
\begin{aligned}
&\limsup_{n\to\infty}\Big\{\sup_{0\le t\le 1-h_n} \pm d_n^{-1}\zeta_n(h_n;t;1)\Big\}\\
&\quad= \limsup_{n\to\infty}\Big\{\sup_{0\le t\le 1-h_n}\sup_{0\le s\le 1} \pm d_n^{-1}\zeta_n(h_n,t;s)\Big\} = 1 \quad \text{a.s.},
\end{aligned}
\tag{1.4}
$$

and

$$
\begin{aligned}
&\limsup_{n\to\infty}\Big\{\sup_{0\le t\le 1-h_n} \pm d_n^{-1}\xi_n(h_n;t;1)\Big\}\\
&\quad= \limsup_{n\to\infty}\Big\{\sup_{0\le t\le 1-h_n}\sup_{0\le s\le 1} \pm d_n^{-1}\xi_n(h_n,t;s)\Big\} = 1 \quad \text{a.s.}
\end{aligned}
\tag{1.5}
$$

A simple argument [see, e.g., Deheuvels and Mason (1994a)] extends the LIL in (1.3) to the following description of the *increments of size* h_n of α_n. Under (H.1) and (H.2), for each specified $t_0 \in [0,1)$,

$$
\begin{aligned}
&\limsup_{n\to\infty}\big\{\pm a_n^{-1}\xi_n(h_n,t_0;1)\big\}\\
&\quad= \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\xi_n(h_n,t_0;s)\Big\} = 1 \quad \text{a.s.}
\end{aligned}
\tag{1.6}
$$

The aim of this paper is to obtain the versions of (1.6) holding when ξ_n is replaced by ζ_n. In view of (1.2), (1.3) and (1.4), (1.5), one could expect this replacement to be possible without any further change in the statement of the results. In Theorem 1.1 below, we establish the unexpected fact that such is not the case when $t_0 \neq 0$.

THEOREM 1.1. *Under* (H.1), *and* (H.5), (H.6), *for any specified* $t_0 \in (0, 1)$, *we have*

$$\begin{aligned}(1.7)\qquad &\limsup_{n\to\infty}\{\pm b_n^{-1}\zeta_n(h_n, t_0; 1)\}\\ &\quad = \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm b_n^{-1}\zeta_n(h_n, t_0; s)\Big\} = 1 \quad a.s.\end{aligned}$$

REMARK 1.1. (i) The definitions (1.1) of a_n and b_n, allow us to distinguish the following three ranges of interest for $\{h_n\colon n \geq 1\}$ depending upon the relative magnitude of b_n relatively to a_n. Below, we give the corresponding versions of (1.7) in Theorem 1.1.

(a) The *large increment* case is that (H.6) holds with $d \in [-\infty, 0]$. It implies (H.2)–(H.5). Moreover, it entails that $(\log_+(1/(h_n\sqrt{n})))/\log_2 n \to 0$ and, for each $\kappa > 0$, ultimately as $n \to \infty$,

$$(1.8)\qquad h_n \geq n^{-1/2}(\log n)^{-\kappa}.$$

If we assume in addition that (H.1) is satisfied, we have

$$\begin{aligned}(1.9)\qquad &\limsup_{n\to\infty}\{\pm a_n^{-1}\zeta_n(h_n, t_0; 1)\}\\ &\quad = \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\zeta_n(h_n, t_0; s)\Big\} = 1 \quad \text{a.s.}\end{aligned}$$

(b) The *intermediate increment* case is that (H.6) holds with $d \in (0, \infty)$, or equivalently, when

$$(1.10)\qquad h_n = n^{-1/2}(\log n)^{-d+o(1)},$$

as $n \to \infty$. It implies (H.2)–(H.5). Under the additional assumption (H.1), we have

$$\begin{aligned}(1.11)\qquad &\limsup_{n\to\infty}\{\pm a_n^{-1}\zeta_n(h_n, t_0; 1)\}\\ &\quad = \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\zeta_n(h_n, t_0; s)\Big\} = (d+1)^{1/2} \quad \text{a.s.}\end{aligned}$$

(c) The *small increment* case is that (H.6) holds with $d = \infty$, or equivalently, when, for each $\kappa > 0$, we have ultimately as $n \to \infty$

$$(1.12)\qquad h_n \leq n^{-1/2}(\log n)^{-\kappa}.$$

Under this assumption, (H.5) becomes equivalent to

$$nh_n/\log n \to \infty, \tag{1.13}$$

so that (H.5) and (H.6) jointly imply (H.2)–(H.4). If, in addition, (H.1) holds, then

$$\begin{aligned}&\limsup_{n\to\infty}\{\pm a_n^{-1}\zeta_n(h_n, t_0; 1)\}\\ &\quad = \limsup_{n\to\infty}\Big\{\sup_{0\le s\le 1} \pm a_n^{-1}\zeta_n(h_n, t_0; s)\Big\} = \infty \quad \text{a.s.}\end{aligned} \tag{1.14}$$

(ii) In each of cases (a)–(c) above, (H.5) and (H.6) jointly imply (H.2)–(H.4) and (1.13).

(iii) In view of (1.6), (1.9), (1.11) and (1.14), for $t_0 \in (0, 1)$, the almost sure asymptotic rates of $\zeta_n(h_n, t_0; 1)$ and $\xi_n(h_n, t_0; 1)$ coincide for *large increments* but differ from each other for *small increments*. For *intermediate increments*, rates are identical, but limiting constants are different.

REMARK 1.2. Theorem 1.1 allows us to give the following answer to Open Question 2, page 495 in Shorack and Wellner (1986). Define a Kiefer process $\{K(n, t)\colon n \ge 1, t \ge 0\}$ [Kiefer (1972b)] by

$$K(n, t) = \sum_{i=1}^{n} \big(W_i(t) - tW_i(1)\big),$$

where $\{W_n(t)\colon t \ge 0\}$, $n = 1, 2, \dots,$ are i.i.d. standard Wiener processes. Komlós, Major and Tusnády (1975a, b, 1976) showed that $\{U_n\colon n \ge 1\}$ and $\{K(n, t)\colon n \ge 1, t \ge 0\}$ may be defined on a probability space $(\Omega_1, A_1, \mathbb{P}_1)$, in such a way that, with probability 1,

$$\sup_{0\le t\le 1} \big|\alpha_n(t) - n^{-1/2}K(n, t)\big| = O\Big(n^{-1/2}(\log n)^2\Big) \quad \text{as } n \to \infty. \tag{1.15}$$

A version of (1.15) for β_n is obtained via the uniform Bahadur–Kiefer representation [see, e.g., Bahadur (1966), Kiefer (1967, 1970), and Deheuvels and Mason (1990a)]

$$\begin{aligned}&\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\Big\{\sup_{0\le t\le 1}\big|\alpha_n(t) + \beta_n(t)\big|\Big\}\\ &\quad = 2^{-1/4} \quad \text{a.s.}\end{aligned} \tag{1.16}$$

By combining (1.15) and (1.16) with the observation that $K'(n, t) = K(n, t)$ is a Kiefer process, we obtain the following result due to Csörgő and Révész (1975). On $(\Omega_1, A_1, \mathbb{P}_1)$, we have

$$\begin{aligned}&\sup_{0\le t\le 1} \big|\beta_n(t) - n^{-1/2}K'(n, t)\big|\\ &\quad = O\Big(n^{-1/4}(\log n)^{1/2}(\log_2 n)^{1/4}\Big) \quad \text{as } n \to \infty.\end{aligned} \tag{1.17}$$

It is natural to investigate the optimality of (1.17) by the following question. *Does there exist a probability space* $(\Omega_2, A_2, \mathbb{P}_2)$ *carrying* $\{\beta_n(t)\colon 0 \le t \le 1\}$ *and a Kiefer process* $K''(n,t)$, *with*

$$\sup_{0\le t\le 1} \left|\beta_n(t) - n^{-1/2}K''(n,t)\right| = O\left(n^{-1/4}(\log n)^{-\varepsilon}\right) \quad \textit{as } n\to\infty, \tag{1.18}$$

for some $\varepsilon > 0$? Csörgő and Révész (1975, 1981) (see their Remark 4.5.1, page 147) conjectured that the rate in (1.17) is "probably far from being the best." The following corollary of Theorem 1.1 disproves in part this conjecture by showing that (1.18) is impossible.

COROLLARY 1.1. *For any Kiefer process* $\{K''(n,t)\colon n\ge 1,\ 0\le t\le 1\}$ *defined on the same probability space as* $\{\beta_n(t)\colon 0\le t\le 1\}$, *we have, with probability* 1 *for each* $\varepsilon > 0$,

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{\varepsilon}\left\{\sup_{0\le t\le 1}\left|\beta_n(t) - n^{-1/2}K''(n,t)\right|\right\} = \infty. \tag{1.19}$$

PROOF. Let β_n and $K''(n,t)$ be defined on the same probability space. Fix any $\varepsilon > 0$ and select a $t_0 \in (0,1)$. Put $d = \varepsilon/2$, $h_n = n^{-1/2}(\log n)^{-d}$ and $\rho_n = n^{1/4}(\log n)^{d/2}(\log_2 n)^{-1/2}$. Set

$$\eta_n(h_n, t; s) = n^{-1/2}\left(K(n, t+h_n s) - K(n,t)\right),$$
$$\eta_n''(h_n, t; s) = n^{-1/2}\left(K''(n, t+h_n s) - K''(n,t)\right),$$

for $s \ge 0$ and $0 \le t \le 1$. It follows from (1.6), (1.15) and our choice of h_n that

$$\begin{aligned}\limsup_{n\to\infty}\rho_n\left|\xi_n(h_n, t_0; 1)\right| &= \limsup_{n\to\infty}\rho_n\left|\eta_n(h_n, t_0; 1)\right| \\ &= \limsup_{n\to\infty}\rho_n\left|\eta_n''(h_n, t_0; 1)\right| = 2^{1/2} \quad \text{a.s.}\end{aligned} \tag{1.20}$$

On the other hand, it follows from (1.11) that

$$\limsup_{n\to\infty}\rho_n\left|\zeta_n(h_n, t_0; 1)\right| = \left(2(d+1)\right)^{1/2} \quad \text{a.s.} \tag{1.21}$$

An easy argument based upon (1.20) and (1.21) and the inequalities

$$\begin{aligned}\left|\zeta_n(h_n, t_0; 1)\right| - \left|\eta_n''(h_n, t_0; 1)\right| &\le \left|\zeta_n(h_n, t_0; 1) - \eta_n''(h_n, t_0; 1)\right| \\ &\le 2\sup_{0\le t\le 1}\left|\beta_n(t) - n^{-1/2}K''(n,t)\right|,\end{aligned}$$

shows that, almost surely

$$\limsup_{n\to\infty}\rho_n\left\{\sup_{0\le t\le 1}\left|\beta_n(t) - n^{-1/2}K''(n,t)\right|\right\} \ge \tfrac{1}{2}\left(\left(2(d+1)\right)^{1/2} - 2^{1/2}\right) > 0,$$

which, since $d = \varepsilon/2$, readily implies (1.19). □

REMARK 1.3. (i) The above given proof of Corollary 1.1 becomes invalid for $\varepsilon = 0$, since then $d = \varepsilon/2 = 0$ and the constants in the RHS of (1.20) and (1.21) are identical.

(ii) By (1.17) and (1.19), there exists a constant $\delta \in [0, 1/2]$ such that the best possible uniform almost sure rate of approximation of $\beta_n(t)$ by a normed Kiefer process is $o(n^{-1/4}(\log n)^{\delta+\varepsilon})$ and $\infty(n^{-1/4}(\log n)^{\delta-\varepsilon})$ for each $\varepsilon > 0$. The value of δ will be investigated elsewhere.

In the remainder of our paper, we will prove Theorem 1.1 and related results, shedding light on the unexpected mechanism which allows, at times, the strong limiting behavior of the *local quantile process* $\zeta_n(h_n, t_0; \cdot)$ to differ from that of the *local empirical process* $\xi_n(h_n, t_0; \cdot)$. By anticipating the exposition of these arguments, we may give a heuristical explanation of the origin of this phenomenon, limiting ourselves, for the sake of simplicity, to $h_n = 1/n^\lambda$ with $\lambda \in (0, 1)$. We will establish in the sequel [see (2.29)] that the limiting behavior of $\pm\zeta_n(h_n, t_0; \cdot)$ coincides essentially with that of $\mp\xi_n(h_n, \mathbb{V}_n(t_0); \cdot)$. This will allow us to show that the latter sequence behaves *in the same way as* or *differently from* $\mp\xi_n(h_n, t_0; \cdot)$, according as $|\mathbb{V}_n(t_0) - t_0|$ is of *smaller* or *higher* order of magnitude than h_n. Since, for $0 < t_0 < 1$, $|\mathbb{V}_n(t_0) - t_0| \to 0$ at an optimal almost sure rate of $O(n^{-1/2}(\log_2 n)^{1/2})$, it will follow that $\zeta_n(h_n, t_0; \cdot)$ and $\xi_n(h_n, t_0; \cdot)$ have different almost sure rates of convergence to 0 when $0 < \lambda < 1/2$. On the other hand, when either $t_0 = 0$, $t_0 = 1$ or $1/2 < \lambda < 1$, $|\mathbb{V}_n(t_0) - t_0|$ is negligible with respect to h_n, and the almost sure rates of $\pm\zeta_n(h_n, t_0; \cdot)$ and $\mp\xi_n(h_n, t_0; \cdot)$ are identical.

2. Proofs—outer bounds.

2.1. *A more general framework.* The results in Section 1 will be shown to follow from a description of the limiting behavior of $\{b_n^{-1}\xi_n(h_n, t; \cdot)$: $t \in [t_0 - c_n, t_0 + c_n]\}$ and $\{c_n^{-1}(\mathbb{V}_{\lfloor n\cdot \rfloor}(t_0 - t_0)\}$, with b_n and c_n given by (1.1). The following notation will be needed. Let $\lfloor u \rfloor \le u < \lfloor u \rfloor + 1$ (respectively $\lceil u \rceil \ge u > \lceil u \rceil - 1$) denote the lower (respectively upper) integer part of u. For each $-\infty < a < b < \infty$, we denote by $(B[a, b], U)$ the set $B[a, b]$ of all bounded functions f on $[a, b]$, endowed with the uniform topology U, generated by $\|f\| = \|f\|_a^b = \sup_{s \in [a, b]} |f(s)|$. For any $\varepsilon > 0$, $f \in B[a, b]$ and $A \subseteq B[a, b]$, $A \neq \varnothing$, we set

$$N_\varepsilon(f) = \{\phi \in B[a, b]: \|\phi - f\| < \varepsilon\} \quad \text{and} \quad A^\varepsilon = \bigcup_{\phi \in A} N_\varepsilon(\phi),$$

$$|f|_H = \Big\{\int_a^b \dot f^2(s)\, ds\Big\}^{1/2} \quad \text{if } f \text{ is absolutely continuous on } [a, b] \text{ with Lebesgue derivative } \dot f = df/ds \text{ and } f(0) = 0,$$

$$|f|_H = \infty \quad \text{otherwise.}$$

We set $S = \{f \in B[a, b]: |f|_H \le 1\}$ [Strassen (1964)], and, for any $\phi \in B[a, b]$ and $A \subseteq B[a, b]$,

$$J(\phi) = |\phi|_H^2 \quad \text{and} \quad J(A) = \begin{cases} \inf_{\phi \in A} J(\phi) & \text{if } A \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

In the sequel, we only consider the cases where either $[a,b]=[-1,1]$ or $[a,b]=[0,1]$. Therefore, whenever a and b are unambiguously defined, we will use the same notation $\|\cdot\|$, $|f|_H$, S, $J(\cdot)$, N_ε and A^ε, independently of these constants. Throughout, $I(s)=s$ denotes the identity function.

The following fact gives an extended description [with respect to (1.6)] of the local behavior of α_n.

FACT 1. *Under* (H.1) *and* (H.2), *for any* $t_0 \in (0,1)$, *the sequence of functions* $\{a_n^{-1}\xi_n(h_n, t_0; \cdot): \ n\ge 1\}$ *is almost surely relatively compact in* $(B[-1,1],U)$, *with limit set equal to* S.

This result is a particular case of Theorem 1.1 of Deheuvels and Mason (1994a), which extends Corollary 2 of Mason (1988), the latter being written in the setting of $B[0,1]$.

Recall that $\{f_n: \ n\ge 1\}$ is relatively compact in $(B[a,b],U)$ with limit set equal to S if and only if:

1. for each $\varepsilon>0$, there exists an $n(\varepsilon)<\infty$ such that $f_n\in S^\varepsilon$ for all $n\ge n(\varepsilon)$.
2. for each $\varepsilon>0$ and $f\in S$, we have infinitely often $\|f_n-f\|<\varepsilon$.

Since S is a compact subset of $(B[a,b],U)$, an easy argument shows that, whenever the conditions above hold, for each U-continuous functional Θ: $B[a,b]\to\mathbb{R}\cup\{\infty\}$, bounded on S,

$$\limsup_{n\to\infty}\Theta(f_n)=\sup_{f\in S}\Theta(f). \tag{2.2}$$

By applying (2.2) to $\Theta(f)=\pm f(1)$ and $\Theta(f)=\sup_{0\le s\le 1}\pm f(s)$, we readily infer (1.3) and (1.6) from Fact 1. The same argument shows that Theorem 1.1 follows from Theorem 2.1 below.

THEOREM 2.1. *Under* (H.1) *and* (H.5), (H.6), *for any* $t_0\in(0,1)$, *the sequence of functions* $\{b_n^{-1}\zeta_n(h_n,t_0;\cdot): \ n\ge 1\}$ *is almost surely compact in* $(B[-1,1],U)$, *with limit set equal to* S.

The proof of Theorem 2.1 is postponed until the next sections.

REMARK 2.1. (i) Our arguments would allow us to show that Theorems 1.1 and 2.1 remain valid [as well as (1.2)–(1.6)] when (H.1) is replaced by

$$\text{(H.1)}' \qquad \text{(i)}\ \ h_n\to 0, \quad \text{(ii)}\ \ \lim_{\rho\downarrow 1}\left(\limsup_{n\to\infty}\left\{\max_{n/\rho\le p,\, q\le\rho n} h_p/h_q\right\}\right)=1.$$

However, to prove our theorems under (H.1)′ would necessitate rewriting in this setting the proofs of a series of technical facts borrowed from the literature. Since this would greatly increase the length of our paper, we will limit ourselves to the present framework by only considering (H.1).

(ii) Under the assumptions of Theorem 2.1, the equality in (2.2) holds with probability 1 for each continuous functional Θ on $(B[-1,1], U)$, with $f_n = b_n^{-1}\zeta_n(h_n, t_0; \cdot)$. The examples of such applications, corresponding to the various possible choices of interest for Θ, are left to the reader.

2.2. *Local Bahadur–Kiefer type approximations.* We inherit the notation of the previous sections and assume throughout that $t_0 \in (0,1)$ is fixed. Below, we establish local Bahadur–Kiefer type representations [Bahadur (1966), Kiefer (1967, 1970), Deheuvels and Mason (1990a)], stated in Lemmas 2.3 and 2.4, which relate the local fluctuations of β_n to that of α_n. This approach will allow us to derive in Lemma 2.2 the *outer bound* halves of Theorem 1.1. First, we give more notation and facts.

For each $n \geq 1$, denote by $0 < U_{1,n} < \cdots < U_{n,n} < 1$ the order statistics of $U_1, \ldots, U_n$, which are a.s. distinct and in $(0,1)$. Set $U_{0,n} = 0$ and $U_{n+1,n} = 1$ for $n \geq 0$. We have, a.s. for $n \geq 1$,

$$\mathbb{V}_n(t) = U_{\lceil nt \rceil, n} \quad \text{and} \quad t \leq \mathbb{U}_n(\mathbb{V}_n(t)) = n^{-1}\lceil nt \rceil < t + n^{-1} \quad \text{for } 0 \leq t \leq 1. \tag{2.3}$$

FACT 2. *We have, with probability* 1, *for any* $n \geq 1$, $a \geq 0$, $0 \leq t \leq 1$ *and* $0 \leq t + as \leq 1$,

$$\left|\zeta_n(a, t; s) + \left\{\alpha_n(\mathbb{V}_n(t+as)) - \alpha_n(\mathbb{V}_n(t))\right\}\right| \leq 2n^{-1/2}. \tag{2.4}$$

The proof follows readily from (2.3) and the triangle inequality [see (1.6) in Shorack (1982)].

FACT 3. *We have, with probability* 1,

$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}(\log_2 n)^{-1/2}\|\mathbb{V}_n - I\|_0^1 \\ &\qquad = \limsup_{n\to\infty} n^{1/2}(\log_2 n)^{-1/2}\|\mathbb{U}_n - I\|_0^1 = 2^{-1/2}.\end{aligned} \tag{2.5}$$

For one proof, see Chung (1949). Notice that $\|\mathbb{V}_n - I\|_0^1 = \|\mathbb{U}_n - I\|_0^1$ a.s.

Introduce the function (see, e.g., pages 439, 440 in Shorack and Wellner (1986)],

$$\begin{aligned}\mathbf{h}(x) = x\log x - x + 1 \quad \text{for } x > 0, \qquad \mathbf{h}(x) = 1 \quad \text{for } x = 0,\\ \mathbf{h}(x) = \infty \quad \text{for } x < 0.\end{aligned} \tag{2.6}$$

Set further $\delta_C^+ = \inf\{x > 1: \mathbf{h}(x) > 1/C\}$ for $C > 0$, and

$$R_C^+ = (C/2)^{1/2}(\delta_C^+ - 1) \quad \text{for } C > 0, \qquad R_C^+ = 1 \quad \text{for } C = \infty. \tag{2.7}$$

FACT 4. *Let* $t_0 \in (0,1)$ *be fixed, and let* $\{A_n: n \geq 1\}$ *and* $\{C_n: n \geq 1\}$ *be two sequences of constants satisfying the following set of assumptions for some*

$C \in (0, \infty]$ *and* $D \in [0, \infty]$:

$$
\begin{aligned}
&\text{(i)}\quad A_n > 0,\ C_n > 0,\ A_n \downarrow 0,\ C_n \downarrow 0,\ nA_n \uparrow,\ nC_n \uparrow;\\
&\text{(ii)}\quad nA_n / \{\log_+(C_n/A_n) + \log_2 n\} \to C;\\
&\text{(iii)}\quad (\log_+(C_n/A_n))/\log_2 n \to D.
\end{aligned} \tag{2.8}
$$

Then, with probability 1,

$$
\limsup_{n\to\infty} \left\{ \sup_{t\in[t_0 - C_n,\, t_0 + C_n]} \left(2A_n\{\log_+(C_n/A_n) + \log_2 n\}\right)^{-1/2} \times \|\xi_n(A_n, t; \cdot)\|_{-1}^{1} \right\} = R_C^+. \tag{2.9}
$$

In view of Remark 1 of Hong (1992), (2.9) reduces to the conclusion of his Theorem 1.2 when $D \in [0, \infty)$, and to the conclusion of his Theorem 1.4 when $D = \infty$.

