The Annals of Probability
1997, Vol. 25, No. 4, 1995–2006

SHARP EXPLICIT LOWER BOUNDS OF HEAT KERNELS[1]

By Feng-Yu Wang

Beijing Normal University and University of Warwick

By using logarithmic transformations, an explicit lower bound estimate of heat kernels is obtained for diffusion processes on Riemannian manifolds. This estimate is sharp for both short and long times, especially for heat kernels on a compact manifold, and is extended to manifolds with unbounded curvature.

1. Introduction. Let M be a complete Riemannian manifold of dimension d and let $L = \frac{1}{2}(\Delta + Z)$ for some C^1-vector field Z. Denote by $p_t(x, y)$ the heat kernel of L, namely, the transition density of the L-diffusion process with respect to Riemannian volume element.

A large number of papers have studied the heat kernel estimates, for instance, [1], [3], [4], [8], [11], [12], [14]–[17], [20] and references therein. Most of these works treat the case $Z = 0$, but their techniques usually can be extended to more general cases. All the resulting explicit estimates are based on a bounded geometry assumption; say, the Ricci curvature for L is bounded from below. Under this assumption, the study now is already complete. To see this, we would like to recall some sharp results in the literature.

First, the well-known result in [2] states

$$p_t(x, y) \sim (2\pi t)^{-d/2} \exp(-\rho^2(x, y)/(2t)) \tag{1.1}$$

for short times and small $\rho(x, y)$ when $Z = 0$, where ρ denotes the Riemannian distance on M.

Next, let $p_t^k(x, y) = p_t^k(\rho(x, y))$ be the heat kernel of $\frac{1}{2}\Delta$ on M_k, the space form with constant sectional curvature $-k \le 0$. Davies and Mandouvalos [9] presented the following two-sided estimate:

$$c(d)^{-1} h(t, \rho(x, y)) \le p_t^k(x, y) \le c(d) h(t, \rho(x, y)), \tag{1.2}$$

where $c(d)$ depends only on d and

$$\begin{aligned} h(t, r) = (2\pi t)^{-d/2} \exp\left[-\frac{r^2}{2t} - \frac{(d-1)^2 kt}{8} - \frac{(d-1)\sqrt{k}r}{2}\right] \\ \times \left(1 + \sqrt{k}r + \frac{kt}{2}\right)^{(d-1)/2-1} (1 + \sqrt{k}r), \qquad t, r > 0. \end{aligned} \tag{1.3}$$

Although [9] only considered the case $k = 1$, the estimate in the present form follows immediately from a simply conformal change of the metric. Then we

Received August 1996; revised November 1996.

[1]Research supported by a Royal Society KCW Fellowship, NSFC, BNU, Tian-Yuan Foundation and the State Education Commission of China.

AMS 1991 *subject classifications*. 58G11, 60H10.

Key words and phrases. Heat kernel, logarithmic transformation, diffusion process.

obtain a precise lower bound estimate by combining (1.2) with Cheeger and Yau's comparison theorem [6]:

$$p_t(x, y) \geq p_t^k(\rho(x, y)) \quad \text{provided } Z = 0 \text{ and } \mathrm{Ric} \geq -k(d-1). \tag{1.4}$$

As was pointed out to the author by a referee, such a comparison estimate can be extended to the gradient drift case.

Finally, a sharp Harnack inequality has been proved by Yau [20] for a very large class of elliptic operators. This leads to a lower bound estimate of heat kernels which is sharp for short times; see also [1].

From the several results mentioned, we see that the study of lower bound estimate has already reached its final form for noncompact manifolds with bounded geometry. So we will be concerned mainly with the unbounded curvature case, as well as sharp estimates for compact manifolds.

In contrast to the works discussed, this paper adopts a probabilistic approach called "logarithmic transformations" initiated by Fleming [10]. This method was applied successfully in [18] to the study of heat kernels on $\mathbb{R}^d$. The advantage of the method is that it not only provides sharp estimates but also is valid for the unbounded curvature case.

For explanation of the main idea of logarithmic transformations, we will also consider the bounded curvature case, although it is not the case we emphasize. Then we go to the unbounded curvature case, and finally, present a lower bound estimate for heat kernels on a compact manifold which is sharp for short times and is nontrival for long times.

2. Logarithmic transformations for Riemannian manifold setting.

For fixed $y \in M$, simply denote $\rho_y(x) = \rho(x, y)$, $x \in M$. Let

$$F^\alpha(x) = C_\alpha^{-1} \exp(-\rho_y^2(x)/(2\alpha)), \qquad \alpha > 0,\ x \in M,$$

where C_α is the positive constant such that $\int_M F^\alpha(x)\,dx = 1$. Then $F^\alpha(x)\,dx$ is a probability measure on M which converges weakly to δ_y (the single-point measure at y) as $\alpha \to 0$. Take

$$F_t^\alpha(x) = P_t F^\alpha(x) := \int_M p_t(x, z) F^\alpha(z)\,dz,$$
$$J_t^\alpha(x) = -\log F_t^\alpha(x), \qquad t \geq 0,\ \alpha > 0.$$

