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# LOGARITHMIC SOBOLEV INEQUALITY FOR SOME MODELS OF RANDOM WALKS[1]

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We determine the logarithmic Sobolev constant for the Bernoulli–Laplace model and the time to stationarity for the symmetric simple exclusion model up to the leading order. Our method for proving the logarithmic Sobolev inequality is based on a martingale approach and is applied to the random transposition model as well. The proof for the time to stationarity is based on a general observation relating the time to stationarity to the hydrodynamical limit.

**1. Introduction.** Let $S$ denote a space of configurations and $\eta \in S$ denote a configuration. We shall assume that $S$ is a finite set. Define on $S$ a dynamics given by a generator

$$Lf(\eta) = \sum_{\zeta \in S} C(\eta, \zeta)[f(\zeta) - f(\eta)].$$

Assume that the dynamics leaves a probability measure $\pi$ invariant. Define the associated Dirichlet form of $f$ by

$$D(f) = -E^{\pi}[fLf].$$

For any probability measure $\nu$ on $S$, define the entropy by

$$H(\nu/\pi) = E^{\nu} \log\left(\frac{d\nu}{d\pi}\right).$$

By convention, $H(g) = H(g\pi/\pi)$. Define the $V^{(p)}$ norm

$$V^{(p)}(\nu/\pi) = E^{\pi}\left[\left|\frac{d\nu}{d\pi} - 1\right|^{p}\right]^{1/p}.$$

Only $p = 1, 2$ are used in this paper. Notice that $V^{(1)}$ is twice the total variational norm. We are interested in the logarithmic Sobolev constant and the time needed to approach the stationary distribution $\pi$.

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*The logarithmic Sobolev constant.* Define $\alpha$, the logarithmic Sobolev constant, by

$$\alpha = \sup\Big\{H(g)/D\big(\sqrt{g}\big)\colon\ g \geq 0,\ E^{\pi}[g] = 1\Big\}. \tag{1.1}$$

*The time of stationarity.* Let $\nu_t$ be the distribution at time $t$ when the initial distribution is $\nu_0$. Define $\tau$, the time to stationarity, to be

$$\tau = \inf\Big\{t\colon\ t > 0,\ \sup_{\nu_0} V^{(2)}(\nu_t/\pi) \leq e^{-1}\Big\}. \tag{1.2}$$

The choice of the constant $e^{-1}$ is only for convenience; any constant less than 1 can be used and there is no canonical choice. Notice that we can use the $V^{(1)}$ norm instead of $V^{(2)}$. Since most results in this paper concern upper bounds of $\tau$, we choose $V^{(2)}$ to have stronger results in the definition of $\tau$.

The logarithmic Sobolev inequality was initiated in the work of Federbush [8] and Gross [9]. Its connection to the rate of relaxation to equilibrium was studied by, for example, [4, 5, 10, 12, 15]. See [1–3, 14] for a review. In particular, in our context, one has the general relation ([5])

$$\frac{1}{2}\alpha \leq \tau \leq \left(1 + \frac{1}{4}\log\log\frac{1}{\min_{\eta}\pi(\eta)}\right)\alpha. \tag{1.3}$$

Our goal in this paper is to determine $\alpha$ and $\tau$ asymptotically as the size of the configuration space tends to infinity for the three models of random walks. Another relevant quantity is the spectral gap. As it has been extensively studied, we shall not discuss it in this paper.

We first fix some notation for these models. For the random transposition (RT) model, the state space is the permutation group $S_n$ of $n$ objects. We think of $S_n$ as the $n!$ number of ways to place $n$ distinct particles on $n$ distinct sites. Both particles and sites are numbered from 1 to $n$. After a mean 1 exponential waiting time, a particle will select one of the $n$ sites with equal probability and exchange the position with the particle at that site. All the exponential waiting times are assumed to be independent. For $\sigma \in S_n$, we write $\sigma_i$, $1 \leq \sigma_i \leq n$, for the particle in site $i$ and write $\sigma^{ij}$ for the resulting configurations after the switch, that is, $(\sigma^{ij})_i = \sigma_j$, $(\sigma^{ij})_j = \sigma_i$ and $(\sigma^{ij})_k = \sigma_k$ for $k \neq i, j$. With these notations, the RT model is uniquely characterized by an initial distribution on $S_n$ and the Markov generator $K_n$ given by

$$(K_n g)(\sigma) = \frac{1}{n}\sum_{i,j=1}^{n}\big[g(\sigma^{ij}) - g(\sigma)\big],$$

where $g$ is a function on $S_n$.

The Bernoulli–Laplace (BL) model has two parameters $n$ and $r$, $1 \leq r \leq n$, the number of distinct sites and the number of identical particles. A site can be occupied by at most one particle. So the state space, denoted by $C_{n,r}$, is the space of all subsets of the $n$ sites with $r$ elements. For $\eta \in C_{n,r}$, denote by $\eta_i$

the number of particles at the site $i$. So, $\eta_i = 0$ or 1 and $\sum_{i=1}^n \eta_i = r$. Each particle waits a mean 1 exponential time to jump to one of the $n$ sites with equal probability $1/n$. The jump is suppressed if the chosen site is occupied. Again, all the exponential waiting times are independent. This Markovian evolution is uniquely characterized by the generator $K_{n,r}$:

$$
\begin{aligned}
(K_{n,r} f)(\eta) &= \frac{1}{n} \sum_{i,j=1}^{n} \eta_i (1 - \eta_j) \left[ f(\eta^{ij}) - f(\eta) \right] \\
&= \frac{1}{n} \sum_{i,j:\, i<j}^{n} \left[ f(\eta^{ij}) - f(\eta) \right],
\end{aligned}
$$

where $f$ is a function on $C_{n,r}$ and

$$
(\eta^{ij})_k = \begin{cases} \eta_j, & \text{if } k = i, \\ \eta_i, & \text{if } k = j, \\ \eta_k, & \text{otherwise.} \end{cases}
$$

The symmetric simple exclusion (SE) model has the same two parameters $n$ and $r$ and the same state space $C_{n,r}$ as in the BL model. The difference is that the $n$ sites are connected using a lattice structure and a particle jumps to one of its nearest neighbors with equal probability. For this article, the $n$ sites are identified as $\mathbb{Z}/n\mathbb{Z}$. So, each site has two nearest-neighbor sites. Using the same notation $\eta$, $\eta^{ij} \in C_{n,r}$ and $f\colon C_{n,r} \to \mathbb{R}$, the generator is $L_{n,r}$:

$$
(L_{n,r} f)(\eta) = \frac{1}{2} \sum_{i=1}^{n} \left[ f(\eta^{i,i+1}) - f(\eta) \right].
$$

This model can be considered in any dimension with $\mathbb{Z}$ replaced by $\mathbb{Z}^d$.

