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FINITE APPROXIMATIONS TO THE CRITICAL REVERSIBLE NEAREST PARTICLE SYSTEM

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Approximating a critical attractive reversible nearest particle system on a finite set from above is not difficult, but approximating it from below is less trivial, as the empty configuration is invariant. We develop a finite state Markov chain that deals with this issue. The rate of convergence for this chain is discovered through a mixing inequality in Jerrum and Sinclair; an application of that spectral gap bound in this case requires the use of "randomized paths from state to state." For applications, we prove distributional results for semiinfinite and infinite critical RNPS.

1. Introduction. We consider attractive reversible critical Feller nearest particle systems (NPS). Nearest particle systems were introduced by Spitzer (1977). They are spin systems on $\{0,1\}^{\mathbf{Z}}$ with transitions as follows: if $\xi(x) = 1$—that is, if site x is occupied—then the flip rate $c(\xi, x)$ is 1; if $\xi(x) = 0$—that is, if site x is vacant—then

$$c(\xi, x) = f(l_x, r_x),$$

where

$$l_x = x - \sup\{y < x \colon \xi(y) = 1\}$$

and

$$r_x = \inf\{y > x \colon \xi(y) = 1\} - x.$$

We are interested in processes that are reversible, attractive, Feller and critical. We now explain what each of these properties requires of the flip rate function $f(l, r)$.

Reversible. A NPS is called reversible if $f(l, r)$ is of the form

$$f(l, r) = \frac{\beta(l)\beta(r)}{\beta(l + r)} \tag{1.1}$$

with $f(\infty, r) = f(r, \infty) = \beta(r)$. Notice that if our initial configuration ξ_0 has $\sum_{x>0} \xi_0(x) = \sum_{x<0} \xi_0(x) = \infty$, then $f(\infty, r)$ and $f(r, \infty)$ are irrelevant and requiring the reversibility condition (1.1) does not fully specify $\beta(\cdot)$.

Attractive. It is easily seen that our process will be attractive iff $\beta(n)/\beta(n+1)$ is a decreasing function on n and it decreases to a limit greater than or equal to one.

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Feller. For an attractive reversible NPS to be Feller (for $\{0,1\}^{\mathbf{Z}}$ equipped with the usual product topology), it is necessary and sufficient that

$$\frac{\beta(n)}{\beta(n+1)} \searrow 1.$$

Critical. Under the above conditions, there exists a nontrivial stationary distribution for the NPS if and only if either $\sum_{n\ge 1}\beta(n) > 1$ or $\sum_{n\ge 1}\beta(n) = 1$ and $\sum_{n\ge 1} n\beta(n) < \infty$. In the former case, the NPS is called supercritical, while in the latter case, it is called critical. In the critical case, the upper invariant measure is the renewal measure on $\{0,1\}^{\mathbf{Z}}$ associated with the probability density β and will be denoted by ν_β.

Detailed discussion of the above issues can be found in Liggett (1985). To summarize, we will consider spin systems on $\{0,1\}^{\mathbf{Z}}$ with the following flip rates: if $\xi(x) = 1$, then $c(\xi, x) = 1$; if $\xi(x) = 0$, then

$$c(\xi, x) = \frac{\beta(l_x)\beta(r_x)}{\beta(l_x + r_x)},$$

where

$$\frac{\beta(n)}{\beta(n+1)} \searrow 1, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \beta(n) = 1 \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} n\beta(n) < \infty.$$

In addition, we will impose a strong regularity condition on β: there exists an increasing function $L(n)$ such that

$$\lim_{n\to\infty} \frac{n}{L(n)}\left(\frac{\beta(n)}{\beta(n+1)} - 1\right) = 1 \tag{1.4}$$

and

$$L(1) > 5.$$

The condition $L(1) > c$ implies the moment condition $\sum n^{c-1}\beta(n) < \infty$. Notice that requiring such a function $L(n)$ with $L(1) > c$ is the same as requiring such an $L(n)$ with $\lim_{n\to\infty} L(n) > c$. We denote the limit of $L(n)$ by L. Condition (1.4) is by no means the weakest to allow our arguments to work, but it is natural, as it includes the cases $\beta(n) = \lambda/n^p$ (for large enough p), and it allows a clear exposition of the underlying ideas.