In the sequel, we will repeatedly make use of Fact 4 with different choices of the auxiliary sequences $\{A_n: n \ge 1\}$ and $\{C_n: n \ge 1\}$ whose definitions will be specified in each application.

The following lemma gives, as a starting point to the proofs of our theorems, crude upper bounds.

LEMMA 2.1. *Under* (H.1), *we have, with probability* 1 *for all large* n,

$$
\|\mathbb{V}_n(t_0 + h_n I) - t_0\|_{-1}^{1} \le h_n + (4/5) n^{-1/2} (\log_2 n)^{1/2} < c_n, \tag{2.10}
$$

$$
\|\mathbb{V}_n(t_0) + h_n I - t_0\|_{-1}^{1} \le h_n + (4/5) n^{-1/2} (\log_2 n)^{1/2} < c_n. \tag{2.11}
$$

PROOF. By (H.1)(i), $0 < t_0 - h_n < t_0 + h_n < 1$ for all large n, whence, by the triangle inequality,

$$
\begin{aligned}
|\mathbb{V}_n(t_0 + h_n s) - t_0| &\le |\mathbb{V}_n(t_0 + h_n s) - (t_0 + h_n s)| + h_n|s|\\
&\le \|\mathbb{V}_n - I\|_0^1 + h_n \quad \text{for } |s| \le 1.
\end{aligned} \tag{2.12}
$$

(2.10) follows from (2.5), (2.12) and $2^{-1/2} < 4/5$. The proof of (2.11) is similar and omitted. □

The next lemma establishes the *upper bound halves* of Theorem 1.1.

LEMMA 2.2. *Assume that* (H.1) *and* (H.5), (H.6) *hold. Then*,

$$
\limsup_{n\to\infty} b_n^{-1} \|\zeta_n(h_n, t_0; \cdot)\|_{-1}^{1} \le 1 \quad a.s. \tag{2.13}
$$

PROOF. Recalling the definitions (1.1) of b_n and c_n, set, for any $\epsilon \ge 0$,

$$
\begin{aligned}
&A_n(\epsilon) = (1+\epsilon) h_n, \qquad c_n = h_n + n^{-1/2}(\log_2 n)^{1/2}\\
&L_n(\epsilon) = \left((1+\epsilon) 2A_n(\epsilon)\{\log_+(c_n/A_n(\epsilon)) + \log_2 n\}\right)^{1/2},\\
&L_n = L_n(0).
\end{aligned} \tag{2.14}
$$

Since $\sup_{u\in\mathbb{R}}|\log_+(1+u)-\log_+ u|<\infty$, we infer from (2.14), that, for any $\epsilon\ge 0$, as $n\to\infty$,

$$\begin{aligned} L_n(\epsilon) &= (1+\epsilon)\Big(2h_n\Big\{\log_+\Big(1/\big(h_n\sqrt{n}\big)\Big)+O(\log_3 n)+\log_2 n\Big\}\Big)^{1/2}\\ &= (1+\epsilon)\Big(2h_n\Big\{\log_+\Big(1/\big(h_n\sqrt{n}\big)\Big)+(1+o(1))\log_2 n\Big\}\Big)^{1/2}\\ &= (1+o(1))(1+\epsilon)b_n. \end{aligned}\tag{2.15}$$

Recall from Remark 1.1 that (H.5), (H.6), entail that $nh_n/\log_2 n\to\infty$. Thus, by (2.15),

$$\begin{aligned} &\frac{n^{-1/2}L_n(\epsilon)}{h_n}\\ &\quad= (1+\epsilon)2^{1/2}\left\{\frac{\log_+\big(1/\big(h_n\sqrt{n}\big)\big)}{nh_n}+(1+o(1))\frac{\log_2 n}{nh_n}\right\}^{1/2}\to 0. \end{aligned}\tag{2.16}$$

Likewise, we infer from (H.1)(ii) and (2.15) that, for any $\epsilon\ge 0$,

$$\begin{aligned} \frac{L_n(\epsilon)}{n^{-1/2}} &= (1+\epsilon)2^{1/2}\Big\{nh_n\Big(\log_+\Big(1/\big(h_n\sqrt{n}\big)\Big)\\ &\qquad +(1+o(1))\log_2 n\Big)\Big\}^{1/2}\to\infty. \end{aligned}\tag{2.17}$$

We now select an arbitrary $\varepsilon>0$, and set, in view of an application of Fact 4, $\epsilon=\varepsilon$, $A_n=A_n(\varepsilon)$ and $C_n=c_n$. By (2.15), (2.16) and (2.17), we have, for all large n and uniformly over $s\in[-1,1]$,

$$\begin{aligned} &|h_n s+n^{-1/2}L_n(\varepsilon)|\le(1+\varepsilon)h_n=A_n\quad\text{and}\\ &-L_n(\varepsilon)+n^{-1/2}\le -L_n(\varepsilon/2). \end{aligned}\tag{2.18}$$

Consider next, for each $s\in[-1,1]$, the events

$$\begin{aligned} E_n^+(\varepsilon,s)&=\big\{\mathbb{V}_n(t_0+h_n s)-\mathbb{V}_n(t_0)-h_n s\ge n^{-1/2}L_n(\varepsilon)\big\},\\ E_n^-(\varepsilon,s)&=\big\{\mathbb{V}_n(t_0+h_n s)-\mathbb{V}_n(t_0)-h_n s\le n^{-1/2}L_n(\varepsilon)\big\}. \end{aligned}\tag{2.19}$$

Recall from (2.3) that $\mathbb{V}_n(u)=U_{\lceil nu\rceil,n}$ for $0\le u\le 1$ and $\mathbb{U}_n(U_{r,n})=n^{-1}r$ with probability 1 for $0\le r\le n$. Set $p_n=\lceil nt_0\rceil$ and $q_n(s)=\lceil n(t_0+h_n s)\rceil$. We infer from (2.15) and (H.1) that, whenever n is so large that $0<t_0-c_n-n^{-1/2}L_n(\varepsilon)<t_0+c_n+n^{-1/2}L_n(\varepsilon)<1$, for all $s\in[-1,1]$,

$$\begin{aligned} E_n^+(\varepsilon,s)&=\big\{U_{q_n(s),n}\ge U_{p_n,n}+h_n s+n^{-1/2}L_n(\varepsilon)\big\}\\ &\subseteq\Big\{\mathbb{U}_n\big(U_{q_n(s),n}\big)-\mathbb{U}_n\big(U_{p_n,n}\big)\\ &\qquad\ge\mathbb{U}_n\big(U_{p_n,n}+h_n s+n^{-1/2}L_n(\varepsilon)\big)-\mathbb{U}_n\big(U_{p_n,n}\big)\Big\} \end{aligned}\tag{2.20}$$

$$= \big\{\mathbb{U}_n\big(\mathbb{V}_n(t_0) + h_n s + n^{-1/2}L_n(\varepsilon)\big) - \mathbb{U}_n\big(\mathbb{V}_n(t_0)\big) \le n^{-1}\big(\lceil n(t_0 + h_n s)\rceil - \lceil nt_0\rceil\big)\big\}$$
$$\subseteq \big\{\alpha_n\big(\mathbb{V}_n(t_0) + h_n s + n^{-1/2}L_n(\varepsilon)\big) - \alpha_n\big(\mathbb{V}_n(t_0)\big) \le -L_n(\varepsilon) + n^{-1/2}\big\}.$$

Since our choice of $C_n = c_n$ implies, via (2.12), that $\mathbb{V}_n(t_0) \in [t_0 - C_n, t_0 + C_n]$ with probability 1 for all large n, (2.20), when combined with (2.18) and (2.19), implies that

$$\mathbb{P}\Big[\bigcup_{s\in[-1,1]} E_n^+(\varepsilon, s)\text{ i.o.}\Big] \le \mathbb{P}\Big[\sup_{t\in[t_0-C_n,\,t_0+C_n]} \|\xi_n(A_n, t; \cdot)\|_{-1}^{1} \ge L_n(\varepsilon/2)\text{ i.o.}\Big]. \tag{2.21}$$

Making use of a similar argument for $E_n^-(\varepsilon, s)$, we obtain likewise that

$$\mathbb{P}\Big[\bigcup_{s\in[-1,1]} E_n^-(\varepsilon, s)\text{ i.o.}\Big] \le \mathbb{P}\Big[\sup_{t\in[t_0-C_n,\,t_0+C_n]} \|\xi_n(A_n, t; \cdot)\|_{-1}^{1} \ge L_n(\varepsilon/2)\text{ i.o.}\Big]. \tag{2.22}$$

Since, by (H.1), (H.5) and (H.6), A_n and C_n fulfill (2.8), with $C = \infty$ and $D = d \vee 0$, we may apply Fact 4 in the present setting, to show, via (2.7), (2.9), (2.14) and (2.15), that

$$\limsup_{n\to\infty}\Big\{\sup_{t\in[t_0-C_n,\,t_0+C_n]} \|\xi_n(A_n, t; \cdot)\|_{-1}^{1}/L_n(\varepsilon/2)\Big\} = (1 + \varepsilon/2)^{-1/2} < 1 \quad \text{a.s.} \tag{2.23}$$

In view of (2.19) and (2.21), (2.22) we readily infer from (2.23) that

$$\limsup_{n\to\infty}\big\{\|\zeta_n(A_n, t_0; \cdot)\|_{-1}^{1}/L_n(\varepsilon)\big\} \le 1 \quad \text{a.s.} \tag{2.24}$$

Recalling from (2.15) that $L_n(\varepsilon) = (1 + o(1))(1 + \varepsilon)b_n$ as $n \to \infty$, we conclude (2.13) by choosing $\varepsilon > 0$ arbitrarily small in (2.24). □

The main result of this section is stated in the next lemma.

LEMMA 2.3. *Under* (H.1) *and* (H.5), (H.6), *we have*

$$\lim_{n\to\infty} b_n^{-1}\|\zeta_n(h_n, t_0; \cdot) + \xi_n(h_n, \mathbb{V}_n(t_0); \cdot)\|_{-1}^{1} = 0 \quad a.s. \tag{2.25}$$

PROOF. Fix an arbitrary $\varepsilon > 0$, and assume that (H.1) and (H.5), (H.6) hold. By combining (2.13) with (2.14), (2.15) and (2.16), we see that, with probability 1 for all large n,

$$\begin{aligned} \|\mathbb{V}_n(t_0+h_nI)-\mathbb{V}_n(t_0)-h_nI\|_{-1}^{1} &= n^{-1/2}\|\zeta_n(h_n,t_0;\cdot)\|_{-1}^{1} \\ &\le (1+\varepsilon)n^{-1/2}b_n \\ &= (1+o(1))n^{-1/2}L_n(\varepsilon)\le \varepsilon h_n. \end{aligned} \tag{2.26}$$

Also, we recall from (1.1), (2.10) and (2.11) that, with probability 1 for all large n,

$$\begin{aligned} &\max\left\{\|\mathbb{V}_n(t_0+h_nI)-t_0\|_{-1}^{1},\mathbb{V}_n(t_0)+h_nI-t_0\|_{-1}^{1}\right\} \\ &\qquad < c_n = h_n+n^{-1/2}(\log_2 n)^{1/2}. \end{aligned} \tag{2.27}$$

We next apply Fact 4 with $A_n=\varepsilon h_n$, $C_n=c_n$, which obviously fulfill (2.8), $C=\infty$ and $D=d\vee 0$. It is readily checked that $(2A_n\{\log_+(C_n/A_n)+\log_2 n\})^{1/2}=(1+o(1))\varepsilon^{1/2}b_n$. Thus, in view of (2.4), we infer from (2.7), (2.9), (2.26) and (2.27), that, with probability 1 for all large n,

$$\begin{aligned} &\|\zeta_n(h_n,t_0;\cdot)+\xi_n(h_n,\mathbb{V}_n(t_0);\cdot)\|_{-1}^{1} \\ &\quad\le 2n^{-1/2}+\|\alpha_n(\mathbb{V}_n(t_0+h_nI))-\alpha_n(\mathbb{V}_n(t_0))-\xi_n(h_n,\mathbb{V}_n(t_0);I)\|_{-1}^{1} \\ &\quad= 2n^{-1/2}+\|\alpha_n(\mathbb{V}_n(t_0)+h_nI)-\alpha_n(\mathbb{V}_n(t_0+h_nI))\|_{-1}^{1} \\ &\quad\le 2n^{-1/2}+\sup_{t\in[t_0-c_n,t_0+c_n]}\|\xi_n(\varepsilon h_n,t;\cdot)\|_{-1}^{1}\le 2n^{-1/2}=2\varepsilon^{1/2}b_n. \end{aligned} \tag{2.28}$$

By (2.14) and (2.17), $n^{-1/2}=o(L_n)=o(b_n)$, whence the RHS of (2.28) is a.s. ultimately less than or equal to $4\varepsilon^{1/2}b_n$. Since we may choose $\varepsilon>0$ arbitrarily small, (2.25) is straightforward. □

The next lemma characterizes in part the range where the a.s. asymptotic rates of $\zeta_n(h_n,t_0;\cdot)$ and $\xi_n(h_n,t_0;\cdot)$ are identical. A different argument will be needed when (H.6) holds with $d=0$.

LEMMA 2.4. *Assume that* (H.1) *and* (H.6) *hold with* $d\in[-\infty,0)$, *Then, we have*

$$\begin{aligned} &\lim_{n\to\infty} a_n^{-1}\|\zeta_n(h_n,t_0;\cdot)+\xi_n(h_n,t_0;\cdot)\|_{-1}^{1} \\ &\qquad=\lim_{n\to\infty} b_n^{-1}\|\zeta_n(h_n,t_0;\cdot)+\xi_n(h_n,t_0;\cdot)\|_{-1}^{1}=0\ a.s. \end{aligned} \tag{2.29}$$

PROOF. (H.6) holds with $d\in(-\infty,0)$ (respectively $d=\infty$) iff $h_n=n^{-1/2}(\log n)^{-d+o(1)}$ [respectively $h_n\ge n^{-1/2}(\log n)^{\kappa}$ ultimately as $n\to\infty$ for each $\kappa>0$]. Thus, in either of these two cases, we have

$$\begin{aligned} b_n &= \left(2h_n\left\{\log_+\left(1/\left(h_n\sqrt{n}\right)\right)+\log_2 n\right\}\right)^{1/2} \\ &= (1+o(1))(2h_n\log_2 n)^{1/2}=(1+o(1))a_n. \end{aligned} \tag{2.30}$$

Fix any $\varepsilon>0$. Set $A_n=n^{-1/2}(\log_n n)^{1/2}$ and $C_n=h_n$. By (2.5), $|\mathbb{V}_n(t_0)-t_0|<A_n$ with probability 1 for all large n. Thus, making use of the inequalities,

for $s \in [-1,1]$,

$$|\zeta_n(h_n,t_0;s)+\xi_n(h_n,t_0;s)| \le |\zeta_n(h_n,t_0;s)+\xi_n(h_n,\mathbb{V}_n(t_0);s)| + |\xi_n(h_n,\mathbb{V}_n(t_0);s)-\xi_n(h_n,t_0;s)|,$$

and

$$\begin{aligned}|\xi_n(h_n,\mathbb{V}_n(t_0);s)-\xi_n(h_n,t_0;s)| &\le |\alpha_n(\mathbb{V}_n(t_0)+h_ns)-\alpha_n(t_0+h_ns)| \\ &\quad + |\alpha_n(\mathbb{V}_n(t_0))-\alpha_n(t_0)| \le \|\xi_n(A_n,t_0+h_ns;\cdot)\|_{-1}^{1}+\|\xi_n(A_n,t_0;\cdot)\|_{-1}^{1} \\ &\le 2\sup_{t\in[t_0-C_n,t_0+C_n]}\|\xi_n(A_n,t;\cdot)\|_{-1}^{1},\end{aligned}$$

we infer from (2.25) that, with probability 1 for all large n,

$$\begin{aligned}&b_n^{-1}\|\zeta_n(h_n,t_0;\cdot)+\xi_n(h_n,t_0;\cdot)\|_{-1}^{1} \\ &\qquad\le \varepsilon/2+2b_n^{-1}\sup_{t\in[t_0-C_n,t_0+C_n]}\|\xi_n(A_n,t;\cdot)\|_{-1}^{1}.\end{aligned}\tag{2.31}$$

Since A_n and C_n fulfill (2.8) with $C=\infty$ and $D=-d$, we apply Fact 4 in the following cases.

(i) When $d\in(-\infty,0)$, $\log(C_n/A_n)=-(1+o(1))d\log_2 n$, so that, by (2.10) and (2.30),

$$2b_n^{-1}\sup_{t\in[t_0-C_n,t_0+C_n]}\|\xi_n(A_n,t;\cdot)\|_{-1}^{1}=O\big((\log n)^{-d/2}\big)\to 0.$$

(ii) When $d=-\infty$, $\log(C_n/A_n)=O(\log n)$, so that by (2.10) and (2.30), for each $\kappa>1$,

$$2b_n^{-1}\sup_{t\in[t_0-C_n,t_0+C_n]}\|\xi_n(A_n,t;\cdot)\|_{-1}^{1}=O\big((\log n)^{(1-\kappa)/2}\big)\to 0.$$

In both cases, we obtain via (2.31) that $\limsup_{n\to\infty} b_n^{-1}\|\zeta_n(h_n,t_0;\cdot)+\xi_n(h_n,t_0;\cdot)\|_{-1}^{1}\le\varepsilon$ a.s. Since $\varepsilon>0$ may be chosen as small as desired, the conclusion (2.29) is straightforward. □

REMARK 2.2. (i) The proof of Lemma 2.4 becomes invalid if we drop the assumption that $n^{-1/2}(\log_2 n)^{1/2}=o(h_n)$. Since this condition always holds for $d\in[-\infty,0)$, but not for $d=0$, there is no hope of establishing (2.29) when $d\notin[-\infty,0)$ without any additional assumption.

(ii) Lemma 2.4 and Fact 1 jointly imply that the conclusions of Theorems 1.1 and 2.1 hold under (H.1) and (H.6) when $d\in[-\infty,0)$. This covers part of the *large increment case.*

REMARK 2.3. Theorem 5 of Einmahl and Mason (1988) shows that the conclusions of Lemmas 2.3 and 2.4 hold under (H.1), (H.2) only if $t_0=0$ or 1, with $\mathbb{V}_n(t_0)=0$.

2.2. *Local functional laws of the iterated logarithm—outer bounds.* The outer bound halves of Theorem 2.1, will be established via Propositions 2.1 and 2.2.

PROPOSITION 2.1. *Under* (H.1) *and* (H.5), (H.6), *for every* $\varepsilon > 0$, *we have with probability* 1 *for all n sufficiently large*,

$$F_n := \{b_n^{-1}\xi_n(h_n, t; \cdot): t \in [t_0 - c_n, t_0 + c_n]\} \subseteq S^{\varepsilon} \subseteq B[-1, 1]. \tag{2.32}$$

Given Proposition 2.1 and Lemma 2.3, a simple argument allows us to prove the following result.

PROPOSITION 2.2. *Under* (H.1) *and* (H.5), (H.6), *for every* $\varepsilon > 0$, *we have with probability* 1 *for all n sufficiently large*,

$$b_n^{-1}\zeta_n(h_n, t_0; \cdot) \in S^{\varepsilon} \subseteq B[-1, 1]. \tag{2.33}$$

PROOF. It follows from (2.25) that, with probability 1, for each $\varepsilon > 0$ and all n sufficiently large,

$$\|b_n^{-1}\zeta_n(h_n, t_0; \cdot) + b_n^{-1}\xi_n(h_n, \mathbb{V}_n(t_0); \cdot)\|_{-1}^{1} < \varepsilon/2.$$

Since (2.11) implies that $\mathbb{V}_n(t_0) \in [t_0 - c_n, t_0 + c_n]$ with probability 1 for all large n, and (2.32) implies that, with probability 1 for all large n, $b_n^{-1}\xi_n(h_n, t; \cdot) \in S^{\varepsilon/2}$ for all $t \in [t_0 - c_n, t_0 + c_n]$, the inequality above implies that $b_n^{-1}\zeta_n(h_n, t_0; \cdot) \in -S^{\varepsilon} = \{-f: f \in S^{\varepsilon}\}$ with probability 1 for all large n. The conclusion (2.33) follows from the observation that $S^{\varepsilon} = -S^{\varepsilon}$. □

In the remainder of this section, we will prove Proposition 2.1, together with a series of technical results of independent interest, which will be used in the forthcoming proof of the inner bound halves of Theorem 2.1. Throughout, we will assume, without loss of generality, that $t_0 \in (0, 1/2]$. Our arguments will apply to the case where $t_0 \in [1/2, 1)$, after being reformulated via the mapping $t_0 \to 1 - t_0$. Moreover, in Fact 5 and Lemmas 2.6–2.9 below *we will assume throughout and unless otherwise specified that the functions we consider vary in* $B[0, 1]$. We will show later how the corresponding results may be modified in the setting of $B[-1, 1]$ to complete the proof of (2.32). A rough outline of our argument, inspired by the proofs of (3.2) in Deheuvels and Mason (1992), and (2.60) in Deheuvels (1992), is as follows. We will show in the forthcoming Lemma 2.7 that the increments ξ_n of α_n behave essentially in the same way as the increments L_n of a Poisson process Π_n [see (2.37), (2.38)], then conclude via blocking arguments and the Borel–Cantelli lemma. The following Lemmas 2.5 and 2.6 evaluate appropriate large deviation probabilities for L_n. We will make use of strong approximations which will allow us to infer these evaluations from similar bounds for Wiener processes, which are stated in Fact 5 below. Let J be as in (2.1).

FACT 5. *Let* $\{W(t): t \geq 0\}$ *be a standard Wiener process and, for any* $\lambda > 0$, *set* $W_{\{\lambda\}}(s) = (2\lambda)^{-1/2} W(s)$ *for* $s \in [0,1]$. *Then, for each closed (resp. open) subset* F *(resp.* G*) of* $(B[0,1], U)$,

$$\begin{aligned} &\limsup_{\lambda \to \infty} \lambda^{-1} \log \mathbb{P}\big(W_{\{\lambda\}} \in F\big) \leq -J(F) \quad \text{and} \\ (2.34)\qquad &\liminf_{\lambda \to \infty} \lambda^{-1} \log \mathbb{P}\big(W_{\{\lambda\}} \in G\big) \geq -J(G). \end{aligned}$$

This result is due to Schilder (1966) [see, e.g., Deuschel and Stroock (1989) page 12].

Denote by $\overline{A}$ the complement in $B[0,1]$ of $A \subseteq B[0,1]$. The next lemma is given in view of an application of (2.34) to the special case where $F = \overline{S^{\varepsilon}}$ and $G = N_{\varepsilon}(f)$.