Since these generators are symmetric and irreducible, all three models have the uniform distribution $\pi$ as the unique stationary distribution. Note the cardinality: $|S_n| = n!$, $|C_{n,r}| = \binom{n}{r} = n!/r!(n-r)!$. The associated expectations will be denoted $E_n^{RT}$, $E_{n,r}^{BL}$ and $E_{n,r}^{SE}$ for the RT, BL and SE models, respectively. Let $g$ ($f$, resp.) be the probability mass function (pmf) relative to the uniform distribution $\pi$ on $S_n$ ($C_{n,r}$, resp.). The relative entropies are given explicitly by

$$
\begin{aligned}
H_n^{RT}(g) &= E_n^{RT}[g \log g] = \frac{1}{n!} \sum_{\sigma \in S_n} g(\sigma) \log g(\sigma), \\
H_{n,r}(f) &= E_{n,r}^{BL}[f \log f] \\
&= E_{n,r}^{SE}[f \log f] = \binom{n}{r}^{-1} \sum_{\eta \in C_{n,r}} f(\eta) \log f(\eta).
\end{aligned}
$$

The Dirichlet forms are explicitly given by

$$D_n^{RT}(g) = \frac{1}{2n} E_n^{RT}\left\{ \sum_{i,j=1}^{n} \left[ g(\sigma^{ij}) - g(\sigma) \right]^2 \right\},$$

$$D_{n,r}^{BL}(f) = \frac{1}{2n} E_{n,r}^{BL}\left\{ \sum_{i,j:\ i<j} \left[ f(\eta^{ij}) - f(\eta) \right]^2 \right\},$$

$$D_{n,r}^{SE}(f) = \frac{1}{4} E_{n,r}^{SE}\left\{ \sum_{i=1}^{n} \left[ f(\eta^{i,i+1}) - f(\eta) \right]^2 \right\}.$$

Denote the logarithmic Sobolev constants for these models (RT, BL and SE) by $a_n$, $a_{n,r}$ and $b_{n,r}$. Denote the times to stationarity of these models (RT, BL and SE) by $\tau_n^{RT}$, $\tau_{n,r}^{BL}$ and $\tau_{n,r}^{SE}$, respectively. Due to the particle–hole symmetry, it suffices to treat the case $1 \le r \le n/2$. For simplicity of presentation, the number of sites $n$ is an even number. The logarithmic Sobolev constant for the random transposition model and its time to stationarity have been determined completely by [6] and [5].

THEOREM 1 (The RT model) (Diaconis–Saloff–Coste). *There exists a constant $c$ independent of $n$, such that $c^{-1} \log n < a_n < c \log n$, $n \ge 2$.*

One has for the RT model with $n$ sites $[\min_\eta \pi(\eta)]^{-1} = |S_n| = n!$ and (1.3) means

$$\frac{1}{2} a_n \le \tau_n^{RT} \le \left[ 1 + \frac{1}{4} \log\log(n!) \right] a_n. \tag{1.4}$$

Since $\log\log(n!)$ is the order of $\log n$, the last inequality and the knowledge for $a_n$ (Theorem 1) determine the time to stationarity up to a factor of $\log n$ in the RT model. It turns out that the lower bound in (1.4) captures the correct order of the time to stationarity in the RT model [6].

THEOREM 2 (Diaconis–Shahshahani). *For the random transposition model, we have*

$$0 < \underline{\lim}_{n\to\infty} \tau_n^{RT}/\log n \le \overline{\lim}_{n\to\infty} \tau_n^{RT}/\log n < \infty.$$

The basic known results concerning the BL and SE models are the following two theorems.

THEOREM 3 (The BL model) (Diaconis–Shahshahani).

(i) $$\tau_{n,n/2}^{BL} \ge \frac{1}{2} \log n.$$

(ii) *There exists $\delta > 0$, independent of $n$, $r$, $1 \le r \le n/2$, such that*

$$\tau_{n,r}^{BL} \le \frac{2r}{n}\left(1 - \frac{r}{n}\right)[\log n + \delta].$$

THEOREM 4 (The SE model). *There exists a positive constant $c$, independent of $n$ and $r$, $1 \le r \le n-1$, such that*

$$c^{-1} n^2 < b_{n,r} < cn^2, \qquad n \ge 2, \quad 1 \le r \le n-1.$$

The first theorem is proved in [7] using Fourier analysis; the second in [17] using a martingale approach in the general framework of lattice gas dynamics and in [7] and [5] with a logarithmic correction. See the remark after Theorem 6 for the methods used in [7] and [5].

We can now use (1.3) to determine $a_{n,r}$ from Theorem 3 [5] and $\tau_{n,r}^{SE}$ from Theorem 4. Notice that the upper and lower bounds differ by a factor of order $\log\log\binom{n}{r}$ which ranges from $\log n$ (if $r = n/2$) to $\log\log n$ (if $r = 1$). The correct answers in the region $0 < \log r/\log n < 1$ for some fixed constant $C$ are given by the following two theorems, which are our main results.

THEOREM 5 (The BL model). *There exists a positive constant $\varepsilon$, independent of $n$, $1 \le r \le n-1$, such that*

$$\varepsilon \log \frac{n^2}{r(n-r)} < a_{n,r} \le \frac{2}{\log 2} \log \frac{n^2}{r(n-r)}, \qquad n \ge 2.$$

THEOREM 6 (The SE model). (i) *There exists a constant $\delta > 0$, independent of $n$, $r$, $1 \le r \le n/2$, such that*

$$\tau_{n,r}^{SE} \ge \delta n^2 (1 + \log r).$$

(ii) *Furthermore, there exists a constant $K$, independent of $n$, $r$, $1 \le r \le n/2$, such that*

$$\tau_{n,r}^{SE} \le K n^2 \log n.$$

(iii) *Combining* (i) *and* (ii), *the time to stationarity is the order of* $n^2 \log n$ *for the SE model provided* $0 < \liminf(\log r/\log n) \le \limsup(\log r/\log n) < 1$.