It is natural to consider critical NPS as Mountford (1997a) provides a complete convergence theorem for supercritical reversible attractive NPS, while Mountford (1995) shows that subcritical attractive reversible NPS converge exponentially in distribution to $\delta_{\mathbf{0}}$, where $\mathbf{0}$ is the null configuration given by $\mathbf{0}(i) = 0$. For the critical case, long term distributional behavior was unresolved in the case where ξ_0 is Bernoulli(p) for small p, for instance. Our finite approximations allow one to prove (under strong moment conditions on β) that the limiting distribution is ν_β in that case [Mountford (1997b)].

In considering a NPS ξ_t on the a site interval $(-n, n)$, one can make a simple comparison with a finite state Markov chain Z_t^n whose initial state is given by

$$Z_0^n(x) = \begin{cases} 1, & \text{if } |x| \geq n, \\ \xi_0(x), & \text{if } |x| < n \end{cases}$$

and where the evolution proceeds according to the Harris construction of ξ_t subject to the values of $Z_t^n(x)$ remaining fixed through time for $|x| \geq n$. By the Harris construction of the process, we mean the following: each integer has a Poisson process of rate 1 corresponding to deaths at that site if a particle is there at the time. Each integer has an independent Poisson process of rate $\beta(1)^2/\beta(2)$—the maximum flip rate—or of rate 1, whichever is larger; each integer also has an independent sequence of uniform i.i.d. r.v.'s over $[0, 1]$. If, at the ith point of the Poisson process, the site is vacant and the ith uniform r.v. is less than the ratio between the flip rate at that site and the Poisson process rate, the site becomes occupied. By attractiveness, the processes are coupled in this standard construction so that $\xi_t \leq Z_t^n$ [in the pointwise sense that $\xi_t(x) \leq Z_t^n(x)$ for all x] for all t.

For a fixed finite interval I we can take n so large that the stationary distribution of Z^n, restricted to I, is as close to ν_β as required. The natural question "For fixed n, how large does t have to be so that Z_t^n has distribution close to its stationary distribution?" was essentially answered in Sweet (1997).

To meaningfully approximate ξ_t on I from below by a finite state NPS is not so easy since δ_0 is invariant. To create such a finite system, we must keep the nearest particles to points in the interval inside the interval. To do this, we fix particles at $-n$ and n. In creating Z^n, this is all that is done. It has the effect of "moving the nearest particle closer" which leads to an upper approximation. What we would like to do here is "move the nearest particle further away" which would lead to a lower approximation. To do this, we ignore the particles near the boundary on the inside of the interval. That is, while we ignore deaths on $-n$ and n, we also ignore births on $(-n, -n + n^\alpha) \cup (n - n^\alpha, n)$.

Specifically, the approximation that we explore here is the following [for $I \subset (-n/2, n/2)$]: fix $\alpha \in (0, 1/2)$ (we will discuss this parameter shortly) and let X^n be given by:

1. $X_t^n(x) = 1$ for all $|x| \geq n$ and all $t \geq 0$;
2. $X_t^n(x) = 0$ for all $x \in (-n, -n + n^\alpha) \cup (n - n^\alpha, n)$ and all $t \geq 0$;
3. $X_0^n(x) = \xi_0(x)$ for all $x \in [-n + n^\alpha, n - n^\alpha]$;
4. X_t^n evolves on $[-n + n^\alpha, n - n^\alpha]$ according to the Harris construction of ξ_t for $t > 0$.

At times, it will be convenient to regard X^n (or Z^n) as a process on $\{0,1\}^{[-n,n]}$, which we will do without further comment. Obviously, we cannot make the global statement $X_t^n \le \xi_t$ [on $(-n, n)$] for all t because the flip rates become better for X^n if, for example, ξ_t becomes empty on $[-n, n]$. However, attractiveness does tell us that the processes are coupled so that

$$X_t^n \le \xi \quad \text{on } I$$

for $t \le \tau$ where

$$\tau = \inf\{t: \xi_t = 0 \text{ on either } [-n, -n + n^\alpha) \text{ or } (n - n^\alpha, n]\}.$$

To see this, simply notice that no jump before time τ can make a nearest particle to a point in I closer under X_t^n than under ξ_t. In other words, for $t \le \tau$, the process does exactly what we wanted: it "moves the nearest particle" of a point in I to n from somewhere inside $(n - n^\alpha, n]$ (if the nearest particle was there). Once there are no particles to "move," which happens at time τ, this breaks down.

