

## ORNSTEIN–ZERNIKE THEORY FOR THE BERNOULLI BOND PERCOLATION ON $\mathbb{Z}^d$

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We derive a precise Ornstein–Zernike asymptotic formula for the decay of the two-point function  $\mathbb{P}_p(0 \leftrightarrow x)$  of the Bernoulli bond percolation on the integer lattice  $\mathbb{Z}^d$  in any dimension  $d \geq 2$ , in any direction  $x$  and for any subcritical value of  $p < p_c(d)$ .

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## 1. Introduction and results.

1.1. *Ornstein–Zernike theory.* The Ornstein–Zernike theory [16] gives a sharp asymptotic description of density correlation functions in classical fluids away from the critical point. In their original work Ornstein and Zernike clearly perceived the mathematical structure of the model and, as often happens in the physical literature, gave a convincing derivation of the asymptotic result, assuming, though, the most formidable issue to be proven. We refer to Chapter 5 of [18] for a very clean and stimulating discussion of the physical background.

The abovementioned crucial property is a certain mass-gap condition (see Section 1.3), and the focal point of our work here is to establish it in the context of the subcritical Bernoulli bond percolation on  $\mathbb{Z}^d$ , which, for that matter, could be thought of as a spatially discretized model of fluids.

An excellent reference for the percolation models is [11]. To set up notation, let  $\{\eta(b)\}$  be a family of Bernoulli i.i.d. random variables, indexed by the nearest neighbor bonds of the integer lattice  $\mathbb{Z}^d$ . We use  $\mathbb{P}_p$  to denote the corresponding joint probability distribution,

$$\mathbb{P}_p(\eta(b) = 1) = p.$$

Given a realization  $\eta$ , we say that a bond  $b$  is open if  $\eta(b) = 1$ ; otherwise we shall call it closed. Two points  $x, y \in \mathbb{Z}^d$  are said to be connected,  $\{x \leftrightarrow y\}$ , if there exists a chain of open bonds leading from  $x$  to  $y$ . The event  $\{x \leftrightarrow y\}$  is, obviously, measurable and, due to the symmetries of the lattice,

$$\mathbb{P}_p(x \leftrightarrow y) = \mathbb{P}_p(0 \leftrightarrow y - x).$$

In the fluid interpretation the connectivity function  $\mathbb{P}_p(x \leftrightarrow y)$  is supposed to describe the truncated correlation function between the particle densities at  $x$  and  $y$ . Short range order of fluctuations corresponds, then, to the requirement that the influence from the origin does not propagate along the lattice:

$$(1.1) \quad \mathbb{P}_p(0 \leftrightarrow \infty) = 0,$$

where the event  $\{0 \leftrightarrow \infty\}$  naturally means that the open cluster of the origin is infinite. The percolation threshold  $p_c = p_c(d)$  is defined as

$$p_c = \sup\{p : (1.1) \text{ holds}\}.$$

We say that the Bernoulli bond percolation model on  $\mathbb{Z}^d$  is subcritical if  $p < p_c(d)$ .

A fundamental result by Menshikov [15] and Aizenman and Barsky [1] states that subcriticality is always reinforced with strong decay properties of connectivities. Namely, in any dimension  $d$

$$(1.2) \quad \chi_d(p) \triangleq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x) < \infty,$$

whenever  $p < p_c(d)$ . An application of the BK inequality (cf. the proof of Hammersley's Theorem 5.1 in [11]) shows that (1.2) actually implies exponential decay of connectivities: for every  $p < p_c$  there exists  $c_1 = c_1(p) > 0$  such that

$$(1.3) \quad \mathbb{P}_p(0 \leftrightarrow x) \leq e^{-c_1 \|x\|}.$$

On the other hand, the FKG property of the Bernoulli bond percolation implies that the inverse correlation length  $\xi_p$ ,

$$(1.4) \quad \xi_p(x) \triangleq - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow [nx]) \quad \text{or} \quad \mathbb{P}_p(0 \leftrightarrow [x]) \asymp e^{-\xi_p(x)},$$

is always defined and, moreover, is a finite, convex and homogeneous-of-order-1 function on  $\mathbb{R}^d$ . By the subadditivity argument, which again follows from the FKG property of  $\mathbb{P}_p$ ,

$$(1.5) \quad \mathbb{P}_p(0 \leftrightarrow x) \leq e^{-\xi_p(x)},$$

for every  $x \in \mathbb{Z}^d$ . Thus, the Hammersley estimate (1.3) asserts that, for every  $p < p_c$ , the inverse correlation length  $\xi_p$  is a strictly positive function on  $\mathbb{R}^d \setminus \{0\}$ . In other words, for subcritical percolation models the inverse correlation length  $\xi_p$  captures the nontrivial leading asymptotics of decay of point-to-point connectivities  $\mathbb{P}_p(0 \leftrightarrow x)$  on the logarithmic scale.

In this paper we derive a rigorous version of the Ornstein–Zernike theory which gives a precise asymptotic description of connectivities up to the zero-order terms.

**THEOREM A.** *Let  $d \geq 2$  and  $p < p_c(d)$ . Then, uniformly in  $x \in \mathbb{Z}^d$ ,  $\|x\| \rightarrow \infty$ ,*

$$(1.6) \quad \mathbb{P}_p(0 \leftrightarrow x) = \frac{\Psi_p(\mathbf{n}(x))}{\sqrt{(2\pi \|x\|)^{d-1}}} e^{-\xi_p(x)} (1 + o(1)),$$

where  $\mathbf{n}(x)$  denotes the unit vector in the direction of  $x$  and  $\Psi_p$  is a positive real analytic function on  $\mathbb{S}^{d-1}$ .

**REMARK.** The explicit expression for  $\Psi_p$  will be given in Section 3.5. The coefficient  $\sqrt{(2\pi)^{d-1}}$  in the denominator of the prefactor is, of course, superfluous—we put it there only to stress that the result is a local limit-type theorem for connectivities.

The relevant local limit behavior will be read from peculiar renewal structures of the probabilities  $\mathbb{P}_p(0 \leftrightarrow x)$ . The corresponding analytic properties of multi-dimensional moment generating functions already lie in the heart of the original paper [16]. In our setup we follow [7], where an analog of Theorem A has been established for on-axis directions of  $x$ . The lattice symmetries, therefore, played an

essential role in the latter work. To treat off-axis directions we rely on general results on the local limit structure of multidimensional renewal arrays, as developed in [12].

As we have already mentioned, the whole Ornstein–Zernike theory hinges on the validity of a certain mass-gap condition. In the case of self-avoiding walks [12] such a condition happens to be technically insensitive to various tiltings of connectivities needed to explore decays in off-axis directions. Thus it could be verified by an almost literal application of “on-axis” methods, which have been developed earlier in [9] and [14].

Contrary to this, in the case of percolation the verification of the on-axis mass-gap condition already requires tedious computation (see [7], Section 5); an introduction of an additional tilt would have complicated the method employed there beyond reason.

A very different approach to the problem of refining subadditive bounds on connectivity-type functions has been developed in a series of papers by Alexander [2, 3, 5, 4]. The renormalization ideas he has introduced in these works were not designed to furnish exact local limit descriptions as in Theorem A, but helped to illuminate the coarse-grained structure of the model to the extent of providing complementary lower bounds with correct order of pre-factors near the decay exponents. It should be mentioned that, unlike the refined renewal methods, Alexander’s techniques require much less structure and apply to a large variety of other models.

Our main observation in this work could now be formulated as follows: the mass-gap condition in question is a coarse property in the realm of renormalization estimates. Furthermore, an appropriate modification of the renormalization ideas of Alexander leads to a relatively short proof. In other words, our version of the Ornstein–Zernike theory comprises two steps; in the first stage “heavy-duty” renormalization techniques are used to clean up the model from exponentially improbable events; then the restructured model is tuned up with the help of more delicate local-limit-type methods, based on the specific renewal properties of the Bernoulli bond percolation.

Strict exponential decay of the two-point function, which, by the results of [15] and [1], gives a sharp characterization of the subcritical percolation models, is absolutely indispensable for our renormalization approach. On the other hand the nearest neighbor structure of the bonds plays no role. A straightforward adjustment of the methods we develop here would yield results similar to (1.6) in any subcritical translation invariant Bernoulli bond percolation model with finite range of bonds or in subcritical site percolation models. For the sake of the exposition, however, we shall stick to the case of nearest neighbor bond percolation.

Ornstein–Zernike theory for high-temperature Ising models has been developed in [8] and will appear elsewhere. While the renormalization procedures in the latter work are built on those we employ here, the local limit part of the analysis

has to be substantially modified—the random line representation of the Ising two-point function does not enjoy factorization properties of independent percolation models. Nevertheless, at the end of the day, the relevant local limit result for the endpoints of random lines has exactly the same classical analytic nature as in the independent case we consider here.

NOTATION. The constants  $c_1, c_2, \dots > 0$  are updated with each section. We use  $\|\cdot\|_d$  and  $\|\cdot\|_{d-1}$  to denote the Euclidean norm on  $\mathbb{R}^d$  and, respectively,  $\mathbb{R}^{d-1}$ . Similarly  $(\cdot, \cdot)_d$  and  $(\cdot, \cdot)_{d-1}$  are used to denote the corresponding scalar products.

1.2. *Renewal structure of connectivities.* The aforementioned renewal properties could be recorded in several different ways. In fact, we give here two alternative proofs of Theorem A, which correspond to two different renewal setups. The first approach is a “parameterized” one, and it has been previously introduced in [7] and developed, to the state we are using it this work, in [12]. An advantage of the parameterized approach is that it contains an explicit treatment of the relevant  $(d-1)$ -dimensional local limit result. Also it illuminates several related geometric issues (see Section 1.4) in a natural way. Most important, the parameterized approach is well suited for studying various related problems, such as scaling analysis of percolation paths or, in the case of two dimensions, the refined fluctuation analysis of phase boundaries. The corresponding results will appear elsewhere.

In the concluding Section 4 we work out an alternative “direct” proof. In this direct approach both the underlying  $(d-1)$ -dimensional local limit structure and the intrinsic geometry of shapes are implicit. However, the proof itself is technically more straightforward, and, moreover, it does not rely on the lattice symmetries of the model and clearly illustrates the generality and the limitations of the method. In particular, whereas the independence of bond variables and, to a lesser extent, shift invariance are rather important for the whole approach, the nearest neighbor structure of the bonds and the lattice symmetries of bond variables play no role. As has been mentioned, we could have generalized our techniques to the case of shift invariant Bernoulli percolation on  $\mathbb{Z}^d$  with, for example, finite range of connecting bonds. For the sake of the clarity of the exposition, however, we refrain from such an exercise.

The renormalization estimates, which are crucial for both parameterized and direct approaches, are developed in Section 2.

Let us proceed and set up the notation for the parameterized approach. It would be enough to prove the result for the lattice cone of points  $x \in \mathbb{Z}^d$  satisfying  $x_1 \geq \|x\|_d/\sqrt{d}$ . Hence the motivation for a parameterization: fix a unit axis direction  $\mathbf{e}_1$  and write  $\mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z}^{d-1}$ . Accordingly, we write  $x = (n, k)$  for a point  $x \in \mathbb{Z}^d$ . We shall prove that, for every  $\alpha > 0$ , the claim of Theorem A holds uniformly over  $x$  belonging to the cone  $\mathcal{C}_\alpha$ :

$$(1.7) \quad \mathcal{C}_\alpha \triangleq \{x = (n, k) : \|k\|_{d-1} \leq \alpha n\}.$$

In view of the above parameterization, it happens to be convenient to establish an analog of Theorem A first for modified connectivities, the so-called cylindrical ones. We follow [7] and [12] for the notation and general setup:

Define lattice  $(d - 1)$ -dimensional hyperplanes  $\mathcal{H}_n, n = 0, 1, \dots$ , as

$$(1.8) \quad \mathcal{H}_n \triangleq \{x \in \mathbb{Z}^d : x = (n, k)\}.$$

Similarly, given  $m, n \in \mathbb{N}$ , define lattice strip  $\mathcal{S}_{m,n}$  as

$$(1.9) \quad \mathcal{S}_{m,n} \triangleq \bigcup_{r=m}^n \mathcal{H}_r = \{x \in \mathbb{Z}^d : x = (r, k) \text{ with } m \leq r \leq n\}.$$

DEFINITION. We say that a point  $x \in \mathcal{H}_n$  is  $h$ -connected to the origin,  $\{0 \overset{h}{\leftrightarrow} x\}$ , if the following hold:

(i) The point  $x$  is connected with 0 in the restriction of the percolation configuration  $\eta$  to the strip  $\mathcal{S}_{0,n}$ .