From Theorem 5 and (1.3), for all $n$, $r$, $1 \le r \le n-1$, we have

$$\tau_{n,r}^{BL} \le \frac{2}{\log 2} \log \frac{n^2}{r(n-r)} \left[1 + \frac{1}{4} \log\log\binom{n}{r}\right].$$

This gives the same order as Theorem 3 when $r \sim n$ ($r$ is of order $n$). Notice that while the correct order of the time to stationarity for the RT model is captured by the lower bound of (1.3) [see (1.4), Theorems 1 and 2], it is the upper bound of (1.3) which provides the correct order for the SE and BL models when, for example, $r$ is of order $n$.

There is a general inequality [11, 13]

$$D_{n,r}^{BL}(f) \le 2n^2 D_{n,r}^{SE}(f).$$

From this inequality, we have

$$b_{n,r} \le 2n^2 a_{n,r}.$$

This relation follows also from a general graph comparison method [5]. Together with Theorem 5, this gives an independent proof of the upper bound of Theorem 4 in case $r \sim n$. Notice that Theorem 5 was known [5] only up to a $\log n$ factor (in case $r \sim n$) and thus Theorem 4 was proved only up to a $\log n$ factor in [5].

We now remark on the methods used in this note. The lower bounds of the logarithmic Sobolev constants are easy to derive by using test functions, as will be shown briefly in Sections 3 and 4. The upper bound of Theorem 5 will be proved in Section 4 using the martingale method of [17]. For completeness, we shall also prove in Section 3 the upper bound of Theorem 1 using the same method. Therefore, all upper bounds of the logarithmic Sobolev constants can now be proved by the same approach.

Theorem 6 is proved in Section 5. The upper bound for the time to stationarity in Theorem 6 follows from (1.3) and the estimate in Theorem 4 for the logarithmic Sobolev constant. It should be pointed out that the lower bound will actually be proved for the total variational norm, or, equivalently, the $V^{(1)}$ norm. Our proof of the lower bound is based on an observation relating the time to stationarity to the hydrodynamical limit. Section 2 gives a brief account of this connection.

**2. Time to stationarity and hydrodynamical limit.** We now explain the connection between the time to stationarity and the hydrodynamic limit. We shall take as an example the symmetric simple exclusion process. The method described in this section is technically not needed for this paper since the hydrodynamical limit of the symmetric simple exclusion process is the trivial heat equation and explicit computation can be performed. This method, however, is important to determine the correct lower bound for the time to stationarity for many particle systems and is very general in natural. We first review some basic definitions from hydrodynamic limits.

Let $f$ be an initial density relative to $\pi$, the uniform distribution on the space $C_{n,r}$, and $P^f$ the corresponding probability measure. We call $m$ the macroscopic density of the profile determined by $f$ if, for any $\delta$ and $J$,

$$P^f[A_{n,J,\delta,C}] \to 1, \qquad C = n^{-1} \sum_{i=0}^{n} J(i/n)\, m(i/n)$$

as $n \to \infty$, where

$$A_{n,J,\delta,C} = \left\{ \eta: \left| n^{-1} \sum_{i=0}^{n} J(i/n) \eta_i - C \right| \leq \delta \right\}.$$

Let $f_s$ denote the density relative to $\pi$ at time $s$ given $f_0 = f$. The hydrodynamic limit states that for any $t$, $J$, $\delta$ fixed,

$$P^{f_{n^2 t}}\left[A_{n,J,\delta,C_t}\right] \to 1$$

provided that

$$C_t = n^{-1} \sum_{i=0}^{n} J(i/n)\, m(i/n, t)$$

and $m$ is the solution of the hydrodynamical equation

$$\partial_t m(x, t) = \Delta m(x, t)/2.$$

Here $\Delta$ is the Laplacian operator for the circle and the hydrodynamical equation is the simple heat equation. For general reversible dynamics, it will be a nonlinear diffusion equation. Note that the space integral of $m(x, t)$ is conserved and $\int m(x, t)\, dx = r/n$ for all $t$. Since the heat equation approaches equilibrium as $t \to \infty$, we have

$$C_t \to C_\infty = n^{-1} \sum_{i=0}^{n} J(i/n)(r/n)$$

as $t \to \infty$. Indeed, let $\kappa > 0$ denote the speed of the hydrodynamical equation to approach the constant function. Then

$$|C_t - C_\infty| = \left| n^{-1} \sum_{i=0}^{n} J(i/n)\, m(i/n, t) - C_\infty \right| \ge \exp[-\kappa t].$$

Suppose we have proved the following stronger estimate

$$P^{f_{n^2 t}}(A_{n, J, \delta, C_t}) \ge 9/10 \tag{2.1}$$

for $t = \gamma \log n$ and

$$\delta = n^{-\varepsilon}$$

for some small constants $\gamma, \varepsilon$, such that

$$\kappa\gamma + \varepsilon < 1/2.$$

Then we have

$$A_{n, J, \beta n^{-1/2}, C_t} \cap A_{n, J, \beta n^{-1/2}, C_\infty} = \varnothing.$$

Hence,

$$P^{f_{n^2 t}}(A_{n, J, \beta n^{-1/2}, C_\infty}) \le 1/10$$

for any $\beta$ fixed. Here and throughout the section, the statements apply to sufficiently large $n$. From the central limit theorem,

$$P^{\pi}(A_{n, J, \beta n^{-1/2}, C_\infty}) \ge 9/10$$

if $\beta$ is large enough. We have thus proved that

$$V^{(1)}(f_{n^2 t}) \ge \left| P^{\pi}(A_{n, J, \beta n^{-1/2}, C_\infty}) - P^{f_{n^2 t}}(A_{n, J, \beta n^{-1/2}, C_\infty}) \right| \ge 4/5.$$

This proves that $\tau \ge \gamma n^2 \log n$.

The estimate (2.1) is a stronger statement of the law of large numbers than one typically proves in the hydrodynamical limit. In the special case of symmetric simple exclusion processes, it is an explicit computation, as will be done in Section 5. In general, more sophisticated methods will be needed. There are several methods available. A simple and reasonably general one is the relative entropy method from [16]. We first introduce the notion of local Gibbs states.