Of course, if X^n gives away too much, then it will not be very useful. Detailed balance tells us that the invariant measure is simply renewal measure conditioned on the particular occupancies and vacancies that have been enforced. If α is too large, then we are conditioning on too many vacancies to expect this invariant measure to look like the ν_β near the origin. Specifically, if $\alpha \ge 1/2$, then the conditioning may persuade the renewal measure to simply makes one very big jump from $-n$ to n rather than two big jumps of size at least n^α. We choose $\alpha < 1/2$ to avoid this problem.

So, the key now is whether the coupling holds long enough, that is, whether τ is big enough, for X^n to serve its purpose. While we need $\alpha < 1/2$ for the stationary measure to be useful, we want α as large as possible so that the coupling will hold for as long as possible. Take $\varepsilon > 0$ and let $\alpha = 1/2 - \varepsilon$. In the proofs, we will choose ε as small as is necessary.

We will actually require an auxiliary chain to analyze X^n. Let $\tilde{X}^n$ be a similar Markov chain on the state space $\{0,1\}^{[-n,n]} \setminus \{\mathbf{0}\}$ where (in a minor abuse of notation) the null configuration $\mathbf{0}$ here is given by $\mathbf{0}(i) = \delta_n(i) + \delta_{-n}(i)$. Then $\tilde{X}^n$ has identical flip rates to X^n subject only to the configuration $\mathbf{0}$ being forbidden. This process is introduced because it has good spectral gap properties (Proposition 3.1), which is vital for a (temporary) lower approximation to be useful. However, X^n does not behave as well; it takes a long time to leave $\mathbf{0}$ since the sites that would flip relatively quickly are forced to remain vacant.

Before ending the section with an outline of subsequent sections, we make a couple of definitions. Given a configuration η and site x, we let η^x be the configuration obtained by changing the value of η at x: $\eta^x(y) = \eta^x(y)$ for $y \ne x$ while $\eta^x(x) = 1 - \eta(x)$. We write $f(n) \preccurlyeq g(n)$ iff there exists a constant k such that $f(n) \le kg(n)$ for all n. We define $f(n) \succcurlyeq g(n)$ similarly. Finally, we write $f \asymp g$ iff $f(n) \preccurlyeq g(n)$ and $f(n) \succcurlyeq g(n)$.

The next section contains a couple of lemmas that relate the finite state processes to the particle system on $\mathbf{Z}$. Section 3 contains the main result of the paper; we will prove that the spectral gap for $\tilde{X}^n$ is at least of order n^{-2}. The given order is the correct one; the proof of the converse—the gap is at most of order n^{-2}—is in Sweet (1997). This means that if the distribution of $\tilde{X}_0^n$ is not too extreme, then $\tilde{X}_{n^{2+\varepsilon}}^n$ should be very close in distribution to $\tilde{\pi}^n$, which is itself (by Proposition 2.3 in Section 2) close to ν_β. In Section 4, we justify the use of $\tilde{X}^n$ instead of X^n by establishing that the chance of X_t^n hitting the null configuration $\mathbf{0}$ in time $n^{2+\varepsilon}$ (starting from stationary) is very small. This and the preceding work will allow us to conclude that $X_{n^{2+\varepsilon}}$ is close to ν_β in distribution when starting from a "reasonable" configuration.

The last two sections each contain an application of this finite state approximation and its rate of convergence. It is here that we state results dealing with how long the coupling will hold (i.e., lower bounds for τ in probability). We wait until the application sections to do this because the particular manner of dealing with this issue will depend on the application at hand. In Section 5 we use the preceding results to prove the following result.