(ii) Let  $\mathbf{C}_{\{0,x\}}^n$  be the corresponding common open cluster of  $x$  and 0 in  $\mathcal{S}_{0,n}$ . Then

$$\mathbf{C}_{\{0,x\}}^n \cap \mathcal{H}_0 = \{0\} \quad \text{and} \quad \mathbf{C}_{\{0,x\}}^n \cap \mathcal{H}_n = \{x\}.$$

For every  $n \in \mathbb{N}$  and every  $x = (n, k) \in \mathcal{H}_n$ , define

$$h(n, k) \triangleq \mathbb{P}_p(0 \overset{h}{\leftrightarrow} (n, k)).$$

Of course,  $h(n, k) < \mathbb{P}_p(0 \leftrightarrow x)$ , for every  $n \in \mathbb{N}$  and  $x = (n, k) \in \mathcal{H}_n$ . On the other hand, it takes a soft argument (see [12], Proposition 3.2) to show that both  $h(n, k)$  and  $\mathbb{P}_p(0 \leftrightarrow x)$  have the same leading asymptotics on the logarithmic scale: namely, for every  $\alpha > 0$ ,

$$(1.10) \quad \xi_p\left(1, \frac{k}{n}\right) + \frac{1}{n} \log h(n, k) = o(1),$$

uniformly (as  $n \rightarrow \infty$ ) in  $x = (n, k) \in \mathcal{C}_\alpha$ , where  $\mathcal{C}_\alpha$  is the cone defined in (1.7).

In fact, as we shall prove in Section 3.5,  $h$ -connectivities approximate the full ones in a much more stringent way:

LEMMA 1.1. *Let  $d \geq 2, p < p_c(d)$  and  $\alpha \in \mathbb{R}_+$  be fixed. Then, uniformly in  $x = (n, k) \in \mathcal{C}_\alpha$ ,*

$$(1.11) \quad \mathbb{P}_p(0 \leftrightarrow x) = \Phi_p(\vec{\mathbf{n}}(x))h(n, k)(1 + o(1)),$$

where  $\Phi_p$  is a positive real analytic function on  $\mathbb{S}_+^{d-1} \triangleq \{\vec{\mathbf{n}} = (\mathbf{n}_1, \dots, \mathbf{n}_d) \in \mathbb{S}^{d-1} : \mathbf{n}_1 > 0\}$ .

Thus, by the above lemma, and in view of the  $\mathbb{Z}^d$ -lattice symmetries of  $\mathbb{P}_p$ , it would be enough to restrict our attention to the asymptotic behavior of  $h$ -connectivities. For every  $n \in \mathbb{N}$  these induce a probability distribution  $\mathbb{P}_n$  on  $\mathbb{Z}^{d-1}$  specified by the weights  $h(n, k)$ ,

$$\mathbb{P}_n(k) \triangleq \frac{h(n, k)}{\sum_{j \in \mathbb{Z}^{d-1}} h(n, j)}.$$

Our study of the local limit properties of  $\mathbb{P}_n$  is based on the specific renewal structure of  $h$ -connectivities, which, following [7], we proceed to describe:

DEFINITION. Given  $n \geq 1$ , the restriction of a percolation configuration  $\eta$  on  $\mathcal{S}_{0,n}$  and a point  $x = (n, k) \in \mathcal{H}_n$  let us say that  $x$  is  $f$ -connected to the origin,  $\{0 \overset{f}{\leftrightarrow} x\}$ , if the following hold:

- (i)  $x$  is  $h$ -connected to the origin;
- (ii) for every  $m = 1, \dots, n - 1$ ,

$$\#(\mathbf{C}_{\{0,x\}}^n \cap \mathcal{H}_m) > 1.$$

Notice that for  $n = 1$  the notions of  $f$ - and  $h$ -connectivities coincide.

Define

$$f(n, k) \triangleq \mathbb{P}_p(0 \overset{f}{\leftrightarrow} x).$$

The event  $\{0 \overset{h}{\leftrightarrow} x\}$  depends only on the percolation configuration inside the strip  $\mathcal{S}_{0,n}$ . Using the disjoint decomposition of  $\{0 \overset{h}{\leftrightarrow} x\}$  with respect to the smallest index  $m$  satisfying  $\#(\mathbf{C}_{\{0,x\}}^n \cap \mathcal{H}_m) = 1$ , we, in view of the shift invariance of  $\mathbb{P}_p$ , obtain [7]

$$(1.12) \quad h(n, k) = \frac{1}{(1-p)^{2(d-1)}} \sum_{m=1}^n \sum_{l \in \mathbb{Z}^d} f(m, l) h(n-m, k-l),$$

$$h(0, k) \triangleq (1-p)^{2(d-1)} \delta_0(k).$$

Normalizing  $\tilde{h} = h/(1-p)^{2(d-1)}$  and  $\tilde{f} = f/(1-p)^{2(d-1)}$ , we arrive at the usual  $(d-1)$ -dimensional renewal relation

$$(1.13) \quad \tilde{h}(n, k) = \sum_{m=1}^n \sum_{l \in \mathbb{Z}^d} \tilde{f}(m, l) \tilde{h}(n-m, k-l) \quad \text{and} \quad \tilde{h}(0, k) = \delta_0(k).$$

1.3. *Local limit results and separation of masses.* An appropriate general local limit theorem for  $d$ -dimensional renewal arrays (1.13) has been established in [12]. Given  $\hat{t} \in \mathbb{R}^{d-1}$ , define moment generating functions

$$\mathbb{H}_n(\hat{t}) \triangleq \sum_{k \in \mathbb{Z}^{d-1}} \tilde{h}(n, k) e^{(\hat{t}, k)_{d-1}} \quad \text{and} \quad \mathbb{F}_n(\hat{t}) \triangleq \sum_{k \in \mathbb{Z}^{d-1}} \tilde{f}(n, k) e^{(\hat{t}, k)_{d-1}}.$$

Of course,  $\mathbb{H}_n(\hat{t}) > \mathbb{F}_n(\hat{t})$ , and both sums above diverge for sufficiently large values of  $\|\hat{t}\|_{d-1}$ . In any case, however,

$$(1.14) \quad \mathbb{H}_n(\hat{t}) = \sum_{m=1}^n \mathbb{F}_m(\hat{t}) \mathbb{H}_{n-m}(\hat{t}).$$

DEFINITION. For every  $\hat{t} \in \mathbb{R}^{d-1}$  the masses  $m_{\mathbb{H}}(\hat{t})$  and  $m_{\mathbb{F}}(\hat{t})$  are defined as

$$m_{\mathbb{H}}(\hat{t}) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{H}_n(\hat{t}) \quad \text{and} \quad m_{\mathbb{F}}(\hat{t}) \triangleq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{F}_n(\hat{t}).$$

Notice that, by the renewal property (1.14), the limit in the above definition of  $m_{\mathbb{H}}$  always exists (though it could be infinite). Furthermore, both  $m_{\mathbb{H}}$  and  $m_{\mathbb{F}}$  are convex functions on  $\mathbb{R}^{d-1}$ , and, of course,  $m_{\mathbb{F}} \leq m_{\mathbb{H}}$ . Let us use  $\mathcal{D}_{\mathbb{H}}$  to denote the effective domain of  $m_{\mathbb{H}}$ ,

$$\mathcal{D}_{\mathbb{H}} \triangleq \{\hat{t} \in \mathbb{R}^{d-1} : m_{\mathbb{H}}(\hat{t}) < \infty\}.$$

Because of (1.3), (1.4) and an obvious bound  $[x = (n, k)]$

$$e^{-c_2 \|x\|_d} \leq h(n, k) \leq \mathbb{P}_p(0 \leftrightarrow x),$$

the convex set  $\mathcal{D}_{\mathbb{H}}$  is bounded and has a nonempty interior  $\text{int}(\mathcal{D}_{\mathbb{H}})$ ,  $0 \in \text{int}(\mathcal{D}_{\mathbb{H}})$ .

We finally formulate the separation-of-masses-type condition to which we have referred on several occasions in the first subsection.

DEFINITION. Let us say that the mass-gap condition is satisfied at a point  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$  if

$$(1.15) \quad m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t}).$$

We rely on the following local limit theorem [12] for multidimensional renewal arrays:

THEOREM B. Assume that, for every point  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ , the following hold:

- (i) The mass-gap condition (1.15) is satisfied.
- (ii) The Hessian  $D^2 m_{\mathbb{H}}(\hat{t})$  is nondegenerate,

$$(1.16) \quad \det(D^2 m_{\mathbb{H}}(\hat{t})) \neq 0.$$



Then, for every  $\alpha \in \mathbb{R}_+$ ,

$$(1.17) \quad h(n, k) = \frac{\Lambda_p(\vec{n}(x))}{\sqrt{(2\pi \|x\|_d)^{d-1}}} e^{-\xi_p(x)} (1 + o(1)),$$

uniformly in  $x = (n, k) \in \mathcal{C}_\alpha$ . As before  $\vec{n}(x)$  is the unit vector in the direction of  $x$ , and  $\Lambda_p$  is a positive real analytic function on  $\mathbb{S}_+^{d-1}$ .

REMARK. The proof of Theorem B as stated above relies on the lattice symmetries of  $\mathbb{P}_p$  [cf. (1.20) below, the general Theorem 2.1 in [12] and the argument on pages 343–344 there].

The main impact of the mass-gap condition (1.15) at  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$  is the validity of the Lee–Yang type analyticity structure of moment generating functions  $\mathbb{H}_n$  in a  $(d - 1)$ -dimensional complex neighborhood of  $\hat{t}$ . In particular [12], for every  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ ,

$$(1.18) \quad m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t}) \implies m_{\mathbb{H}}(\hat{z}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{H}_n(\hat{z}),$$

in the sense of analytic functions on a  $\mathbb{C}^{d-1}$ -neighborhood of  $\hat{t}$ . Thus,  $m_{\mathbb{H}}$  is real analytic in an  $\mathbb{R}^{d-1}$ -neighborhood of  $\hat{t}$ , as soon as the mass-gap condition (1.15) is satisfied. Together with the nondegeneracy condition (1.16) such results enable classical local limit analysis of the  $\hat{t}$ -tilted measure (cf. [12], Section 2).

The main technical result of this paper is stated as follows.

THEOREM C. *Let  $d \geq 2$  and  $p < p_c(d)$ . Then both the mass-gap condition (1.15) and the nondegeneracy condition (1.16) hold for every  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ .*

The crux of the matter is to prove the mass-gap. Once this is accomplished, the nondegeneracy follows by a simple conditional variance bound.

1.4. *Geometry of Wulff shapes and equidecay profiles.* Analytic properties of connectivities have useful geometric counterparts: by (1.3),  $\xi_p$  is an equivalent norm on  $\mathbb{R}^d$ ,

$$(1.19) \quad 0 < \min_{x \in \mathbb{S}^{d-1}} \xi_p(x) \leq \max_{x \in \mathbb{S}^{d-1}} \xi_p(x) < \infty.$$

Let us denote the corresponding  $\xi_p$ -unit ball as  $\mathbf{U}^p$ ,

$$\mathbf{U}^p = \{x \in \mathbb{R}^d : \xi_p(x) \leq 1\}.$$

We use the term equidecay profiles  $a \partial \mathbf{U}^p$  for the boundaries of the  $\xi_p$ -balls

$$a \mathbf{U}^p = \{x \in \mathbb{R}^d : \xi_p(x) \leq a\}.$$

Similarly, because of (1.19),  $\xi_p$  is the support function of the compact convex set

$$\mathbf{K}^p \triangleq \bigcap_{n \in \mathbb{S}^{d-1}} \{t \in \mathbb{R}^d : (t, n)_d \leq \xi_p(n)\},$$

with nonempty interior  $\text{int}\{\mathbf{K}^p\}$ ,  $0 \in \text{int}\{\mathbf{K}^p\}$ .