LEMMA 2.5. *For each* $\varepsilon \in (0,1)$ *and* $f \in S \subseteq B[0,1]$ *such that* $0 < \varepsilon < |f|_H \leq 1$, *we have*

$$(2.35)\qquad \begin{aligned} &\text{(i)}\quad J\big(\overline{S^{\varepsilon}}\big) \geq (1+\varepsilon)^2; \\ &\text{(ii)}\quad J\big(N_{\varepsilon}(f)\big) \leq (|f|_H - \varepsilon)^2 \leq |f|_H^2 (1-\varepsilon)^2. \end{aligned}$$

PROOF. Let $\psi \in B[0,1]$. The inequality $\|\psi\| \leq |\psi|_H$ is trivial when $|\psi|_H = \infty$ and holds likewise when $|\psi|_H < \infty$, since the Schwarz inequality entails that

$$(2.36)\qquad \|\psi\| = \sup_{0 \leq s \leq 1} \left| \int_0^s \dot{\psi}(s)\, ds \right| \leq \left\{ \int_0^1 \dot{\psi}(s)^2\, ds \right\}^{1/2} = |\psi|_H.$$

For each $g \notin S^{\varepsilon}$, $1 < C := |g|_H \leq \infty$. If $C < \infty$, then $\phi := C^{-1} g \in S$ and $|g - \phi|_H = C - 1$. Therefore, by setting $\psi = g - \phi$ in (2.36), we see that $\varepsilon \leq \|g - \phi\| \leq |g - \phi|_H = C - 1$, which implies that $C = |g|_H \geq 1 + \varepsilon$. Since, by (2.1), $J(\overline{S^{\varepsilon}}) = \inf_{g \notin S^{\varepsilon}} |g|_H^2$, we conclude (2.35)(i). Next, our assumptions imply that $0 < |f|_H \leq 1$, and hence, that $\|\rho f - f\| = (1-\rho)\|f\| \leq (1-\rho)|f|_H$ for all $0 \leq \rho \leq 1$. Therefore, by choosing $\rho = 1 - \varepsilon' |f|_H^{-1}$ with $0 < \varepsilon' < \varepsilon$, we obtain that $g := \rho f \in N_{\varepsilon}(f)$ and $|g|_H = |f|_H - \varepsilon' \leq |f|_H (1 - \varepsilon')$. This readily implies (2.35)(ii) by letting $\varepsilon' \uparrow \varepsilon$. □

Given a unit-rate homogeneous Poisson process $\Xi(\cdot)$ on $\mathbb{R}^2$, we set for $n \geq 0$ and $t \in [0,1]$

$$(2.37)\qquad \Pi_n(t) = \Xi\big((0,t] \times (0,n]\big),$$

so that $\{\Pi_n(t): t \geq 0\}$ is a (right-continuous) Poisson process with $\mathbb{E}(\Pi_n(t)) = nt$ for $t \geq 0$ and $n \geq 0$. We denote by $\{\Pi(t): t \geq 0\} = \{\Pi_1(t): t \geq 0\}$ a (right-continuous) standard Poisson process, and set, for $n \geq 1$, $a \geq 0$ and $-\infty < s$, $t < \infty$,

$$(2.38)\qquad L_n(a,t;s) = n^{-1/2}\big(\Pi_n(t+sa) - \Pi_n(t) - nsa\big).$$

LEMMA 2.6. *Assume that* (H.1) *and* (H.5), (H.6) *hold. Then, for any* $\varepsilon \in (0,1)$ *and* $f \in S$ *with* $0 < \varepsilon < |f|_H < 1$, *there exists an* $\eta = \eta(\varepsilon) \ge 0$ *such that, for all large* n,

$$\left(1 + \frac{c_n}{h_n}\right)\mathbb{P}\left(b_n^{-1}L_n(h_n, t_0; \cdot) \notin S^{\varepsilon}\right) \le \frac{(1 + c_n/h_n)^{-\eta}}{(\log n)^{1+\eta}} \le (\log n)^{-(1+\eta)}, \tag{2.39}$$

$$\left(1 + \frac{c_n}{h_n}\right)\mathbb{P}\left(b_n^{-1}L_n(h_n, t_0; \cdot) \in N_\varepsilon(f)\right) \ge \frac{(1 + c_n/h_n)^{1-|f|_H^2(1-\eta)}}{(\log n)^{|f|_H^2(1-\eta)}} \ge \frac{(1 + c_n/h_n)^{\eta}}{(\log n)^{1-\eta}}. \tag{2.40}$$

PROOF. Assume that (H.1) and (H.5), (H.6) hold. Fix $\varepsilon \in (0, |f|_H)$ and $f \in S$. By Fact 4 in Deheuvels and Mason (1994b) and Komlós, Major and Tusnády (1975a, b, 1976), we may construct jointly a standard Poisson process $\{\Pi(t)\colon t \ge 0\}$, and a standard Wiener process $\{W(t)\colon t \ge 0\}$, such that, for universal constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$,

$$\mathbb{P}\left(\sup_{0 \le x \le T} |\Pi(x) - x - W(x)| \ge C_1 \log T + z\right) \le C_2 \exp(-C_3 z) \quad \text{for } T > 0,\ -\infty < z < \infty. \tag{2.41}$$

Given (2.41) and recalling (1.1) and (2.37), (2.38), we make use of the triangle inequality to write

$$\begin{aligned} &\mathbb{P}\left(b_n^{-1}L_n(h_n, t_0; \cdot) \notin S^{\varepsilon}\right) \\ &\quad = \mathbb{P}\left(b_n^{-1}n^{-1/2}(\Pi(nh_n \cdot) - nh_n \cdot) \notin S^{\varepsilon}\right) \\ &\quad \le \mathbb{P}\left(\|\Pi(nh_n \cdot) - nh_n \cdot - W(nh_n \cdot)\|_0^1 \ge (\varepsilon/2)n^{1/2}b_n\right) \\ &\qquad + \mathbb{P}\left(b_n^{-1}n^{-1/2}W(nh_n \cdot) \notin S^{\varepsilon/2}\right) \\ &\quad =: P_{1,n} + P_{2,n}. \end{aligned} \tag{2.42}$$

It follows from (H.1)(ii) that, for all large n,

$$2 \le 1 + c_n/h_n = 2 + \left(1/\left(h_n\sqrt{n}\right)\right)(\log_2 n)^{1/2} \le 2n^{1/2}(\log_2 n)^{1/2}. \tag{2.43}$$

Set $z = z_n = (\varepsilon/2)n^{1/2}b_n - C_1 \log(nh_n)$ and $T = nh_n$ in (2.41). We have, for all large n,

$$\begin{aligned} z_n &\ge (\varepsilon/4)n^{1/2}b_n \\ &= (\varepsilon/4)\left(2nh_n\left\{\log_+\left(1/\left(h_n\sqrt{n}\right)\right) + \log_2 n\right\}\right)^{1/2} \ge z_n' \\ &:= (3/C_3)\log n. \end{aligned} \tag{2.44}$$

It follows from (2.41), (2.42), (2.43) and (2.44) that, for each specified $\Delta \in (0,3]$ and all large n,

$$
\begin{aligned}
P_{1,n} &\le C_2 \exp(-C_3 z_n) \le C_2 \exp(-C_3 z'_n) \le C_2/n^3 \\
&\le \big(1/(2e^2)\big)/\big((\log n)^{\Delta}(1+c_n/h_n)^{\Delta}\big).
\end{aligned}
\tag{2.45}
$$

We next observe that the complement $\overline{S^{\varepsilon/2}}$ of $S^{\varepsilon/2}$ in $B[0,1]$ is closed in $(B[0,1],U)$. By (2.35)(i), $\delta = \frac{1}{4}((1+\varepsilon/2)^2-1) \in (0,\frac{5}{16})$ fulfills $1 < 1 + 2\delta < 1+4\delta \le J(\overline{S^{\varepsilon/2}})$. Therefore, by setting $F = \overline{S^{\varepsilon/2}}$ in (2.34)(i), we easily infer from (2.14), (2.15) and (2.30) that, for all large n,

$$
\begin{aligned}
P_{2,n} &= \mathbb{P}\Big(W_{\{h_n^{-1}b_n^2/2\}} \in \overline{S^{\varepsilon/2}}\Big) \le \exp\big(-(1+2\delta)\,h_n^{-1}b_n^2/2\big) \\
&\le \exp\big(-(1+\delta)\,h_n^{-1}L_n^2/2\big) \\
&= \big((c_n/h_n)\vee e\big)^{-1-\delta}/(\log n)^{1+\delta} \\
&\le (1+1/e)^2/\big((\log n)^{1+\delta}(1+c_n/h_n)^{1+\delta}\big).
\end{aligned}
\tag{2.46}
$$

Here, we have used the inequalities $1+\delta<2$ and $(1+u)/(1+1/e) \le u \vee e$ for $u = c_n/h_n \ge 1$. By combining (2.42) with (2.46) and (2.45) taken with $\Delta = 1+\delta$, we obtain readily that the LHS of (2.39) is ultimately less than $\{(1+1/e)^2 + (1/(2e^2))\}(1+c_n/h_n)^{-\delta}(\log n)^{-1-\delta}$. This last inequality entails that (2.39) holds ultimately as $n\to\infty$ for each $\eta \in (0,\delta)$.

By a similar argument as that used in (2.42), we may write

$$
\begin{aligned}
&\mathbb{P}\big(b_n^{-1}L_n(h_n,t_0;\cdot) \in N_\varepsilon(f)\big) \\
&\quad = \mathbb{P}\big(b_n^{-1}n^{-1/2}\big(\Pi(nh_n\cdot) - nh_n\cdot\big) \in N_\varepsilon(f)\big) \\
&\quad \ge \mathbb{P}\big(b_n^{-1}n^{-1/2}W(nh_n\cdot) \in N_{\varepsilon/2}(f)\big) \\
&\qquad - \mathbb{P}\big(\|\Pi(nh_n\cdot) - nh_n\cdot - W(nh_n\cdot)\|_0^1 \ge (\varepsilon/2)\,n^{1/2}b_n\big) \\
&\quad =: P_{3,n} - P_{1,n}.
\end{aligned}
\tag{2.47}
$$

Since $|f|_H \le 1$, by (2.35)(ii), $J(N_{\varepsilon/2}(f)) \le 1-4\delta' < 1-2\delta' < 1$ for $\delta' = \frac{1}{4}(1-(1-\varepsilon/2)^2) \in (0,\frac{3}{16})$. $N_{\varepsilon/2}(f)$ being open in $(B([0,1],U)$, by setting $G = N_{\varepsilon/2}(f)$ in (2.34), we see that, for all large n,

$$
\begin{aligned}
P_{3,n} &= \mathbb{P}\Big(W_{\{h_n^{-1}b_n^2/2\}} \in N_{\varepsilon/2}(f)\Big) \ge \exp\big(-(1-2\delta')|f|_H^2\,h_n^{-1}b_n^2/2\big) \\
&\ge \exp\big(-(1-\delta')|f|_H^2\,h_n^{-1}L_n^2/2\big) \\
&= \big((c_n/h_n)\vee e)(\log n)\big)^{-(1-\delta')|f|_H^2} \\
&\ge (1/e^2)/\big((\log n)(1+c_n/h_n)\big)^{(1-\delta')|f|_H^2}.
\end{aligned}
\tag{2.48}
$$

Here, we have used (2.14), (2.15), (2.30) and the inequality $e(1+u) \ge u\vee e$ for $u = c_n/h_n \ge 1$. By combining (2.47) and (2.48) with (2.45) taken with $\Delta = (1-\delta')|f|_H^2$, we obtain that the LHS of (2.40) is ultimately greater than or equal to $(1/(2e^2))(1+c_n/h_n)^{1-(1-\delta')|f|_H^2}(\log n)^{-(1-\delta')|f|_H^2}$. This, in turn, implies that the first inequality in (2.40) holds ultimately in $n \to \infty$ for each $\eta \in (0, \delta \wedge \delta')$. The second inequality in (2.40) is always satisfied when $\eta \in [0,1]$, $0 \le |f|_H \le 1$ and $n \ge 3$. □

LEMMA 2.7. *Fix* $\rho \ge 1$ *and* $N \ge 1$. *For any* $\{t_1, \dots, t_m\} \subseteq [0, t_0 + c_n] \subseteq [0, \frac{3}{4} - h_n]$, $B_1, \dots, B_m$, *Borel subsets of* $(B[0,1], U)$, *set*

$$E_1 = \bigcap_{j=1}^{m} \{\xi_n(h_n, t_j; \cdot) \in B_j\}, \tag{2.49}$$

and

$$E_2 = \bigcap_{j=1}^{m} \{L_n(h_n, t_j; \cdot) \in B_j\}. \tag{2.50}$$

Then, there exists an absolute constant C_4 *such that, for all large* n,

$$\mathbb{P}(E_1) \le C_4 \mathbb{P}(E_2). \tag{2.51}$$

PROOF. This is an extended version of Lemmas 2.1 and 3.1 of Deheuvels and Mason (1992), who showed that $\mathbb{P}(E_1) \le 2\mathbb{P}(E_2)$ for $n \ge 5$, when $[0, t_0 + c_n] \subseteq [0, \frac{1}{2} - h_n]$. The proof is very similar, and obtained by combining the following observations.

First, we note that if $P_\lambda(r) = (r^\lambda / r!) e^{-\lambda}$, $r = 0, 1, \dots$ is a Poisson distribution, then, for integer λ,

$$\sup_{r \ge 0} P_\lambda(r) = P_\lambda(\lambda) = \frac{1 + o(1)}{\sqrt{2\pi\lambda}} \quad \text{as } \lambda \to \infty.$$

This, in turn, readily implies that, as $\lambda + \mu \to \infty$ with $\lambda/(\lambda + \mu) \ge c > 0$,

$$\limsup \left\{ \frac{1}{P_{\lambda+\mu}(\lfloor \lambda + \mu \rfloor)} \left(\sup_{r \ge 0} P_\lambda(r) \right) \right\} \le \frac{1}{\sqrt{c}}.$$

Second, for each $n \ge 1$, the Poisson process $\{\Pi_n(t) : 0 \le t \le 1\}$ follows, conditionally on $\Pi_n(1) = n$, the same distribution as $\{n\mathbb{U}_n(t) : 0 \le t \le 1\}$. It follows that, if we fix $0 < d < 1$ and consider two measurable events of the form

$$E_1 = \{\{\Pi_n(t) : 0 \le t \le d\} \in B\} \quad \text{and} \quad E_2 = \{\{n\mathbb{U}_n(t) : 0 \le t \le d\} \in B\},$$

then, letting $R = \Pi_n(d)$, $\overline{R} = \Pi_n(1) - \Pi_n(d)$, $\lambda = nd$ and $\mu = n(1-d)$, we see that there exists a constant C depending upon d only ($C = 2d^{-1/2}$ will do), such that, for all large n,

$$\begin{aligned} \mathbb{P}(E_2) &= \mathbb{P}(E_1 | R + \overline{R} = n) = \sum_{k=0}^{n} \mathbb{P}(E_1 \cap \{R = n - k\}) \frac{\mathbb{P}(\overline{R} = k)}{\mathbb{P}(R + \overline{R} = n)} \\ &\le \mathbb{P}(E_1) \frac{1}{\mathbb{P}(R + \overline{R} = n)} \sup_{k \ge 0} \mathbb{P}(\overline{R} = k) \le C\mathbb{P}(E_1). \end{aligned}$$

The conclusion (2.51) is readily achieved by suitable choices of B and d. □

Recall the definitions (1.1) of b_n and c_n, and (2.14) of L_n. By (2.15), under (H.1) and (H.5), (H.6),

$$L_n = \big(2h_n\{\log_+(c_n/h_n) + \log_2 n\}\big)^{1/2} = (1+o(1))\,b_n \to 0 \quad \text{as } n \to \infty. \tag{2.52}$$

Let $\gamma > 0$ and $\theta \in (0,1]$ be two constants which will be given precisely later on. Set $\nu_k = \lfloor (1+\gamma)^k \rfloor$ for $k \geq 0$, and $t_n(i,\theta) = t_0 + i\theta h_n$ for $-M_n \leq i \leq M_n := \lceil 2c_n/(\theta h_n) \rceil$. Since (1.1) implies that $0 < \theta h_n \leq h_n \leq c_n$, we have the inequalities

$$|t_n(i,\theta) - t_0| \leq M_n \theta h_n = \lceil 2c_n/(\theta h_n)\rceil \theta h_n \in [2c_n, 3c_n] \quad \text{for } -M_n \leq i \leq M_n, \tag{2.53}$$

Set $T_k(\theta) = \{t_{\nu_{k+1}}(i,\theta) \colon -M_{\nu_{k+1}} \leq i \leq M_{\nu_{k+1}}\}$, and introduce the events, for $\epsilon > 0$ and $k \geq 1$,

$$C_k(\epsilon,\gamma) = \Big\{(n/\nu_{k+1})^{1/2} b_{\nu_{k+1}}^{-1} \xi_n(h_{\nu_{k+1}}, t; \cdot) \notin S^{\epsilon} \quad \text{for some } t \in T_k(\theta) \text{ and } n \in (\nu_k, \nu_{k+1}]\Big\}, \tag{2.54}$$

$$D_k(\epsilon,\gamma) = \Big\{b_{\nu_{k+1}}^{-1} \xi_{\nu_{k+1}}(h_{\nu_{k+1}}, t; \cdot) \notin S^{\epsilon} \quad \text{for some } t \in T_k(\theta)\Big\}. \tag{2.55}$$

LEMMA 2.8. *For each $\epsilon > 0$, $\theta \in (0,1]$ and $\gamma > 0$, there exists a K', such that, for all $k \geq K'$,*

$$\mathbb{P}(C_k(\epsilon,\gamma)) \leq 2\mathbb{P}(D_k(\epsilon/2,\gamma)). \tag{2.56}$$

PROOF. This is a version of Lemma 3.4, pp. 1268–1269 in Deheuvels and Mason (1992), with small changes of notation. The proof is achieved by exactly the same arguments, after checking that

$$(\nu_{k+1} - \nu_k) h_{\nu_{k+1}} / \big(\nu_{k+1} b_{\nu_{k+1}}^2\big) \leq 1 \Big/ \Big\{\log_+\Big(1/\Big(h_{\nu_{k+1}} \sqrt{\nu_{k+1}}\Big)\Big) + \log_2 \nu_{k+1}\Big\} \to 0.$$

Therefore, we omit details. □

LEMMA 2.9. *Under* (H.1) *and* (H.5), (H.6), *for every $\varepsilon \in (0,1]$, there exists with probability* 1 *an $n(\varepsilon) < \infty$ such that, for all $n \geq n(\varepsilon)$,*

$$F_n = \big\{b_n^{-1} \xi_n(h_n, t; \cdot) \colon t \in [t_0 - c_n, t_0 + c_n]\big\} \subseteq S^{\varepsilon} \subseteq B[0,1]. \tag{2.57}$$

PROOF. Fix $\epsilon \in (0,1]$, $\theta \in (0,1]$ and $\gamma \in (0,1/2]$. Recall (2.53), (2.54), (2.55), and observe that $2M_n + 1 = 2\lceil 2c_n/(\theta h_n)\rceil + 1 \leq (4/\theta)(1 + c_n/h_n)$. It follows therefore from (2.39), (2.51), (2.56) and the Bonferroni inequalities that there exists an $\eta > 0$ such that, for all large k,

$$\begin{aligned}
\mathbb{P}(C_k(\epsilon,\gamma)) &\leq 2\mathbb{P}(D_k(\epsilon/2,\gamma)) \\
&\leq 2(2M_{\nu_k} + 1)\mathbb{P}\Big(b_{\nu_{k+1}}^{-1} \xi_{\nu_{k+1}}(h_{\nu_{k+1}}, t_0; \cdot) \notin S^{\epsilon/2}\Big) \\
&\leq (8/\theta)(1 + c_{\nu_{k+1}}/h_{\nu_{k+1}})\mathbb{P}\Big(b_{\nu_{k+1}}^{-1} \xi_{\nu_{k+1}}(h_{\nu_{k+1}}, t_0; \cdot) \notin S^{\epsilon/2}\Big) \\
&\leq (8C_4/\theta)(1 + c_{\nu_{k+1}}/h_{\nu_{k+1}})\mathbb{P}\Big(b_{\nu_{k+1}}^{-1} L_{\nu_{k+1}}(h_{\nu_{k+1}}, t_0; \cdot) \notin S^{\epsilon/2}\Big) \\
&\leq (8C_4/\theta)(\log \nu_k)^{-(1+\eta)} = O(k^{-(1+\eta)}),
\end{aligned} \tag{2.58}$$

which is summable in k. Thus, by (2.54), (2.58) and the Borel–Cantelli lemma, a.s. for all large k,

$$b_n^{-1}\,\xi_n\big(h_{\nu_{k+1}},t;\cdot\big)\in\big(b_{\nu_{k+1}}/b_n\big)\big(\nu_{k+1}/n\big)^{1/2}\,S^{\epsilon},\tag{2.59}$$

uniformly over $t\in T_k(\theta)=\{t_{\nu_{k+1}}(i,\theta)\colon -M_{\nu_{k+1}}\le i\le M_{\nu_{k+1}}\}$ and $\nu_k<n\le\nu_{k+1}$. By the inequality $1-(1+\gamma)^{-1/2}<\gamma$ for $0<\gamma\le 1$ and (H.1) we have, for all large k, and $\nu_k\le n\le\nu_{k+1}$,

$$\begin{aligned}0&\le h_n-h_{\nu_{k+1}}\le h_n\big(1-h_{\nu_{k+1}}/h_{\nu_k}\big)\le h_n(1-\nu_k/\nu_{k+1})\\&=(1+o(1))\,h_n\gamma/(1+\gamma)<\gamma h_n,\end{aligned}\tag{2.60}$$

$$\begin{aligned}0&\le h_{\nu_k}-h_n\le h_n\big(h_{\nu_k}/h_{\nu_{k+1}}-1\big)\le h_n(\nu_{k+1}/\nu_k-1)\\&=(1+o(1))\,h_n\gamma<2\gamma h_n,\end{aligned}\tag{2.61}$$

$$\begin{aligned}0&\le c_n-c_{\nu_{k+1}}\\&=h_n-h_{\nu_{k+1}}+(n/\log_2 n)^{-1/2}-(\nu_{k+1}/\log_2\nu_{k+1})^{-1/2}<\gamma c_n.\end{aligned}\tag{2.62}$$

By combining the definition (1.1) of b_n with (2.60) and (2.61), we obtain that, for all large k,

$$\begin{aligned}\max_{\nu_k\le n\le\nu_{k+1}}\max&\Big\{\Big|\big(b_{\nu_{k+1}}/b_n\big)(\nu_{k+1}/n)^{1/2}-1\Big|,\Big|\big(b_n/b_{\nu_k}\big)(n/\nu_k)^{1/2}-1\Big|\Big\}\\&\le 2\gamma.\end{aligned}\tag{2.63}$$