Let $\lambda$ be a smooth function. Define the local Gibbs state $\psi^\lambda$ by

$$\psi^\lambda = \exp(\lambda \cdot \eta)/M(\lambda), \qquad \lambda \cdot \eta = \sum_{x=1}^{n} \eta_x \lambda(x/n), \tag{2.2}$$

where $M(\lambda)$ is the normalization constant $E^\pi \exp[\lambda \cdot \eta]$. The density is well defined by the formula

$$m(x/n) = (dp/d\lambda)(\lambda(x/n)), \tag{2.3}$$

where the pressure $p$ is defined by

$$p(\lambda) = \log E^\pi \exp[\lambda \cdot \eta]. \tag{2.4}$$

We shall choose $\lambda$ depending on $t$ in such a way that the corresponding $m(x/n, t)$ is the solution of the hydrodynamical equation. Denote the corresponding local Gibbs state by $\psi_t$. The standard result of local limit theorem states that

$$\overline{\lim}_{n \to \infty} n^{-1+\varepsilon} \log P^{\psi_t}(A_{n, J, n^{-\varepsilon}, C_t}) < -C \tag{2.5}$$

for some positive constant $C$ for any small positive constant $\varepsilon$. Recall the following inequality concerning relative entropy:

$$P^f(A) \le \frac{\log 2 + H(f/g)}{\log(1 + 1/P^g(A))}. \tag{2.6}$$

Applying this inequality with $f = f_{n^2 t}$ and $g = \psi_{n^2 t}$, we have

$$P^{f_{n^2 t}}(A_{n, J, \beta n^{-1/2}, C_\infty}) \le n^{-1+\varepsilon} H(f_{n^2 t}/\psi_{n^2 t})/C.$$

Let $h_t = n^{-1} H(f_{n^2 t}/\psi_{n^2 t})$. Hence (2.1) follows if we have $h_t \le o(n^{-\varepsilon})$.

A typical method to estimate the relative entropy is [16], where the following estimate is established: there exist $\delta > 0$ such that

$$\frac{d}{dt} h(t) \le \delta h(t) + \Omega_n, \tag{2.7}$$

where $\Omega_n$ denote the error. Assuming that $h(0) = 0$, we have

$$h(t) \le e^{\delta t} \Omega_n.$$

For the hydrodynamical limit, we need only $\Omega_n \to 0$ as $n \to \infty$. This is proved in [16] for Ginzburg–Landau models and can be generalized to many gradient models. Since our goal is to prove $h_t \le o(n^{-\varepsilon})$ for $t = \gamma \log n$, we need

$$n^{\gamma+\varepsilon} \Omega_n \to 0 \quad \text{as } n \to 0.$$

In other words, we need a precise error estimate for the hydrodynamical limit. Notice that the error terms in [16] appear in eigenvalue estimates. Hence we obtain better error estimates if some spectral gap or logarithmic

Sobolev inequality estimates are given. We will not carry out this approach because for the models considered in this paper, the hydrodynamical limit is trivially given by, for example, the heat equation. In this case, the error terms come from some martingale and can be estimated directly. But the connection with the relative entropy will be needed for more general models with nontrivial hydrodynamical equations.

**3. Proof of Theorem 1.** Since the present section concerns only the RT model, all the superscripts RT will be omitted. The lower bound of $a_n$ is easier and is proved using a test function $g$. Since the relaxation to stationarity is the slowest when the initial distribution is deterministic, we use the test function

$$g = n!\delta_{id},$$

where $\delta_{id}$ is the indicator function of the identity permutation. From the definition of $a_n$, we have

$$\begin{aligned} a_n &\geq H_n^{RT}(g)/D_n\left(\sqrt{g}\right) \\ &= \log(n!) \Bigg/ \left\{ \left[ \pi(id)\, n(n-1)(n!) + \sum_{i,j,\, i \neq j} \pi((i,j))(n!) \right] \frac{1}{2n} \right\} \\ &= \log(n!)/(n-1), \end{aligned}$$

which, by Stirling's formula, has the desired lower bound $c^{-1}\log n$.

To prove the upper bound for $a_n$, we shall perform conditioning on $\sigma_s$ and averaging over $1 \leq s \leq n+1$ to establish an inequality involving $a_{n+1}$ and $a_n$. Note that, fixing $s$, $\{X_t \equiv \sigma_s(t),\ t \geq 0\}$ is itself a continuous-time Markov chain. It is a $\{1, 2, \dots, n+1\}$-valued jump process of rate 2 and it jumps to state $1, 2, \dots, n+1$ with equal probability $1/(n+1)$. The corresponding logarithmic Sobolev inequality is known [5].

LEMMA 1. *For the above-mentioned continuous-time, $(n+1)$-state Markov chain $X_t$, there exists a constant $K$, independent of $n$, such that*

$$\begin{aligned} &\frac{1}{n+1} \sum_{x=1}^{n+1} h(x) \log h(x) \\ &\quad \leq K \frac{\log n}{(n-1)(n+1)} \sum_{x,y=1}^{n+1} \left[ h(y)^{1/2} - h(x)^{1/2} \right]^2, \qquad n \geq 1, \end{aligned} \tag{3.1}$$

*for all pmf $h$, relative to the uniform distribution on $\{1, \dots, n+1\}$. In fact, the best constant is $K = \frac{1}{2}$.*

To prove an inequality involving $a_{n+1}$ and $a_n$, let $g$ be a pmf relative to the uniform distribution $\pi$ on $S_{n+1}$ and define $(1 \le s \le n+1)$

$$g_s = \text{the marginal pmf of } \sigma_s, \text{ relative to } \pi,$$

$$g(\cdot|\sigma_s) = \text{the conditional pmf of } \sigma, \text{ given } \sigma_s, \text{ relative to } \pi,$$

$$\begin{aligned} \mathrm{I}_{1,s}(g) &\equiv E_{n+1}\Big\{ g_s(\sigma_s)\, E_{n+1} \\ &\qquad \times \Big[ \sum_{i,j:\ i\neq s,\ j\neq s} \big[ g(\sigma^{ij}|\sigma_s)^{1/2} - g(\sigma|\sigma_s)^{1/2} \big]^2 \big|\sigma_s \Big]\Big\} \\ &= E_{n+1}\Big\{ \sum_{i,j\neq s} \big[ g(\sigma^{ij})^{1/2} - g(\sigma)^{1/2}\big]^2 \Big\}, \end{aligned} \tag{3.2}$$

$$\mathrm{I}_{2,s}(g) \equiv E_{n+1}[g_s(\sigma_s) \log g_s(\sigma_s)],$$

$$\mathrm{I}_i(g) \equiv \frac{1}{n+1} \sum_{s=1}^{n+1} \mathrm{I}_{i,s}(g), \qquad i = 1, 2.$$