Furthermore, the function  $\xi_p$  is symmetric with respect to permutations and reflections across coordinate hyperplanes,

$$(1.20) \quad \xi_p(\varepsilon_1 x_{\pi(1)}, \dots, \varepsilon_d x_{\pi(d)}) = \xi_p(x_1, \dots, x_d),$$

for every  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , every permutation  $\{\pi(1), \dots, \pi(d)\}$  of  $\{1, \dots, d\}$  and every collection of numbers  $\varepsilon_i = \pm 1$ ,  $i = 1, \dots, d$ . Consequently, both the polar shape  $\mathbf{K}^p$  and the equidecay profile  $\mathbf{U}^p$  enjoy the same symmetries as  $\xi_p$ . These symmetries will substantially facilitate some of our arguments below.

We shall refer to  $\mathbf{K}^p$  as the polar body or as the Wulff shape (which it really is in the case of two dimensions). The first appellation, however, is justified by the fact that the convex bodies  $\mathbf{U}^p$  and  $\mathbf{K}^p$  are in the polar relation: for every  $t \in \partial\mathbf{K}^p$  and  $x \in \partial\mathbf{U}^p$ ,

$$(1.21) \quad \xi_p(x) = 1 = \max_{y \in \mathbf{U}^p} (t, y)_d = \max_{s \in \mathbf{K}^p} (s, x)_d.$$

Given  $x \in \mathbb{R}^d \setminus \{0\}$ , let us say that a point  $t \in \partial\mathbf{K}^p$  is polar to  $x$  if

$$(1.22) \quad (t, x)_d = \xi_p(x) = \max_{s \in \mathbf{K}^p} (s, x)_d.$$

Geometrically,  $t$  is orthogonal to a tangent hyperplane to the equidecay profile  $\xi_p(x) \partial\mathbf{U}^p$  passing through  $x$ . A priori  $x$  might have many different polar points. The collection of these points, however, always forms a convex set. Therefore nonuniqueness of polar points at  $x$  is tantamount to an existence of a flat facet on  $\partial\mathbf{K}^p$ .

There is, of course, an intimate relation between the geometry of polar sets and the mass  $m_{\mathbb{H}}$ .

PROPOSITION 1.2. *If  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}}) \subset \mathbb{R}^{d-1}$ , then  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t}) \in \partial\mathbf{K}^p \subset \mathbb{R}^d$ .*

PROOF. First of all, by the usual LD-style application of the Hölder inequality, there exist  $\delta > 0$  and a constant  $A_\delta < \infty$  such that

$$(1.23) \quad \sum_{\|k\|_{d-1} \geq A_\delta n} e^{(k, \hat{t})_{d-1}} h(n, k) \leq e^{n(m_{\mathbb{H}}(\hat{t}) - \delta)}.$$

Thus, using (1.10) with  $\alpha = A_\delta$ , we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\{ \sum_{\|k\|_{d-1} \leq A_\delta n} e^{-\xi_p(n, k) + (k, \hat{t})_{d-1} - nm_{\mathbb{H}}(\hat{t})} \right\} = 0.$$

It follows that there exists  $x \in \mathbb{R}^d$  with

$$\sqrt{\sum_{i=2}^d x_i^2} \leq A_\delta x_1,$$

satisfying

$$\xi_p(x) = (t, x)_d, \quad \text{where } t = (-m_{\mathbb{H}}(\hat{t}), \hat{t}).$$

Hence,  $t \in \partial \mathbf{K}^p$ .  $\square$

REMARK. Notice that because of the lattice symmetries (1.20) the above proposition implies that the point  $\tilde{t} \triangleq (0, \hat{t})$  belongs to the convex set  $\mathbf{K}^p$  whenever  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ . Since, by (1.3),  $\mathbf{K}^p$  has a nonempty interior, we conclude

$$(1.24) \quad \hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}}) \iff \tilde{t} \triangleq (0, \hat{t}) \in \text{int}(\mathbf{K}^p).$$

In particular,  $-m_{\mathbb{H}}(\hat{t}) > 0$  for every interior point  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ .

Local validity of the assumptions of Theorem A leads to nice analytic properties of the boundary  $\partial \mathbf{K}^p$ .

PROPOSITION 1.3. *Let  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ . Assume that  $m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t})$  and that  $\det(D^2 m_{\mathbb{H}}(\hat{t})) \neq 0$ . Then  $\partial \mathbf{K}^p$  is analytic and strictly convex in a neighborhood of  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$ .*

As in [12], Theorem C implies the following result on the geometry of sets  $\mathbf{K}^p$  and  $\mathbf{U}^p$ :

THEOREM D. *Assume that the assumptions of Theorem B are satisfied. Then  $m_{\mathbb{H}}$  is a real analytic function on  $\text{int}(\mathcal{D}_{\mathbb{H}})$ . In addition,  $m_{\mathbb{H}}$  is strictly convex and steep:*

$$(1.25) \quad \bigcup_{\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})} \nabla m_{\mathbb{H}}(\hat{t}) = \mathbb{R}^{d-1}.$$

Furthermore, both  $\mathbf{K}^p$  and  $\mathbf{U}^p$  are strictly convex bodies with analytic boundaries  $\partial \mathbf{K}^p$  and  $\partial \mathbf{U}^p$ . Finally, the Gaussian curvatures of both  $\partial \mathbf{U}^p$  and  $\partial \mathbf{K}^p$  are everywhere strictly positive.

REMARK. In particular, Theorem D implies that the two-dimensional Wulff shape is strictly convex and analytic for every supercritical (dual) value  $p^* > 1/2$ . This enables us to refine the results of [6] and [3] along the lines of [13].

**2. Coarse graining.** Assume that 0 and  $x$  are connected. We use  $\mathbf{C}_{\{0,x\}}$  to denote the corresponding common cluster. In this section we study the coarse-grained structure of  $\mathbf{C}_{\{0,x\}}$ . The coarse graining is constructed around a self-avoiding path  $\gamma: 0 \rightarrow x$  (trunk) lying inside  $\mathbf{C}_{\{0,x\}}$  and disjoint leaves growing from this trunk. What we roughly show is that on appropriate renormalization scales the corresponding tree skeleton has, with an overwhelming probability, a nice localized structure, in the sense that the trunk skeleton of  $\gamma$  goes in a more or less straight way, with only a negligible number of backtracks, from 0 to  $x$ , whereas the renormalized leaves become very sparse.

The coarse graining itself will boil down to a specific way to cover the cluster  $\mathbf{C}_{\{0,x\}}$  by the balls of the type  $k\mathbf{U}^p(y)$ , where  $k$  is the current renormalization scale, and

$$(2.1) \quad k\mathbf{U}^p(y) \triangleq (y + k\mathbf{U}^p) \cap \mathbb{Z}^d.$$

2.1. *Alexander’s surcharge function.* Given  $t \in \partial\mathbf{K}^p$  define the surcharge function  $\mathfrak{s}_t$  in the direction  $t$  as

$$\mathfrak{s}_t(x) \triangleq \xi_p(x) - (t, x)_d.$$

By (1.21)  $\mathfrak{s}_t$  is always nonnegative and  $\mathfrak{s}_t(x) = 0$  only if  $t$  is polar to  $x$ . Notice that our definition here differs from the original one given in [2], but our approach is definitely inspired by Alexander’s point of view on the relevant renormalization procedures.

PROPOSITION 2.1. *Let  $x \in \mathbb{Z}^d$ . Set  $y_0 = 0$  and  $y_n = x$ . Then, for every  $t \in \partial\mathbf{K}^p$  and for any collection  $\{y_1, \dots, y_{n-1}\}$  of points from  $\mathbb{Z}^d$ ,*

$$(2.2) \quad \begin{aligned} &\mathbb{P}_p(0 \leftrightarrow y_1 \circ y_1 \leftrightarrow y_2 \circ \dots \circ y_{n-1} \leftrightarrow x) \\ &\leq \exp \left\{ - \sum_{k=0}^{n-1} \mathfrak{s}_t(y_{k+1} - y_k) - (t, x)_d \right\}. \end{aligned}$$

PROOF. By the BK inequality,

$$\begin{aligned} &\mathbb{P}_p(0 \leftrightarrow y_1 \circ y_1 \leftrightarrow y_2 \circ \dots \circ y_{n-1} \leftrightarrow x) \\ &\leq \exp \left\{ - \sum_{k=0}^n \xi_p(y_{k+1} - y_k) \right\} \\ &= \exp \left\{ - \sum_{k=0}^{n-1} \mathfrak{s}_t(y_{k+1} - y_k) - (t, x)_d \right\}. \quad \square \end{aligned}$$

2.2. *Coarse graining of a self-avoiding path  $\gamma$ .* Fix a number  $k \in \mathbb{R}_+$ . Given  $x \in \mathbb{Z}^d$  and a self-avoiding lattice path  $\gamma: 0 \leftrightarrow x$ ,

$$\gamma = \{\gamma(1), \dots, \gamma(n)\}, \quad \gamma(0) = 0 \quad \text{and} \quad \gamma(n) = x,$$

we construct the  $k$ -skeleton  $\gamma^{(k)} = \{y_0, \dots, y_m\}$  of  $\gamma$  as follows:

*Step 0.*  $y_0 = 0$ .

*Step 1.*  $n_1 = \min\{l: \gamma(l) \notin k\mathbf{U}^p(y_0)\} \wedge n$ . If  $n_1 = n$ , then  $m = 1$ ,  $y_m = x$  and the process terminates. Otherwise set  $y_1 = \gamma(n_1)$ ,  $i = 2$  and proceed to the next step.

*Step  $i$ .*  $n_i = \min\{l > n_{i-1}: \gamma(l) \notin k\mathbf{U}^p(y_{i-1})\} \wedge n$ . If  $n_i = n$ , then  $m = i$ ,  $y_m = x$  and the process terminates. Otherwise set  $y_i = \gamma(n_i)$ , and proceed to *Step* ( $i + 1$ ).

Let us use the symbol  $\gamma \sim \gamma^{(k)}$  to denote the fact that  $\gamma^{(k)}$  is the  $k$ -skeleton of  $\gamma$ . Given a  $k$ -skeleton  $\gamma^{(k)}$  define the event

$$\{0 \overset{\gamma^{(k)}}{\leftrightarrow} x\} \triangleq \{0 \text{ is connected to } x \text{ by a self-avoiding path } \gamma; \gamma \sim \gamma^{(k)}\}.$$

As follows immediately from Proposition 2.1, for every  $x \in \mathbb{Z}^d$ , every  $t \in \partial\mathbf{K}^p$ , each renormalization scale  $k \in \mathbb{R}_+$  and every  $k$ -skeleton  $\gamma^{(k)}$ ,

$$(2.3) \quad \mathbb{P}_p(0 \overset{\gamma^{(k)}}{\leftrightarrow} x) \leq \exp\left\{-\sum_{l=1}^m \mathfrak{s}_t(y_{l+1} - y_l) - (t, x)_d\right\}.$$

2.3. *Surcharge cones and typical  $k$ -trunks.* Given  $t \in \partial\mathbf{K}^p$  and  $\varepsilon > 0$  let us define the surcharge cone  $\mathcal{C}_\varepsilon(t)$  as

$$(2.4) \quad \mathcal{C}_\varepsilon(t) \triangleq \{x \in \mathbb{R}^d: \mathfrak{s}_t(x) \leq \varepsilon \xi_p(x)\} = \{x \in \mathbb{R}^d: (t, x) \geq (1 - \varepsilon) \xi_p(x)\}.$$