By definition, $f\in S^{\epsilon}$ if $\|f-g\|<\epsilon$ for some g with $|g|_H\le 1$. By (2.36), $\|g\|\le|g|_H$, so that, for each $\rho>0$, $\|\rho f-g\|\le\rho\|f-g\|+|\rho-1|\times\|g\|<\rho\epsilon+|\rho-1|$. Thus, $\rho f\in S^{\rho\epsilon+|\rho-1|}$, and

$$\rho\,S^{\epsilon}=\{\rho f\colon f\in S^{\epsilon}\}\subseteq S^{\rho\epsilon+|\rho-1|}.\tag{2.64}$$

By combining (2.64) taken with $\rho=1+2\gamma$, with (2.59) and (2.63), we obtain that, a.s. for all large k, uniformly over $t\in T_k(\theta)$ and $\nu_k<n\le\nu_{k+1}$,

$$b_n^{-1}\,\xi_n\big(h_{\nu_{k+1}},t;\cdot\big)\in\big(b_{\nu_{k+1}}/b_n\big)(\nu_{k+1}/n)^{1/2}\,S^{\epsilon}\subseteq S^{(1+2\gamma)\epsilon+2\gamma}.\tag{2.65}$$

By Fact 4, taken with $A_n=\gamma h_n$ and $C_n=3c_n$, (2.53), (2.54) and (2.64), a.s. for all large k,

$$\begin{aligned}&\limsup_{k\to\infty}\Big\{\max_{\nu_k<n\le\nu_{k+1}}\sup_{t\in T_k(\theta)}b_n^{-1}\|\xi_n\big(h_{\nu_{k+1}},t;\cdot\big)-\xi_n(h_n,t;\cdot)\|_{-1}^{1}\Big\}\\&\qquad\le 2\lim_{n\to\infty}\Big\{\sup_{t\in[t_0-3c_n,\,t_0+3c_n]}b_n^{-1}\|\xi_n(\gamma h_n,t;\cdot)\|_{-1}^{1}\Big\}=2\gamma^{1/2}.\end{aligned}$$

Thus, by (2.64), (2.65), we have a.s. for all large k, uniformly over $t\in T_k(\theta)$ and $\nu_k<n\le\nu_{k+1}$,

$$b_n^{-1}\,\xi_n(h_n,t;\cdot)\in S^{\{(1+2\gamma)\epsilon+2\gamma+2\gamma^{1/2}\}}.\tag{2.66}$$

Next, we infer from (2.53), (2.62) and $\gamma \in (0, 1/2]$, that, for all large k and $\nu_k < n \le \nu_{k+1}$,

$$
\begin{aligned}
[t_0 - c_n, t_0 + c_n] &\subseteq \left[t_0 - (1-\gamma)^{-1} c_{\nu_{k+1}}, t_0 + (1-\gamma)^{-1} c_{\nu_{k+1}}\right] \\
&\subseteq \left[t_0 - 2c_{\nu_{k+1}}, t_0 + 2c_{\nu_{k+1}}\right] \\
&\subseteq \left[t_{\nu_{k+1}}(-M_{\nu_{k+1}}, \theta), t_{\nu_{k+1}}(M_{\nu_{k+1}}, \theta)\right] \\
&\subseteq \left[t_0 - 3c_{\nu_{k+1}}, t_0 + 3c_{\nu_{k+1}}\right] \subseteq [t_0 - 3c_n, t_0 + 3c_n]. \qquad (2.67)
\end{aligned}
$$

In view of (2.52), an application of Fact 4, taken with $A_n = \theta h_n$ and $C_n = 3c_n$, shows that

$$
\begin{aligned}
&\limsup_{n\to\infty} \left\{ \sup_{t\in[t_0-c_n, t_0+c_n]} \sup_{|t'-t|\le\theta h_n} b_n^{-1} \| \xi_n(h_n, t; \cdot) - \xi_n(h_n, t'; \cdot)\|_{-1}^1 \right\} \\
&\quad \le 2 \limsup_{n\to\infty} \left\{ \sup_{t\in[t_0-3c_n, t_0+3c_n]} b_n^{-1} \|\xi_n(\theta h_n, t; \cdot)\|_{-1}^1 \right\} = 2\theta^{1/2} \quad \text{a.s.} \qquad (2.68)
\end{aligned}
$$

Since (H.1)(i) implies that $t_{\nu_{k+1}}(i+1, \theta) - t_{\nu_{k+1}}(i, \theta) = \theta h_{\nu_{k+1}} \le \theta h_n$ for all $\nu_k \le n \le \nu_{k+1}$, it follows from (2.66), (2.67) and (2.68), that, with probability 1 for all n sufficiently large,

$$
\begin{aligned}
F_n &= \left\{ b_n^{-1} \xi_n(h_n, t; \cdot) : t \in [t_0 - c_n, t_0 + c_n] \right\} \\
&\subseteq S^{\{(1+2\gamma)\varepsilon + 2\gamma + 2\gamma^{1/2} + 2\theta^{1/2}\}}. \qquad (2.69)
\end{aligned}
$$

By (2.69), the observation that $(1+2\gamma)\varepsilon + 2\gamma + 2\gamma^{1/2} + 2\theta^{1/2} < \varepsilon$, subject to an initial choice of $\varepsilon \in (0,1]$, $\varepsilon = \varepsilon/4$, $\gamma \in (0, \varepsilon^2/64]$ and $\theta \in (0, \varepsilon^2/64]$, yields readily (2.57). □

Proof of Proposition 2.1. In view of (1.1) and (2.64), we apply (2.57) with the formal replacements of h_n and c_n by $2h_n$ and $c_n + h_n$ respectively. We readily obtain that, for each $\varepsilon > 0$, almost surely for all large n and $t' \in [t_0 - c_n - h_n, t_0 + c_n + h_n]$, there exists an $f = f_{n,\varepsilon,t'} \in S$ with

$$
\|\xi_n(2h_n, t'; \cdot) - 2^{1/2} f\|_0^1 < \varepsilon/2. \qquad (2.70)
$$

Observe that $\xi_n(h_n, t; s) = \xi_n(2h_n, t - h_n; (s+1)/2) - \xi_n(2h_n, t - h_n, 1/2)$ for $s \in [-1, 1]$ and $t \in [t_0 - c_n, t_0 + c_n]$. Thus, by setting $t' = t - h_n$, we infer from (2.70) that

$$
\begin{aligned}
&\|\xi_n(h_n, t; \cdot) - g\|_{-1}^1 \\
&\quad = \|\xi_n(h_n, t; \cdot) - 2^{1/2}\{f((\cdot + 1)/2) - f(1/2)\}\|_{-1}^1 < \varepsilon, \qquad (2.71)
\end{aligned}
$$

where the function $g(s) := 2^{1/2}\{f((s+1)/2) - f(1/2)\}$ of $s \in [-1, 1]$ satisfies $g(0) = 0$ and $\dot{g}(s) = 2^{-1/2} \dot{f}((s+1)/2)$. The change of variable $s = 2v - 1$ allows us write $|g|_H^2 = \int_{-1}^1 \dot{g}(s)^2\, ds = \int_0^1 \dot{f}(v)^2\, dv = |f|_H^2 \le 1$. Given (2.71) and this last inequality, (2.32) is straightforward. □

3. Proofs—inner bounds.

3.1. *Introduction.* In the present section we establish the inner bounds of Theorem 2.1. In view of (2.25) and $S = -S$, this amounts to showing that, for each $\varepsilon > 0$ and $f \in S$, we have almost surely

$$\liminf_{n\to\infty} \| b_n^{-1}\, \xi_n(h_n, \mathbb{V}_n(t_0); \cdot) - f \|_{-1}^{1} \le \varepsilon. \tag{3.1}$$

For each $f \in B[-1,1]$, define $f^{\pm} \in B[0,1]$ and $|f^{\pm}|_H$ (when $|f|_H < \infty$) by letting

$$f^{\pm}(s) = \pm f(\pm s) \quad \text{for } s \in [0,1] \quad \text{and} \quad |f^{\pm}|_H = \left\{ \int_0^1 \dot{f}(\pm s)^2\, ds \right\}^{1/2}. \tag{3.2}$$

We will limit ourselves to proving that (3.1) holds when f varies in the set $S_0 \subseteq B[-1,1]$ [with closure equal to S in $(B[-1,1], U)$], which is composed of all $f \in S \in B[-1,1]$ such that

$$\begin{gathered} 0 < |f^{+}|_H^2,\ 0 < |f^{-}|_H^2, \quad \text{and} \\ |f|_H^2 = \int_{-1}^{1} \dot{f}(\pm s)^2\, ds = |f^{-}|_H^2 + |f^{+}|_H^2 < 1. \end{gathered} \tag{3.3}$$

We will consider successively the case of *small*, *large* and *intermediate* increments, corresponding respectively to $d = \infty$, $d \in [-\infty, 0]$ and $d \in (0, \infty)$ in (H.6). A rough outline of the arguments of our proofs in each of these cases is as follows. Let $f \in S_0$.

1. For *small* and *intermediate* increments, we show that, for each small $\rho > 0$ and $\gamma > 0$, there exists almost surely a sequence $1 \le R_1 < R_2 < \cdots$, together with a sequence $t_k \in [t_0 + \rho c_{R_k}, 2\rho c_{R_k}]$, $k \ge 1$, such that, for infinitely many indices k, we have jointly $\| b_n^{-1}\, \xi_n(h_n, t_k; \cdot) - f \|_{-1}^{1} \le \varepsilon/2$ for all integers $n \in [R_k, (1+\gamma)R_k]$, and $b_n^{-1} \| \xi_n(h_n, t_k; \cdot) - \xi_n(h_n, \mathbb{V}_n(t_0); \cdot) \|_{-1}^{1} \le \varepsilon/2$ for some integers $n \in [R_k, (1+\gamma)R_k]$. The conclusion (3.1) follows by combining these inequalities.
2. For *large* increments with $d \in [-\infty, 0)$, we combine Fact 1 and Lemma 2.4. In the remaining case where $d = 0$, we introduce a sequence $1 \le n_1 < n_2 < \cdots$ such that, almost surely for infinitely many indices k, $\| b_{n_k}^{-1} \xi_{n_k}(h_{n_k}, t_0; \cdot) - f \|_{-1}^{1} \le \varepsilon/2$ and $|\mathbb{V}_{n_k}(t_0) - t_0| \le n_k^{-1/2} (\log n_k)^{-\kappa}$. Here, $\kappa > 0$ is a constant chosen in such a way that $b_{n_k}^{-1} \| \xi_{n_k}(h_{n_k}, t_0; \cdot) - \xi_{n_k}(h_{n_k}, \mathbb{V}_{n_k}(t_0); \cdot) \| \le \varepsilon/2$ for all large k. The conclusion (3.1) follows from these combined facts.

We start by giving a series of technical lemmas which will be useful for our needs. Recall (2.38).

LEMMA 3.1. *Under* (H.1) *and* (H.5), (H.6), *for any* $\varepsilon > 0$ *and* $f \in S_0$, *there exists an* $\eta \in (0,1)$ *such that, for all large* n,

$$\left(1+\frac{c_n}{h_n}\right)\mathbb{P}\left(b_n^{-1}L_n(h_n,t_0;\cdot) \in N_\varepsilon(f)\right) \geq \frac{(1+c_n/h_n)^{1-|f|_H^2(1-\eta)}}{(\log n)^{|f|_H^2(1-\eta)}} \geq \frac{(1+c_n/h_n)^{\eta}}{(\log n)^{1-\eta}}. \tag{3.4}$$

PROOF. Set $L_n^{\pm}(h_n,t_0;s) = \pm L_n(h_n,t_0;\pm s)$ for $s \in [0,1]$. Observe that, whenever n is so large that $0 < t_0 - h_n < t_0 + h_n < 1$, then, $L_n^{+}(h_n,t_0;\cdot) \in B[0,1]$ and $L_n^{-}(h_n,t_0;\cdot) \in B[0,1]$ are independent and identically distributed. By (2.40), it follows that, for all large n,

$$\begin{aligned}
&\left(1+\frac{c_n}{h_n}\right)\mathbb{P}\left(b_n^{-1}L_n(h_n,t_0;\cdot) \in N_\varepsilon(f)\right)\\
&\quad= \left(1+\frac{c_n}{h_n}\right)\mathbb{P}\Big(\left\{b_n^{-1}L_n^{+}(h_n,t_0;\cdot) \in N_\varepsilon(f^{+})\right\}\\
&\qquad\qquad \cap\left\{b_n^{-1}L_n^{-}(h_n,t_0;\cdot) \in N_\varepsilon(f^{-})\right\}\Big)\\
&\quad\geq \frac{(1+c_n/h_n)^{1-(|f^{+}|_H^2+|f^{-}|_H^2)(1-\eta)}}{(\log n)^{(|f^{+}|_H^2+|f^{-}|_H^2)(1-\eta)}}\\
&\quad= \frac{(1+c_n/h_n)^{1-|f|_H^2(1-\eta)}}{(\log n)^{|f|_H^2(1-\eta)}} \geq \frac{(1+c_n/h_n)^{\eta}}{(\log n)^{1-\eta}},
\end{aligned}$$

which is (3.4) □

In our proofs, we will follow the conventions of Section 2.2, by assuming, without loss of generality, that $t_0 \in (0,1/2]$. The following additional notation will be needed. Set

$$\begin{aligned}
\tau_n(\theta) &= (2-\theta)t_0 - \mathbb{U}_n(t_0(1-\theta)) = t_0 - n^{-1/2}\alpha_n(t_0(1-\theta)) \\
&\qquad\qquad \text{for } \theta \in [0,1],\\
\tau_n &= \tau_n(0) = 2t_0 - \mathbb{U}_n(t_0) = t_0 - n^{-1/2}\alpha_n(t_0).
\end{aligned} \tag{3.5}$$

LEMMA 3.2. *For each* $\theta \in [0,1]$, *we have, with probability* 1,

$$\limsup_{n\to\infty} n(\log n)^{-1}|\mathbb{V}_{n+1}(t_0) - \mathbb{V}_n(t_0)| \leq 1, \tag{3.6}$$

$$\limsup_{n\to\infty} n^{1/2}(\log_2 n)^{-1/2}|\tau_n(\theta) - \tau_n| = 2^{1/2}\left(\theta t_0(1-\theta t_0)\right)^{1/2}, \tag{3.7}$$

$$\limsup_{n\to\infty} n^{3/4}(\log_2 n)^{-3/4}|\mathbb{V}_n(t_0) - \tau_n| = 2^{5/4}3^{-3/4}\left(t_0(1-t_0)\right)^{1/4}, \tag{3.8}$$

$$\limsup_{n\to\infty} n^{1/2}(\log_2 n)^{-1/2}|\mathbb{V}_n(t_0) - t_0| = 2^{1/2}\left(t_0(1-t_0)\right)^{1/2}. \tag{3.9}$$

PROOF. Note from (2.3) that $|\mathbb{V}_{n+1}(t_0) - \mathbb{V}_n(t_0)| = |U_{j,n+1} - U_{i,n}|$ with $i := \lceil nt_0 \rceil$, $j := \lceil (n+1)t_0 \rceil$, and $i \le j \le i+1$ a.s. Since $U_{j,n+1} = U_{j,n}$ when $U_{n+1} > U_{j,n}$, and $U_{j,n+1} \in [U_{j-1,n}, U_{j,n}]$ when $U_{n+1} \le U_{j,n}$, in either case, $U_{j,n+1} \in [U_{i-1,n}, U_{i+1,n}]$. It follows that, with probability 1,

$$|\mathbb{V}_{n+1}(t_0) - \mathbb{V}_n(t_0)| \le \max_{1 \le m \le n+1} \{U_{m,n} - U_{m-1,n}\} =: K_n.$$

Since $n(\log n)^{-1} K_n \to 1$ a.s. [Devroye (1981, 1982)], the inequality above suffices for (3.6). Making use of the assumption that $t_0 \in (0, 1/2]$, we see that $0 \le t_0\theta \le t_0 \le 1/2$, which enables to derive (3.7) from Corollary 2.1 of Deheuvels (1992). Equation (3.8) is due to Kiefer (1967) [see, e.g., Deheuvels and Mason (1990a) and Section 3.4 of Deheuvels and Mason (1994b)]. Finally, we obtain readily (3.9) by combining (3.8) with the law of the iterated logarithm for $n\mathbb{U}_n(t_0)$, considered as the partial sum of order n of an i.i.d. sequence of Bernoulli random variables with expectation t_0. □

LEMMA 3.3. *Under* (H.5), (H.6), *we have*, *with probability* 1,

$$\lim_{n\to\infty} b_n^{-1} \sup_{0 \le t \le 1} \|\xi_n(2n^{-1}\log n, t; \cdot)\|_0^1 = 0. \tag{3.10}$$

PROOF. Recall the definition (3.7) of R_C^+ for $C > 0$. By Theorem 1(I) of Mason, Shorack and Wellner (1983) [see, e.g., (4.1.1)–(4.1.6) in Deheuvels and Mason (1992)], we have

$$\lim_{n\to\infty} \left(n^{1/2}/2\log n\right) b_n^{-1} \sup_{0 \le t \le 1} \|\xi_n(2n^{-1}\log n, t; \cdot)\|_0^1 = R_2^+ < \infty \quad \text{a.s.}$$

By Remark 1.1(ii), (H.5), (H.6) imply (1.13) so that $b_n^{-1}(\log n)/n^{1/2} = O((nh_n/\log n)^{-1/2}) \to 0$. This, when combined with the above inequality, yields (3.10). □

3.2. *Small increments* $(d = \infty)$. The main result of this section is stated in the following proposition.

PROPOSITION 3.1. *Assume that* (H.1) *and* (H.5), (H.6) *hold with* $d = \infty$. *Then*, *the sequence* $\{b_n^{-1}\zeta_n(h_n, t_0; \cdot)\colon\ n \ge 1\}$ *is almost surely compact in* $(B[-1,1], U)$, *with limit set equal to* S.

The following arguments are oriented towards proving Proposition 3.1. We will assume throughout that (H.1) and (H.5), (H.6) hold with $d = \infty$. By (1.12) and (1.13), this implies that, for each $\kappa > 0$, we have, ultimately in $n \to \infty$,

$$nh_n/\log n \to \infty \quad \text{and} \quad h_n \le n^{-1/2}(\log n)^{-\kappa}. \tag{3.11}$$

Recalling the notation of Sections 1 and 2, we let Π_n, Π and $L_n(h_n, t; \cdot)$ be as in (2.37), (2.38) and $\mathbf{h}$ be as in (2.6). We let $\gamma > 0$ denote a constant which will be specified later on, and set $\nu_k = \lfloor (1+\gamma)^k \rfloor$ for $k \ge 0$. The following fact will be useful.

FACT 6. *Let* $\{\Pi(t)\colon t \ge 0\}$ *be a standard Poisson process. Then, for any* $T > 0$ *and* $x \ge 0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le u\le T}|\Pi(u) - u| \ge Tx\Big) \\ \le \exp(-T\mathbf{h}(1+x)) + \exp(-T\mathbf{h}(1-x)) \\ \le 2\exp(-T\mathbf{h}(1-x)).\end{aligned} \tag{3.12}$$

Since $\mathbf{h}(1+x) \le \mathbf{h}(1-x)$ for $x \ge 0$, $\mathbf{h}(1-x) = \infty$ for $x > 1$, (3.12) follows from Inequality 1, page 569 in Shorack and Wellner (1986) and Lemma 2.1 in Deheuvels and Mason (1994b).

LEMMA 3.4. *Assume that* (H.1) *and* (H.5), (H.6) *hold with* $d = \infty$. *Select an arbitrary* $f \in S_0$. *Fix any* $\varepsilon \in (0,1]$, *any* $\rho \in (0, 1/2]$, *and choose a* $\gamma \in (0, (\varepsilon/64)^2]$. *Then, with probability* 1, *for all large* k, *there exists a* $t_k = t_k(f, \varepsilon, \rho) \in [t_0 + \rho c_{\nu_k}, t_0 + 2\rho c_{\nu_k}]$ *such that*

$$\| b_{\nu_k}^{-1}\ \xi_{\nu_k}\big(h_{\nu_k}, t_k; \cdot\big) - f\|_{-1}^{1} \le \varepsilon/2, \tag{3.13}$$

and

$$\max_{\nu_k < n \le \nu_{k+1}} \| b_n^{-1}\ \xi_n\big(h_{\nu_k}, t_k; \cdot\big) - b_{\nu_k}^{-1}\ \xi_{\nu_k}\big(h_{\nu_k}, t_k; \cdot\big)\|_{-1}^{1} \le \varepsilon/2. \tag{3.14}$$

PROOF. The proof will be achieved in the following three steps.