By an elementary calculation of conditional expectations,

$$H_{n+1}(g) = E_{n+1}\{ g_s(\sigma_s)\, E_{n+1}[g(\cdot|\sigma_s)\log g(\cdot|\sigma_s)|\sigma_s]\} + \mathrm{I}_{2,s}.$$

Since the uniform distribution on $S_{n+1}$, given $\sigma_s$, is the uniform distribution on $S_n$, the definition of $a_n$ then implies that

$$H_{n+1}(g) \le \frac{a_n}{2n} \mathrm{I}_{1,s} + \mathrm{I}_{2,s},$$

thus

$$H_{n+1}(g) \le \frac{a_n}{2n} \mathrm{I}_1 + \mathrm{I}_2. \tag{3.3}$$

It follows from a simple combinatorial calculation that

$$\sum_{s=1}^{n+1} \mathrm{I}_{1,s} = (n-1)\, E_{n+1}\Big\{ \sum_{i,j} \big[ g(\sigma^{ij})^{1/2} - g(\sigma)^{1/2} \big]^2 \Big\},$$

and therefore

$$\mathrm{I}_1 = 2(n-1)\, D_{n+1}(\sqrt{g}). \tag{3.4}$$

To estimate $\mathrm{I}_{2,s}$, we apply (3.1) (Lemma 1) to $h = g_s$ to get

$$\mathrm{I}_{2,s} \le K \frac{\log n}{(n-1)(n+1)} \sum_{x=1}^{n+1} \sum_{y=1}^{n+1} \big[ g_s(y)^{1/2} - g_s(x)^{1/2} \big]^2. \tag{3.5}$$

Let $a = \sigma_x^{-1}$, $b = \sigma_y^{-1}$, $\tilde{\sigma}^{xy} = \sigma^{ab}$. Since $g_s(x) = E_{n+1}[g(\sigma)|\sigma_s = x]$ and the function $(a, b) \to (a^{1/2} - b^{1/2})^2$ is convex for $a, b \ge 0$, Jensen's inequality implies

$$
\begin{aligned}
&\left[g_s(y)^{1/2} - g_s(x)^{1/2}\right]^2 \\
&\quad = \left[E_{n+1}[g(\sigma)|\sigma_s = y]^{1/2} - E_{n+1}[g(\sigma)|\sigma_s = x]^{1/2}\right]^2 \\
&\quad = \left[E_{n+1}[g(\tilde{\sigma}^{xy})|\sigma_s = x]^{1/2} - E_{n+1}[g(\sigma)|\sigma_s = x]^{1/2}\right]^2 \\
&\quad \le (n+1)\, E_{n+1}\left\{\left[g(\tilde{\sigma}^{xy})^{1/2} - g(\sigma)^{1/2}\right]^2;\ \sigma_s = x\right\}.
\end{aligned}
\tag{3.6}
$$

Combining (3.5) and (3.6), we have

$$
\mathrm{I}_{2,s} \le K \frac{\log n}{n-1} E_{n+1}\left\{\sum_{x,y=1}^{n+1}\left[g(\tilde{\sigma}^{xy})^{1/2} - g(\sigma)^{1/2}\right]^2\right\}.
$$

Recall the definition of $D_{n+1}(\sqrt{g})$. Averaging the $E_{n+1}\{\ \}$ part of the right-hand side gives $2D_{n+1}(\sqrt{g})$. Thus,

$$
\mathrm{I}_2 \le 2K \frac{\log n}{n-1} D_{n+1}\left(\sqrt{g}\right).
\tag{3.7}
$$

By the definition of the logarithmic Sobolev constant $a_{n+1}$, the three inequalities (3.3), (3.4) and (3.7) now imply

$$
a_{n+1} \le \frac{a_n}{n}(n-1) + 2K\frac{\log n}{n-1}.
$$

It remains to prove the desired upper bound by an induction on $n \ge 2$. The initial check for $n = 2$ follows from the $n = 1$ case of Lemma 1. Assuming that $a_n \le c \log n$, the last inequality yields

$$
\begin{aligned}
a_{n+1} &\le c(n-1)\frac{\log n}{n} + 2K\frac{\log n}{n-1} \\
&\le \left(c\frac{n-1}{n} + 2K\frac{1}{n-1}\right)\log n \\
&< c\log(n+1),
\end{aligned}
$$

provided $c = 4K = 2$ or larger. The proof is complete. □

**4. The proof of Theorem 5.** The proof resembles, in its structure, that of Theorem 1. Note that we have one more parameter $r$, the number of particles. Thus, $a_{n,r}$ will be treated as a function of $r$, $1 \le r \le n-1$, for each $n$.

Since any probability distribution on $C_{n,r}$ is a mixture of distributions associated with deterministic configurations, it is conceivable that the BL model will approach the uniform distribution the slowest when the initial is deterministic. We therefore choose the test function

$$f = \binom{n}{r} \delta_*,$$

where $\delta_*$ is the indicator function of a fixed configuration. However, it should be pointed out that a deterministic distribution in general does not produce the logarithmic Sobolev constant.