We quantify  $k$ -skeletons  $\gamma^{(k)}: 0 \overset{\gamma^{(k)}}{\leftrightarrow} x$ ,  $\gamma^{(k)} = \{y_0, \dots, y_m\}$ , by the number of  $k$ -increments of  $\gamma^{(k)}$ ,

$$\mathfrak{g}^{(k)} \triangleq m = \#\{\gamma^{(k)}\},$$

and by the number of costly  $\varepsilon$ -backtracking full increments with respect to the surcharge cone  $\mathcal{C}_\varepsilon(t)$ ,

$$\#_{t,\varepsilon}(\gamma^{(k)}) = \#\{1 \leq l \leq \mathfrak{g}^{(k)} - 1: y_l - y_{l-1} \notin \mathcal{C}_\varepsilon(t)\}.$$

Notice that on the  $k$ th renormalization scale a “bad” increment  $y_l - y_{l-1} \notin \mathcal{C}_\varepsilon(t)$  automatically satisfies

$$(2.5) \quad \mathfrak{s}_t(y_l - y_{l-1}) \geq \varepsilon k.$$

LEMMA 2.2. For every  $x \in \mathbb{Z}^d$ ,  $\varepsilon > 0$ ,  $t \in \partial\mathbf{K}^p$ ,  $\delta \in \mathbb{R}_+$  and every coarse-graining scale  $k$ ,

$$(2.6) \quad \begin{aligned} \mathbb{P}_p\left(\gamma^{(k)}: \#_{t,\varepsilon}(\gamma^{(k)}) \geq \frac{\delta\|x\|_d}{k}; 0 \overset{\gamma^{(k)}}{\leftrightarrow} x\right) \\ \leq \exp\left\{c_1 \frac{\log k}{k} \|x\|_d - \delta\varepsilon\|x\|_d - (x, t)_d\right\}. \end{aligned}$$

PROOF. By the very construction of  $k$ -skeletons and by the BK inequality (2.3),

$$\mathbb{P}_p(0 \overset{\gamma^{(k)}}{\leftrightarrow} x) \leq \exp\{-\mathfrak{g}^{(k)}k\},$$

uniformly in  $k$ -skeletons  $\gamma^{(k)}$ .

On the other hand, the number of skeletons

$$\#\{\gamma^{(k)}: 0 \overset{\gamma^{(k)}}{\leftrightarrow} x \text{ and } \#(\gamma^{(k)}) = m\}$$

is bounded above by

$$(2.7) \quad (c_1 k^{d-1})^m = \exp\{c_2 m \log k\}.$$

Consequently,

$$\mathbb{P}_p(\gamma^{(k)}: 0 \overset{\gamma^{(k)}}{\leftrightarrow} x; \mathfrak{g}^{(k)} \geq m) \leq \exp\left\{-\frac{m}{2}k\right\},$$

as soon as  $k$  is sufficiently large. Since, by (1.19),  $\xi_p(x) \leq c_3\|x\|_d$ , we infer that there exist two positive constants  $c_4 = c_4(d, p)$  and  $c_5 = c_5(d, p)$  such that

$$(2.8) \quad \begin{aligned} \mathbb{P}_p\left(\gamma^{(k)}: 0 \overset{\gamma^{(k)}}{\leftrightarrow} x; \mathfrak{g}^{(k)} \geq c_4 \frac{\|x\|_d}{k}\right) &\leq \exp\{-c_5\|x\|_d - \xi_p(x)\} \\ &\leq \exp\{-c_5\|x\|_d - (x, t)_d\}, \end{aligned}$$

uniformly in  $x \in \mathbb{Z}^d$  and  $t \in \partial\mathbf{K}^p$ . As a result we can restrict attention only to the  $k$ -skeletons  $\gamma^{(k)}$  satisfying

$$(2.9) \quad \mathfrak{g}^{(k)} = \#(\gamma^{(k)}) \leq c_4\|x\|_d/k.$$

By (2.7) the number of such skeletons is bounded above by

$$\exp\left\{c_6 \frac{\log k}{k} \|x\|_d\right\}.$$

Finally, (2.3) and the lower bound (2.5) on the surcharge value of bad increments imply that, for any  $t \in \partial\mathbf{K}^p$ ,  $\varepsilon > 0$  and any skeleton  $\gamma^{(k)}$ ,

$$\mathbb{P}_p(0 \overset{\gamma^{(k)}}{\leftrightarrow} x) \leq \exp\{-c_7\varepsilon k\#_{t,\varepsilon}(\gamma^{(k)}) - (t, x)_d\}.$$

Patching the latter two estimates together we arrive at the conclusion (2.6) of Lemma 2.2.  $\square$

2.4. *Coarse graining of clusters*  $\mathbf{C}_{\{0,x\}}$ . Recall that if 0 is connected to  $x$ , then we denote the corresponding common connected cluster as  $\mathbf{C}_{\{0,x\}}$ . With each realization of  $\mathbf{C}_{\{0,x\}}$  we associate, on every coarse-graining scale  $k$ , a treelike subset  $\mathbf{C}_{\{0,x\}}^{(k)} \subset \mathbb{Z}^d$  such that  $\mathbf{C}_{\{0,x\}}$  lies inside the  $k$ -neighborhood of  $\mathbf{C}_{\{0,x\}}^{(k)}$  in the sense of the  $\xi_p$ -distance, that is,

$$(2.10) \quad \forall y \in \mathbf{C}_{\{0,x\}}, \quad \min_{z \in \mathbf{C}_{\{0,x\}}^{(k)}} \xi_p(y - z) \leq k.$$

The sets  $\mathbf{C}_{\{0,x\}}^{(k)}$  will always be composed of shifts of the lattice  $\xi_p$ -balls  $k\mathbf{U}^p$ :

$$(2.11) \quad \mathbf{C}_{\{0,x\}}^{(k)} = \bigcup_{y \in \mathfrak{T}^{(k)}} k\mathbf{U}^p(y),$$

where  $k\mathbf{U}^p(y)$  has been defined in (2.1).

It is convenient to describe the construction of  $\mathbf{C}_{\{0,x\}}^{(k)}$  or, equivalently, of its tree skeleton  $\mathfrak{T}^{(k)}$  algorithmically. For technical reasons we would like to construct  $\mathfrak{T}^{(k)}$  unambiguously; later this will enable a disjoint splitting of the relevant percolation events with respect to different possible tree skeletons.

*Choosing the self-avoiding trunk*  $\gamma^{(k)}$ . Consider all possible self-avoiding paths  $\gamma$  leading from 0 to  $x$  within  $\mathbf{C}_{\{0,x\}}$ , and let  $\gamma^{(k)} = \{y_0, \dots, y_m\}$  be the corresponding  $k$ -skeletons of  $\gamma$ . Of all these  $\gamma^{(k)}$  we first choose skeletons of minimal length  $\mathfrak{g}^{(k)}$  and, provided that there are several such minimal-length skeletons, we further choose the minimal one of them, say in the sense of lexicographical order.

We shall refer to the resulting  $\gamma^{(k)} = (y_0, \dots, y_{\mathfrak{g}^{(k)}})$  as the self-avoiding trunk of  $\mathbf{C}_{\{0,x\}}$  on the  $k$ th renormalization scale.

Define  $\mathfrak{T}^{(k)} = \{y_0, \dots, y_m\}$  and, accordingly, define  $\mathbf{C}_{\{0,x\}}^{(k)}$  by (2.11). If (2.10) is already satisfied, then stop.

Otherwise proceed to the following update step:

*Update step.* Reorder all the sites of  $\mathfrak{T}^{(k)}$ , for instance again according to lexicographical order;  $\mathfrak{T}^{(k)} = \{z_1, \dots, z_{\mathfrak{t}^{(k)}}\}$ , where  $\mathfrak{t}^{(k)}$  is used to denote the cardinality of  $\mathfrak{T}^{(k)}$ ,

$$\mathfrak{t}^{(k)} \triangleq \#\{\mathfrak{T}^{(k)}\}.$$

Set  $l := 1$ .

*Step l* ( $l \leq \mathfrak{t}^{(k)}$ ). Screen the  $\mathbb{Z}^d$  lattice points attached to  $k\partial\mathbf{U}^p(z_l)$  in the lexicographical order. If there exists  $z \in k\partial\mathbf{U}^p(z_l)$  such that one can find a self-avoiding open path  $\gamma_z$  leading from  $z$  to  $\partial k\mathbf{U}^p(z)$  inside  $\mathbb{Z}^d \setminus \mathbf{C}_{\{0,x\}}^{(k)}$ , then add  $z$  to  $\mathfrak{T}^{(k)}$ , that is, set

$$\mathfrak{T}^{(k)} := \mathfrak{T}^{(k)} \cup \{z\} \quad \text{and} \quad \mathbf{C}_{\{0,x\}}^{(k)} := \mathbf{C}_{\{0,x\}}^{(k)} \cup k\mathbf{U}^p(z),$$

and return to the update step.

Otherwise set  $l := l + 1$  and proceed to Step 1.

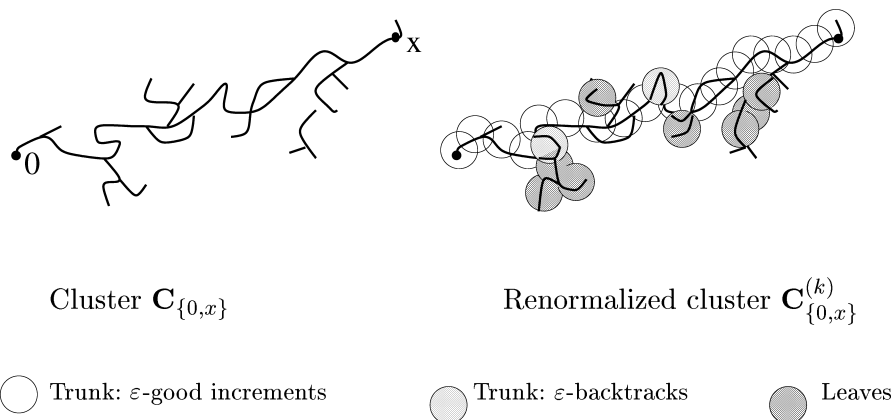


FIG. 1. Construction of the renormalized clusters  $\mathbf{C}_{\{0,x\}}^{(k)}$ .

Step  $(t^{(k)} + 1)$ . Stop. We shall say that the resulting  $\mathfrak{T}^{(k)}$  is the tree skeleton of  $\mathbf{C}_{\{0,x\}}$  on the  $k$ th renormalization scale, and denote the corresponding percolation event either as  $\mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}$  or as  $\{0 \overset{\mathfrak{T}^{(k)}}{\leftrightarrow} x\}$ .

Clearly, once Step  $(t^{(k)} + 1)$  is reached, condition (2.10) is satisfied.

We should still provide an argument that with probability 1 the process terminates in a finite number of steps. This will follow from a much more precise estimate on the number  $t^{(k)}$ , which we proceed to derive.

2.5. *Renormalization: typical tree skeletons.* By construction, the tree skeleton  $\mathfrak{T}^{(k)}$  is composed of the original self-avoiding trunk  $\gamma^{(k)}$  and of the set of additional leaves  $\mathfrak{L}^{(k)}$ ,

$$\mathfrak{T}^{(k)} = \gamma^{(k)} \cup \mathfrak{L}^{(k)}.$$

Thus, the corresponding set  $\mathbf{C}_{\{0,x\}}^{(k)}$  [defined in (2.11)] contains the following:

1. a self-avoiding path  $\gamma: 0 \leftrightarrow x$  with a  $k$ -trunk  $\gamma^{(k)}$ ;
2. for each leaf  $z \in \mathfrak{L}^{(k)}$ , a self-avoiding path  $\gamma_z$  leading from  $z$  to  $k \partial \mathbf{U}^p(z)$ ;
3. by construction all these  $\gamma_z$  are disjoint and, moreover, each such  $\gamma_z$  is disjoint from  $\gamma$ .