THEOREM 1.1. *Let ξ_t be a critical attractive reversible Feller NPS satisfying* (1.4) *with* $L(1) > 6$. *Let ξ_0 be distributed by renewal measure* (*with* β) *on* $(-\infty, 0]$ [*with* $\xi_0(0) = 1$] *and let* $\xi_0 = 0$ *on* $(0, \infty)$. *Then, as* $t \to \infty$,

$$\xi_t \to_D \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\nu_\beta.$$

This result was conjectured by Schinazi (1992) (though we have made large assumptions on the regularity of β and the corresponding moments). With ξ_0 distributed as in the theorem, Schinazi (1992) proved that (if β has a finite second moment) the rightmost particle of the process started from ξ_0, renewal measure on the half line, has Brownian fluctuations; that is, $r_{N^2 t}/N$ converges in distribution to a Brownian motion, where $r_s = \sup\{x\colon \xi_s(x) = 1\}$.

From this fact it is clear that any limiting measure μ of ξ_t must be stochastically less than $\frac{1}{2}\delta_0 + \frac{1}{2}\nu_\beta$. The difficulty lies in establishing that if $r_t > 0$ then ξ_t looks like ν_β around the origin. Since renewal measure on the half line is invariant for the process seen from the right edge [Schinazi (1992)], attractiveness implies that on $\{r_t > 0\}$ the configuration $\bar{\xi}_t(x) = \xi_t(x + r_t)$ should be at least renewal measure. However, we are interested in the behavior of ξ_t around the origin, that is, the configuration $\bar{\xi}_t$ shifted a random amount relative to r_t. To put this another way, we are interested in ξ_t given that $r_t = x$; this conditioning event is no longer increasing (and hence the attractiveness argument does not apply).

A second use in applying our comparison methods is to establish that for ν_β-almost all ξ_0, $\xi_t \to_D \nu_\beta$. In Section 6, we sketch a proof of the following.

THEOREM 1.2. *Let ξ_t be a critical, reversible, attractive, Feller NPS satisfying* (1.4) *with* $L(1) > 5$. *Then for almost all* ξ_0 (*with respect to* ν_β),

$$S(t)\xi_0 \to_D \nu_\beta.$$

where $S(t)$ *is the semigroup for the process.*

If, instead of $S(t)\xi_0$, we considered $t^{-1}\int_0^t S(s)\xi_0\,ds$, then soft ergodic arguments yield the desired convergence without the added conditions on β. Theorem 1.2 is similar to Theorem 1.13 of Liggett (1991). Our moment conditions are weaker and we do not require that β be totally positive of order 3, but apart from this we require more regularity for β.

2. The finite processes. In this section we present some elementary calculations concerning the stationary distributions π^n and $\tilde{\pi}^n$ for X^n and $\tilde{X}^n$, respectively. First we require a few definitions. Given a sequence of β-renewal points $0 = z_0 < z_1 < z_2 < \cdots$, let $A(i) = P(\exists\ m,\ z_m = i)$. It is well known that $\lim_{i\to\infty} A(i) = \mu^{-1} = (\sum n\beta(n))^{-1}$ and therefore that $\sup A(i) < 1$ and $\inf A(i) > 0$. Let

$$\bar{\beta}(m) = \sum_{i=m}^{\infty} \beta(i).$$

This notation is useful since

$$\nu_\beta(\xi(0) = 1, \xi(x) = 0 \text{ for } x \in [1, m-1]) = \mu^{-1}\bar{\beta}(m).$$

Notice that if $L < \infty$, then

$$m^{-(L+\delta)} \preccurlyeq \beta(m) \preccurlyeq m^{-(L-\delta)} \tag{2.1}$$

and

$$m^{-(L+\delta-1)} \preccurlyeq \bar{\beta}(m) \preccurlyeq m^{-(L-\delta-1)} \tag{2.2}$$

for any $\delta > 0$, while if $L = \infty$, then

$$\beta(m) \preccurlyeq m^{-M} \tag{2.3}$$

for any $M > 0$.

As noted in the Introduction, a configuration ξ can be identified with its occupied sites. In the following, given a configuration ξ, $-n = x_0 < x_1 < \cdots < x_r = n$ will denote the occupied sites of ξ.

LEMMA 2.1.

$$\pi^n(\xi) = \frac{1}{C(n)} \prod_{i=1}^{r} \beta(x_i - x_{i-1})$$

where $C(n) \asymp \bar{\beta}(n^\alpha)^2$. A similar statement holds for $\tilde{\pi}^n(\xi)$.