Then

$$\lim_{\alpha \to 0} J_t^\alpha(x) = -\log p_t(x, y), \qquad t > 0,\ x \in M. \tag{2.1}$$

Since $\int_M p_t(x, z)\,dz = 1$ for each $t > 0$,

$$\begin{aligned} |F_t^\alpha(x) - F_{t-\alpha}^\alpha(x)| &\leq 2C_\alpha^{-1} \exp(-1/(2\alpha)) + \int_{B(y,1)} |p_{t-\alpha}(x, z) - p_t(x, z)| F^\alpha(z)\,dz \\ &\leq 2C_\alpha^{-1} \exp(-1/(2\alpha)) + c_\alpha(t), \qquad \alpha < t, \end{aligned}$$

where $c_\alpha(t) := \sup_{\rho_y(z)\le 1} |p_t(x,z) - p_{t-\alpha}(x,z)|$ goes to zero as $\alpha \to 0$. Hence, by (2.1),

$$\lim_{\alpha\to 0} J^\alpha_{t-\alpha}(x) = -\log p_t(x,y), \qquad t>0,\, x\in M. \tag{2.2}$$

It is easy to check that

$$\begin{aligned}\frac{\partial}{\partial t} J^\alpha_t &= \frac{1}{2}(\Delta + Z)J^\alpha_t - \frac{1}{2}|\nabla J^\alpha_t|^2\\ &= \inf_{U\in T_xM}\left\{\frac{1}{2}(\Delta + Z + 2U)J^\alpha_t + \frac{1}{2}|U|^2\right\},\end{aligned} \tag{2.3}$$

where the inf can be attained at $U = -\nabla J^\alpha_t$.

For fixed $T>0$, define

$$\mathscr{F}_T = \big\{U \in C^1([0,T);\mathscr{X}(M)):\ \text{there exists } c\in C([0,T)) \text{ such that } |U(t,x)| \le c(t)\rho_y(x)\big\},$$

where $\mathscr{X}(M)$ denotes the set of C^1-vector fields on M. Now, we are ready to prove the following lemma, which is key to this paper.

LEMMA 2.1. *Suppose that $(\Delta + Z)\rho_y^2(x) \le c(1+\rho_y^2(x))$ for some $c>0$ and all $x\notin \mathrm{cut}(y)$. For any $T>0$ and $\varepsilon\in(0,T)$, we have*

$$J^\alpha_{T-\varepsilon}(x) \le \inf_{U\in\mathscr{F}_T} E^x\big\{\tfrac{1}{2}\int_0^{T-\varepsilon} |U(t,\eta_t)|^2\,dt + J^\alpha_0(\eta_{T-\varepsilon})\big\}, \tag{2.4}$$

where $\eta_t (t<T)$ is the $\{\frac{1}{2}(\Delta + Z + 2U)\}$-diffusion process which solves the following SDE up to time T:

$$\begin{aligned} d\eta_t &= \Phi_t \circ dB_t + \tfrac{1}{2}\big(Z(\eta_t) + 2U(t,\eta_t)\big)\,dt,\\ d\Phi_t &= H_{\Phi_t}\circ d\eta_t, \qquad \eta_0 = x,\ \Phi_0\in O_x(M).\end{aligned}$$

Here, H is the horizontal lift and $O_x(M)$ the set of orthonormal frames at x.

PROOF. For any $U\in\mathscr{F}_T$, we have

$$\tfrac{1}{2}(\Delta + Z + 2U(t,\cdot))\rho_y^2(x) \le c_1 + c_2\rho_y^2(x), \qquad t\le T-\varepsilon$$

for some constants $c_1, c_2 > 0$ and all $x\notin\mathrm{cut}(y)$. Then η_t is not explosive before time $T-\varepsilon$. Since ε can be arbitrary, the lifetime of η_t is T. Let $\tau_n = \{t\in[0,T]: \rho_y(\eta_t)\ge n\}$. Since ∇J^α is bounded on $[0,T]\times\Omega$ for any compact

$\Omega \subset M$, by Itô's formula we obtain

$$
\begin{aligned}
J^\alpha_{T-\varepsilon}(x) &= E^x J^\alpha_{(T-\varepsilon-\tau_n)^+}(\eta_{(T-\varepsilon)\wedge\tau_n}) - E^x \int_0^{(T-\varepsilon)\wedge\tau_n} dJ^\alpha_{T-t-\varepsilon}(\eta_t) \\
&= E^x J^\alpha_{(T-\varepsilon-\tau_n)^+}(\eta_{(T-\varepsilon)\wedge\tau_n}) - E^x \int_0^{(T-\varepsilon)\wedge\tau_n} \left\{\frac{\partial}{\partial t} J^\alpha_{T-t-\varepsilon}\right\}(\eta_t)\,dt \\
&\quad - E^x \int_0^{(T-\varepsilon)\wedge\tau_n} \frac{1}{2}(\Delta + Z + 2U(t,\cdot)) J^\alpha_{T-t-\varepsilon}(\eta_t)\,dt \\
&\le E^x J^\alpha_{(T-\varepsilon-\tau_n)^+}(\eta_{(T-\varepsilon)\wedge\tau_n}) + \frac{1}{2}\int_0^{T-\varepsilon} E^x |U(t,\eta_t)|^2\,dt.
\end{aligned} \tag{2.5}
$$