By the BK inequality the probability of a given tree skeleton  $\mathfrak{T}^{(k)}$  is bounded above as

$$(2.12) \quad \mathbb{P}_p(\mathfrak{T}^{(k)}) \leq \mathbb{P}_p(\gamma^{(k)}) e^{-c_7 l^{(k)} k},$$

where the number of leaves  $l^{(k)}$  is defined by

$$l^{(k)} \triangleq \#(\mathfrak{L}^{(k)}) = t^{(k)} - g^{(k)}.$$



LEMMA 2.3. *There exists a constant  $c_8 = c_8(d, p)$ , such that, for every  $\delta > 0$  fixed,*

$$(2.13) \quad \mathbb{P}_p\left(\mathfrak{L}^{(k)}: \iota^{(k)} \geq \frac{\delta \|x\|_d}{k}; 0 \leftrightarrow x\right) \leq \exp\{-\xi_p(x) - c_8 \delta \|x\|_d\},$$

*uniformly in  $k$  and for  $x$  sufficiently large.*

PROOF. Given the skeleton  $\gamma^{(k)}$  of a self-avoiding path  $\gamma: 0 \leftrightarrow x$ , let us estimate the number of ways one can attach  $l$  leaves to the “trunk”  $\gamma = \{y_0, \dots, y_m\}$ :

To each point  $y \in \gamma^{(k)}$  one can attach at most

$$n = n(k, d) \triangleq c_9 k^{d-1}$$

new points from  $k \partial U^p$ . To each of these new points one can attach another point again in, at most,  $n$  ways, and so on. Thus, the number of ways to attach  $l_i$  leaves to a point  $y_i \in \gamma^{(k)}$  is bounded above by a number of connected trees with  $l_i$  vertices and branching ratio  $n$ . By the well-known estimate of Kesten on the number of lattice animals (see, e.g., [11]) the latter is bounded above by

$$(2.14) \quad \left\{ \max_{q \in (0,1)} q^{l_i} (1 - q)^{n l_i} \right\}^{-1} \leq \exp\{c_{10} l_i \log n\} \leq \exp\{c_{11} l_i \log k\}.$$

Finally, the number of ways to distribute  $l$  leaves to  $m$  different branches of the trunk  $\gamma^{(k)}$  is bounded above by

$$(2.15) \quad \binom{m+l}{l} \approx \exp\left\{l \log\left(1 + \frac{m}{l}\right) + m \log\left(1 + \frac{l}{m}\right)\right\}.$$

By (2.8) it is no loss to assume that the the cardinality  $m$  of the  $k$ -skeleton  $\gamma^{(k)}$  satisfies  $m \leq c_5 \|x\|/k$ . Thus,

$$\binom{m+l}{l} \leq \exp\left\{c_{12} l \log \frac{1}{\delta}\right\},$$

whenever  $l \geq \delta \|x\|/k$ .

The bounds (2.12), (2.14) and (2.15) and the estimate (2.7) on the number of different trunks  $\gamma^{(k)}$  of the maximal cardinality  $c_5 \|x\|_d/k$  readily imply that

$$\begin{aligned} & \mathbb{P}_p\left(\#\mathfrak{L}^{(k)} \geq \frac{\delta \|x\|_d}{k}; 0 \leftrightarrow x\right) \\ & \leq \exp\left\{-\xi_p(x) - c_7 \delta \|x\|_d + c_{11} \frac{\log(1/\delta)}{k} + c_{13} \frac{\log k}{k} \|x\|_d\right\}, \end{aligned}$$

and the conclusion (2.13) of the lemma follows.  $\square$

**3. Separation of masses.** As we have seen in Proposition 1.2, any point  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$  gives rise to the point  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$  on the boundary  $\partial\mathbf{K}^p$ . To prove the mass-gap at  $\hat{t}$  we shall fix this  $t$  and use it to quantify the surcharge costs of the increments, as defined in the framework of the renormalization results of the preceding section. Such an approach happens to be useful, if at some surcharge value  $\varepsilon > 0$ , all  $\varepsilon$ -good increments have strictly positive  $\mathbf{e}_1$ -component. We shall then argue that such a “forward” structure of typical  $\mathfrak{T}^{(k)}$ -tree skeletons necessarily decouples the event  $\{0 \overset{f}{\leftrightarrow} x\}$  into intersection of many localized independent subevents, each of the latter having probability strictly less than 1.

The appropriate forward condition on the increments is formulated in Section 3.1. Geometrically it boils down to certain strict convexity requirements on the connectivity function  $\xi_p$ . Because of lattice symmetries (1.20) of  $\xi_p$  and, accordingly, of  $\mathbf{K}^p$  the proof that such a condition always holds for the interior points  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$  is essentially trivial; see Lemma 3.1 below.

A more robust approach, which does not rely on lattice symmetries, is explained and worked out in Section 4 in the course of giving a “direct” proof of Theorem A.

3.1. *Positive cone property.* Let  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ . We say that  $\hat{t}$  satisfies the positive cone property if there exists  $\varepsilon > 0$  such that

$$(3.1) \quad \alpha(t, \varepsilon) \triangleq \min_{x \in \mathcal{C}_\varepsilon(t) \setminus \{0\}} \frac{x_1}{\|x\|_d} > 0,$$

where  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$  and  $\mathcal{C}_\varepsilon(t)$  is the surcharge cone defined in (2.4).

Informally, the positive cone condition is satisfied if all  $\mathfrak{s}_t$ -reasonable increments  $x$  have a nontrivial forward component in the direction of the axis  $\mathbf{e}_1$ . By continuity, the positive cone condition is satisfied, iff

$$(3.2) \quad \min\{x_1 : x \in \partial\mathbf{U}^p \text{ and } \mathfrak{s}_t(x) = 0\} > 0.$$

LEMMA 3.1. *The positive cone property is satisfied for every  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ .*

PROOF. By the remark following the proof of Proposition 1.2, the point  $\tilde{t} \triangleq (0, \hat{t})$  belongs to  $\text{int}(\mathbf{K}^p)$ , as soon as  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ . With  $t$  defined as  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$ , let us assume that there exists  $x = (x_1, \dots, x_d) \in \partial\mathbf{U}^p$  such that  $(x, t)_d = \xi_p(x)$  and  $x_1 = 0$ . In this case, however,  $(x, \tilde{t})_d = \xi_p(x)$ , as well. This, by (1.22), implies that  $\tilde{t} \in \partial\mathbf{K}^p$ , a contradiction.  $\square$

Our main result in this section is stated as follows:

LEMMA 3.2. *If  $\hat{t}$  satisfies the positive cone property, then  $m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t})$ .*

Consequently, the mass-gap condition is satisfied at any interior point  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ . This sets the stage for local limit analysis of the multidimensional renewal relation (1.12) along the lines of Theorem B; see Section 3.4.

3.2. *Reduction to regular tree skeletons.* We wish to show that if  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$  satisfies the positive cone condition (3.1), then there exists  $\nu > 0$  such that

$$(3.3) \quad e^{-nm_{\mathbb{H}}(\hat{t})} \sum_k \mathbb{P}_p(0 \overset{f}{\leftrightarrow} (n, k)) e^{(\hat{t}, k)_{d-1}} \leq e^{-\nu n},$$

uniformly in  $n$  sufficiently large.

By Proposition 1.2 the point  $t \triangleq (-m_{\mathbb{H}}(\hat{t}), \hat{t}) \in \partial \mathbf{K}^p$ . With the renormalization estimates of the preceding section in mind, we rewrite (3.3) as

$$(3.4) \quad \sum_{x \in \mathcal{H}_n} \mathbb{P}_p(0 \overset{f}{\leftrightarrow} x) e^{(t, x)_d} \leq e^{-\nu n},$$

where, as before,  $\mathcal{H}_n$  denotes the lattice hyperplane (1.8).

In fact, we shall prove a slightly more general claim [see (3.5) below]:

DEFINITION. Given a point  $x = (n, k) \in \mathcal{H}_n$ , let us say that it is  $d$ -connected to the origin,  $\{0 \overset{d}{\leftrightarrow} x\}$ , if  $\{0 \leftrightarrow x\}$ , and the common cluster  $\mathbf{C}_{\{0, x\}}$  satisfies

$$\#(\mathbf{C}_{\{0, x\}} \cap \mathcal{H}_m) > 1 \quad \forall m = 1, \dots, n-1.$$

Set

$$d(n, k) = \mathbb{P}_p(0 \overset{d}{\leftrightarrow} x).$$

DEFINITION. Given  $x = (n, k)$  and a cluster  $\mathbf{C}_{\{0, x\}}$ , let us say that a point  $y = (m, l)$  is a regeneration point for  $\mathbf{C}_{\{0, x\}}$ , if  $1 \leq m \leq n-1$  and  $\mathbf{C}_{\{0, x\}} \cap \mathcal{H}_m = \{y\}$ .

Notice that clusters  $\mathbf{C}_{\{0, x\}}$  corresponding to  $d$ -connections could be defined as those which have no regeneration points. Clearly,  $d$ -connectivities dominate the  $f$ -connectivities,  $d(n, k) > f(n, k)$ . We claim that there exists  $\nu > 0$  such that

$$(3.5) \quad \sum_{x \in \mathcal{H}_n} \mathbb{P}_p(0 \overset{d}{\leftrightarrow} x) e^{(t, x)_d} \leq e^{-\nu n}.$$

By (2.8) and (1.10),

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{x \in \mathcal{H}_n} \mathbb{P}_p(0 \leftrightarrow x) e^{(t, x)_d} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{k \in \mathbb{Z}^{d-1}} h(n, k) e^{(t, x)_d} \right) = 0.$$

Consequently, for the purpose of proving (3.5), one is entitled to work with full clusters  $\mathbf{C}_{\{0, x\}}$  instead of the restricted clusters  $\mathbf{C}_{\{0, x\}}^n$  which appear in the definition of  $h(n, k)$ . The renormalization estimates of the previous subsections imply that a nonnegligible contribution to the left-hand side of (3.5) could come only from the clusters  $\mathbf{C}_{\{0, x\}}$  which are compatible with sufficiently well behaved tree skeletons  $\mathfrak{T}^{(k)}$ . This enables several reductions in the sum (3.4). Specifically, we have the following:

*Reduction in  $\|x\|_d$ .* By (1.23) it would be enough to restrict summation only to those  $x \in \mathcal{H}_n$  which satisfy  $\|x\|_n \leq c_1 n$ .

*Reduction in  $\mathfrak{g}^{(k)} = \#(\gamma^{(k)})$ .* In view of the reduction in  $\|x\|$  and by (2.8) it would be enough to restrict the summation only to the case of  $k$ -skeleton ( $k$  large enough) connections, which satisfy  $\mathfrak{g}^{(k)} \leq c_2 n/k$ .

Now fix  $\varepsilon = \varepsilon(t)$  and  $\alpha = \alpha(t, \varepsilon)$  as in (3.1).

*Reduction in  $\#_{t,\varepsilon}(\gamma^{(k)})$ .* Fix any  $\delta > 0$ . By Lemma 2.2,

$$\mathbb{P}_p\left(\gamma^{(k)}: \#_{t,\varepsilon}(\gamma^{(k)}) \geq \delta \frac{n}{k}; 0 \overset{\gamma^{(k)}}{\leftrightarrow} x\right) \leq \exp\{-c_3 \delta \varepsilon n - (t, x)_d\},$$

uniformly in  $n$  and  $x \in \mathcal{H}_n$ ,  $\|x\|_d \leq c_1 n$ , as soon as the scale  $k$  is chosen to be sufficiently large, that is,  $k \geq k_0(\delta, \varepsilon, \hat{t})$ .

Consequently, for all such scales  $k$  we obtain

$$(3.6) \quad \sum_{x \in \mathcal{H}_n} e^{(t,x)_d} \mathbb{P}_p\left(\mathfrak{T}^{(k)}: \#_{t,\varepsilon}(\gamma^{(k)}) \geq \delta \frac{n}{k}; 0 \overset{\mathfrak{T}^{(k)}}{\leftrightarrow} x\right) \leq e^{-c_4 \delta \varepsilon n}.$$

*Reduction in  $\mathfrak{l}^{(k)} = \#(\mathfrak{L}^{(k)})$ .* Similarly, Lemma 2.3 implies that for every  $\delta > 0$  fixed

$$(3.7) \quad \sum_{x \in \mathcal{H}_n} e^{(t,x)_d} \mathbb{P}_p\left(\mathfrak{T}^{(k)}: \mathfrak{l}^{(k)} \geq \delta \frac{n}{k}; 0 \overset{\mathfrak{T}^{(k)}}{\leftrightarrow} x\right) \leq e^{-c_5 \delta n},$$

uniformly in  $n$  and for sufficiently large renormalization scales  $k$ .