We then have

$$\begin{aligned} H_{n,r}(f) &= \log\binom{n}{r}, \\ D_{n,r}(\sqrt{f}) &= \frac{1}{2n}\binom{n}{r}^{-1}\left[\binom{n}{r} r(n-r)\right] = \frac{r(n-r)}{2n}, \\ a_{n,r} &\geq \frac{H_{n,r}(f)}{D_{n,r}(\sqrt{f})} = \frac{2n \log\binom{n}{r}}{r(n-r)}. \end{aligned}$$

The superscript BL in $D_{n,r}$ is omitted here and throughout Section 4. One needs to show the existence of $c$ such that

$$\frac{n \log\binom{n}{r}}{r(n-r)} > c^{-1} \log \frac{n^2}{r(n-r)}.$$

It suffices to show that a positive number $\varepsilon$ exists such that

$$\frac{1}{r} \log\binom{n}{r} > \varepsilon \log \frac{n}{r}, \qquad 1 \leq r \leq \frac{n}{2}. \tag{4.1}$$

There are two simple lower bounds. The first is

$$\frac{1}{r} \log\binom{n}{r} \geq \frac{1}{r} \log\left[\left(\frac{n-r}{r}\right)^r\right] = \log\left(\frac{n}{r} - 1\right), \qquad 1 \leq r \leq \frac{n}{3}.$$

The second one is that, if $n/2 \geq r \geq n/3$, then

$$\frac{1}{r} \log\binom{n}{r} \geq \frac{2}{n} \log\left(\begin{array}{c} n \\ \left[\frac{n}{3}\right] \end{array}\right) \geq \frac{2}{n}\left[\frac{n}{3}\right] \log 3,$$

where $[x]$ is the integer part of $x$. These two bounds yield (4.1), thus, the desired lower bound of the theorem.

We will derive an upper bound of $a_{n+1,r}$ in terms of $a_{n,r}$ and $a_{n,r-1}$. To this end, let us write $f_s$ for the marginal pmf of $\eta_s$, and $f(\cdot|\eta_s)$ for the conditional pmf given $\eta_s$, $1 \leq s \leq n+1$, both relative to $\pi$.

Similar to the proof of Theorem 1, define

$$
\begin{aligned}
\mathrm{I}_{1,s} &\equiv E_{n+1,r}\biggl\{ f_s(\eta_s)\, a_{n,r-\eta_s} E_{n+1,r} \\
&\qquad \times \biggl( \sum_{i,j:\ i\neq s,\ j\neq s} \Bigl[ f(\eta^{ij}|\eta_s)^{1/2} - f(\eta|\eta_s)^{1/2} \Bigr]^2 |\eta_s \biggr) \biggr\} \\
&= E_{n+1,r}\biggl\{ a_{n,r-\eta_s} \sum_{i,j\neq s} \Bigl[ f(\eta^{ij})^{1/2} - f(\eta)^{1/2} \Bigr]^2 \biggr\}, \qquad (4.2)\\
\mathrm{I}_{2,s} &\equiv E_{n+1,r}\{ f_s(\eta_s) \log f_s(\eta_s)\}, \\
\mathrm{I}_i &= \frac{1}{n+1} \sum_{s=1}^{n+1} \mathrm{I}_{i,s}, \qquad i = 1, 2.
\end{aligned}
$$

Simple calculation of conditional expectations gives

$$H_{n+1,r}(f) = E_{n+1,r}\{ f_s(\eta_s)\, E_{n+1,r}\{ f(\cdot|\eta_s) \log f(\cdot|\eta_s)|\eta_s\}\} + \mathrm{I}_{2,s}.$$

Observe that, if $\eta_s = 0$ (1, respectively), then the inner expectation concerns a BL model with parameters $n, r$ ($r-1$, respectively). By the definition of $a_{n,r}$, $a_{n,r-1}$ and $\mathrm{I}_{1,s}$, we now have

$$H_{n+1,r}(f) \le \frac{1}{2n} \mathrm{I}_{1,s} + \mathrm{I}_{2,s},$$

thus

$$H_{n+1,r}(f) \le \frac{1}{2n} \mathrm{I}_1 + \mathrm{I}_2, \qquad 2 \le r \le n-1. \tag{4.3}$$

A simple combinatorial calculation shows that

$$\sum_{s=1}^{n+1} \mathrm{I}_{1,s} = \bigl[ a_{n,r}(n-r) + a_{n,r-1}(r-1) \bigr] 2(n+1) D_{n+1,r}\bigl(\sqrt{f}\bigr),$$

where the two factors, $(n-r)$ and $(r-1)$, are simply the number of times that each term $[f(\eta^{ij})^{1/2} - f(\eta)^{1/2}]^2$ appears. Therefore,

$$\mathrm{I}_1 = 2\bigl[ a_{n,r}(n-r) + a_{n,r-1}(r-1) \bigr] D_{n+1,r}\bigl(\sqrt{f}\bigr). \tag{4.4}$$

Note that, for each $1 \le s \le n+1$, the process $\{\eta_s(t);\ t \ge 0\}$ is itself a continuous-time Markov chain on the state space $\{0, 1\}$. The jump rates $A_{ij}$ (from state $i$ to $j$) are $A_{0,1} = r/(n+1)$ and $A_{1,0} = (n+1-r)/(n+1)$. In order to estimate $\mathrm{I}_{2,s}$, we need a logarithmic Sobolev inequality for this simple Markov chain. This inequality is proved in [9, 5] and we stated it as Lemma 2.

LEMMA 2. *Let $h$ be a pmf on $\{0, 1\}$, relative to the distribution $(1-\rho, \rho)$, $0 < \rho < 1$. There exists a constant $B$, independent of $\rho$ and $h$, such that*

$$\begin{aligned}(1-\rho)&h(0)\log h(0) + \rho h(1)\log h(1)\\ &\le \frac{\log(1-\rho) - \log \rho}{(1-\rho)-\rho}\rho(1-\rho)\left[h(0)^{1/2} - h(1)^{1/2}\right]^2\\ &\le B\left(\log\frac{1}{\rho(1-\rho)}\right)\rho(1-\rho)\left[h(0)^{1/2} - h(1)^{1/2}\right]^2.\end{aligned}$$

*In fact, the best constant is $B = 1/\log 2$.*

Applying Lemma 2, with $\rho = r/(n+1)$, to $h = f_s$, we get

$$\mathrm{I}_{2,s} \le B\frac{r(n+1-r)}{(n+1)^2}\log\frac{(n+1)^2}{r(n+1-r)}\left[f_s(0)^{1/2} - f_s(1)^{1/2}\right]^2. \tag{4.5}$$