Step 1. Fix t, $i \ge 1$ and $N \ge 1$ in such a way that $0 \le t \le t + iN^{-1}h_n \le 1$. The random variables

$$\begin{aligned}X_m = (n+m)\mathbb{U}_{n+m}\big(t + iN^{-1}h_n\big) - (n+m)\mathbb{U}_{n+m}(t) \\ -(n+m-1)\mathbb{U}_{n+m-1}\big(t + iN^{-1}h_n\big) + (n+m-1)\mathbb{U}_{n+m-1}(t), \\ m \ge 1,\end{aligned}$$

are independent with $\mathbb{P}(X_m = 1) = 1 - \mathbb{P}(X_m = 0) = p := iN^{-1}h_n$. On a probability space enlarged by products, define a sequence of independent random variables $\{Y_m\colon m \ge 1\}$, independent of $\{X_m\colon m \ge 1\}$, and such that, for each $m \ge 1$,

$$\begin{aligned}\mathbb{P}(Y_m = 0) &= p^{-1}(e^{-p} - 1 + p), \\ \mathbb{P}(Y_m = k) &= \frac{p^{k-1}}{k!}e^{-p} \quad \text{for integer } k \ge 1.\end{aligned}$$

Observe that $\{Z_m = X_m Y_m\colon m \ge 1\}$ are independent Poisson random variables with $\mathbb{E}(Z_m) = p$ for $m \ge 1$. Moreover, for each $M \ge 1$,

$$\begin{aligned}\mathbb{P}(X_m = Z_m, \forall\, 1 \le m \le M) &= \big(1 - \mathbb{P}(X_1 = 1)\mathbb{P}(Y_1 \ne 1)\big)^M \\ &= \big(1 - p(1 - e^{-p})\big)^M \ge 1 - Mp^2.\end{aligned}$$

Since $X_1+\cdots+X_m-mp=(n+m)^{1/2}\xi_{n+m}(h_n,t;iN^{-1})-n^{1/2}\xi_n(h_n,t;iN^{-1})$, we infer from this inequality taken with $n=\nu_k$, $p=iN^{-1}h_{\nu_k}$ and $M=\nu_{k+1}-\nu_k$, that, for each $\varepsilon>0$, $-N\le i\le N$ and $t\in[h_{\nu_k},1-h_{\nu_k}]$,

$$\begin{aligned}\mathbb{P}\Big(&\max_{\nu_k<n\le\nu_{k+1}} b_{\nu_k}^{-1}|(n/\nu_k)^{1/2}\xi_n(h_{\nu_k},t;iN^{-1})-\xi_{\nu_k}(h_{\nu_k},t;iN^{-1})|\ge\varepsilon/4\Big)\\&\le\mathbb{P}\Big(\max_{\nu_k<n\le\nu_{k+1}} b_{\nu_k}^{-1}|(n/\nu_k)^{1/2}L_n(h_{\nu_k},t;iN^{-1})-L_{\nu_k}(h_{\nu_k},t;iN^{-1})|\ge\varepsilon/4\Big)\\&\quad+(\nu_{k+1}-\nu_k)(i/N)^2h_{\nu_k}^2.\end{aligned}$$

Fix now $N\ge(64/\varepsilon)^2$. By (2.38), it is readily verified that, independently of $t\in[h_{\nu_k},1-h_{\nu_k}]$,

$$\begin{aligned}Q_{k,N}(t,\varepsilon/4)&:=\mathbb{P}\Big(\max_{-N\le i\le N}\max_{\nu_k<n\le\nu_{k+1}} b_{\nu_k}^{-1}\Big|(n/\nu_k)^{1/2}\xi_n(h_{\nu_k},t;iN^{-1})\\&\qquad\qquad-\xi_{\nu_k}(h_{\nu_k},t;iN^{-1})\Big|\ge\varepsilon/4\Big)\\&\le 2N(\nu_{k+1}-\nu_k)h_{\nu_k}^2\\&\quad+2\sum_{i=1}^N\mathbb{P}\Big(\max_{\nu_k<n\le\nu_{k+1}} b_{\nu_k}^{-1}\Big|(n/\nu_k)^{1/2}L_n(h_{\nu_k},t;iN^{-1})\\&\qquad\qquad-L_{\nu_k}(h_{\nu_k},t;iN^{-1})\Big|\ge\varepsilon/4\Big)\\&\le 2N(\nu_{k+1}-\nu_k)h_{\nu_k}^2\\&\quad+2\sum_{i=1}^N\mathbb{P}\Big(\sup_{0\le u\le iN^{-1}(\nu_{k+1}-\nu_k)h_{\nu_k}}|\Pi(u)-u|\ge(\varepsilon/4)\nu_k^{1/2}b_{\nu_k}\Big).\end{aligned}\tag{3.15}$$

Assumptions (H.5) and (H.6) jointly imply that $n^{1/2}h_n/b_n\to\infty$. Moreover, $(\nu_{k+1}-\nu_k)/\nu_k\to\gamma$ as $k\to\infty$. Therefore, each $i=1,\dots,N$, we have, as $k\to\infty$,

$$\begin{aligned}x_{k,N}(i)&:=\Big((\varepsilon/4)\nu_k^{1/2}b_{\nu_k}\Big)/\Big(iN^{-1}(\nu_{k+1}-\nu_k)h_{\nu_k}\Big)\\&=(1+o(1))\varepsilon Nb_{\nu_k}/\Big(4i\gamma\nu_k^{1/2}h_{\nu_k}\Big)\to 0.\end{aligned}$$

Since $\mathbf{h}(1-u)=(1+o(1))u^2/2$ as $|u|\to 0$ and $0<\gamma<(\varepsilon/64)^2$, it follows that, for each $i=1,\dots,N$, as $k\to\infty$,

$$\begin{aligned}iN^{-1}(\nu_{k+1}-\nu_k)h_{\nu_k}\mathbf{h}(1-x_{k,N}(i))&=(1+o(1))\frac{i\gamma\nu_k h_{\nu_k}x_{k,N}^2(i)}{2N}\\&=(1+o(1))\frac{\varepsilon^2Nb_{\nu_k}^2}{32i\gamma h_{\nu_k}}\\&\ge\frac{4b_{\nu_k}^2}{h_{\nu_k}}\ge 2\log\left(\frac{1}{h_{\nu_k}\sqrt{\nu_k}}\right)\to\infty.\end{aligned}$$

By (3.12) and (3.15), we obtain therefore that, uniformly over $t \in [h_{\nu_k}, 1 - h_{\nu_k}]$, as $k \to \infty$,

$$
\begin{aligned}
Q_{k,N}(t, \varepsilon/4) &\le 4N\gamma\nu_k h_{\nu_k}^2 \\
&\quad + 4\sum_{i=1}^{N} \exp\bigl(-iN^{-1}(\nu_{k+1} - \nu_k) h_{\nu_k}\mathbf{h}\bigl(1 - x_{k,N}(i)\bigr)\bigr) \qquad (3.16)\\
&\le 4N(\gamma + 1)\nu_k h_{\nu_k}^2.
\end{aligned}
$$

Step 2. Fix $f \in S_0$ and choose any $\rho \in (0, 1/2]$. By (3.11) we have, for each $\kappa > 0$ and all large n,

$$
\begin{aligned}
\frac{(\log n)^{\kappa/4}}{n^{1/2} h_n} &\ge \lfloor \rho c_n / 2h_n \rfloor \ge \rho c_n / 2h_n - 1 \qquad (3.17)\\
&> (\rho/4)(1 + c_n/h_n) \ge (\rho/4)(\log n)^{\kappa/2}.
\end{aligned}
$$

Set $M_k' = \lfloor \rho c_{\nu_k} / 2h_{\nu_k} \rfloor$ for $k \ge 1$. We infer from (2.51) that, for all large k,

$$
\begin{aligned}
P_k &:= \mathbb{P}\Biggl(\bigcap_{j=M_k'+1}^{2M_k'} \bigl\{b_{\nu_k}^{-1}\, \xi_{\nu_k}\bigl(h_{\nu_k}, t_0 + 2jh_{\nu_k}; \cdot\bigr) \notin N_{\varepsilon/2}(f)\bigr\}\Biggr) \\
&\quad + \mathbb{P}\Biggl(\bigcup_{j=M_k'+1}^{2M_k'} \Bigl\{\max_{-N \le i \le N}\ \max_{\nu_k < n \le \nu_{k+1}} b_{\nu_k}^{-1}\Bigl|(n/\nu_k)^{1/2}\xi_n\bigl(h_{\nu_k}, t_0 + 2jh_{\nu_k}; iN^{-1}\bigr) \\
&\qquad\qquad - \xi_{\nu_k}\bigl(h_{\nu_k}, t_0 + 2jh_{\nu_k}; iN^{-1}\bigr)\Bigr| \ge \varepsilon/4\Bigr\}\Biggr) \\
&\le C_4\mathbb{P}\Biggl(\bigcap_{j=M_k'+1}^{2M_k'} \bigl\{b_{\nu_k}^{-1} L_{\nu_k}\bigl(h_{\nu_k}, t_0 + 2jh_{\nu_k}; \cdot\bigr) \notin N_{\varepsilon/2}(f)\bigr\}\Biggr) \\
&\quad + \sum_{j=M_k'+1}^{2M_k'} Q_{k,N}\bigl(t_0 + 2jh_{\nu_k}, \varepsilon/4\bigr).
\end{aligned}
$$

By combining (2.37) and the independence of the increments of Π_n on nonoverlapping intervals with (3.4), (3.11), (3.15), (3.16) and (3.17), we infer from the inequality above that there exists an $\eta > 0$ such that, for each $\kappa > 0$ and all k sufficiently large,

$$
\begin{aligned}
P_k &\le C_4\Bigl(1 - \mathbb{P}\bigl(b_{\nu_k}^{-1} L_{\nu_k}\bigl(h_{\nu_k}, t_0; \cdot\bigr) \in N_{\varepsilon/2}(f)\bigr)\Bigr)^{M_k'} + 4N(\gamma + 1)\, M_k' \nu_k h_{\nu_k}^2 \\
&\le C_4 \exp\Biggl(-\biggl\lfloor \frac{\rho c_{\nu_k}}{2h_{\nu_k}} \biggr\rfloor \mathbb{P}\bigl(b_{\nu_k}^{-1} L_{\nu_k}\bigl(h_{\nu_k}, t_0; \cdot\bigr) \in N_{\varepsilon/2}(f)\bigr)\Biggr) \\
&\quad + 4N(\gamma + 1)(\log \nu_k)^{\kappa/4} \nu_k^{1/2} h_{\nu_k} \qquad (3.18)
\end{aligned}
$$

$$\leq C_4 \exp\left(-(\rho/4)\frac{(1+c_{\nu_k}/h_{\nu_k})^{\eta}}{(\log \nu_k)^{1-\eta}}\right) + (\log \nu_k)^{-\kappa/2}$$

$$\leq C_4 \exp\left(-(\rho/4)(\log \nu_k)^{\eta(1+\kappa/2)-1}\right) + (\log \nu_k)^{-\kappa/2}.$$

Since $\log \nu_k = (1+o(1))k\log(1+\gamma)$, the choice of $\kappa = 4(1 \vee \eta^{-1})$ in (3.17) and (3.18) entails that $\sum_k P_k < \infty$. The Borel–Cantelli lemma implies therefore that for all k sufficiently large there exists a $t_k \in \{t_0 + 2jh_{\nu_k}\colon\ M'_k < j \leq 2\,M'_k\} \subseteq [t_0 + \rho c_{\nu_k}, t_0 + 2\rho c_{\nu_k}]$ fulfilling (3.13), together with

$$\max_{\nu_k < n \leq \nu_{k+1}} \left\{ \max_{-N \leq i \leq N} b_{\nu_k}^{-1} \left| (n/\nu_k)^{1/2} \xi_n\left(h_{\nu_k}, t_k; iN^{-1}\right) - \xi_{\nu_k}\left(h_{\nu_k}, t_k; iN^{-1}\right)\right| \right\} < \varepsilon/4. \tag{3.19}$$

Step 3. Since $\rho \in (0, 1/2]$, by Fact 4, taken with $A_n = N^{-1}h_n$, $C_n = 2c_n$, we obtain readily that

$$\begin{aligned}
&\limsup_{k\to\infty} b_{\nu_k}^{-1} \left\| \xi_{\nu_k}\left(h_{\nu_k}, t_k; \pm I\right) - \xi_{\nu_k}\left(h_{\nu_k}, t_k; \pm \lfloor NI \rfloor N^{-1}\right)\right\|_0^1 \\
&\quad\leq \limsup_{n\to\infty} b_n^{-1} \sup_{t\in[t_0-c_n,\, t_0+c_n]} \left\| \xi_n(h_n, t; \pm I) - \xi_n\left(h_n, t; \pm \lfloor NI \rfloor N^{-1}\right)\right\|_0^1 \\
&\quad\leq 2\limsup_{n\to\infty} b_n^{-1} \sup_{t\in[t_0-2c_n,\, t_0+2c_n]} \left\| \xi_n\left(N^{-1}h_n, t; \cdot\right)\right\|_{-1}^1 = 2\,N^{-1/2} \quad \text{a.s.}
\end{aligned} \tag{3.20}$$

By (2.61) and (2.63), we have, for all k sufficiently large, uniformly over $\nu_k \leq n \leq \nu_{k+1}$

$$\left(b_n/b_{\nu_k}\right)(n/\nu_k)^{1/2} \leq 1 + 2\gamma \quad \text{and} \quad |h_n - h_{\nu_k}| \leq 2\gamma h_n.$$

By combining (3.20) with Fact 4, taken with $A_n = 2\gamma h_n$ and $C_n = 2c_n$, we obtain therefore that

$$\begin{aligned}
&\limsup_{k\to\infty} b_{\nu_k}^{-1} \max_{\nu_k < n \leq \nu_{k+1}} (n/\nu_k)^{1/2} \left\| \xi_n(h_n, t_k; \pm I) - \xi_n\left(h_{\nu_k}, t_k; \pm \lfloor NI \rfloor N^{-1}\right)\right\|_0^1 \\
&\quad\leq (1+2\gamma)\left\{ \limsup_{n\to\infty} b_n^{-1} \sup_{t\in[t_0-c_n,\, t_0+c_n]} \left\| \xi_n(h_n, t; \pm I) - \xi_n\left(h_n, t; \pm \lfloor NI \rfloor N^{-1}\right)\right\|_0^1 \right. \\
&\qquad\left. + \limsup_{k\to\infty} \max_{\nu_k < n \leq \nu_{k+1}} b_n^{-1} \sup_{t\in[t_0-c_n,\, t_0+c_n]} \left\| \xi_n\left(h_n, t; \pm \lfloor NI \rfloor N^{-1}\right) - \xi_n\left(h_{\nu_k}, t; \pm \lfloor NI \rfloor N^{-1}\right)\right\|_0^1 \right\} \\
&\quad\leq (1+2\gamma)\left\{ 2\,N^{-1/2} + 2\limsup_{n\to\infty} b_n^{-1} \sup_{t\in[t_0-2c_n,\, t_0+2c_n]} \left\| \xi_n(2\gamma h_n, t; \cdot)\right\|_{-1}^1 \right\} \\
&\quad= 2(1+2\gamma)\left(N^{-1/2} + (2\gamma)^{1/2}\right) \quad \text{a.s.}
\end{aligned}$$

This, when combined with (3.19) and (3.20), shows that, with probability 1 for all large k,

$$\begin{aligned}
&\max_{\nu_k < n \le \nu_{k+1}} b_{\nu_k}^{-1} \left\| (n/\nu_k)^{1/2} \xi_n(h_n, t_k; \cdot) - \xi_{\nu_k}(h_{\nu_k}, t_k; \cdot) \right\|_{-1}^{1} \\
(3.21)\qquad &\le \varepsilon/4 + 2N^{-1/2} + 2(1+2\gamma)\left(N^{-1/2} + (2\gamma)^{1/2}\right) \\
&< \varepsilon/4 + 8\left(N^{-1/2} + \gamma^{1/2}\right) < 3\varepsilon/8,
\end{aligned}$$

where we have used $\gamma \le (\varepsilon/64)^2 < 1/2$, $0 < \varepsilon < 1$ and $N \ge (64/\varepsilon)^2$. By (2.63) and (3.20),

$$\begin{aligned}
&\limsup_{k\to\infty} \left(\max_{\nu_k < n \le \nu_{k+1}} |(b_n/b_{\nu_k})(n/\nu_k)^{1/2} - 1| \times b_n^{-1} \| \xi_n(h_n, t_k; \cdot) \|_{-1}^{1} \right) \\
&\quad < 2\gamma \limsup_{n\to\infty} \left(\sup_{t \in [t_0 - c_n, t_0 + c_n]} b_n^{-1} \| \xi_n(h_n, t; \cdot) \|_{-1}^{1} \right) = 2\gamma < \varepsilon/8,
\end{aligned}$$

which, when combined with (3.21), yields (3.14). □

PROOF OF PROPOSITION 3.1. By Proposition 2.2 and the discussion in Section 3.1, we need only prove that (3.1) holds for each specified $\varepsilon \in (0,1]$ and $f \in S_0$. Towards this aim, we apply Lemma 3.4 with the formal replacement of ε by $\varepsilon/2$, $\gamma = (\varepsilon/128)^2$ and $\rho = \frac{1}{3}\gamma(1+\gamma)^{-1/2}(2t_0(1-t_0))^{1/2} \in (0, 1/2]$. By (3.13), (3.14), there exists almost surely for each large k a $t_k \in [t_0 + \rho c_{\nu_k}, t_0 + 2\rho c_{\nu_k}]$ such that

$$b_n^{-1}\xi_n(h_n, t_k; \cdot) \in N_{\varepsilon/2}(f) \quad \text{for all } \nu_k \le n \le \nu_{k+1}. \tag{3.22}$$

We will now prove that, with probability 1, we have, infinitely often in k,

$$\mathbb{V}_{\nu_k}(t_0) - t_0 > 2\rho c_{\nu_k} \quad \text{and} \quad \mathbb{V}_{\nu_{k+1}}(t_0) - t_0 < \rho c_{\nu_k}. \tag{3.23}$$

For this, we observe that $n^{1/2}\alpha_n(t_0) = n\mathbb{U}_n(t_0) - nt_0$ is the partial sum of order n from a sequence of independent centered Bernoulli random variables with parameter t_0. Making use of the functional version of the Weber (1990) law of the iterated logarithm for subsequences [see (1.3), (1.4) and Theorem 1.1 in Deheuvels and Lifshits (1993)], we readily obtain that the sequence $g^*_{\nu_{k+1}}(s) = (2t_0(1-t_0))^{-1/2}(\log_2 \nu_k)^{-1/2} s^{1/2} \alpha_{\lfloor \nu_k s \rfloor}(t_0)$ of functions of $s \in [0,1]$, is almost surely relatively compact in $(B[0,1], U)$, with limit set equal to S. By setting for $s \in [0,1]$ $g_n(s) = (2t_0(1-t_0))^{-1/2}(\log_2 n)^{-1/2} s^{1/2} \beta_{\lfloor ns \rfloor}(t_0)$, we infer from (1.16), that $\|g^*_n + g_n\|_0^1 \to 0$. This, in turn, implies that $\{g_{\nu_{k+1}}: k \ge 1\}$ is a.s. relatively compact in $(B[0,1], U)$, with limit set equal to $S = -S$. Since the function $g(s) = \min(s, 1-s)$ for $0 \le s \le 1$, belongs to S, there exists therefore a.s. a sequence $1 \le k(1) < k(2) < \cdots$, such that $\|g_{\nu_{k(m)+1}} - g\|_0^1 \to 0$ as $m \to \infty$. In particular, for each $\varepsilon_0 \in (0, \frac{1}{4}\gamma(1+\gamma)^{-1/2})$, there exists a.s. an $m_0(\varepsilon_0)$ such that

$$\|g_{\nu_{k(m)+1}} - g\|_0^1 \le \varepsilon_0/4 \quad \text{for } m \ge m_0(\varepsilon_0).$$

Moreover, since $\nu_{k(m)}/\nu_{k(m)+1} \to (1+\gamma)^{-1}$, there exists an $m_1(\varepsilon_0)$ such that for $m \geq m_1(\varepsilon_0)$,

$$\Big|\big(\nu_{k(m)+1}/\nu_{k(m)}\big)^{1/2} g\big(\nu_{k(m)}/\nu_{k(m)+1}\big) - (1+\gamma)^{1/2} g\big((1+\gamma)^{-1}\big)\Big|$$
$$= \Big|\big(\nu_{k(m)+1}/\nu_{k(m)}\big)^{1/2} g\big(\nu_{k(m)}/\nu_{k(m)+1}\big) - \gamma(1+\gamma)^{-1/2}\Big| \leq \varepsilon_0/4.$$

Since $g(1) = 0$, the above two inequalities imply that, for all $m \geq m_0(\varepsilon_0) \vee m_1(\varepsilon_0)$,

$$\max\Big\{\big|g_{\nu_{k(m)+1}}(1)\big|, \Big|\big(\nu_{k(m)+1}/\nu_{k(m)}\big)^{1/2} g_{\nu_{k(m)+1}}\big(\nu_{k(m)}/\nu_{k(m)+1}\big) - \gamma(1+\gamma)^{-1/2}\Big|\Big\}$$
$$\leq \varepsilon_0/2.$$

Since (3.11) implies that $c_n = (1+o(1))n^{-1/2}(\log_2 n)^{1/2} \geq n^{-1/2}(\log_2 n)^{1/2}/2$ for all large n, this implies in turn that there exists an $m_2(\varepsilon_0) \geq m_0(\varepsilon_0) \vee m_1(\varepsilon_0)$ such that for all $m \geq m_2(\varepsilon_0)$,

$$(3.24)\quad \begin{aligned} \mathbb{V}_{\nu_{k(m)+1}}(t_0) - t_0 &\leq \varepsilon_0\big(2t_0(1-t_0)\big)^{1/2} c_{\nu_{k(m)+1}} \\ &\leq \tfrac{1}{4}\gamma(1+\gamma)^{-1/2}\big(2t_0(1-t_0)\big)^{1/2} c_{\nu_{k(m)+1}} < \rho c_{\nu_{k(m)}}, \\ \mathbb{V}_{\nu_{k(m)}}(t_0) - t_0 &\geq \Big(\gamma(1+\gamma)^{-1/2} - \varepsilon_0\Big)\big(2t_0(1-t_0)\big)^{1/2} c_{\nu_{k(m)+1}} \\ &\geq \tfrac{3}{4}\gamma(1+\gamma)^{-1/2}\big(2t_0(1-t_0)\big)^{1/2} c_{\nu_{k(m)+1}} > 2\rho c_{\nu_{k(m)}}, \end{aligned}$$

which shows that (3.23) holds for $k = k(m)$. We infer from (3.22) and (3.23) that there exists a.s. for all large m an $n(m) \in [\nu_{k(m)}, \nu_{k(m)+1}]$, such that $\mathbb{V}_{n(m)}(t_0) \geq t_{k(m)}$ and $\mathbb{V}_{n(m)+1}(t_0) < t_{k(m)}$. By (3.6) and (3.10), it follows that, a.s. for all large m,

$$b_{n(m)}^{-1}\Big\| \xi_{n(m)}\big(h_{n(m)}, \mathbb{V}_{n(m)}(t_0), \cdot\big) - \xi_{n(m)}\big(h_{n(m)}, t_{k(m)}, \cdot\big)\Big\|_{-1}^{1} \leq \varepsilon/2.$$

By (3.22) and the triangle inequality, this implies that

$$b_{n(m)}^{-1}\xi_{n(m)}\big(h_{n(m)}, \mathbb{V}_{n(m)}(t_0), \cdot\big) \in N_\varepsilon(f),$$

which is (3.1). The proof of Proposition 3.1 is therefore complete. □

3.3. *Large increments* ($d \in [-\infty, 0]$). In the *large increment* case where (H.6) holds with $d \in [-\infty, 0]$, the following proposition holds.

PROPOSITION 3.2. *Assume that* (H.1) *and* (H.6) *hold with* $d \in [-\infty, 0]$. *Then, the sequence* $\{b_n^{-1}\zeta_n(h_n, t_0; \cdot)\colon\ n \geq 1\}$ *is almost surely compact in* $(B[-1,1], U)$, *with limit set equal to* S.

The proof of Proposition 3.2 is postponed until the end of this section. The following Lemmas 3.5 and 3.6 hold for $d = 0$ as well as for intermediate increments with $d \in (0, \infty)$. Let τ_n be as in (3.5).