To summarize all the reductions above, for every  $\delta > 0$  one can restrict summation in (3.4) to the case of  $\|x\|_d \leq c_1 n$  and the percolation clusters  $\mathbf{C}_{\{0,x\}}$  which are, on sufficiently large renormalization scales  $k$ , compatible with tree skeletons  $\mathfrak{T}^{(k)} = \gamma^{(k)} \cup \mathfrak{L}^{(k)}$  satisfying

$$(3.8) \quad \#(\gamma^{(k)}) \leq c_2 \frac{n}{k}, \quad \#_{t,\varepsilon}(\gamma^{(k)}) \leq \delta \frac{n}{k} \quad \text{and} \quad \#(\mathfrak{L}^{(k)}) \leq \delta \frac{n}{k}.$$

Let us say that  $\mathfrak{T}^{(k)}$  is a  $\delta$ -regular tree skeleton if it complies with (3.8). Similarly, let us say that a cluster  $\mathbf{C}_{\{0,x\}}$  is  $(k, \delta)$ -regular if its tree skeleton  $\mathfrak{T}^{(k)}$  on the  $k$ th renormalization scale is  $\delta$ -regular.

Thus, it remains to show that, for an appropriate choice of  $k$  and  $\delta$ ,

$$(3.9) \quad \sum_{x \in \mathcal{H}_n} \mathbb{P}_p(0 \overset{d}{\leftrightarrow} x; \mathbf{C}_{\{0,x\}} \text{ is } (k, \delta)\text{-regular}) e^{(t,x)_d} \leq e^{-\nu n}.$$

3.3. *Proof of the mass-gap.* There is a transparent logic behind the latter estimate: the condition on the cluster  $\mathbf{C}_{\{0,x\}}$  to be  $(k, \delta)$ -regular forces most of  $\mathbf{C}_{\{0,x\}}$  to be localized within chunks of  $k\mathbf{U}^p$ -balls centered around vertices of the trunk  $\gamma^{(k)}$  with successive  $\varepsilon$ -good increments. Notice that by the positive cone condition (3.1) each  $\varepsilon$ -good increment shifts the  $e_1$ -projection of the corresponding endpoints by a fixed fraction of  $k$ , which gives rise to decoupling properties along finite sequences of such successive vertices. Small values of  $\delta$  insure a fixed fraction of  $n/k$  of such disjoint sequences which, already, leads to the target bound (3.9). Let us proceed with a rigorous implementation of the above idea:

For every  $x \in \mathcal{H}_n$ ,  $\|x\|_d \leq c_1 n$ ,

$$\mathbb{P}_p(0 \overset{d}{\leftrightarrow} x; \mathbf{C}_{\{0,x\}} \text{ is } (\delta, k)\text{-regular}) = \sum_{\mathfrak{T}^{(k)} \text{ is } \delta\text{-regular}} \mathbb{P}_p(0 \overset{d}{\leftrightarrow} x; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}).$$

Similarly,

$$\mathbb{P}_p(0 \leftrightarrow x; \mathbf{C}_{\{0,x\}} \text{ is } (\delta, k)\text{-regular}) = \sum_{\mathfrak{T}^{(k)} \text{ is } \delta\text{-regular}} \mathbb{P}_p(0 \leftrightarrow x; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}).$$

We claim that there exists  $\nu > 0$  such that

$$(3.10) \quad \mathbb{P}_p(0 \overset{d}{\leftrightarrow} x; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}) \leq e^{-\nu n} \mathbb{P}_p(0 \leftrightarrow x; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}),$$

uniformly in all sufficiently large renormalization scales  $k$  and in all  $\delta$ -regular tree skeletons  $\mathfrak{T}^{(k)}$ .

Indeed, let us choose a sufficiently large number  $r \in \mathbb{N}$ ; below we shall specify an appropriate choice; eventually it will depend on the value of  $\alpha$  in the positive cone condition (3.1), but not on the particular renormalization scale  $k$ . Given  $r$  we associate, on every coarse-graining scale  $k$ , a sequence of slabs  $\mathcal{S}_{k,r}^j$ ,  $j = 1, 2, \dots$ ,

$$(3.11) \quad \mathcal{S}_{k,r}^j \triangleq \{x \in \mathbb{Z}^d : |x_1 - 4jkr| \leq rk\}.$$

In other words,  $\mathcal{S}_{k,r}^j$  is the lattice slab of width  $2kr$  centered at the point  $4jrk\mathbf{e}_1 = (4jrk, 0)$ . For a given tree skeleton  $\mathfrak{T}^{(k)}$ , let us say that a slab  $\mathcal{S}_{k,r}^j$  is good if  $4jrk < n$ , and

$$\mathcal{S}_{k,r}^j \cap \bigcup_{z \in \mathfrak{T}_{\text{bad}}^{(k)}} k\mathbf{U}^p(z) = \emptyset,$$

where the bad part of  $\mathfrak{T}^{(k)}$  is defined via

$$\mathfrak{T}_{\text{bad}}^{(k)} \triangleq \mathfrak{L}^{(k)} \cup \{\gamma^{(k)}(i) : \gamma^{(k)}(i+1) - \gamma^{(k)}(i) \notin \mathcal{C}_\varepsilon(t)\}.$$

By (3.8) the number of good slabs  $\mathcal{S}_{k,r}^j$  is, uniformly in all  $\delta$ -regular tree skeletons  $\mathfrak{T}^{(k)}$ , bounded below:

$$(3.12) \quad \{j : \mathcal{S}_{k,r}^j \text{ is good}\} \geq \frac{n}{8rk},$$

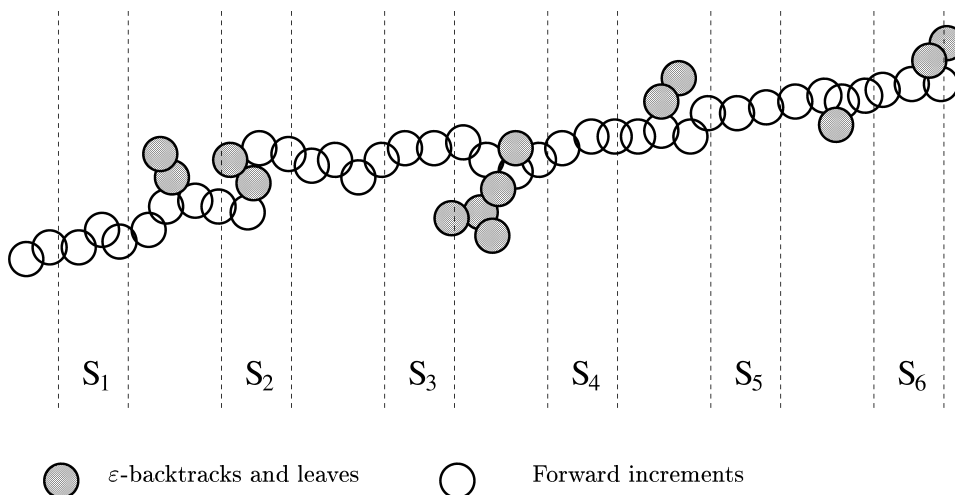


FIG. 2. The renormalized cluster  $C_{\{0,x\}}^{(k)}$ : the slabs  $S_1$ ,  $S_4$  and  $S_5$  are good; the slabs  $S_2$ ,  $S_3$  and  $S_6$  are bad.

as soon as  $\delta$  is sufficiently small, which, again by (3.8), amounts to choosing a sufficiently large coarse-graining scale  $k$ .

So let us fix a  $\delta$ -regular tree skeleton  $\mathfrak{T}^{(k)}$ , and let us renumber all good slabs of  $\mathfrak{T}^{(k)}$  as  $\mathfrak{g}_{k,r}^{j_1}, \dots, \mathfrak{g}_{k,r}^{j_m}$ ,  $m \geq n/8rk$ . For every good slab  $\mathfrak{g}_{k,r}^{j_l}$  and every cluster  $C_{\{0,x\}}$  compatible with  $\mathfrak{T}^{(k)}$ , the intersection  $C_{\{0,x\}} \cap \mathfrak{g}_{k,r}^{j_l}$  is confined to the set  $\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}$ ,

$$C_{\{0,x\}} \cap \mathfrak{g}_{k,r}^{j_l} \subseteq \mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l} \triangleq \bigcup_{\gamma^{(k)}(i) \in \mathfrak{g}_{k,r}^{j_l}} 2k\mathbf{U}^p(\gamma^{(k)}(i)).$$

By construction, for every  $j_l$ ,  $l = 1, \dots, m$ , the number of points  $\#(\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}) \leq c_3(rk)^d$ . Also all the  $\gamma^{(k)}$ -increments inside  $\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}$  are  $\varepsilon$ -good. Thus, if  $r > c_4/\alpha$ , then, in view of (3.1), one can locally modify at most  $c_5(rk)^d$  bonds inside  $\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}$  in such a way that the following hold:

1. the modified cluster is still compatible with  $\mathfrak{T}^{(k)}$ ;
2. there is at least one regeneration point inside  $\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}$ .

Since these modifications could be performed independently in each of the sets  $\mathbf{R}_{\mathfrak{T}^{(k)}}^{j_l}$ ,  $l = 1, \dots, m$ ,  $m \geq n/2rk$ , the inequality (3.10) follows.

3.4. *Asymptotics of  $h$ -connectivities.* With the mass-gap condition (1.15) verified at all the points  $\hat{t} \in \text{int}(\mathcal{D}_{\mathbb{H}})$ , we literally proceed as in [12]. In particular,

the nondegeneracy condition (1.16) follows by the conditional variance argument as in [12], pages 341–342.

Similarly, following the proof of Lemma 4.1 in [12], let us describe the analytic function  $\Lambda_p$ , which appears in the right-hand side of (1.17):

Given  $x = (n, k)$ , choose the unique point  $\hat{t} = \hat{t}(n, k) = \hat{t}(\vec{n}(x)) \in \mathcal{D}_{\mathbb{H}}$  which satisfies

$$\frac{k}{n} = \nabla m_{\mathbb{H}}(\hat{t}).$$

The existence and uniqueness of such  $\hat{t}$  follows from Theorem D. Set  $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t}) \in \partial \mathbf{K}^p$  and

$$\kappa(\hat{t}) = \sum_{n=1}^{\infty} n \mathbb{F}_n(\hat{t}) e^{-nm_{\mathbb{H}}(\hat{t})}.$$

Then

$$(3.13) \quad \Lambda_p(\vec{n}(x)) = \frac{(1-p)^{d-1}}{\kappa(\hat{t}) \sqrt{(1 + \|k/n\|_{d-1}^2) \rho_{\partial \mathbf{K}^p}(t)}},$$

where, as before,  $\rho_{\partial \mathbf{K}^p}(t)$  is the Gaussian curvature of the polar shape  $\partial \mathbf{K}^p$  at the point  $t$ .

*3.5. Asymptotics of full connectivities.* Because of the exponential bound (3.5) we actually have all the data to proceed as in [7] and [12] (see the detailed computation in [12], pages 347–349).

Namely, let  $\alpha \in \mathbb{R}_+$  be fixed, and let  $x = (n, k) \in \mathcal{H}_n \cap \mathcal{C}_\alpha$ . Choose  $t = t(x) = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$  as in the preceding subsection. Of course,  $t$  is polar to  $x$ ,  $(t, x)_d = \xi_p(x)$ .

Decomposing the cluster  $\mathbf{C}_{\{0,x\}}$  with respect to the leftmost and rightmost regeneration points on the interval  $[1, \dots, n-1]$ , we obtain

$$\mathbb{P}_p(0 \leftrightarrow x) = d(n, k) + \sum_{r=1}^n \sum_{l \in \mathbb{Z}^{d-1}} u(r, l) \tilde{h}(n-r, k-l),$$

where  $u$  is the connectivity function along the clusters with exactly one regeneration point, that is, by definition,

$$\{0 \overset{u}{\leftrightarrow} x\} \iff \exists \text{ unique } r \in [1, \dots, m-1] : \#(\mathbf{C}_{\{0,x\}} \cap \mathcal{H}_r) = 1.$$

We set  $u(r, l) = \mathbb{P}_p(0 \overset{u}{\leftrightarrow} (r, l))$ .