Here we have used (2.3) in the last step. Note that $P_t F^\alpha \le \|F^\alpha\|_\infty = C_\alpha^{-1}$ and we have $J_t \ge \log C_\alpha$. Now, by Fatou's lemma, the proof is completed by letting $n \to \infty$ in (2.5). $\square$

We remark that the equality in (2.4) holds when $U(t,x) = -\nabla J^\alpha_{T-t-\varepsilon}(x) \in \mathscr{F}_{T-\varepsilon}$. From Lemma 2.1 we see that a good lower bound estimate of the heat kernel follows from a suitable choice of U. But to compute $E^x J^\alpha_0(\eta_{T-\varepsilon})$, we need to estimate J^α_0 first.

LEMMA 2.2. *For* $\alpha > 0$ *and* $l \in (0, 1/2\alpha)$, *we have*

$$
J^\alpha_0(x) \le \frac{d}{2}\log\frac{2\pi\alpha}{1-2\alpha l} + \frac{(d-1)^2 k}{4l} + \frac{\rho_y^2(x)}{2\alpha}, \qquad x \in M.
$$

PROOF. It is well known that the volume element on M can be characterized in polar coordinates at y as (see [5], page 67)

$$
dz = \sqrt{g(r,\xi)}\,dr\,d\xi,
$$

where $z = \exp[r\xi] \notin \mathrm{cut}(y)$ for $r \ge 0$ and $\xi \in \mathbb{S}_y^{d-1} := \{X \in T_y M\colon |X| = 1\}$ such that $\exp(s\xi)|_{s\in[0,r]}$ is the minimal geodesic from y to z, and $d\xi$ is the standard measure on $\mathbb{S}_y^{d-1}$.

By Bishop's comparison theorem (see [5], page 71),

$$
\sqrt{g(r,\xi)} \le k^{-(d-1)/2}\sinh^{d-1}(\sqrt{k}r), \qquad r \ge 0,\ \xi \in \mathbb{S}_y^{d-1}, \tag{2.6}
$$

where the right-hand side of (2.6) is understood as r^{d-1} when $k = 0$.

Since the volume of $\mathrm{cut}(y)$ is zero,

$$
\begin{aligned}
C_\alpha &= \int_M \exp(-\rho_y^2(z)/(2\alpha))\,dz = \int_{M\setminus \mathrm{cut}(y)} \exp(-\rho_y^2(z)/(2\alpha))\,dz \\
&\le |\mathbb{S}^{d-1}| \int_0^\infty \exp(-r^2/(2\alpha))\,k^{-(d-1)/2}\sinh^{d-1}(\sqrt{k}r)\,dr,
\end{aligned}
$$

where $|\mathbb{S}^{d-1}|$ denotes the $(d-1)$-volume of the unit sphere in $\mathbb{R}^d$. Note that $\sinh r \le r\exp(r)$ for $r \ge 0$, then

$$\begin{aligned} C_\alpha &\le |\mathbb{S}^{d-1}| \int_0^\infty \exp(-r^2/(2\alpha) + (d-1)\sqrt{k} r) r^{d-1}\,dr \\ &\le \exp((d-1)^2 k/(4l)) |\mathbb{S}^{d-1}| \int_0^\infty \exp\{(l - 1/(2\alpha)) r^2\} r^{d-1}\,dr \\ &= \exp((d-1)^2 k/(4l)) \int_{\mathbb{R}^d} \exp\{(l-1/(2\alpha))|z|^2\}\,dz \\ &= \exp((d-1)^2 k/4l) \biggl(\frac{2\pi\alpha}{1-2\alpha l}\biggr)^{d/2}. \end{aligned}$$

Now the proof is complete by the fact that $J_0^\alpha(x) = \log C_\alpha + \rho_y^2(x)/(2\alpha)$. □

3. Manifolds with Ricci curvature bounded from below. In this section, we assume that Ricci $\ge -k(d-1)$ for some $k \ge 0$.

THEOREM 3.1. *Suppose that $|Z|$ is bounded from above by b, let $\beta = \sqrt{k}(d-1)+b$. For any $t, \sigma > 0$ and $x, y \in M$, we have*

$$p_t(x,y) \ge (2\pi t)^{-d/2} \exp\biggl[-\biggl(\frac{1}{2t} + \frac{\sigma}{3\sqrt{t}}\biggr)\rho^2(x,y) - \frac{\beta^2 t}{8} - \biggl(\frac{\beta^2}{4\sigma} + \frac{2d\sigma}{3}\biggr)\sqrt{t}\biggr].$$

We remark that when $Z = 0$ (or more generally, Z is a gradient), the above estimate is not as precise as that obtained by (1.2), (1.3) and (1.4). But it should contain interest for general Z. Moreover, this estimate is sharp for short times (take $\sigma = c\sqrt{t}$) by (1.1), and when $M = M_k$ and $Z = 0$, it provides the exact main order $\exp(-t(d-1)^2k/8)$ for long times.