LEMMA 3.5. *Assume that* (H.1) *and* (H.6) *hold with* $d \in [0, \infty)$. *Then, we have*

$$\lim_{n\to\infty} b_n^{-1} \| \zeta_n(h_n, t_0; \cdot) + \xi_n(h_n, \tau_n; \cdot)\|_{-1}^{1} = 0 \quad a.s. \tag{3.25}$$

PROOF. By (1.8) and (1.10), our assumptions imply that, as $n \to \infty$,

$$b_n = (1 + o(1))\big(2h_n(d+1)\log_2 n\big)^{1/2}. \tag{3.26}$$

Moreover, for any $\kappa > d$, we have for all n sufficiently large

$$\text{(i)}\quad h_n \geq n^{-1/2}(\log n)^{-\kappa} \quad \text{and} \quad \text{(ii)}\quad b_n^{-1} \leq n^{1/4}(\log n)^{\kappa/2}. \tag{3.27}$$

Since $2^{5/4}3^{-3/4}(t_0(1-t_0))^{1/4} < 3$, (3.8) implies that $|\mathbb{V}_n(t_0) - \tau_n| \leq h_n^* := 3n^{-3/4}(\log_2 n)^{3/4}$ a.s. for all large n. The replacement of h_n, d_n by h_n^*, $d_n^* = (2h_n^*\{\log(1/h_n^*) + \log_2 n\})^{1/2}$ in (1.5) yields

$$\begin{aligned}&\limsup_{n\to\infty} (d_n^*)^{-1}\|\xi_n(h_n, \mathbb{V}_n(t_0); \cdot) - \xi_n(h_n, \tau_n; \cdot)\|_{-1}^{1}\\ &\qquad \leq 2 \limsup_{n\to\infty} (d_n^*)^{-1} \sup_{0\leq t\leq 1-h_n^*} \|\xi_n(h_n^*; t; u)\|_0^1 = 2 \quad \text{a.s.}\end{aligned} \tag{3.28}$$

Since (3.27) implies that $b_n^{-1} d_n^* \to 0$, as $n \to \infty$, the conclusion (3.25) follows readily from (3.28). □

Let $\gamma > 0$ denote a constant which will be specified later on. Set

$$\begin{aligned}&n_k = \lfloor \exp(k \log^2 k)\rfloor, \qquad m_k = \lfloor (1+\gamma) n_k \rfloor \quad \text{for } k \geq 1,\\ &N_k = n_k - n_{k-1} \quad \text{and} \quad M_k = m_k - n_{k-1} \quad \text{for } k \geq 2.\end{aligned} \tag{3.29a}$$

For $N_k \leq m \leq M_k$ (equivalently, for $n_k \leq n_{k-1} + m \leq (1+\gamma)n_k$), $m \geq 1$, $n \geq 0$ and $t \in \mathbb{R}$, set

$$\begin{aligned}&\mathbb{U}_{m;k}(t) = m^{-1}\big((n_{k-1} + m)\mathbb{U}_{n_{k-1}+m}(t) - n_{k-1}\mathbb{U}_{n_{k-1}}(t)\big),\\ &\tau_n = \tau_n(0) = 2t_0 - \mathbb{U}_n(t_0), \qquad T_{m;k} = 2t_0 - \mathbb{U}_{m;k}(t_0),\\ &\xi_{m;k}(h_{n_k}, t; s)\\ &\qquad = m^{-1/2}\Big\{(n_{k-1} + m)^{1/2}\xi_{n_{k-1}+m}(h_{n_k}, t; s) - n_{k-1}^{1/2}\xi_{n_{k-1}}(h_{n_k}, t; s)\Big\}.\end{aligned} \tag{3.29b}$$

REMARK 3.1. (i) Let k_0 be so large that $n_k < m_k < n_{k+1}$ for all $k \geq k_0$. It is noteworthy that, for each $k \geq k_0$, $\{(\mathbb{U}_{m;k}, T_{m,k}, \xi_{m,k})\colon N_k \leq m \leq M_k\}$ and $\{(\mathbb{U}_m, \tau_m, \xi_m)\colon N_k \leq m \leq M_k\}$ follow the same distribution. Moreover, the $\{(\mathbb{U}_{m;2q}, T_{m,2q}, \xi_{m,2q})\colon N_{2q} \leq m \leq M_{2q}\}$, $q \geq \lceil k_0/2 \rceil$ constitute a sequence of independent random objects.

(ii) For each specified $K > 0$, we have, ultimately as $k \to \infty$,

$$\begin{aligned}&1 - N_k/n_k = n_{k-1}/n_k = \exp\big(-(1+o(1))(\log_2 n_k)^2\big) \leq (\log n_k)^{-K} \to 0,\\ &\log N_k = (1+o(1))\log n_k = (1+o(1))\log n_{k-1},\\ &\log_2 n_k = (1+o(1))\log_2 n_{k-1}.\end{aligned} \tag{3.30}$$

LEMMA 3.6. *Under* (H.1) *and* (H.6) *with* $d \in [0, \infty)$, *for each* $\kappa > d$, *we have with probability* 1,

$$\lim_{k\to\infty}\left\{b_{n_k}^{-1}\max_{N_k\le m\le M_k}\|\xi_{m;k}(h_{n_k}, T_{m;k};\cdot) - \xi_{n_{k-1}+m}(h_{n_k}, \tau_{n_{k-1}+m};\cdot)\|_{-1}^{1}\right\} = 0, \tag{3.31}$$

and

$$\lim_{k\to\infty}\left\{\sup_{t:\,|t-t_0|\le n_k^{-1/2}(\log n_k)^{-\kappa}} b_{n_k}^{-1}\|\xi_{N_k;k}(h_{n_k}, t;\cdot) - \xi_{N_k;k}(h_{n_k}, t_0;\cdot)\|_{-1}^{1})\right\} = 0. \tag{3.32}$$

PROOF. By (3.29), for $N_k \le m \le M_k$, we have $n_k \le n_{k-1} + m \le (1+\gamma)n_k$, and hence, by (H.1),

$$(1+\gamma)^{-1}h_{n_k} \le (n_k/(n_{k-1}+m))h_{n_k} \le h_{n_{k-1}+m} \le h_{n_k}. \tag{3.33}$$

Making use of (1.10) (which holds for $d \in [0, \infty)$), we obtain likewise that

$$\begin{aligned} 1 \le h_{n_{k-1}}/h_{n_k} &= (n_k/n_{k-1})^{(1/2)+o(1)} \\ &= \exp\big((1+o(1))(\log_2 n_k)^2/2\big) \to \infty \quad \text{as } k\to\infty. \end{aligned} \tag{3.34}$$

By (3.26), (3.33) and (1.1), this implies that, for all $N_k \le m \le M_k$, ultimately as $k \to \infty$,

$$\begin{aligned} (1+\gamma)^{-1}c_{n_k} < c_{m_k} \le c_{n_{k-1}+m} \le c_{n_k} \le c_{n_{k-1}} \quad \text{and} \\ (1+\gamma)^{-1}b_{n_k} \le b_{n_{k-1}+m} \le 2b_{n_k}. \end{aligned} \tag{3.35}$$

Our assumptions imply (see Remark 1.1) that (H.1)–(H.4) and (1.5) hold. Thus, by (3.26), (3.30) and (2.9), we obtain that, uniformly over $N_k \le m \le M_k = m_k - n_{k-1}$, as $k \to \infty$,

$$\begin{aligned} &b_{n_k}^{-1}(m/n_{k-1})^{-1/2}\sup_{t\in[t_0-c_{n_k},\, t_0+c_{n_k}]}\big\|\xi_{n_{k-1}}(h_{n_k}, t;\cdot)\big\|_{-1}^{1} \\ &\quad\le (b_{n_{k-1}}/b_{n_k})(n_{k-1}/n_k)^{1/2} \\ &\qquad\times(1-n_{k-1}/n_k)^{-1/2}\sup_{t\in[t_0-c_{n_{k-1}},\, t_0+c_{n_{k-1}}]} b_{n_{k-1}}^{-1}\big\|\xi_{n_{k-1}}(h_{n_{k-1}}, t;\cdot)\big\|_{-1}^{1} \\ &\quad\le 2(h_{n_{k-1}}/h_{n_k})^{1/2}(n_{k-1}/n_k)^{1/2} = (n_{k-1}/n_k)^{(1/4)+o(1)} \to 0 \quad \text{a.s.} \end{aligned} \tag{3.36}$$

By an easy argument based upon (2.2) and (2.32), we infer from (3.29), (3.30) and (3.33), (3.34) that with probability 1 as $k \to \infty$, we have uniformly over $N_k \le m \le M_k$,

$$\begin{aligned} &\max_{N_k \le m \le M_k} \Big\{ |m^{-1/2}(n_{k-1}+m)^{1/2} - 1| \big(b_{n_{k-1}+m}/b_{n_k}\big) \\ &\qquad\qquad \times \sup_{t \in [t_0 - c_{n_k}, t_0 + c_{n_k}]} b_{n_{k-1}+m}^{-1} \big\| \xi_{n_{k-1}+m}(h_{n_k}, t; \cdot) \big\|_{-1}^{1} \Big\} \\ &\quad = O(n_{k-1}/n_k) \to 0. \end{aligned} \tag{3.37}$$

By combining the definition (3.29) of $\xi_{m;k}$ with (3.30), (3.36) and (3.37), we obtain that, as $k \to \infty$,

$$\begin{aligned} &b_{n_k}^{-1} \sup_{t \in [t_0 - c_{n_k}, t_0 + c_{n_k}]} \max_{N_k \le m \le M_k} \big\| \xi_{m;k}(h_{n_k}, t; \cdot) - \xi_{n_{k-1}+m}(h_{n_k}, t; \cdot) \big\| \\ &\quad \to 0 \quad \text{a.s.} \end{aligned} \tag{3.38}$$

By (2.5), (3.29), (3.30) and (3.34), we see that, a.s. as $k \to \infty$, uniformly over $N_k \le m \le M_k$,

$$\begin{aligned} &|\tau_{n_{k-1}+m} - T_{m;k}| \\ &\quad = \left(\frac{n_{k-1}}{m}\right) \Big| \big(\mathbb{U}_{n_{k-1}+m}(t_0) - t_0\big) - \big(\mathbb{U}_{n_{k-1}}(t_0) - t_0\big) \Big| \\ &\quad \le 2 \left(\frac{n_{k-1}}{n_k}\right) \left(\frac{\log_2 n_k}{n_k}\right)^{1/2} \left\{ \left(\frac{2 n_k \log_2(n_{k-1}+m)}{(n_{k-1}+m)\log_2 n_k}\right)^{1/2} \right. \\ &\qquad\qquad \times \left(\frac{|\alpha_{n_{k-1}+m}(t_0)|}{(2\log_2(n_{k-1}+m))^{1/2}}\right) \\ &\qquad\qquad \left. + \left(\frac{2 n_k \log_2 n_{k-1}}{n_{k-1}\log_2 n_k}\right)^{1/2} \left(\frac{|\alpha_{n_{k-1}}(t_0)|}{(2\log_2 n_{k-1})^{1/2}}\right) \right\} \\ &\quad \le 4 \left(\frac{n_{k-1}}{n_k}\right)^{1/2} \left(\frac{\log_2 n_k}{n_k}\right)^{1/2} \\ &\quad = 4 \left(\frac{\log_2 n_k}{n_k}\right)^{1/2} \exp\big(-(1+o(1))(\log_2 n_k)^2/2\big) \\ &\quad \le n_k^{-1/2}(\log n_k)^{-(2d+3)} = o\big(n_k^{-1/2}(\log_2 n_k)^{1/2}\big). \end{aligned} \tag{3.39}$$

By combining (2.5) with (3.33)–(3.35) and (3.39), we obtain readily that, a.s. for all large k, $\tau_{n_{k-1}+m} \in [t_0 - c_{n_k}, t_0 + c_{n_k}]$ and $T_{k;m} \in [t_0 - c_{n_k}, t_0 + c_{n_k}]$ for all $N_k \le m \le M_k$. Thus, by (3.38) and (3.35), the assertions (3.31) and (3.32) are implied by

$$\lim_{k\to\infty}\left\{\max_{N_k\le m\le M_k} b^{-1}_{n_{k-1}+m}\left\|\xi_{n_{k-1}+m}(h_{n_k},T_{m;k};\cdot) - \xi_{n_{k-1}+m}(h_{n_k},\tau_{n_{k-1}+m};\cdot)\right\|^1_{-1}\right\} = 0, \tag{3.40}$$

and

$$\lim_{k\to\infty}\left\{\sup_{t:\,|t-t_0|\le n_k^{-1/2}(\log n_k)^{-\kappa}} b^{-1}_{n_k}\left\|\xi_{n_k}(h_{n_k},t;\cdot) - \xi_{n_k}(h_{n_k},t_0;s)\right\|^1_{-1}\right\} = 0 \quad \text{a.s.} \tag{3.41}$$

By applying (1.5) with the formal replacement of h_n by $h'_n := 2(1+\gamma)^2 n^{-1/2}(\log n)^{-(2d+3)}$, we infer from (3.39), in combination with (3.27) taken with $\kappa = 2d$, that, a.s.,

$$\begin{aligned}
&\lim_{k\to\infty}\left\{\max_{N_k\le m\le M_k} b^{-1}_{n_{k-1}+m}\left\|\xi_{n_{k-1}+m}(h_{n_k},T_{m;k};\cdot) - \xi_{n_{k-1}+m}(h_{n_k},\tau_{n_{k-1}+m};\cdot)\right\|^1_{-1}\right\}\\
&\quad\le 2\limsup_{n\to\infty}\left\{b_n^{-1}\sup_{t\in[h'_n,1-h'_n]}\left\|\xi_n(h'_n,t;\cdot)\right\|^1_{-1}\right\}\\
&\quad\le 4(1+\gamma)\lim_{n\to\infty}\left\{b_n^{-1}n^{-1/4}(\log n)^{-(d+1)}\right\} = 0,
\end{aligned}$$

which is (3.40). Likewise, by combining (1.5), taken with the formal replacement of h_n by $h''_n := n^{-1/2}(\log n)^{-\kappa}$, with (3.27), and our assumption that $\kappa > d$, we obtain that, a.s.,

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{\sup_{t:\,|t-t_0|\le n^{-1/2}(\log n)^{-\kappa}} b_n^{-1}\left\|\xi_n(h_n,t;\cdot) - \xi_n(h_n,t_0;\cdot)\right\|^1_{-1}\right\}\\
&\quad\le 2\limsup_{n\to\infty}\left\{b_n^{-1}\sup_{t\in[h''_n,1-h''_n]}\left\|\xi_n(h''_n,t;\cdot)\right\|^1_{-1}\right\}\\
&\quad= 2\lim_{n\to\infty}\left\{b_n^{-1}n^{-1/4}(\log n)^{(1-\kappa)/2}\right\} = 0,
\end{aligned}$$

which implies (3.41). □

LEMMA 3.7. *There exists a constant* $K = K(t_0)$ *such that, for each* $\kappa > 0$ *and all large* k,

$$\begin{aligned}
&\mathbb{P}\left(n_k^{1/2}(T_{N_k;k} - t_0) \in \left(0,(\log n_k)^{-\kappa}\right)\right)\\
&\quad= \mathbb{P}\left(n_k^{1/2}(\tau_{N_k} - t_0) \in \left(0,(\log n_k)^{-\kappa}\right)\right) \ge K(\log n_k)^{-\kappa}.
\end{aligned} \tag{3.42}$$

PROOF. Set $Z_k := N_k^{1/2}(\tau_{N_k} - t_0)$. Since $N_k^{1/2}Z_k$ is the sum of N_k independent centered Bernoulli random variables with parameter t_0, the following Berry–Esseen type theorem holds [Berry (1941), Esseen (1945); see, e.g., Chow and Teicher (1988), page 305]. There exists a constant Γ such that

$$\sup_{-\infty < x \le y < \infty} \Big| \mathbb{P}\Big(\big(t_0(1-t_0)\big)^{-1/2} Z_k \in (x, y) \Big) - (2\pi)^{-1/2} \int_x^y \exp\big(-t^2/2\big)\, dt \Big| \le \Gamma N_k^{-1/2}. \tag{3.43}$$

Set $x = 0$ and $y = (N_k/n_k)^{1/2}(t_0(1-t_0))^{-1/2}(\log n_k)^{-\kappa}$ in (3.43). By (3.29), (3.30), for all large k, we have $(N_k/n_k)^{1/2} \le 4/3$ and $\exp(-t^2/2) \ge 1/2$ for all $t \in (x, y)$. Thus, by (3.30) and (3.43),

$$\begin{aligned} &\mathbb{P}\Big(\tau_{N_k} - t_0 \in \Big(0, n_k^{-1/2}(\log n_k)^{-\kappa}\Big)\Big) \\ &\quad \ge (2\pi)^{-1/2} 2^{-1} (N_k/n_k)^{1/2} \big(t_0(1-t_0)\big)^{-1/2} (\log n_k)^{-\kappa} - \Gamma N_k^{-1/2} \\ &\quad \ge (1/6)\big(t_0(1-t_0)\big)^{-1/2} (\log n_k)^{-\kappa}, \end{aligned}$$

which yields (3.42) after setting $K = (1/6)(t_0(1-t_0))^{-1/2}$. □

LEMMA 3.8. *Assume that* (H.1) *and* (H.6) *hold with* $d = 0$. *Let* $\varepsilon \in (0, 1]$, $\kappa > 0$ *and* $f \in S_0$ *be arbitrary. Then, for* $K = K(t_0) > 0$ *as in Lemma* 3.7, *we have, for all large* k,

$$\begin{aligned} &\mathbb{P}\Big(b_{n_k}^{-1} \xi_{N_k}(h_{n_k}, t_0; \cdot) \in N_\varepsilon(f), |\tau_{N_k} - t_0| \le n_k^{-1/2}(\log n_k)^{-\kappa} \Big) \\ &\quad \ge K(\log n_k)^{-\kappa - (1-\varepsilon/8)|f|_H^2}. \end{aligned} \tag{3.44}$$

PROOF. When X and Y are jointly defined on the same probability space, denote by $L(X)$ (respectively $L(X \mid Y)$) the distribution of X (respectively the conditional distribution of X given Y). Denote by $\{\alpha'_N(t) = N^{1/2}(\mathbb{U}'_N(t) - t)$: $t \in \mathbb{R}\}$ and $\{\alpha''_N(t) = N^{1/2}(\mathbb{U}''_N(t) - t)$: $t \in \mathbb{R}\}$ two independent replicas of $\{\alpha_N(t) = N^{1/2}(\mathbb{U}_N(t) - t)$: $t \in \mathbb{R}\}$. Observe that, for $0 < a < 1$ and $1 \le m \le N - 1$,

$$\begin{aligned} &L\Big(\Big\{N\big(\mathbb{U}_N(a+I) - \mathbb{U}_N(a)\big), N\big(\mathbb{U}_N(a) - \mathbb{U}_N(a-I)\big)\Big\} \Big| N\mathbb{U}_N(a) = m\Big) \\ &\qquad = L\Big(\Big\{(N-m)\big(\mathbb{U}'_{N-m}(I/(1-a))\big), m\big(\mathbb{U}''_m(I/a)\big)\Big\}\Big) \end{aligned}$$

We apply this equality for $a = t_0$ and $N = N_k$. By setting $K_0 = N_k \mathbb{U}_{N_k}(t_0) = N_k(2t_0 - \tau_{N_k})$ and $\xi_{N_k}^{\pm}(h_{n_k}, t_0; s) = \xi_{N_k}(h_{n_k}, t_0; \pm s)$ for $s \in [0, 1]$, we see that, when k is so large that $h_{n_k} \le \min\{t_0, 1 - t_0\}$, we have, for each integer $1 \le m \le N_k - 1$,

$$\begin{aligned} L\Bigg(&\Bigg\{ b_{n_k}^{-1} \xi_{N_k}^{+}\Bigg(h_{n_k}, t_0; \frac{u}{h_{n_k}}\Bigg) : u \in \big[0, h_{n_k}\big] \Bigg\}, \\ &\Bigg\{ b_{n_k}^{-1} \xi_{N_k}^{-}\Bigg(h_{n_k}, t_0; \frac{v}{h_{n_k}}\Bigg) : v \in \big[0, h_{n_k}\big] \Bigg\} \Bigg| K_0 = m \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= L\Bigg(\Bigg\{b_{n_k}^{-1}\left(\frac{m}{N_k t_0}\right)^{1/2} t_0^{1/2}\alpha'_m\left(\frac{u}{t_0}\right)\\
&\qquad - u b_{n_k}^{-1}\left(\frac{N_k t_0 - m}{N_k t_0}\right) N_k^{1/2} \colon u \in [0, h_{n_k}]\Bigg\},\\
&\qquad \Bigg\{b_{n_k}^{-1}\left(\frac{N_k - m}{N_k(1-t_0)}\right)^{1/2}(1-t_0)^{1/2}\alpha''_{N_k - m}\left(\frac{v}{(1-t_0)}\right)\\
&\qquad\qquad - v b_{n_k}^{-1}\left(\frac{m - N_k t_0}{N_k(1-t_0)}\right) N_k^{1/2} \colon v \in [0, h_{n_k}]\Bigg\}\Bigg)\\
&=: L\Big(\big\{(1+\lambda_{k,1})\Lambda_{k,1}(u) + \mu_{k,1}(u) \colon u \in [0, h_{n_k}]\big\},\\
&\qquad \big\{(1+\lambda_{k,2})\Lambda_{k,2}(v) + \mu_{k,2}(v) \colon v \in [0, h_{n_k}]\big\}\Big),
\end{aligned}
\tag{3.45}
$$

where $\Lambda_{k,1}(u) = b_{n_k}^{-1} t_0^{1/2}\alpha'_m(u/t_0)$, $\Lambda_{k,2}(v) = b_{n_k}^{-1}(1-t_0)^{1/2}\alpha''_{N_k-m}(v/(1-t_0))$,

$$
\lambda_{k,1} = \left(\frac{m}{N_k t_0}\right)^{1/2} - 1, \qquad \mu_{k,1}(u) = -u b_{n_k}^{-1}\left(\frac{N_k t_0 - m}{N_k t_0}\right) N_k^{1/2} \quad \text{for } u \in [0, h_{n_k}],
$$

$$
\lambda_{k,2} = \left(\frac{N_k - m}{N_k(1-t_0)}\right)^{1/2} - 1, \qquad \mu_{k,2}(v) = -v b_{n_k}^{-1}\left(\frac{m - N_k t_0}{N_k(1-t_0)}\right) N_k^{1/2} \quad \text{for } v \in [0, h_{n_k}].
$$

Set $C_5 = 2\max\{t_0^{-1}, (1-t_0)^{-1}\}$. By (3.30), for all large k, and m with $|N_k t_0 - m| \le n_k^{1/2}(\log n_k)^{-\kappa}$,

$$
\begin{aligned}
&\max\left\{\left|\frac{N_k t_0 - m}{N_k t_0}\right|, \left|\frac{m - N_k t_0}{N_k(1-t_0)}\right|\right\}\\
&\qquad = \max\left\{\left|1 - \frac{m}{N_k t_0}\right|, \left|1 - \frac{N_k - m}{N_k(1-t_0)}\right|\right\}\\
&\qquad \le \max\{t_0^{-1}, (1-t_0)^{-1}\}\left(\frac{n_k}{N_k}\right) n_k^{-1/2}(\log n_k)^{-\kappa}\\
&\qquad \le C_5 n_k^{-1/2}(\log n_k)^{-\kappa}.
\end{aligned}
\tag{3.46}
$$

Fix any $\varepsilon \in (0, 1]$. By combining (3.26)–(3.35) with (3.45), (3.46) and (3.27), we readily obtain that, for all large k and m with $|N_k t_0 - m| \le n_k^{1/2}(\log n_k)^{-\kappa}$,

$$
\begin{aligned}
&\max\{|\lambda_{k,1}|, |\lambda_{k,2}|\} \le C_5 n_k^{-1/2}(\log n_k)^{-\kappa} \le \varepsilon/8,\\
&\max\left\{\|\mu_{k,1}(h_{n_k}\cdot)\|_0^1, \|\mu_{k,2}(h_{n_k}\cdot)\|_0^1\right\} \le C_5 h_{n_k} b_{n_k}^{-1}(\log n_k)^{-\kappa}\\
&\qquad \le C_5 h_{n_k}^{1/2}(\log n_k)^{-\kappa} \le \varepsilon/8.
\end{aligned}
\tag{3.47}
$$