As in the case of  $d$ -connectivities the results of Section 3 imply that there exists  $v' > 0$  such that

$$\mathbb{U}_r(\hat{t}) \triangleq \sum_{l \in \mathbb{Z}^{d-1}} u(r, l) e^{(l, l)_{d-1}} \leq e^{-v'r + rm_{\mathbb{H}}(\hat{t})},$$

uniformly in  $r \in \mathbb{N}$ . Using the local asymptotics (1.17) of  $h$ -connectivities, we, therefore, arrive at (1.11) of Lemma 1.1 with

$$\Phi_p(\vec{n}(x)) = \sum_{r=1}^{\infty} r \mathbb{U}_r(\hat{t}) e^{-rm_{\mathbb{H}}(\hat{t})}.$$

**4. Direct approach.**

4.1. *Renewal along generic directions.* Let  $t \in \partial \mathbf{K}^p$ . Given  $x, y \in \mathbb{Z}^d$  we define the hyperplane

$$\mathcal{H}_x^t = \{z \in \mathbb{R}^d \mid (t, z)_d = (t, x)_d\}$$

and the slab

$$\mathcal{S}_{x,y}^t = \{z \in \mathbb{R}^d \mid (t, x)_d \leq (t, z)_d \leq (t, y)_d\}.$$

If  $(t, x)_d > (t, y)_d$ , we set  $\mathcal{S}_{x,y}^t = \emptyset$ .

We shall define connectivity functions  $h_t, f_t$  associated with  $t$ : Let  $\mathbf{e}$  be a unit vector in the direction of one of the axes such that the scalar product of  $\mathbf{e}$  with  $t$  is maximal.

DEFINITION. For  $x, y \in \mathbb{Z}^d$  let  $\{x \overset{h_t}{\leftrightarrow} y\}$  denote the (possibly empty) event that:

1.  $x$  and  $y$  are connected in the restriction of the percolation configuration to the slab  $\mathcal{S}_{x,y}^t$ . Let  $\mathbf{C}_{x,y}^t$  denote the corresponding common cluster. If  $x \neq y$ , then in addition
2.  $\mathbf{C}_{x,y}^t \cap \mathcal{S}_{x,x+\mathbf{e}}^t = \{x, x + \mathbf{e}\}$  and  $\mathbf{C}_{x,y}^t \cap \mathcal{S}_{y-\mathbf{e},y}^t = \{y - \mathbf{e}, y\}$ .

Set

$$h_t(x) \triangleq \mathbb{P}_p(\mathbf{0} \overset{h_t}{\leftrightarrow} x).$$

Notice that  $h_t(0) = 1$  and, by translational invariance, that  $h_t(x) = \mathbb{P}_p(z \overset{h_t}{\leftrightarrow} z + x)$  for every  $x, z \in \mathbb{Z}^d$ . Also, as in the case of the parameterized  $h$ -connectivities, it is easy to show that

$$(4.1) \quad \xi_p(x) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log h_t([nx]),$$

for any  $t$  and  $x$  satisfying  $(t, x)_d > 0$ .

The  $d$ -dimensional array of  $h_t$ -connectivities possesses a natural renewal structure, which we proceed to describe:

DEFINITION. For  $x, y \in \mathbb{Z}^d$  let us say that they are  $f_t$ -connected,  $\{x \overset{f_t}{\leftrightarrow} y\}$ , if the following hold:



- (i) the event  $\{x \overset{h_t}{\leftrightarrow} y\}$  happens, and  $x \neq y$ ;
- (ii) for no  $z \in \mathbb{Z}^d \setminus \{x, y\}$  both

$$(4.2) \quad \{x \overset{h_t}{\leftrightarrow} z\} \quad \text{and} \quad \{z \overset{h_t}{\leftrightarrow} y\}$$

take place.

Set

$$f_t(x) \triangleq \mathbb{P}_p(0 \overset{f_t}{\leftrightarrow} x).$$

Notice that  $f_t(0) = 0$ , and, moreover, the  $f_t$ -connectivities are  $\mathbb{Z}^d$ -shift invariant.

The parameterized construction of the preceding sections corresponds to the choice of  $t$  along one of the axis directions. The point we are making here is that it is natural to relate the asymptotics of  $\mathbb{P}_p(0 \leftrightarrow x)$  to the asymptotics of the  $h_{t_0}$ -connectivities, where  $t_0 \in \partial \mathbf{K}^p$  is essentially chosen to be polar to  $x$ .

From now on we assume that  $t_0 \in \partial \mathbf{K}^p$  is not orthogonal to any of the axis directions. We shall adjust the notion of regeneration point to the direction  $t_0$ : Let  $y \in \mathbb{Z}^d$  and assume that  $y$  is connected to the origin. We say that  $z \in \mathbb{Z}^d$  is a regeneration point of  $\mathbf{C}_{\{0,y\}}$  if the following hold:

1.  $(t_0, \mathbf{e})_d \leq (t_0, z)_d \leq (t_0, y)_d - (t_0, \mathbf{e})_d$ ;
2.  $\mathcal{S}_{z-\mathbf{e}, z+\mathbf{e}}^t \cap \mathbf{C}_{\{0,y\}}$  contains exactly three points  $z - \mathbf{e}$ ,  $z$  and  $z + \mathbf{e}$ , where  $\mathbf{e}$  is a unit axis direction, such that the scalar product  $(t_0, \mathbf{e})_d$  is maximal.

For any point  $y$  and for any realization of the cluster  $\mathbf{C}_{0,y}^{t_0}$  there are at most a finite number of regeneration points. Notice that if  $0$  and  $y$  are  $h_{t_0}$ -connected and  $z$  is a regeneration point, then (4.2) is satisfied (with  $x = 0$ ). If there is no such point at all, then, by definition,  $0$  is  $f_t$ -connected to  $x$ . Otherwise,  $0$  is  $f_t$ -connected to the regeneration point  $z$ , which has the minimal  $t$ -projection. Using the corresponding decomposition of the clusters  $\mathbf{C}_{0,x}^{t_0}$  one gets the following “renewal type” equation:

$$(4.3) \quad h_{t_0}(y) = \sum_{z \in \mathbb{Z}^d} f_{t_0}(z) h_{t_0}(y - z).$$

4.2. *Regeneration points.* By compactness of  $\partial \mathbf{K}^p$  for every  $\varepsilon \in (0, \frac{1}{2})$  one can choose  $\lambda = \lambda(\varepsilon) > 0$  such that, for every  $t_0 \in \partial \mathbf{K}^p$ ,

$$(4.4) \quad t \in B_\lambda(t_0) \cap \partial \mathbf{K}^p \triangleq \{s \in \mathbb{R}^d : \|s - t_0\|_d \leq \lambda\} \cap \partial \mathbf{K}^p$$

implies that  $\mathcal{C}_\varepsilon(t) \subset \mathcal{C}_{2\varepsilon}(t_0)$

uniformly in  $t_0 \in \partial \mathbf{K}^p$ .

This is the appropriate version of the positive cone condition for the  $t_0$ -adjusted renewal structure. Notice that the forthcoming proofs do not rely on the lattice

symmetries of  $\mathbb{Z}^d$ . Let us use  $\mathcal{R}_x^{t_0}$  to denote the (random) set of  $t_0$ -regeneration points of  $\mathbf{C}_{\{0,x\}}$ . The following lemma gives a uniform probabilistic estimate on the typical size of  $\mathcal{R}_x^{t_0}$  as  $\|x\|_d$  increases.

LEMMA 4.1. *For every  $\varepsilon \in (0, \frac{1}{2})$  there exist  $\delta > 0$  and  $\nu > 0$  such that*

$$(4.5) \quad \mathbb{P}_p \left( \#(\mathcal{R}_x^{t_0}) < \delta \|x\|_d; 0 \overset{h_{t_0}}{\leftrightarrow} x \right) \leq \exp(-(t, x)_d - \nu \|x\|_d)$$

*uniformly in  $t_0 \in \partial \mathbf{K}^p$ ,  $t \in B_\lambda(t_0) \cap \partial \mathbf{K}^p$  and  $x \in \mathcal{C}_\varepsilon(t)$ , where  $\lambda = \lambda(\varepsilon)$  is defined in (4.4).*

PROOF. The proof is identical to that of the estimate (3.10). The only required modification is to redefine the slabs  $\mathcal{S}_{k,r}^j$  as

$$\mathcal{S}_{k,r}^j \triangleq \{y \in \mathbb{Z}^d : |(t_0) - 4jkr| \leq rk\},$$

instead of the parameterized definition employed there.  $\square$

For  $t_0 \in \partial \mathbf{K}^p$  and  $t \in \mathbb{R}^d$  define

$$\begin{aligned} \mathbb{H}_{t_0}(t) &\triangleq \sum_{x \in \mathbb{Z}^d} h_{t_0}(x) e^{(t,x)_d}, \\ \mathbb{F}_{t_0}(t) &\triangleq \sum_{x \in \mathbb{Z}^d} f_{t_0}(x) e^{(t,x)_d}. \end{aligned}$$

An almost immediate consequence of Lemma 4.1 is the mass-gap-type condition for the  $t_0$ -adjusted connectivities.

LEMMA 4.2. *For every  $\varepsilon \in (0, \frac{1}{2})$  there exists  $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$  such that, uniformly in  $t_0 \in \partial \mathbf{K}^p$ ,*

$$(4.6) \quad \mathbb{F}_{t_0}(t) < \infty \quad \text{on } B_{\bar{\lambda}}(t_0).$$

*Furthermore, for every  $t_0 \in \partial \mathbf{K}^p$ , the implicit description of  $\partial \mathbf{K}^p$  in the  $\partial \mathbf{K}^p \cap B_{\bar{\lambda}}(t_0)$  neighborhood of  $t_0$  is given by*

$$(4.7) \quad t \in \partial \mathbf{K}^p \cap B_{\bar{\lambda}}(t_0) \iff \mathbb{F}_{t_0}(t) = 1.$$

*$\partial \mathbf{K}^p$  is a real analytic surface and it is strictly convex with Gaussian curvature uniformly bounded away from 0.*

PROOF. Fix  $\varepsilon \in (0, \frac{1}{2})$ . For every  $t \in \mathbb{R}^d \setminus 0$  define  $\eta_t$  to be the unique point of the boundary  $\partial \mathbf{K}^p$  in the direction of  $t$ ;  $\eta_t = t/\xi_p^*(t)$ , where  $\xi_p^*$  is the support function of  $\mathbf{U}^p$ . Of course,

$$(4.8) \quad \mathbb{P}_p \left( 0 \overset{f_{t_0}}{\leftrightarrow} x \right) \leq \mathbb{P}_p(0 \leftrightarrow x) \leq \exp(-(\eta_t, x)_d - c_1 \varepsilon \|x\|_d),$$

whenever  $x \notin \mathcal{C}_\varepsilon(\eta_t)$ . Consequently, there exists  $\lambda_1 = \lambda_1(\varepsilon)$  such that

$$(4.9) \quad \mathbb{P}_p \left( 0 \overset{f_{t_0}}{\leftrightarrow} x \right) \leq \exp(-(t, x)_d - c_1 \varepsilon \|x\|_d)$$

uniformly in  $t_0 \in \partial \mathbf{K}^p$ ,  $t \in B_{\lambda_1}(t_0)$  and  $x \notin \mathcal{C}_\varepsilon(\eta_t)$ .

On the other hand, using the fact that  $f_{t_0}$ -clusters have no regeneration point at all we infer from (4.5) that there exist  $\nu' > 0$  and  $\lambda_2 > 0$  such that

$$(4.10) \quad \mathbb{P}_p \left( 0 \overset{f_{t_0}}{\leftrightarrow} x \right) \leq \exp(-(t, x)_d - \nu' \|x\|_d)$$

uniformly over  $t_0 \in \partial \mathbf{K}^p$ ,  $t \in B_{\lambda_2}(t_0)$  and  $x \in \mathcal{C}_\varepsilon(\eta_t)$ .