PROOF. Without loss of generality, we assume that $k > 0$. For $\alpha > 0$, choose $f \in C^\infty(\mathbb{R}_+)$ such that $f' \in [0, 1+\alpha]$, $f'(0) = 0$ and $f(r) = r$ for $r \ge \alpha$. For fixed $y \in M$ and $T > 0$, take $U(t,x) = 0$ for $x \in \mathrm{cut}(y)$ and

$$U(t,x) = -\frac{\nabla \rho_y^2(x)}{2(T-t)} - \frac{1}{2}\beta \nabla f \circ \rho_y(x), \quad x \notin \mathrm{cut}(y),\ t < T.$$

We use Cranston's trick [7] to treat $\mathrm{cut}(y)$ and then obtain the following formula as in [13]:

$$d\rho_y^2(\eta_t) = 2\rho_y(\eta_t)\,db_t + \tfrac{1}{2}(\Delta + Z + 2U(t,\cdot))\rho_y^2(\eta_t)\,dt - dL_t, \tag{3.1}$$

where b_t is a one-dimensional Brownian motion and L_t an increasing process with support contained in $\{t \ge 0 : \eta_t \in \mathrm{cut}(y)\}$. Moreover, $(\Delta + Z + 2U)\rho_y(\eta_t)$ is taken to be zero whenever $\eta_t \in \mathrm{cut}(y)$.

Since Ricci $\ge -k(d-1)$, we have (see [5])

$$\begin{aligned} \tfrac{1}{2}\Delta\rho_y^2(x) &\le 1 + \sqrt{k}(d-1)\rho_y(x)\coth[\sqrt{k}\rho_y(x)] \\ &= d + (d-1)\bigl\{\sqrt{k}\rho_y(x)\coth[\sqrt{k}\rho_y(x)] - 1\bigr\}. \end{aligned} \tag{3.2}$$

Let $h(r) = r\coth r - 1 - r$, $r \geq 0$, then $h(0) = 0$ and

$$h'(r) = \sinh^{-2} r(\sinh r \cosh r - r - \sinh^2 r) := \sinh^{-2} r h_1(r).$$

We see that $h_1(0) = 0$ and $h_1'(r) = \exp(-2r) - 1 \leq 0$ for $r \geq 0$. So $h(r) \leq 0$ for $r \geq 0$. Therefore, by (3.2),

$$\tfrac{1}{2}(\Delta + Z)\rho_y^2(x) \leq d + \beta\rho_y(x). \tag{3.3}$$

On the other hand,

$$\begin{aligned} U(t,x)\rho_y^2(x) &= -\frac{2\rho_y^2(x)}{T-t} - \beta\rho_y(x) f' \circ \rho_y(x) \\ &\leq -\frac{2\rho_y^2(x)}{T-t} - \beta\rho_y(x) + \beta\alpha, \qquad t < T. \end{aligned}$$

We have

$$\frac{1}{2}(\Delta + Z + U(t,\cdot))\rho_y^2(x) \leq d + \beta\alpha - \frac{2\rho_y^2(x)}{T-t}. \tag{3.4}$$

Let $\tau_n = \{t \geq 0\colon \rho_y(\eta_t) \geq n\}$; by (3.1) and (3.4) we obtain

$$E^x \rho_y^2(\eta_{t\wedge\tau_n}) \leq (d + \beta\alpha)t + \rho_y^2(x), \qquad t < T.$$

Then

$$g(t) := E^x \rho^2(\eta_t) = E^x \rho_y^2(\eta_{t\wedge\tau_\infty}) \leq (d + \beta\alpha)t + \rho_y^2(x)$$

which is bounded on $[0, T)$. This means that $2\int_0^t \rho_y(\eta_s)\, db_s$ is a (square integrable) martingale before time T and hence Lemma 2.1 holds if we put the present U into $\mathscr{F}_T$. Now, it follows from (3.1) and (3.4) that

$$g'(t) \leq d + \beta\alpha - \frac{2g(t)}{T-t}, \qquad g(0) = \rho_y^2(x), \qquad t \in [0, T). \tag{3.5}$$

Let $G(t) = g(t)/(T-t)^2$, then (3.5) is equivalent to $G'(t) \leq (d+\beta\alpha)/(T-t)^2$. This implies that

$$E^x \rho_y^2(\eta_t) \leq \frac{(T-t)^2}{T^2}\rho_y^2(x) + \frac{t}{T}(T-t)(d+\beta\alpha), \qquad t < T. \tag{3.6}$$