For $\eta$ with $\eta_s = 1$, let $\eta^{s*}$ be the random configuration obtained from $\eta$ by moving the particle in $s$ to an unoccupied site at random. Write $M_s$ for the corresponding expectation. Then

$$\begin{aligned} f_s(0) &= E_{n+1,r}[f|\eta_s = 0] = M_s E_{n+1,r}[f(\eta^{s*})|\eta_s = 1],\\ \left(f_s(0)^{1/2} - f_s(1)^{1/2}\right)^2 &= \Big\{M_s E_{n+1,r}[f(\eta^{s*})|\eta_s = 1]^{1/2}\\ &\qquad - E_{n+1,r}[f(\eta)|\eta_s = 1]^{1/2}\Big\}^2\\ &\le M_s E_{n+1,r}\left\{\left[f(\eta^{s*})^{1/2} - f(\eta)^{1/2}\right]^2|\eta_s = 1\right\},\end{aligned}$$

where the last inequality follows from the convexity of the function $(a, b) \to (a^{1/2} - b^{1/2})^2$, $a, b \ge 0$, and Jensen's inequality. Carrying out the $M_s$ expectation and summing over $s$, we get

$$\begin{aligned}\sum_{s=1}^{n+1}&\left[f_s(0)^{1/2} - f_s(1)^{1/2}\right]^2\\ &\le \frac{1}{n+1-r}\left(\frac{r}{n+1}\right)^{-1}\sum_{s=1}^{n+1}E_{n+1,r}\\ &\qquad\times\left\{\sum_{\eta^{s*}}\left[f(\eta^{s*})^{1/2} - f(\eta)^{1/2}\right]^2;\ \eta_s = 1\right\}\\ &= \frac{1}{n+1-r}\left(\frac{r}{n+1}\right)^{-1}2(n+1)D_{n+1,r}\left(\sqrt{f}\right)\\ &= \frac{2(n+1)^2}{(r(n+1-r)}D_{n+1,r}\left(\sqrt{f}\right).\end{aligned} \tag{4.6}$$

It follows from (4.2), (4.5) and (4.6) that

$$\mathrm{I}_2 \le \frac{2B}{n+1} \log \frac{(n+1)^2}{r(n+1-r)} D_{n+1,r}\big(\sqrt{f}\big). \tag{4.7}$$

Therefore, by (4.3), (4.4) and (4.7),

$$\begin{aligned} &H_{n+1,r}(f)/D_{n+1,r}\big(\sqrt{f}\big) \\ &\qquad \le \frac{1}{n}\big[a_{n,r}(n-r) + a_{n,r-1}(r-1)\big] + \frac{2B}{n+1}\log\frac{(n+1)^2}{r(n+1-r)}. \end{aligned}$$

By the definition of $a_{n+1,r}$, we have obtained

$$\begin{aligned} a_{n+1,r} &\le \frac{1}{n}\big[a_{n,r}(n-r) + a_{n,r-1}(r-1)\big] \\ &\qquad + \frac{2B}{(n+1)}\log\frac{(n+1)^2}{r(n+1-r)}, \qquad 2 \le r \le n-1. \end{aligned} \tag{4.8}$$

It remains to prove, by (4.8) and an induction on $n$, the upper bound in the theorem. The initial case $n = 2$, $r = 1$ follows from Lemma 2. Now, suppose $a_{n,k} \le 2B\log[n^2/k(n-k)]$, $1 \le k \le n-1$, $B = 1/\log 2$. Then, for $2 \le r \le n - 1$, (4.8) implies

$$\begin{aligned} a_{n+1,r} \le \frac{2B}{n}&\left[(n-r)\log\frac{n^2}{r(n-r)} + (r-1)\log\frac{n^2}{(r-1)(n+1-r)}\right] \\ &+ \frac{2B}{n+1}\log\frac{(n+1)^2}{r(n+1-r)}. \end{aligned}$$

Since the log function is concave, Jensen's inequality implies

$$\begin{aligned} a_{n+1,r} &\le \frac{2B(n-1)}{n}\log\frac{n^2(n+1)}{(n-1)r(n+1-r)} + \frac{2B}{n+1}\log\frac{(n+1)^2}{r(n+1-r)} \\ &= 2B\log\frac{(n+1)^2}{r(n+1-r)} \\ &\qquad \times \left\{\left(1-\frac{1}{n}\right)\left[1 + \left(\log\frac{n^2}{n^2-1}\right)\Big/\left(\log\frac{(n+1)^2}{r(n+1-r)}\right)\right] + \frac{1}{n+1}\right\} \\ &\le 2B\log\frac{(n+1)^2}{r(n+1-r)}\left\{\left(1-\frac{1}{n}\right)\left(1 + \left[(n^2-1)\log 4\right]^{-1}\right) + \frac{1}{n+1}\right\}. \end{aligned}$$

A simple calculation shows that the last bracket equals $1 - [1/n(n+1)](1 - 1/(2\log 2))$, and hence is less than 1.

The upper bound is proved except for the case $r = 1$ or $n$. Due to the particle–hole symmetry, two cases are identical. Take $r = 1$, which means that there is only one particle and there are $n+1$ sites. It follows from (3.1) and the Dirichlet form of the BL model that the logarithmic Sobolev constant is bounded by $[(n+1)/2(n-1)]\log n$, hence less than $2B\log[(n+1)^2/1\cdot n]$. The proof is completed.

**5. Proof of Theorem 6.** The upper bound (ii) is a consequence of Theorem 3 and (1.3) since $\log\log(2^n) = \log n + \log\log 2$.

For $\eta \in C_{n,r}$, we will write $P_\eta$ for the probability measure associated with the SE model with $\eta$ initially and $E_\eta, V_\eta$ for the corresponding expectation and variance. The subscript $\pi$ is used if the SE system is in equilibrium.

For the lower bound, we need the following.

LEMMA 3. *Let $N_T = \sum_i J(i/n)\eta_i(T)$. Then, there exist a function $J$, a configuration $\eta \in C_{n,r}$ and three positive constants $\varepsilon, \lambda, c$, independent of $n, r$, such that*

$$E_\eta(N_T) - E_\pi(N_T) \ge \varepsilon r e^{-\lambda T/n^2},$$
$$V_\eta(N_T) \le c^2 r,$$
$$V_\pi(N_T) \le c^2 r,$$

*for all $T > 0$. A possible value for $\lambda$ is $\lambda = 2\pi^2$.*

The proof of this lemma will be given at the end. We will use this lemma to prove the lower bound and, indeed, a stronger statement in which the $V^{(1)}$ norm replaces the $V^{(2)}$ norm in defining the time to stationarity in (1.2).