Thus (4.6) follows with  $\bar{\lambda}(\varepsilon) = \lambda_1(\varepsilon) \wedge \lambda_2(\varepsilon)$ .

Since for  $t \in \text{int}(\mathbf{K}^p) \cap B_{\bar{\lambda}}(t_0)$  the moment generating function  $\mathbb{H}_{t_0}$  is finite and, moreover,

$$(4.11) \quad \mathbb{H}_{t_0}(t) = \frac{1}{1 - \mathbb{F}_{t_0}(t)},$$

(4.7) follows by the continuity of  $\mathbb{F}_{t_0}$  and the fact that  $\mathbb{H}_{t_0}$  diverges on  $B_{\bar{\lambda}}(t_0) \setminus \mathbf{K}^p$ .

Given  $t_0 \in \partial \mathbf{K}^p$ ,  $t \in \mathbb{R}^d$  and  $x \in \mathbb{Z}^d$  let us define the measure

$$(4.12) \quad \mathbb{Q}_{t_0}^t(x) = f_{t_0}(x) e^{(t,x)_d}.$$

By (4.6) and (4.7),  $\mathbb{Q}_{t_0}^t$  is a probability measure with exponentially decaying tails whenever  $t \in B_{\bar{\lambda}}(t_0) \cap \partial \mathbf{K}^p$ .

In the latter case let  $\mu_{t_0}(t) \in \mathbb{R}^d$  be the expectation of a random variable  $X$  under the probability distribution  $\mathbb{Q}_{t_0}^t$ ,

$$(4.13) \quad \mu_{t_0}(t) = \mathbb{E}_{t_0}^t X = \sum_{x \in \mathbb{Z}^d} x \mathbb{Q}_{t_0}^t(x) = \nabla \log \mathbb{F}_{t_0}(t).$$

Let  $A_{t_0}(t) = \text{Hess}(\log \mathbb{F}_{t_0}(t))$  be the corresponding covariance matrix. It is straightforward to check that  $A_{t_0}(t)$  is uniformly [in  $t_0 \in \partial \mathbf{K}^p$  and  $t \in B_{\bar{\lambda}}(t_0) \cap \partial \mathbf{K}^p$ ] nondegenerate. Consequently, as the measure  $\mathbb{Q}_{t_0}^t$  is concentrated on one side of a hyperplane containing the origin,  $\mu_{t_0}^t \neq 0$ . By the analytic implicit function theorem we infer that  $\partial \mathbf{K}^p$  is a real analytic surface in a neighborhood of  $t_0$ . Similarly, strict convexity of and positive Gaussian curvature of  $\partial \mathbf{K}^p$  at  $t_0$  follow from the strict convexity of  $\log \mathbb{F}_{t_0}$  and nondegeneracy of  $A_{t_0}$  in a neighborhood of this point.  $\square$

Using the general theory of convex bodies [12, 17] we can obtain a corresponding result for the surface  $\partial \mathbf{U}^p$  which is polar to  $\partial \mathbf{K}^p$ .

LEMMA 4.3. *The surface  $\partial \mathbf{U}^p$  is an analytic convex surface with Gaussian curvature uniformly bounded away from 0. The Gaussian curvatures of  $\partial \mathbf{U}^p$  and  $\partial \mathbf{K}^p$  at two conjugate points  $x$  and  $t$  are reciprocals of one other.*

REMARK. It should be mentioned that our choice of the renewal relation (4.3), and, accordingly, of the connectivity type functions  $h_t$  and  $f_t$ , is certainly not the only possible one. This is, however, not that important—whatever notion of regeneration points one employs, the localized structure of percolation clusters (in the sense of renormalization results of Section 2) will lead to an appropriate version of the entropic bound (4.5).

4.3. *Tilted measures and strict convexity.* It is instructive to give a direct proof of the strict convexity of the norm  $\xi_p$ . Notice that the argument below suggests that this property is, in fact, a purely entropic phenomenon.

LEMMA 4.4. *The norm  $\xi_p$  is strictly convex.*

PROOF. First of all we shall show that, for every  $t_0 \in \partial\mathbf{K}^P$  and each  $t \in B_{\bar{\lambda}}(t_0) \cap \partial\mathbf{K}^P$ , the points  $t$  and  $\mu_{t_0}(t)$  [defined in (4.13)] are in polar relation:

$$(4.14) \quad (t, \mu_{t_0}(t))_d = \xi_p(\mu_{t_0}(t)).$$

Moreover, we claim that, for every  $\mu \neq \mu_{t_0}(t)$  with  $\|\mu - \mu_{t_0}(t)\|_d \leq 1$  and  $(\mu, t)_d = (\mu_{t_0}(t), t)_d$ ,

$$(4.15) \quad \xi_p(\mu) \geq (\mu, t)_d + c_4 \|\mu - \mu_{t_0}(t)\|_d^2 = \xi_p(\mu_{t_0}(t)) + c_4 \|\mu - \mu_{t_0}(t)\|_d^2,$$

for some strictly positive constant  $c_4$ . Strict convexity of  $\xi_p$  then instantly follows.

Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables distributed according to  $\mathbb{Q}_{t_0}^t$ . One can then rewrite the renewal relation (4.3) as

$$(4.16) \quad \begin{aligned} h_{t_0}([n\mu]) &= \delta_0([n\mu]) \\ &+ \exp(-(t, [n\mu])_d) \sum_1^k \bigotimes_1^k \mathbb{Q}_{t_0}^t(X_1 + \dots + X_k = [n\mu]). \end{aligned}$$

Since  $\mathbb{Q}_{t_0}^t$  is supported by  $\{x \in \mathbb{Z}^d \mid (t_0, x)_d \geq 0\}$ , the expected value

$$(t_0, \mu_{t_0}(t))_d = \mathbb{E}_{t_0}^t(t_0, X_1) > 0.$$

Thus, for  $t \in B_{\lambda}(t_0)$  with  $\lambda$  sufficiently small,  $(t, \mu_{t_0}(t))_d > 0$  as well. For these  $t \in B_{\lambda}(t_0)$  there exist  $c_5, c_6 > 0$  such that

$$\left\| n\mu - \sum_{i=1}^k \mathbb{E}_{t_0}^t X_i \right\|_d = \|n\mu - k\mu_{t_0}(t)\|_d \geq c_5 |n - k| \|\mu_{t_0}\|_d + c_6 n \|\mu - \mu_{t_0}\|_d,$$

for every  $k, n \in \mathbb{N}$ . By the usual large deviation upper bound,

$$(4.17) \quad \begin{aligned} &\bigotimes_1^k \mathbb{Q}_{t_0}^t \left( \sum_{i=1}^k X_i = [n\mu] \right) \\ &\leq \exp \left( -c_7 \frac{(n-k)^2}{k} \wedge |n-k| - c_8 \frac{n^2}{k} \wedge n \|\mu - \mu_{t_0}(t)\|_d^2 \right). \end{aligned}$$

Substituting the latter estimate to (4.16), we obtain

$$h_{t_0}([n\mu]) \leq c_9 \sqrt{n} \exp(-(t, [n\mu])_d - c_{10} n \|\mu - \mu_{t_0}(t)\|_d^2).$$

The claim (4.15) follows from the asymptotic relation (4.1).  $\square$

4.4. *Local limit structure of connectivities.* Let us fix  $\varepsilon > 0$  sufficiently small. We shall give sharp large- $\|x\|_d$  asymptotics of  $h_{t_0}(x)$  uniformly over  $t_0 \in \partial\mathbf{K}^p$  and  $x \in \mathcal{C}_\varepsilon(t_0)$ . As in the parameterized approach of Section 3.5, the passage from the asymptotics of  $h_{t_0}(x)$  to the full asymptotics (1.6) is, in view of the mass-gap assertion of Lemma 4.1, secured by the decomposition of the cluster  $\mathbf{C}_{\{0,x\}}$  with respect to the leftmost and rightmost  $t_0$ -regeneration points. In other words, the claim of Theorem A follows, once we show that the following lemma holds.

LEMMA 4.5. *Uniformly in  $t_0 \in \partial\mathbf{K}^p$  and in  $x \in \mathcal{C}_\varepsilon(t_0) \cap \mathbb{Z}^d$ ,*

$$(4.18) \quad h_{t_0}(x) = \frac{\Lambda_{t_0}(\vec{n}(x))}{\sqrt{(2\pi \|x\|_d)^{d-1}}} e^{-\xi_p(x)} (1 + o(1)).$$

PROOF. We use notation and results from the previous subsections. Since  $A_{t_0}(t) = \text{Hess}(\log \mathbb{F}_{t_0}(t))$  is nondegenerate at  $t = t_0$ , the cone  $\mathcal{C}_\varepsilon(t_0)$  lies inside the cone generated by the vectors

$$\{\mu_{t_0}(t) = \mathbb{E}_{t_0}^t X_1 \mid t \in B_{\bar{\lambda}}(t_0) \cap \partial\mathbf{K}^p\}.$$

In particular, for every  $x \in \mathcal{C}_\varepsilon(t_0)$  there exists  $t \in B_{\bar{\lambda}}(t_0)$  such that  $x$  and  $t$  are in the polar relation, which, by (4.14) and (4.15), means that

$$(4.19) \quad \vec{n}(x) = \frac{\mu_{t_0}(t)}{\|\mu_{t_0}(t)\|_d}.$$

Furthermore, there exists a number  $n = n(x)$  such that

$$(4.20) \quad \|x - n\mu_{t_0}(t)\|_d \leq c_8,$$

the latter estimate being uniform in  $t_0 \in \partial\mathbf{K}^p$  and  $x \in \mathcal{C}_\varepsilon(t_0)$ .

We now rewrite (4.16) as

$$(4.21) \quad \begin{aligned} & h_{t_0}(x) \exp(\xi_p(x)) \\ &= \delta_0(x) + \exp(-(t, x)_d) \sum_{k=1}^{\infty} \bigotimes_{i=1}^k \mathbb{Q}_{t_0}^t(X_1 + \dots + X_k = x). \end{aligned}$$

Fix  $\alpha \in (0, \frac{1}{2})$ . Notice that the support of  $\mathbb{Q}_{t_0}^t$  spans the whole lattice  $\mathbb{Z}^d$ . In view of (4.20), uniform exponential bounds (4.6) on the tails of  $\mathbb{Q}_{t_0}^t$  and uniform

nondegeneracy of  $A_{t_0}$ , we infer, by the usual  $d$ -dimensional local central limit theorem, that

$$(4.22) \quad \prod_{i=1}^k \mathbb{Q}_{t_0}(X_i + \dots + X_k = x) = \frac{\exp\{-(n-k)^2/2n\}(A_{t_0}^{-1}\mu_{t_0}, \mu_{t_0})_d(t)}{\sqrt{(2\pi n)^d \det(A_{t_0}(t))}}(1 + o(1))$$

uniformly over  $t_0 \in \partial\mathbf{K}^P$ ,  $x \in \mathcal{C}_\varepsilon(t_0)$  satisfying (4.20) and  $k$  in the range  $|n - k| < n^{1/2+\alpha}$ .

In the remaining range of  $k$ 's one has, proceeding as in (4.17),

$$(4.23) \quad \sum_{|k-n| \geq n^{1/2+\alpha}} \prod_{i=1}^k \mathbb{Q}_{t_0}(X_i + \dots + X_k = x) \leq \exp(-c_9 n^{2\alpha}).$$

Substituting (4.22) and (4.23) into (4.21) we, using (4.20), recover (4.18) with

$$\Lambda_{t_0}(\bar{\mathbf{n}}(x)) = \sqrt{\frac{\|\mu_{t_0}(t)\|_d^{d-1}}{(A_{t_0}^{-1}\mu_{t_0}, \mu_{t_0})_d(t) \det(A_{t_0}(t))}}.$$

Finally, the analyticity of  $\Lambda_{t_0}$  follows from the relation (4.19) and the analyticity of  $\log \mathbb{F}_{t_0}$ , which has been discussed in Section 4.2.  $\square$

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