Next, for any $\sigma > 0$, we have

$$\begin{aligned} E^x|U(t,\eta_t)|^2 &\leq E^x\left\{\frac{\rho_y(\eta_t)}{T-t} + \frac{1}{2}(1+\alpha)\beta\right\}^2 \\ &\leq \big(1 + \sigma\sqrt{T-t}\big)E^x \frac{\rho_y^2(\eta_t)}{(T-t)^2} + \frac{1}{4}(1+\alpha)^2\beta^2\left(1 + \frac{1}{\sigma\sqrt{T-t}}\right) \\ &\leq \big(1 + \sigma\sqrt{T-t}\big)\left\{\frac{\rho_y^2(x)}{T^2} + \frac{(d+\beta\alpha)t}{T(T-t)}\right\} \\ &\quad + \frac{1}{4}(1+\alpha)^2\beta^2\left(1 + \frac{1}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Note that

$$\int_0^{T-\alpha} \sqrt{T-t}\,dt = \frac{2}{3}(T^{3/2}-\alpha^{3/2}), \qquad \int_0^{T-\alpha} \frac{1}{\sqrt{T-t}}\,dt = 2(\sqrt{T}-\sqrt{\alpha}),$$

$$\int_0^{T-\alpha} \frac{t}{\sqrt{T-t}}\,dt = 2T(\sqrt{T}-\sqrt{\alpha}) - \frac{2}{3}(T^{3/2}-\alpha^{3/2}) \le \frac{4}{3}T^{3/2},$$

$$\int_0^{T-\alpha} \frac{t}{T-t}\,dt = -T+\alpha+T\log\frac{T}{\alpha}.$$

Then

$$\begin{aligned}\int_0^{T-\alpha} & E^x|U(t,\eta_t)|^2\,dt \\ &\le \left(\frac{1}{T}+\frac{2\sigma}{3\sqrt{T}}\right)\rho_y^2(x) + \frac{1}{4}(1+\alpha)^2\beta^2\left(T+\frac{2}{\sigma}\sqrt{T}\right) \\ &\quad + (d+\beta\alpha)\left(\frac{\alpha}{T}-1+\log\frac{T}{\alpha}+\frac{4}{3}\sigma\sqrt{T}\right).\end{aligned} \tag{3.7}$$

By Lemmas 2.1 and 2.2, (3.6) and (3.7),

$$\begin{aligned}J^\alpha_{T-\alpha}(x) &\le E^x J_0^\alpha(\eta_{T-\alpha}) + \frac{1}{2}\int_0^{T-\alpha} E^x|U(t,\eta_t)|^2\,dt \\ &\le \frac{d}{2}\log\frac{2\pi\alpha}{1-2\alpha l} + \frac{(d-1)^2k}{4l} + \frac{1}{2\alpha}E^x\rho_y^2(\eta_{T-\alpha}) \\ &\quad + \frac{1}{2}\int_0^{T-\alpha} E^x|U(t,\eta_t)|^2\,dt \\ &\le \frac{d}{2}\log\frac{2\pi}{1-2\alpha l} + \frac{(d-1)^2k}{4l} + \frac{1}{2}\left(\frac{1}{T}+\frac{2\sigma}{3\sqrt{T}}+\frac{\alpha}{T^2}\right)\rho_y^2(x) \\ &\quad + \frac{1}{2}(d+\beta\alpha)\left(\frac{2\alpha}{T}+\log T+\frac{4}{3}\sigma\sqrt{T}\right) + \frac{\beta^2}{8}\left(T+\frac{2}{\sigma}\sqrt{T}\right).\end{aligned}$$

Therefore

$$\begin{aligned}-\log p_t(x,y) &= \lim_{\alpha\to 0} J^\alpha_{T-\alpha}(x) \\ &\le \frac{d}{2}\log(2\pi T) + \frac{(d-1)^2k}{4l} + \frac{1}{2}\left(\frac{1}{T}+\frac{2\sigma}{3\sqrt{T}}\right)\rho_y^2(x) \\ &\quad + \frac{2d\sigma}{3}\sqrt{T} + \frac{1}{8}\beta^2\left(T+\frac{2}{\sigma}\sqrt{T}\right), \qquad l>0,\ \sigma>0.\end{aligned}$$

Now the proof is complete by letting $l\to\infty$. □

4. Manifolds with unbounded curvature. In this section, we extend Theorem 3.1 to manifolds with unbounded curvature. Suppose that for fixed $y\in M$, there exist $k_1, k_2 > 0$ (which may depend on y) such that

$$\mathrm{Ricci}(X,X) \ge -(k_1+k_2\rho_y(x))^2(d-1)|X|^2, \qquad X\in T_xM. \tag{4.1}$$

Next, let $b_1 = |Z(y)|$ and suppose that

$$b_2 := \sup\{\langle \nabla_U Z, U\rangle : U \in TM,\ |U| = 1\} < \infty.$$

Let $\beta_i = \max\{0, (d-1)k_i + b_i\}$, $i = 1, 2$.

For any $x \notin \text{cut}(y)$, let $U_s (s \in [0, \rho_y(x)])$ be the unit tangent vector field along the minimal geodesic from y to x. Then

$$Z\rho_y(x) = \int_0^{\rho_y(x)} \frac{d}{ds}\langle U, Z\rangle\, ds + \langle U_0, Z\rangle \le b_1 + b_2\rho_y(x). \tag{4.2}$$

Recalling that $r \coth r - 1 \le r$ for $r \ge 0$, by Bishop's comparison theorem we obtain

$$\begin{aligned} \tfrac{1}{2}(\Delta + Z)\rho_y^2(x) &\le 1 + (d-1)(k_1\rho_y(x) + k_2\rho_y^2(x)) \coth[k_1\rho_y(x) + k_2\rho_y^2(x)] \\ &\qquad + b_1\rho_y(x) + b_2\rho_y^2(x) \\ &\le d + \beta_1\rho_y(x) + \beta_2\rho_y^2(x). \end{aligned} \tag{4.3}$$

Then Lemma 2.1 holds for the present case.