Since $f \in S_0$, (3.3) and (2.36) imply that $\|f\| < 1$. Thus, by the triangle inequality, whenever $\lambda \in \mathbb{R}$, $|\lambda| \le \varepsilon/8$, $\mu \in B[0,1]$, $\|\mu\| \le \varepsilon/8$, $(1+\lambda)g + \mu \in N_{\varepsilon/2}(f)$ and $\varepsilon \in (0,1]$,

$$\begin{aligned}\|g - f\| &\le |1+\lambda|^{-1}(\|(1+\lambda)g + \mu - f\| + |\lambda| \times \|f\| + \|\mu\|)\\ &\le (1-\varepsilon/8)^{-1}(3\varepsilon/4) \le 6\varepsilon/7 < \varepsilon,\end{aligned}$$

so that $g \in N_\varepsilon(f)$. In the particular case where $g(s) = \Lambda_{k,j}(h_{n_k}s)$, $\lambda = \lambda_{k,j}$ and $\mu(s) = \mu_{k,j}(h_{n_k}s)$ for $j = 1, 2$ and $s \in [0,1]$, we infer from (3.45), (3.46) and (3.47) that, for all k sufficiently large, we have uniformly over all m with $|N_k t_0 - m| \le n_k^{1/2}(\log n_k)^{-\kappa}$

$$\begin{aligned}&\mathbb{P}\Big(b_{n_k}^{-1}\xi_{N_k}\big(h_{n_k}, t_0; \cdot\big) \in N_\varepsilon(f)|K_0 = m\Big)\\ &\quad= \mathbb{P}\Big(\Big\{b_{n_k}^{-1}\xi_{N_k}^{+}\big(h_{n_k}, t_0; \cdot\big) \in N_\varepsilon(f^+)\Big\}\\ &\qquad\qquad \cap\Big\{b_{n_k}^{-1}\xi_{N_k}^{-}\big(h_{n_k}, t_0; \cdot\big) \in N_\varepsilon(f^-)\Big\}|K_0 = m\Big)\\ &\quad\ge \mathbb{P}\Big(b_{n_k}^{-1}t_0^{1/2}\alpha_m\big(h_{n_k}I/t_0\big) \in N_{\varepsilon/2}(f^+)\Big)\\ &\qquad \times \mathbb{P}\Big(b_{n_k}^{-1}(1-t_0)^{1/2}\alpha_{N_k - m}\big(h_{n_k}I/(1-t_0)\big) \in N_{\varepsilon/2}(f^-)\Big).\end{aligned} \tag{3.48}$$

By Theorem 1 of Bretagnolle and Massart (1989), we may define $\{\alpha_N\colon N \ge 1\}$ jointly with a sequence $\{W^{(N)}\colon N \ge 1\}$ of Wiener processes, in such a way that, for all $z \ge 0$ and $N \ge 2$,

$$\begin{aligned}&\mathbb{P}\Big(\|\alpha_N - W^{(N)} + IW^{(N)}(1)\|_0^1 \ge N^{-1/2}(z + 12\log N)\Big)\\ &\qquad \le 2\exp(-z/6).\end{aligned} \tag{3.49}$$

We apply (3.49), with $N = m$ and $z = (\varepsilon/8)m^{1/2}b_{n_k}h_{n_k}^{-1/2} - 12\log m$. By (3.33)–(3.35) and (3.46), we have, ultimately as $k \to \infty$, uniformly over all m such that $|N_k t_0 - m| \le n_k^{1/2}(\log n_k)^{-\kappa}$,

$$\begin{aligned}m &= (1+o(1))\,N_k t_0 = (1+o(1))\,n_k t_0 \to \infty,\\ z &= (1+o(1))(\varepsilon/8)t_0^{1/2}\Big(2n_k\Big\{\log_+\Big(1/\Big(h_{n_k}\sqrt{n_k}\Big)\Big) + \log_2 n_k\Big\}\Big)^{1/2}\\ &\ge 6n_k^{1/4},\\ x_k &:= (\varepsilon/8)t_0 b_{n_k}h_{n_k}^{-1}\\ &= (\varepsilon/8)t_0\Big(2h_{n_k}^{-1}\Big\{\log_+\Big(1/\Big(h_{n_k}\sqrt{n_k}\Big)\Big) + \log_2 n_k\Big\}\Big)^{1/2}\\ &\ge 2n_k^{1/8}.\end{aligned} \tag{3.50}$$

Let $W = W^{(1)}$. By combining (3.49), (3.50) with the inequality [see, e.g., (1.1.1), page 23 in Csörgő and Révész (1981)] $\mathbb{P}(|W(1)| \ge x) \le \exp(-x^2/2)$ for $x \ge 1$, we see that, for all large k,

$$\begin{aligned}(3.51)\qquad &\mathbb{P}\Big(b_{n_k}^{-1}t_0^{1/2}\alpha_m\big(h_{n_k}I/t_0\big)\in N_{\varepsilon/2}(f^+)\Big)\\ &\quad\geq \mathbb{P}\Big(b_{n_k}^{-1}t_0^{1/2}W\big(h_{n_k}I/t_0\big)\in N_{\varepsilon/4}(f^+)\Big)\\ &\qquad - \mathbb{P}\Big(|W(1)|\geq(\varepsilon/8)t_0 b_{n_k}h_{n_k}^{-1}\Big)-2e^{-n_k^{1/4}}\\ &\quad\geq \mathbb{P}\Big(W_{\{h_{n_k}^{-1}b_{n_k}^2/2\}}\in N_{\varepsilon/4}(f^+)\Big)-3e^{-n_k^{1/4}}.\end{aligned}$$

By letting $d=0$ in (3.26), we see that, ultimately in $n\to\infty$,

$$(3.52)\qquad h_n^{-1}b_n^2/2=(1+o(1))\log_2 n\leq\big((1-\varepsilon/8)/(1-\varepsilon/4)\big)\log_2 n.$$

By (2.35)(ii) and (3.3), $J(N_{\varepsilon/4}(f^+))<(1-\varepsilon/4)|f^+|_H^2$. Thus, by setting $G=N_{\varepsilon/4}(f)$ in (2.34), we infer from (3.51), (3.52) that for all large k and uniformly over $|N_k t_0-m|\leq n_k^{1/2}(\log n_k)^{-\kappa}$,

$$\mathbb{P}\Big(b_{n_k}^{-1}t_0^{1/2}\alpha_m\big(h_{n_k}I/t_0\big)\in N_{\varepsilon/2}(f^+)\Big)\geq\exp\Big(-(1-\varepsilon/8)|f^+|_H^2\log_2 n_k\Big).$$

We obtain likewise that

$$\begin{aligned}&\mathbb{P}\Big(b_{n_k}^{-1}(1-t_0)^{1/2}\alpha_{N_k-m}\big(h_{n_k}I/(1-t_0)\big)\in N_{\varepsilon/2}(f^-)\Big)\\ &\qquad\geq\exp\Big(-(1-\varepsilon/8)|f^-|_H^2\log_2 n_k\Big),\end{aligned}$$

whence, by (3.3) and (3.48), for all large k and m with $|N_k t_0-m|\leq n_k^{1/2}(\log n_k)^{-\kappa}$,

$$\begin{aligned}(3.53)\qquad &\mathbb{P}\Big(b_{n_k}^{-1}\xi_{N_k}\big(h_{n_k},t_0;\cdot\big)\in N_{\varepsilon/4}(f)|K_0=m\Big)\\ &\quad\geq\exp\Big(-(1-\varepsilon/8)\big(|f^+|_H^2+|f^-|_H^2\big)\log_2 n_k\Big)\\ &\quad=(\log n_k)^{-(1-\varepsilon/8)|f|_H^2}.\end{aligned}$$

Recalling that $N_k t_0-K_0=N_k(\tau_{N_k}-t_0)$, we readily infer from (3.42) that for all large k,

$$\begin{aligned}&\mathbb{P}\Big(|N_k t_0-K_0|\leq n_k^{1/2}(\log n_k)^{-\kappa}\Big)\\ &\qquad\geq\mathbb{P}\Big(n_k^{1/2}\big(\tau_{n_k}-t_0\big)\in\big(0,(\log n_k)^{-\kappa}\big)\Big)\geq K(\log n_k)^{-\kappa}.\end{aligned}$$

This, when combined with (3.53), readily yields (3.44). □

PROOF OF PROPOSITION 3.2. When $d\in(-\infty,0)$, the result we seek is obtained by combining Fact 1 with Lemma 2.4. When $d=0$, the following arguments are needed. By (3.25), Proposition 2.1 and (3.1), our proof boils down to showing that for an arbitrary $f\in S_0$,

$$(3.54)\qquad \liminf_{k\to\infty}\|b_{n_k}^{-1}\xi_{n_k}\big(h_{n_k},\tau_{n_k};\cdot\big)-f\|=0\quad\text{a.s.}$$

In view of (3.31), (3.32), we see that (3.54) holds if, for each $\varepsilon > 0$, there exists a $\kappa > 0$ such that

$$\mathbb{P}\Big(b_{n_k}^{-1}\xi_{N_k};\, k\big(h_{n_k}, t_0;\cdot\big) \in N_\varepsilon(f),\quad |T_{N_k;k} - t_0| \le n_k^{-1/2}(\log n_k)^{-\kappa} \text{ i.o. (in } k)\Big) = 1. \tag{3.55}$$

By the Borel–Cantelli lemma and Remark 3.1, the assertion (3.55) is equivalent to

$$\sum_{q=1}^{\infty} P_{2q} := \sum_{q=1}^{\infty} \mathbb{P}\Big(b_{n_{2q}}^{-1}\xi_{N_{2q}}\big(h_{n_{2q}}, t_0;\cdot\big) \in N_\varepsilon(f),\quad |\tau_{N_{2q}} - t_0| \le n_{2q}^{-1/2}\big(\log n_{2q}\big)^{-\kappa}\Big) = \infty. \tag{3.56}$$

Since (3.44) entails that $P_k \ge K(\log n_k)^{-\kappa-(1-\varepsilon/8)|f|_H^2} = (1 + o(1))(k \log k)^{-\kappa-(1-\varepsilon/8)|f|_H^2}$ for all large k, (3.56) holds when $0 < \kappa < 1 - (1 - \varepsilon/8)|f|_H^2$, which is allowed by our assumptions. □

3.4. *Intermediate increments* ($d \in (0,\infty)$). The main result of this section is captured in the next proposition.

PROPOSITION 3.3. *Under* (H.1) *and* (H.6) *with* $d \in (0,\infty)$, *the sequence* $\{b_n^{-1}\zeta_n(h_n, t_0;\cdot)\colon\ n \ge 1\}$ *is almost surely relatively compact in* $(B[-1,1], U)$ *with limit set equal to* S.

The following arguments are oriented towards proving Proposition 3.3. We will assume throughout that (H.1) and (H.6) hold with $d \in (0,\infty)$. By (1.1), (1.10) and (3.26), this implies that, as $n \to \infty$,

$$\begin{aligned} h_n &= n^{-1/2}(\log n)^{-d+o(1)} = o(c_n),\\ c_n &= (1 + o(1))\, n^{-1/2}(\log_2 n)^{1/2},\\ b_n &= (1 + o(1))\big(2h_n(d+1)\log_2 n\big)^{1/2} = n^{-1/4}(\log n)^{-d/2+o(1)}. \end{aligned} \tag{3.57}$$

Recall from (3.29) the definitions of $n_k = \lfloor \exp(k \log^2 k)\rfloor$, $m_k = \lfloor (1+\gamma)n_k \rfloor$ for $k \ge 1$, $N_k = n_k - n_{k-1}$ and $M_k = m_k - n_{k-1}$ for $k \ge 2$, where $\gamma > 0$ is an auxiliary constant. Following Komlós, Major and Tusnády (1975a, b), we assume, without loss of generality, that $\{U_n\colon n \ge 1\}$ sits on a probability space on which is defined a two-parameter Wiener process $\{W(x, y)\colon\ x \ge 0,\ y \ge 0\}$ such that the Kiefer process $K(x,t) = W(x,t) - tW(x,1)$ fulfills (1.15). We will make use of the following additional notation. For any (possibly noninteger) $r > 0$, we will set $w_r(x) = r^{-1/2}W(r, x)$ and rewrite for convenience (1.15) into

$$\|\alpha_n - \{w_n - Iw_n(1)\}\| = O\big(n^{-1/2}(\log n)^2\big) \quad \text{a.s. as } n \to \infty. \tag{3.58}$$

Keeping in mind that, for each $r > 0$, $\{w_r(x)\colon\ x \ge 0\}$ is a standard Wiener process, we set, for $n \ge 1$, $h \le t \le 1 - h$, $s \in [-1, 1]$,

$$\begin{aligned} \eta_n^{(1)}(h, t; s) &= w_n(t + hs) - w_n(t) - hsw_n(1), \\ \eta_n^{(2)}(h, t; s) &= w_n(t + hs) - w_n(t), \end{aligned} \tag{3.59}$$

and, for $k \ge 2$, $N_k \le m \le M_k$ (equivalently, for $n_k \le n_{k-1} + m \le m_k$), and $m \ge 1$,

$$\begin{aligned} &\eta_{m;k}^{(j)}(h, t; s) \\ &\quad = m^{-1/2}\Big\{(n_{k-1} + m)^{1/2}\eta_{n_{k-1}+m}^{(j)}(h, t; s) - n_{k-1}^{1/2}\eta_{n_{k-1}}^{(j)}(h, t; s)\Big\}, \\ &\qquad\qquad j = 1, 2. \end{aligned} \tag{3.60}$$

Moreover, for $k \ge 2$ and $N_k \le m \le M_k$ (or equivalently, for $n_k \le n_{k-1} + m \le m_k$), we let $T_{m,k}$ be as in (3.29), and set, for each (possibly noninteger) $r, s > 0$, $M_k \le s \le N_k$ and $0 \le \theta \le 1$,

$$\begin{aligned} \Sigma_r^{(1)}(\theta) &= t_0 - r^{-1/2}\big\{w_r\big(t_0(1-\theta)\big) - t_0(1-\theta)w_r(1)\big\} \\ &= t_0 - r^{-1}\big\{W\big(r, t_0(1-\theta)\big) - t_0(1-\theta)W(r,1)\big\}, \\ \Sigma_r^{(2)}(\theta) &= t_0 - r^{-1/2}\big\{w_r\big(T_0(1-\theta)\big) - t_0(1-\theta) \\ &\qquad \times\big\{w_r(1) - w_r\big(t_0(1+\theta)\big) + w_r\big(t_0(1-\theta)\big)\big\}\big\}, \\ \Sigma_{s;k}^{(j)}(\theta) &= t_0 - s^{-1}\Big\{(n_{k-1} + s)\big(t_0 - \Sigma_{n_{k-1}+s}^{(j)}(\theta)\big) \\ &\qquad - n_{k-1}\big(t_0 - \Sigma_{n_{k-1}}^{(j)}(\theta)\big)\Big\}, \qquad j = 1, 2. \end{aligned} \tag{3.61}$$

REMARK 3.2. (i) For $j = 1, 2$, $M_k \le s \le N_k$ and $s > 0$, $\Sigma_{s,k}^{(j)}$ follows the distribution of $\Sigma_s^{(j)}$. For all large q_0, $\{\Sigma_{s,2q}^{(j)}\colon\ M_{2q} \le s \le N_{2q}\}$, $q \ge q_0$ constitutes a sequence of independent processes.

(ii) Let $\theta \in (0, 1)$ and $h > 0$ be such that $0 < t_0(1 - \theta) \le t_0 - h \le t_0 + h \le t_0(1 + \theta) < 1$. Then, for any Wiener process $\{W(t)\colon\ t \ge 0\}$, we have independence of $\{W(t_0 + hI) - W(t_0)\} \in B[-1, 1]$ and $W(1) - W(t_0(1 + \theta)) + W(t_0(1 - \theta))$. It follows that, whenever $|t - t_0| + h \le \theta t_0$, we have independence of $\Sigma_n^{(2)}(\theta)$ and $\eta_n^{(2)}(h, t; \cdot)$, and likewise of $\Sigma_{m;k}^{(2)}(\theta)$ and $\eta_{m;k}^{(2)}(h, t; \cdot)$, for all large n and k, uniformly over $m > 0$ fulfilling $M_k \le m \le N_k$.

(iii) It follows from (3.58) and the definitions (3.29), (3.59), and (3.61) of $\eta_n^{(1)}(h, t; s)$, $\tau_n(\theta)$ and $\Sigma_n^{(1)}(\theta)$ that there exists a constant $C_6 < \infty$ such that, a.s. for all large n and $h \in (0, 1)$,

$$\begin{aligned} \sup_{t\in[h,1-h]} \|\xi_n(h, t; \cdot) - \eta_n^{(1)}(h, t; \cdot)\|_{-1}^{1} &\le C_6 n^{-1/2}\log^2 n, \\ \sup_{\theta\in[0,1]} |\Sigma_n^{(1)}(\theta) - \tau_n(\theta)| &\le C_6 n^{-1}\log^2 n. \end{aligned} \tag{3.62}$$

LEMMA 3.9. *Let* (H.1) *and* (H.6) *hold with* $d \in (0, \infty)$. *Then, we have almost surely*

$$\lim_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{n_k\le n\le m_k}\|\xi_n(h_{n_k}, \tau_n; \cdot) - \eta_{n-n_{k-1};k}^{(2)}(h_{n_k}, T_{n-n_{k-1}}; k; \cdot)\|_{-1}^{1}\Big\} = 0, \tag{3.63}$$

$$\limsup_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{n_k\le n\le m_k}\|\xi_n(h_{n_k},\tau_n;\cdot)-\xi_n(h_n,\tau_n;\cdot)\|_{-1}^{1}\Big\}\le 2\gamma^{1/2}, \tag{3.64}$$

$$\limsup_{k\to\infty}\Big\{\max_{n_k\le n\le m_k}|b_n^{-1}-b_{n_k}^{-1}|\times\|\xi_n(h_n,\tau_n;\cdot)\|_{-1}^{1}\Big\}\le 2\gamma, \tag{3.65}$$

$$\lim_{k\to\infty}\Big\{\sup_{\theta\in[0,1]}c_{n_k}^{-1}\max_{n_k\le n\le m_k}|\Sigma_{n-n_{k-1};k}^{(1)}(\theta)-\tau_n(\theta)|\Big\}=0, \tag{3.66}$$

$$\lim_{k\to\infty}\Big\{c_{n_k}^{-1}\max_{n_k\le n\le m_k}|\Sigma_{n-n_{k-1};k}^{(1)}(0)-T_{n-n_{k-1};k}|\Big\}=0, \tag{3.67}$$

$$\limsup_{k\to\infty}\Big\{c_{n_k}^{-1}\max_{n_k\le n\le m_k}|\Sigma_{n-n_{k-1};k}^{(1)}(\theta)-\Sigma_{n-n_{k-1};k}^{(2)}(\theta)|\Big\}\le 2(\theta\gamma)^{1/2}. \tag{3.68}$$

PROOF. In view of (3.29) and (3.31), the proof of (3.63) reduces to showing that

$$\begin{aligned}&\lim_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{N_k\le m\le M_k}\|\xi_{m;k}(h_{n_k},T_{m;k};\cdot)-\eta_{m;k}^{(2)}(h_{n_k},T_{m;k};\cdot)\|_{-1}^{1}\Big\}\\&\qquad=0\quad\text{a.s.}\end{aligned} \tag{3.69}$$

By the triangle inequality, we write

$$\begin{aligned}&b_{n_k}^{-1}\max_{N_k\le m\le M_k}\|\xi_{m;k}(h_{n_k},T_{m;k};\cdot)-\eta_{m;k}^{(2)}(h_{n_k},T_{m;k};\cdot)\|_{-1}^{1}\\&\quad\le b_{n_k}^{-1}\max_{N_k\le m\le M_k}\|\xi_{m;k}(h_{n_k},T_{m;k};\cdot)-\eta_{m;k}^{(1)}(h_{n_k},T_{m;k};\cdot)\|_{-1}^{1}\\&\qquad+b_{n_k}^{-1}\max_{N_k\le m\le M_k}\|\eta_{m;k}^{(1)}(h_{n_k},T_{m;k};\cdot)-\eta_{m;k}^{(2)}(h_{n_k},T_{m;k};\cdot)\|_{-1}^{1}\\&\quad=:E_{1,k}+E_{2,k}.\end{aligned} \tag{3.70}$$

By (3.29), (3.30), (3.57) and (3.60)–(3.62), with probability 1 for all large k and $N_k\le m\le N_k$,

$$\begin{aligned}E_{1,k}&\le C_6 b_{n_k}^{-1}\max_{N_k\le m\le M_k}m^{-1/2}\big(\log^2(n_{k-1}+m)+\log^2(n_{k-1})\big)\\&\le 3C_6N_k^{-1/2}b_{n_k}^{-1}\log^2 n_k=n_k^{-1/4}(\log n_k)^{2+d/2+o(1)}\to 0.\end{aligned} \tag{3.71}$$

The law of the iterated logarithm for Wiener processes, in combination with (3.29), (3.57) and (3.59) shows that, ultimately,

$$\begin{aligned}E_{2,k}&\le b_{n_k}^{-1}/h_{n_k}\max_{n_k\le n\le m_k}(n-n_{k-1})^{-1/2}|W(n,1)-W(n_k,1)|\\&\le b_{n_k}^{-1}h_{n_k}(m_k/N_k)^{1/2}\sup_{0\le s,t\le m_k}m_k^{-1/2}|W(s,1)-W(t,1)|\\&\le 2(1+\gamma)^{1/2}b_{n_k}^{-1}h_{n_k}(\log_2 n_k)^{1/2}\\&=n_k^{-1/4}(\log n_k)^{-d/2+o(1)}\to 0\quad\text{a.s.}\end{aligned} \tag{3.72}$$

We obtain readily (3.69), and hence, (3.63), from (3.70), (3.71) and (3.72).