In view of the fact that Lemma 3 gives the variances and the difference of the mean values of $N_T$, it is clear that Chebyshev's inequality can be applied to yield a lower bound. The next paragraph contains the details.

Let $\mu_T$ be the law of the position at time $T$ under $P_\eta$. If $m > 0$, $T = (n^2/\lambda)\log(\varepsilon\sqrt{r}/m) > 0$, we next show that the initial $\eta$ of Lemma 3 gives

$$V^{(1)}(\mu_T/\pi) \ge 2 - 16c^2 m^{-2}. \tag{5.1}$$

Note that Lemma 3 and $T = (n^2/\lambda)\log(\varepsilon\sqrt{r}/m)$ imply that

$$\begin{aligned} E_\eta(N_T) - E_\pi(N_T) &\ge \varepsilon r \exp(-\lambda n^{-2} T) = m\sqrt{r} \\ &\ge m/(2c)V_\eta(N_T)^{1/2} + m/(2c)V_\pi(N_T)^{1/2}. \end{aligned}$$

Therefore the two events $F_\pi = \{|N_T - E_\pi(N_T)| < m/(2c)V_\pi(N_T)^{1/2}\}$ and $F_\eta = \{|N_T - E_\eta(N_T)| < m/(2c)V_\eta(N_T)^{1/2}\}$ are disjoint. Chebyshev's inequality implies

$$\begin{aligned} V^{(1)}(\mu_T/\pi) &\ge \left[P_\eta(F_\eta) - P_\pi(F_\eta)\right] + \left[P_\pi(F_\pi) - P_\eta(F_\pi)\right] \\ &\ge \left[(1 - 4c^2m^{-2}) - 4c^2m^{-2}\right] + \left[(1 - 4c^2m^{-2}) - 4c^2m^2\right] \\ &= 2 - 16c^2m^{-2}. \end{aligned}$$

This is (5.1). If one then chooses $m = 4c(2 - e^{-1})^{-1/2}$, then $2 - 16c^2 m^{-2} = e^{-1}$ and

$$V^{(2)}(\nu_T/\pi) \geq V^{(1)}(\nu_T/\pi) \geq e^{-1}.$$

Thus, $\tau_{n,r}^{SE} \geq T = (n^2/\lambda)\log[\varepsilon(2 - e^{-1})^{1/2} r^{1/2}/4c]$. The desired lower bound of $\tau_{n,r}^{SE}$ then follows, except for small $r$, say $r \leq 64c^2/\varepsilon^2(2 - e^{-1})$. This missing case is easy since $\varepsilon$ does not depend on $n$, and by (1.3) and Theorem 4, the time to stationarity is at least the order of $n^2$ for a fixed number of particles.

PROOF OF LEMMA 3. We point out that if $r$ is the order of $n$, then Lemma 3 verifies (2.1) for the SE model and the above proof of Theorem 6 uses the idea following (2.1).

Lemma 3 is, in fact, valid for nearly all functions $J$. One such function suffices for our purpose of the lower bound. Note that, for a simple random walk on $\mathbb{Z}/n\mathbb{Z}$, the first two eigenvalues are 0 and $-\lambda_n = -(1 - \cos(2\pi/n))$ and the corresponding eigenfunctions are the constant function and $f_n(k) = \sin(2k\pi/n)$, $k \in \mathbb{Z}/n\mathbb{Z}$. The function

$$J(k/n, s) = \exp(-\lambda_n s) f_n(k), \qquad J(k/n) = J(k/n, 0) = f_n(k)$$

is particularly amenable to our end.

Note that $J(i/n, s)$ satisfies

$$(\partial_s - \Delta_n/2)\, J(i/n, s) = 0,$$

where

$$\Delta_n J(i/n, s) = J((i+1)/n, s) - 2J(i/n, s) + J((i-1)/n, s).$$

For any $T > 0$, Itô's calculus for the particle system then implies

$$M_{n,T} = \sum_i J(i/n, 0)\eta_i(T) - \sum_i J(i/n, T)\eta_i(0),$$

is a martingale with mean 0 and variance given by

$$\begin{aligned} E\big(M_{n,T}^2\big) &= E\bigg\{\int_0^T \sum_i [J((i+1)/n, T-s) - J(i/n, T-s)]^2 \eta_i(s)[1 - \eta_{i+1}(s)] \\ &\qquad + [J((i-1)/n, T-s) - J(i/n, T-s)]^2 \\ &\qquad\qquad \times \eta_i(s)[1 - \eta_{i-1}(s)]\big)\, ds\bigg\}\Big/2. \end{aligned}$$

From the definition of $N_T$ and the fact $E(M_{n,T}) = 0$,

$$E(N_T) = \exp(-\lambda_n T) \sum_{k=1}^{n} \sin(2k\pi/n)\eta(k) \geq \varepsilon r \exp(-\lambda_n T),$$

for some positive $\varepsilon$ if we take $\eta$ with the $r$ particles crowding in the middle of the set $\{1, 2, \ldots, n/2\}$. Since $-\lambda_n \geq -2\pi^2/n^2$, the right-hand side gives the desired lower bound with $\lambda = 2\pi^2$.

The variance of $N_T$ equals that of $M_{n,T}$. Hence

$$V_\eta(N_T) = E\big(M_{n,T}^2\big) \le c^2 r,$$

for some constant $c$ since there are $r$ particles and

$$\int_0^T [J((i+1)/n, T-s) - J(i/n, T-s)]^2 \, ds$$
$$\le (2\pi/n)^2 \int_0^T \exp(-2\lambda_n t)\, dt \le 2\pi^2/\big(n^2\lambda_n\big),$$

independent of $T$. Since $E_\pi(N_T) = 0$, $E_\pi(\eta_i\eta_j) = r(r-1)/n(n-1)$, $i \ne j$ and $r/n$, $i = j$, it is not difficult to see that the variance of $N_T$ is of order $r$ in equilibrium. Thus, $c$ can be chosen so that $c^2 r$ bounds both variances. This completes the proof of Lemma 3. □

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