Since Ricci $\ge -(d-1)(k_1 + k_2 r)^2$ in $B(y, r)$, (2.6) holds with $\sqrt{k}$ replaced by $k_1 + k_2 r$. Then

$$\begin{aligned} C_\alpha &\le |\mathbb{S}^{d-1}| \int_0^\infty \exp(-r^2/2\alpha + (d-1)(k_1 + k_2 r)) r^{d-1}\, dr \\ &\le \exp((d-1)k_1^2/4l) \int_{\mathbb{R}^d} \exp\left\{\left(l + (d-1)k_2 - \frac{1}{2\alpha}\right)|z|^2\right\} dz \\ &= \exp((d-1)^2 k_1/4l)\left(\frac{2\pi\alpha}{1 - 2\alpha[l + (d-1)k_2]}\right)^{d/2}, \qquad 1 > 2\alpha[l + (d-1)k_2]. \end{aligned}$$

Therefore

$$J_0^\alpha(x) \le \frac{(d-1)^2 k_1^2}{4l} + \frac{d}{2}\log\frac{2\pi\alpha}{1 - 2\alpha[l + (d-1)k_2]} + \frac{\rho_y^2(x)}{2\alpha}. \tag{4.4}$$

Take $U(t, x) = 0$ for $x \in \text{cut}(y)$ and

$$U(t, x) = -\frac{\nabla\rho_y^2(x)}{2(T-t)} - \frac{\beta_2\nabla\rho_y^2(x)}{4} - \frac{1}{2}\beta_1\nabla f \circ \rho_y(x), \qquad x \notin \text{cut}(y),\ t < T,$$

where f is as in Section 3. Then (3.6) holds with β replaced by β_1.

Finally, from Section 3 we see that

$$\int_0^{T-\alpha} E^x|U(t, \eta_t)|^2\, dt = \int_0^{T-\alpha} E^x\left\{\frac{\rho_y(\eta_t)}{T-t} + \frac{\beta_2\rho_y(\eta_t)}{2} + \frac{\beta_1}{2}\right\}^2 + O(-\alpha\log\alpha)$$

for small α. Now, we obtain the following result by some ordinary calculations as in Section 3.

THEOREM 4.1. *For any $\sigma > 0$, we have*

$$p_t(x, y) \geq (2\pi t)^{-d/2} \exp\{-f_1(\sigma, t)\rho^2(x, y) - f_2(\sigma, t)\}, \qquad t > 0, \ x, y \in M,$$

where

$$f_1(\sigma, t) = \frac{1}{2t} + \frac{\sigma}{3\sqrt{t}} + \frac{\beta_2}{4} + \frac{\sigma\beta_2}{5}\sqrt{t} + \frac{\beta_2^2}{24}t + \frac{\sigma\beta_2^2}{28}t^{3/2},$$
$$f_2(\sigma, t) = \left(\frac{\beta_1^2}{4\sigma} + \frac{2d\sigma}{3}\right)\sqrt{t} + \frac{\beta_1^2 + 2d\beta_2}{8}t + \frac{2\sigma d\beta_2}{15}t^{3/2} + \frac{d\beta_2^2}{48}t^2 + \frac{d\sigma\beta_2^2}{70}t^{5/2}.$$

5. Heat kernels on a compact manifold. As was pointed out in Section 1, there are many estimates which are sharp for short times, and some are sharp as well for long times when the manifold is a noncompact space form. But most of them become zero when $t \to \infty$; this is not the case for heat kernels on a compact manifold. The purpose of this section is to present an explicit lower bound of heat kernels on a compact manifold which is sharp for short times and nontrivial for long times.

Let D be the diameter of M and β as in Theorem 3.1. By (3.3) we have

$$\frac{1}{2}(\Delta + Z)\rho_y^2(x) \leq d + \frac{\sigma\rho_y^2(x)}{(T-t)^{3/4}} + \frac{1}{4}\sigma^{-1}\beta^2(T-t)^{3/4} \tag{5.1}$$

for any $\sigma > 0$.

Take $U(t, x) = 0$ in $\text{cut}(y)$ and

$$U(t, x) = -\frac{\nabla\rho_y^2(x)}{2(T-t)} - \frac{\sigma\nabla\rho_y^2(x)}{4(T-t)^{3/4}}, \qquad x \notin \text{cut}(y), \ t < T.$$

We obtain

$$d\rho_y^2(\eta_t) \leq 2\rho_y(\eta_t)\, db_t + [d + (4\sigma)^{-1}\beta^2(T-t)^{3/4}]\, dt - \frac{2\rho_y^2(\eta_t)}{T-t}\, dt, \qquad t < T.$$