To establish (3.64), we first recall from (3.33) and (3.35) that, for all large k, and uniformly over $n_k\le n\le m_k$, we have $h_n\le h_{n_k}\le(1+\gamma)h_n$ and

$b_{n_k}^{-1} \le 2b_n^{-1}$. This, in turn, implies that for all large k, uniformly over $n_k \le n \le m_k$,

$$\begin{aligned} &b_{n_k}^{-1}\|\xi_n(h_{n_k},\tau_n;\cdot) - \xi_n(h_n,\tau_n;\cdot)\|_{-1}^{1} \\ &\quad\le 2 \sup_{t:\,|t-\tau_n|\le h_n} b_n^{-1}\|\xi_n(\gamma h_n, t;\cdot)\|_{-1}^{1}. \end{aligned} \tag{3.73}$$

Recalling (1.1), (2.5) and (3.5), we see that $\tau_n \pm h_n \in [t_0 - c_n, t_0 + c_n]$ a.s. for all large n. By (3.73), this entails that the LHS of (3.64) is almost surely smaller than or equal to

$$2 \limsup_{n\to\infty}\left\{ \sup_{t\in[t_0-c_n,\,t_0+c_n]} b_n^{-1}\|\xi_n(\gamma h_n, t;\cdot)\|_{-1}^{1}\right\}.$$

An application of Fact 4, taken with $A_n = \gamma h_n$ and $C_n = c_n$, shows readily via (2.9) and (1.1) that this last expression equals $2\gamma^{1/2}$ almost surely, which completes the proof of (3.64). □

We observe, via (3.33)–(3.35) and (3.57), that, for all large k, uniformly over $n_k \le n \le m_k$,

$$\begin{aligned} |b_{n_k}/b_n - 1| &\le 2\left\{(h_{n_k}/h_{m_k})^{1/2} - 1\right\} \le 2\left\{(m_k/n_k)^{1/2} - 1\right\} \\ &\le 2\left\{(1+\gamma)^{1/2} - 1\right\} \le \gamma. \end{aligned}$$

We conclude (3.65) readily by an application of Proposition 2.1 in combination with (2.2).

To establish (3.66), we combine (3.29) and (3.61), then make use of (3.62) in combination with (3.7), (2.5), and (3.30), to obtain that, a.s. for all large k and n with $n_k \le n \le m_k$,

$$\begin{aligned} &|\Sigma^{(1)}_{n-n_{k-1};\,k}(\theta) - \tau_n(\theta)| \\ &\quad = (n - n_{k-1})^{-1}\Big| n\left\{\Sigma^{(1)}_n(\theta) - \tau_n(\theta)\right\} \\ &\qquad - n_{k-1}\left\{\Sigma^{(1)}_{n_{k-1}}(\theta) - \tau_{n_{k-1}}(\theta)\right\} + n_{k-1}\left\{\tau_n(\theta) - t_0\right\} - n_{k-1}\left\{\tau_{n_{k-1}}(\theta) - t_0\right\}\Big| \\ &\quad \le (n - n_{k-1})^{-1}\Big(C_6\left\{\log^2 n + \log^2 n_{k-1}\right\} \\ &\qquad\qquad + 4n_{k-1}\left\{n^{-1/2}(\log_2 n)^{1/2} + n_{k-1}^{-1/2}(\log_2 n_{k-1})^{1/2}\right\}\Big) \\ &\quad \le 8(n_{k-1}/n_k)^{1/2}\left\{n_k^{-1/2}(\log_2 n_k)^{1/2}\right\} = o(c_n), \end{aligned}$$

which yields (3.66).

We observe that (3.67) is implied by (3.39) and (3.62). Finally, the proof of (3.68) is obtained as a consequence of the law of the iterated logarithm, via the same argument as in (3.72). □

LEMMA 3.10. *Under* (H.1) *and* (H.6) *with* $d \in (0,\infty)$, *for any* $N \geq 1$, *we have almost surely*

$$\limsup_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{N_k\leq m\leq M_k}\|\eta^{(2)}_{m;k}(h_{n_k},T_{m;k};\cdot) - \eta^{(2)}_{m;k}(h_{n_k},T_{m;k};\lfloor NI\rfloor N^{-1})\|_{-1}^{1}\Big\} \leq 4\gamma^{1/2}+2N^{-1/2}, \tag{3.74}$$
$$\limsup_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{N_k\leq m\leq M_k}|(m/N_k)^{1/2}-1|\times\|\eta^{(2)}_{m;k}(h_{n_k},T_{m;k};\cdot)\|_{-1}^{1}\Big\}\leq 2\gamma.$$

PROOF. We infer from (3.63) and (3.64), that almost surely

$$\limsup_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{N_k\leq m\leq M_k}\|\eta^{(2)}_{m;k}(h_{n_k,Tm;k};\cdot)-\eta^{(2)}_{m;k}(h_{n_k},T_{m;k};\lfloor NI\rfloor N^{-1})\|_{-1}^{1}\Big\}$$
$$\leq 4\gamma^{1/2}+\limsup_{k\to\infty}\Big\{b_{n_k}^{-1}\max_{n_k\leq n\leq m_k}\big\|\xi_n(h_n,\tau_n;\cdot)-\xi_n(h_n,\tau_n;\lfloor NI\rfloor N^{-1})\big\|_{-1}^{1}\Big\}.$$

Next, we proceed as in (3.20) and (3.73) with the formal replacement of γ by N^{-1}, to obtain that the expression above is almost surely less than or equal to

$$4\gamma^{1/2}+2\limsup_{n\to\infty}\Big\{\sup_{t\in[t_0-c_n,t_0+c_n]}b_n^{-1}\|\xi_n(N^{-1}h_n,t;\cdot)\|_{-1}^{1}\Big\}=4\gamma^{1/2}+2N^{-1/2},$$

which proves the first half of (3.74). The second half follows along the same lines via (3.30). □

Fix $\rho\in(0,1/2]$. Set $M''_k=\lfloor\rho c_{n_k}/2h_{n_k}\rfloor$, and, for $-N\leq i\leq N$ and $M''_k+1\leq j\leq 2M''_k$,

$$A_k(\varepsilon,j)=\big\{b_{n_k}^{-1}\eta^{(2)}_{N_k;m}(h_{n_k},t_0+2jh_{n_k};\cdot)\in N_\varepsilon(f)\big\},$$
$$B_k(\varepsilon,j,i)=\Big\{b_{n_k}^{-1}\max_{N_k\leq m\leq M_k}|(m/N_k)^{1/2}\eta^{(2)}_{m;k}(h_{n_k},t_0+2jh_{n_k};iN^{-1}) - \eta^{(2)}_{N_k;k}(h_{n_k},t_0+2jh_{n_k};iN^{-1})|\geq\varepsilon\Big\},$$
$$C_k(\varepsilon,j)=\bigcap_{i=-N}^{N}\overline{B}_k(\varepsilon,j,i),\qquad D_k(\varepsilon,j)=A_k(\varepsilon/2,j)\cap C_k(\varepsilon/4,j), \tag{3.75}$$
$$E_k(\varepsilon)=\bigcup_{j=M''_k+1}^{2M''_k}D_k(\varepsilon,j).$$

We denote here by $\overline{E}$ the complement of the event E.

LEMMA 3.11. *Assume that* (H.1) *and* (H.6) *hold with* $d\in(0,\infty)$. *Select an arbitrary* $f\in S_0$. *Fix any* $\varepsilon\in(0,1]$, $\rho\in(0,1/2]$, *and* $N\geq 1$. *Then, for all* k *sufficiently large, we have*

$$\mathbb{P}(E_k(\varepsilon))\geq\tfrac12\min\Big\{k^{-1+(d+1)(1-|f|_H^2)+(\varepsilon/8)(d+1)|f|_H^2},\tfrac12\Big\}. \tag{3.76}$$

PROOF. By (3.57), $h_{n_k}^{-1}b_{n_k}^2/2=(1+o(1))(d+1)\log_2 n_k=(1+o(1))(d+1)\log k$ as $k\to\infty$. Thus, by (2.34), (2.35), (3.3) and (3.60), we obtain that, for all large k and $j \in \{M_k'' + 1, \dots, 2M_k''\}$,

$$
\begin{aligned}
P_{1,k}(\varepsilon) &:= \mathbb{P}(A_k(\varepsilon, j)) \\
&= \mathbb{P}\Big(W_{\{h_{n_k}^{-1} b_{n_k}^2/2\}} \in N_\varepsilon(f^+)\Big) \times \mathbb{P}\Big(W_{\{h_{n_k}^{-1} b_{n_k}^2/2\}} \in N_\varepsilon(f^-)\Big) \\
&\geq \exp\Big(-(1-\varepsilon)|f|_H^2 h_{n_k}^{-1} b_{n_k}^2/2\Big) \geq k^{-(1-\varepsilon/2)(d+1)|f|_H^2}.
\end{aligned}
\tag{3.77}
$$

Since $N_k/(N_k - M_k) \to 1/\gamma$, the inequality $\mathbb{P}(\|W\| \geq x) \leq 4\mathbb{P}(W(1) \geq x) \leq 2\exp(-x^2/2)$ for $x \geq 1$ [see (1.5.1) and (1.1.1) in Csörgő and Révész (1981)], when combined with Bonferroni's inequality entails that, independently of $j \in \{M_k'' + 1, \dots, 2M_k''\}$, for all large k,

$$
\begin{aligned}
P_{2,k}(\varepsilon) &:= 1 - \mathbb{P}(C_k(\varepsilon, j))) = \mathbb{P}\left(\bigcup_{i=-N}^{N} B_k(\varepsilon, j, i)\right) \\
&\leq 2\sum_{i=1}^{N} \mathbb{P}\left(\sup_{0 \leq u \leq iN^{-1}(M_k - N_k)h_{n_k}} |W(u)| \geq \varepsilon N_k^{1/2} b_{n_k}\right) \\
&= 2\sum_{i=1}^{N} \mathbb{P}\left(\|W\| \geq \frac{\varepsilon N_k^{1/2} b_{n_k}}{\sqrt{iN^{-1}(M_k - N_k)h_{n_k}}}\right) \\
&\leq 4\sum_{i=1}^{N} \exp\left(-\frac{\varepsilon^2 N_k b_{n_k}^2}{2iN^{-1}(M_k - N_k)h_{n_k}}\right) \\
&= 4\sum_{i=1}^{N} \exp\left(-(1+o(1))\frac{N(d+1)\log_2 n_k}{i\gamma}\right) \to 0.
\end{aligned}
\tag{3.78}
$$

Recalling from (3.75) that the events $A_k(\varepsilon/2, j)$ and $C_k(\varepsilon/4, j)$ are independent, we infer from (3.78) that, for all large k, we have $\mathbb{P}(D_k(\varepsilon, j)) = P_{1,k}(\varepsilon, 2))(1 - P_{2,k}(\varepsilon/4)) \geq \frac{1}{2} P_{1,k}(\varepsilon/2)$, independently of $j \in \{M_k'' + 1, \dots, 2M_k''\}$. Since the events $\{D_k(\varepsilon, j)\colon M_k'' + 1 \leq j \leq 2M_k''\}$ are independent, it follows from (3.77), (3.78) and the inequality $1 - (1-u)^r \geq 1 - e^{-ru}$ for $r > 0$ and $0 \leq u \leq 1$ that

$$
\begin{aligned}
\mathbb{P}(E_k(\varepsilon)) &= \mathbb{P}\left(\bigcup_{j=M_k''+1}^{2M_k''} D_k(\varepsilon, J)\right) \\
&= 1 - \big(1 - P_{1,k}(\varepsilon/2)(1 - P_{2,k}(\varepsilon/4))\big)^{M_k''} \\
&\geq 1 - \exp\big(-M_k'' P_{1,k}(\varepsilon/2)(1 - P_{2,k}(\varepsilon/4))\big) \\
&\geq 1 - \exp\Big(-\tfrac{1}{2} M_k'' k^{-(1-\varepsilon/4)(d+1)|f|_H^2}\Big).
\end{aligned}
\tag{3.79}
$$

By (3.57), we have, ultimately as $k \to \infty$, $c_{n_k}/h_{n_k} = (\log n_k)^{d+o(1)} = k^{d+o(1)} \geq k^{d-(\varepsilon/8)(d+1)|f|_H^2}$ and $M_k'' = \lfloor \rho c_{n_k}/2h_{n_k} \rfloor \geq (\rho/4) c_{n_k}/h_{n_k} = (\log n_k)^{d+o(1)} = k^{d+o(1)} \geq 4k^{d-(\varepsilon/8)(d+1)|f|_H^2}$. By combining this last inequality with (3.79) and $1 - e^{-u} \geq \frac{1}{2}\min\{u, \frac{1}{2}\}$ for $u \geq 0$, we obtain readily (3.76). □

PROOF OF PROPOSITION 3.3.

Step 1. Fix $f \in S_0$ and $\varepsilon \in (0,1]$, with $\varepsilon < 8/\{(d+1)|f|_H^2\}$. Set $\gamma = (\varepsilon/128)^2$, which implies that $2\gamma + 2\gamma^{1/2} < \varepsilon/32$. Proposition 2.1 and (3.1) reduce our proof to show that the event $\{\exists n \in \{n_k, \dots, m_k\}: \|b_n^{-1}\xi_n(h_n, \tau_n; \cdot) - f\|_{-1}^1 \le \varepsilon\}$ holds i.o. in k with probability 1. Making use of (3.63), (3.64) and (3.65), in combination with the triangle inequality and $2\gamma + 2\gamma^{1/2} < \varepsilon/32$, we obtain readily that this property is satisfied whenever the event

$$\|b_{n_k}^{-1}\eta_{m;k}^{(2)}(h_{n_k}, T_{m;k}; \cdot) - f\|_{-1}^1 \le (31/32)\varepsilon \quad \text{for some } m \in \{N_k, \dots, M_k\},$$

holds i.o. in k with probability 1. Let $M_k'' = \lfloor \rho c_{n_k}/2h_{n_k} \rfloor$ be as in (3.75). By the argument used in the proof of Proposition 3.1, we reduce our proof to show that the following statement holds. There exists a $\rho > 0$ such that the events E_k' and E_k'' below hold jointly i.o. in k with probability 1.

$$E_k' = \Big\{\text{For some } j \in \{M_k'' + 1, \dots, 2M_k''\}, \text{ we have}$$
$$\text{(i)(a) } \|b_{n_k}^{-1}\eta_{n_k}^{(2)}(h_{n_k}, t_0 + 2jh_{n_k}; \cdot) - f\|_{-1}^1 \le 8\varepsilon/16,$$
$$\text{(i)(b) } b_{n_k}^{-1} \max_{N_k \le m \le M_k} \|\eta_{m;k}^{(2)}(h_{n_k}, t_0 + 2jh_{n_k}; \cdot) - \eta_{N_k;k}^{(2)}(h_{n_k}, t_0 + 2jh_{n_k}; \cdot)\|_{-1}^1 \le 7\varepsilon/16\Big\}.$$

$$E_k'' = \{T_{N_k;k} \le \rho c_{n_k}/h_{n_k} \text{ and } 2\rho c_{n_k}/h_{n_k} \le T_{M_k;k}\}.$$

Set $N = (128/\varepsilon)^2$, so that $4\gamma^{1/2} + 2\gamma + 2N^{-1/2} \le 6\varepsilon/128 + \varepsilon/64 = \varepsilon/16$. Making use of (3.74) in combination with the definition (3.75) of $E_k(\varepsilon)$, it is readily verified that there exists an event Ω_1 of probability 1, such that $\{\Omega_1 \cap E_k' \text{ i.o.}\} = \{\Omega_1 \cap E_k(\varepsilon) \text{ i.o.}\}$.

The following arguments are needed to conclude our proof by showing that $\mathbb{P}(E_k'' \cap E_k(\varepsilon) \text{ i.o.}) = 1$.

Step 2. Set $\delta = \{(\varepsilon/8)(d+1)|f|_H^2\}^{1/2}$, which, by our choice of ε and f satisfies $0 < \delta < 1$. Recalling the notation (3.61), set for $k \ge 2$, $\theta \in (0,1]$ and $s > 0$,

$$G_k(\theta, s) = s(M_k - n_{k-1})^{1/2}\sigma_\theta^{-1}\big(t_0 - \Sigma_{(M_k - n_{k-1})s,k}^{(2)}(\theta)\big), \tag{3.80}$$

where the choice of $\sigma_\theta^2 = 1/\mathrm{Var}(G_k(\theta,1)) = 1/\mathrm{Var}(\Sigma_1^{(2)}(\theta))$ ensures that $\{G_k(\theta, s): s \ge 0\}$ is a standard Wiener process. Since $\sigma_\theta^2 \to t_0(1-t_0)$ as $\theta \to 0$, there exists a $\theta_0 > 0$ such that $(15/16)(t_0(1-t_0))^{1/2} \le \sigma_\theta \le (17/16)(t_0(1-t_0))^{1/2}$ for all $\theta \in (0, \theta_0]$. Note further that

$$\begin{aligned} &G_k(\theta, N_k/(M_k - n_{k-1})) \\ &\quad = N_k(M_k - n_{k-1})^{-1/2}\sigma_\theta^{-1}\big(t_0 - \Sigma_{N_k;k}^{(2)}(\theta)\big), \end{aligned} \tag{3.81}$$

and

$$G_k(\theta, 1) = (M_k - n_{k-1})^{1/2}\sigma_\theta^{-1}\big(t_0 - \Sigma_{M_k - n_{k-1};k}^{(2)}(\theta)\big). \tag{3.82}$$

Set $\epsilon = (1/16)\delta(1 - (1+\gamma)^{-1}) \le \gamma\delta/16 < 1/16$. The function $g_\delta(s) = -\delta\min(s, 1-s)$ for $s \in [0,1]$ satisfies $|g_\delta|_H^2 = \delta^2 \le 1$. Therefore, by (2.34), (2.35) and (3.57), we obtain that for all large k,

$$\begin{aligned}\mathbb{P}(E_k(\epsilon)) &:= \mathbb{P}\Big(n_k^{-1/2}c_{n_k}^{-1}G_k(\theta,\cdot) \in N_\epsilon(g_\delta)\Big) \\ &= \mathbb{P}\Big(W_{\{n_k c_{n_k}^2/2\}} \in N_\epsilon(g_\delta)\Big) \qquad (3.83)\\ &\geq \exp\Big(-(\log k)|g_\delta|_H^2\Big) = k^{-\delta^2} = k^{-(\varepsilon/8)(d+1)|f|_H^2}.\end{aligned}$$

By (3.30), $N_k/(M_k - n_{k-1}) \to (1+\gamma)^{-1}$, and $g_\delta(N_k/(M_k - n_{k-1})) \to \delta(1 - (1+\gamma)^{-1})$. Therefore, for all large k, on the event $E_k(\epsilon) = \{n_k^{-1/2}c_{n_k}^{-1}G_k(\theta,\cdot) \in N_\epsilon(g_\delta)\}$, it holds that

$$|n_k^{-1/2}c_{n_k}^{-1}G_k(\theta, N_k/(M_k - n_{k-1})) + \delta\Big(1 - (1+\gamma)^{-1}\Big)| < 3\epsilon/2 \quad \text{and}$$

$$|n_k^{-1/2}c_{n_k}^{-1}G_k(\theta,1)| < 3\epsilon/2.$$

By (3.30), we see that $n_k^{1/2}(M_k - n_{k-1})^{1/2}/N_k \to (1+\gamma)^{1/2}$ and $n_k^{1/2}(M_k - n_{k-1})^{-1/2} \to (1+\gamma)^{-1/2}$. Therefore, by (3.81), (3.82) and our choice of γ, for all large k, on the event $E_k(\epsilon)$ it holds that

$$\begin{aligned}&|t_0 - \Sigma_{N_k;k}^{(2)}(\theta) + \delta\Big(1-(1+\gamma)^{-1}\Big)\sigma_\theta c_{n_k}| \\ &\quad < 2\epsilon c_{n_k}\sigma_\theta \qquad (3.84)\\ &\quad = (1/8)\delta\Big(1-(1+\gamma)^{-1}\Big)c_{n_k}\sigma_\theta,\end{aligned}$$

and likewise

$$|t_0 - \Sigma_{M_k;k}^{(2)}(\theta)| < 2\epsilon c_{n_k}\sigma_\theta = (1/8)\delta\Big(1-(1+\gamma)^{-1}\Big)c_{n_k}\sigma_\theta. \qquad (3.85)$$

By combining (3.7) with (3.57), (3.66), (3.67) and (3.68) we see that, a.s. for all large k,

$$|\Sigma_{N_k;k}^{(2)}(\theta) - T_{N_k;k}| \leq \Big(2(\theta\gamma)^{1/2} + 2\{\theta t_0(1-\theta t_0)\}^{1/2}\Big)c_{n_k} \leq 4\theta^{1/2}c_{n_k}. \qquad (3.86)$$

Set now $\theta = \min(\theta_0, \{(1/64)\delta(1-(1+\gamma)^{-1})t_0(1-t_0)\}^2)$. By (3.84), (3.85) and (3.86), we have

$$4\theta^{1/2}c_{n_k} \leq (1/8)\delta\Big(1-(1+\gamma)^{-1}\Big)c_{n_k}\sigma_\theta.$$

Thus, by (3.84), (3.85), there exists an event Ω_2 of probability 1 such that, on $\Omega_2 \cap E_k(\epsilon)$,

$$T_{N_k,k} \geq (3/4)\delta\Big(1-(1+\gamma)^{-1}\Big)\{t_0(1-t_0)\}^{1/2}c_{n_k}, \qquad (3.87)$$

and

$$T_{M_k,k} \leq (1/4)\delta\Big(1-(1+\gamma)^{-1}\Big)\{t_0(1-t_0)\}^{1/2}c_{n_k}, \qquad (3.88)$$

for all large k. In view of the definition of E_k'', we combine (3.69) with (3.87) and (3.88), to obtain that, for some suitable event $\Omega_3 \subseteq \Omega_2$ of probability 1, the inclusion of events $\{\Omega_3 \cap E_k(\epsilon) \text{ i.o.}\} \subseteq \{\Omega_3 \cap E_k'' \text{ i.o.}\}$ holds for $\rho = (1/3)\delta(1-(1+\gamma)^{-1})\{t_0(1-t_0)\}^{1/2}$, which is allowed by our assumptions.

Step 3. By putting together the results of Steps 1 and 2, we see that are done if we can prove that $\mathbb{P}(E_k(\epsilon) \cap E_k(\varepsilon) \text{ i.o.}) = 1$ for the above choices of ε, ϵ, N, γ, δ, θ and ρ. Now, it is easily checked via Remark 3.2 that $E_{2q}(\varepsilon)$ and $E_{2q}(\epsilon)$ are independent for all q so large that $h_{n_{2q}} < \theta t_0$, and moreover, that,

$\{E_{2q}(\varepsilon) \cap E_{2q}(\epsilon)\colon q \geq q_0\}$ is a sequence of independent events. Therefore, the Borel–Cantelli lemma reduces our proof to show that

$$\sum_q \mathbb{P}\big(E_{2q}(\varepsilon)\big)\mathbb{P}\big(E_{2q}(\epsilon)\big) = \infty. \tag{3.89}$$

By (3.76) and (3.83), we have $\mathbb{P}(E_k(\varepsilon))\mathbb{P}(E_k(\varepsilon)) \geq \frac{1}{2}\min\{k^{-1+(d+1)(1-|f|_H^2)}, \frac{1}{2}k^{-(\varepsilon/8)(d+1)|f|_H^2}\}$ for all large k. Since $\delta^2 = (\varepsilon/8)(d+1)|f|_H^2 < 1$, we obtain readily (3.89) as sought. □

For the proof of theorem 2.2, combine Propositions 3.1, 3.2 and 3.3.

Concluding comments. Statistical applications of our theorems, together with investigations of local quantile processes under other sets of assumptions, will be considered elsewhere.

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