This implies that (cf. Section 3)

$$\begin{aligned} E^x\rho_y^2(\eta_t) &\leq \min\left\{D^2, \ \frac{(T-t)^2}{T^2}\rho_y^2(x) + \frac{d(T-t)t}{T}\right. \\ &\qquad\qquad \left. + \frac{\beta^2(T-t)^2}{\sigma}[(T-t)^{-1/4} - T^{-1/4}]\right\} \\ &\leq \min\left\{D^2, \ \frac{(T-t)^2}{T^2}\rho_y^2(x) + \frac{d(T-t)t}{T} + \frac{\beta^2 t(T-t)^{7/4}}{\sigma T}\right\}. \end{aligned} \tag{5.2}$$

Then

$$
\begin{aligned}
E^x \int_0^{T-\alpha} |U(t,\eta_t)|^2\,dt &= \int_0^{T-\alpha}\left[\frac{1}{T-t}+\frac{\sigma}{2(T-t)^{3/4}}\right]^2 E^x\rho_y^2(\eta_t)\,dt \\
&\le \rho_y^2(x)\int_0^{T-\alpha}\frac{(T-t)^2}{T^2}\left[\frac{1}{T-t}+\frac{\sigma}{2(T-t)^{3/4}}\right]^2 dt \\
&\quad + \int_0^{T-\alpha}\left[\frac{1}{T-t}+\frac{\sigma}{2(T-t)^{3/4}}\right]^2 \\
&\qquad \times \min\left\{D^2, \frac{d(T-t)t}{T}+\frac{\beta^2 t(T-t)^{7/4}}{\sigma T}\right\} dt \\
&:= \rho_y^2(x) I_1 + I_2.
\end{aligned} \tag{5.3}
$$

It is easy to see that

$$
I_1 \le \frac{1}{T}+\frac{4\sigma}{5T^{3/4}}+\frac{\sigma^2}{6\sqrt{T}}. \tag{5.4}
$$

On the other hand, for any $p \in (0,T)$, let $q = T-p$. Divide I_2 into three terms:

$$
I_2 = \int_p^q + \int_0^p + \int_q^{T-\alpha} := I_3 + I_4 + I_5. \tag{5.5}
$$

We have

$$
\begin{aligned}
I_3 &\le D^2\int_p^q\left\{\frac{1}{T-t}+\frac{\sigma}{2(T-t)^{3/4}}\right\}^2 dt \\
&\le \left\{D^2\left(p^{-1}+\frac{4\sigma}{3}p^{-3/4}+\frac{\sigma^2}{2}p^{-1/2}\right)\right\}1_{\{q>p\}}.
\end{aligned} \tag{5.6}
$$

Note that $T-t \ge q$ for $t \in [0,p]$, then

$$
\begin{aligned}
I_4 &\le \int_0^p\left[\frac{1}{T-t}+\frac{\sigma}{2(T-t)^{3/4}}\right]^2\left(\frac{d(T-t)t}{T}+\frac{\beta^2 t(T-t)^{7/4}}{\sigma T}\right) dt \\
&\le \frac{p^2}{2T}\left\{dq^{-1}+\sigma^{-1}\beta^2 q^{-1/4}+d\sigma q^{-3/4}+\frac{\sigma^2 d}{4}q^{-1/2}+\beta^2+\frac{\sigma\beta^2}{4}q^{1/4}\right\}.
\end{aligned} \tag{5.7}
$$

Similarly, note that $t/T \le 1$, we have

$$
\begin{aligned}
I_5 &\le \int_0^{T-\alpha}\left\{\frac{d}{T-t}+\frac{\beta^2}{\sigma(T-t)^{1/4}}+\frac{d\sigma}{(T-t)^{3/4}}\right. \\
&\qquad \left. +\beta^2+\frac{\sigma^2 d}{4\sqrt{T-t}}+\frac{\sigma\beta^2}{4}(T-t)^{1/4}\right\} dt \\
&\le d\log\frac{p}{\alpha}+\frac{4\beta^2}{3\sigma}p^{3/4}+4d\sigma p^{1/4}+\beta^2 p+\frac{\sigma^2 d}{2}\sqrt{p}+\frac{\sigma\beta^2}{5}p^{5/4}.
\end{aligned} \tag{5.8}
$$

Now, by Lemmas 2.1 and 2.2 and (5.2)–(5.8), we obtain the following result.

THEOREM 5.1. *Suppose that* $D<\infty$. *For any* $\sigma>0$ *and* $t>p>0$, *we have*

$$p_t(x,y)\ge(2\pi p)^{-d/2}\exp\{-f_3(t,\sigma)\rho^2(x,y)-f_4(t,p,\sigma)\},\qquad x,y\in M,$$

where

$$f_3(t,\sigma)=\frac{1}{2t}+\frac{2\sigma}{4t^{3/4}}+\frac{\sigma^2}{12\sqrt{t}},$$

$$\begin{aligned}f_4(t,p,\sigma)=\frac{d}{2}&+\left\{\frac{D^2}{2}\left(p^{-1}+\frac{4\sigma}{3}p^{-3/4}+\frac{\sigma^2}{2}p^{-1/2}\right)\right\}1_{\{t>2p\}}\\&+\frac{p^2}{4t}\left\{d(t-p)^{-1}+\frac{\beta^2}{\sigma}(t-p)^{-1/4}+d\sigma(t-p)^{-3/4}\right.\\&\qquad\left.+\frac{\sigma^2 d}{4}(t-p)^{-1/2}+\beta^2+\frac{\sigma\beta^2}{4}(t-p)^{1/4}\right\}\\&+\frac{2\beta^2}{3\sigma}p^{3/4}+3d\sigma p^{1/4}+\frac{\beta^2}{2}p+\frac{\sigma^2 d}{4}p^{1/2}+\frac{\sigma\beta^2}{10}p^{5/4}.\end{aligned}$$

If $\beta=0$, *we may take* $\sigma=0$, *then*

$$p_t(x,y)\ge(2\pi p)^{-d/2}\exp\left\{-\frac{d}{2}-\frac{\rho^2(x,y)}{2t}-\frac{dp^2}{4t(t-p)}-\frac{D^2}{2p}1_{\{t>2p\}}\right\}.$$

REMARKS. (a) Obviously, we have $f_3(\infty,\sigma)=0$ and $f_4(\infty,p,\sigma)<\infty$ for all $p,\sigma>0$. Then the lower bound given in Theorem 5.1 is nontrivial as $t\to\infty$. On the other hand, take $p=\frac{1}{2}t$ and $\sigma=t^{3/4}$; then the lower bound is sharp for short times.

(b) Theorems 3.1, 4.1 and 5.1 are also true for Neumann heat kernels on a convex regular domain. The only change made to the proofs is that η_t now becomes a reflecting diffusion process, so the increasing process L_t appearing in (3.1) may also increase whenever η_t visits the boundary. See [19].

Acknowledgments. The author thanks Professors M. F. Chen, K. D. Elworthy and S. J. Yan for their encouragement and helpful suggestions. Thanks also to a referee whose comments improved the quality of the paper.

REFERENCES

[1] BAKRY, D. and QIAN, Z. (1996). Gradient estimates and Harnack inequality on a complete manifold. Preprint.

[2] BERG, M., GAUDUCHON, P. and MAZET, E. (1971). Le spectre d'une variété Riemannienne. *Lecture Notes in Math.* 194. Springer, Berlin.

[3] BENJAMINI, I., CHAVEL, I. and FELDMAN, E. A. (1996). Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash. *Proc. London Math. Soc.* 72 215–240.

[4] BENJAMINI, I., CHAVEL, I. and FELDMAN, E. A. (1995). Heat kernel lower bounds on Riemannian manifolds. *Proc. Sympos. Pure Math.* 57 213–225.

[5] CHAVEL, I. (1984). Eigenvalues in Riemannian geometry. Academic Press, New York.

[6] CHEEGER, J. and YAU, S. T. (1981). A lower bound for heat kernel. *Comm. Pure Applied Math.* 34 465–480.

[7] CRANSTON, M. (1991). Gradient estimates on manifolds using coupling. *J. Funct. Anal.* 99 110–124.

[8] DAVIES, E. B. (1989). *Heat Kernels and Spectral Theory*. Cambridge Univ. Press.

[9] DAVIES, E. B. and MANDOUVALOS, N. (1988). Heat kernel bounds on hyperbolic space and Kleinian groups *Proc. London Math. Soc.* 57 182–208.

[10] FLEMING, W. H. (1982). Logarithmic transformations and stochastic control. *Lecture Notes in Control and Inform. Sci.* 42 131–141.

[11] GRIGOR'YAN, A. A. (1994). Heat kernel upper bounds on a complete non-compact manifold. *Rev. Mat. Iberoamericana* 10 395–452.

[12] GRIGOR'YAN, A. A. (1994). Heat kernel on a manifold with a local Harnack inequality. *Comm. Anal. Geom.* 2 111–138.

[13] KENDALL, W. S. (1987). The radial part of Brownian motion on a manifold: a semimartingale property. *Ann. Probab.* 15 1491–1500.

[14] LI, P. (1986). Large time behaviour of the heat equation on complete manifolds with non-negative curvature. *Ann. Math.* 124 1–21.

[15] LI, P. and YAU, S. T. (1986). On the parabolic heat kernel of the Schrödinger operator. *Acta Math.* 156 153–201.

[16] QIAN, Z. (1995). Gradient estimates and heat kernel estimates. *Proc. Roy. Soc. Edinburgh Sect. A* 125 975–990.

[17] SETTI, A. G. (1992). Gaussian estimates for the heat kernel of the weighted Laplacian and fractal measures. *Canad. J. Math.* 44 1061–1078.

[18] SHEU, S. J. (1991). Some estimates of the transition density of a nondegenerate diffusion Markov process. *Ann. Probab.* 19 538–561.

[19] WANG, F. Y. (1994). Application of coupling method to the Neumann eigenvalue problem. *Probab. Theory Related Fields* 98 299–306.

[20] YAU, S. T. (1995). Harnack inequality for non-self-adjoint evolution equations. *Math. Res. Lett.* 2 387–399.

DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING 100875
PEOPLE'S REPUBLIC OF CHINA
E-MAIL: wangfy@bnu.edu.cn