PROOF. That $(C(n))^{-1}\prod_{i=1}^r \beta(x_i - x_{i-1})$ is a stationary distribution [where $C(n)$ is the normalizing constant] follows simply from the detailed balance equations. It only remains to show that normalizing constant $C(n)$ is of the order claimed:

$$C(n) = \beta(2n) + \sum_{-n+n^\alpha < l \le r < n-n^\alpha} \beta(l+n)A(r-l)\beta(n-r).$$

Now the second term on the right-hand side is clearly of order $\overline{\beta}(n^{\alpha})^2$ and so we must show that

$$\frac{\beta(2n)}{\overline{\beta}(n^{\alpha})^2} \to 0 \quad \text{as } n \to \infty.$$

We will actually prove

$$\frac{\beta(an)}{\beta(n^{\alpha})^2} \preccurlyeq n^{-\varepsilon} \tag{2.4}$$

for any fixed a. For simplicity, we will show this for $\beta(n)/\beta(n^{\alpha})^2$ (though the proof for general a is essentially the same).

Now there exists some γ so that

$$(1-\gamma)^2(1-\alpha) - \alpha(1+\gamma) > 1 - 2\alpha - 9\varepsilon/5 = \varepsilon/5.$$

Clearly there exists a positive integer N such that

$$1 + \frac{(1-\gamma)L(m)}{m} \le \frac{\beta(m)}{\beta(m+1)} \le 1 + \frac{(1+\gamma)L(m)}{m}$$

for $m \ge N$. Since $L(n)/n \to 0$ [as $\beta(n)/\beta(n+1) \to 1$], we can further assume that

$$1 - \frac{L(m)(1-\gamma)}{2m} > 1 - \gamma$$

and

$$\frac{(1+\gamma)L(m)}{m} < 1$$

for $m \ge N$. Now, let $\beta(0) = 1$ for temporary convenience and let K be the constant $2\sum_{i=1}^{i=N-1} \ln[\beta(i+1)/\beta(i)]$. Then for $n > N$,

$$\begin{aligned}
\ln \beta(n^{\alpha}) &= \sum_{i=1}^{n^{\alpha}-1} \ln \frac{\beta(i+1)}{\beta(i)} \\
&\ge -\sum_{i=N}^{n^{\alpha}-1} \ln\left(\frac{(1+\gamma)L(i)}{i} + 1\right) + K \\
&\ge -\sum_{i=N}^{n^{\alpha}-1} \frac{(1+\gamma)L(i)}{i} + K \\
&\ge -(1+\gamma)L(n^{\alpha}) \sum_{i=N}^{n^{\alpha}-1} \frac{1}{i} + K \\
&\ge -\alpha(1+\gamma)L(n^{\alpha})\ln(n) + K,
\end{aligned}$$

while

$$
\begin{aligned}
\ln[\beta(n)/\beta(n^\alpha)] &\le -\sum_{i=n^\alpha}^{n} \ln\left(\frac{(1-\gamma)L(i)}{i}+1\right) \\
&\le -\sum_{i=n^\alpha}^{n}\left[\frac{(1-\gamma)L(i)}{i}-\frac{1}{2}\left(\frac{(1-\gamma)L(i)}{i}\right)^2\right] \\
&\le -\sum_{i=n^\alpha}^{n}\frac{(1-\gamma)L(i)}{i}(1-\gamma) \\
&\le -(1-\gamma)^2 L(n^\alpha)(\ln(n)-\ln(n^\alpha)) \\
&= -(1-\gamma)^2(1-\alpha)L(n^\alpha)\ln(n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\ln\left[\beta(n)/\beta(n^\alpha)^2\right] &\le -K - L(n^\alpha)\ln(n)\left[(1-\gamma)^2(1-\alpha)-\alpha(1+\gamma)\right] \\
&\le \ln(n^{-L(n^\alpha)\varepsilon/5}) - K \\
&\le \ln(n^{-\varepsilon}) - K
\end{aligned}
$$

and so (2.4) follows. This proves the result for π^n and the proof for $\tilde{\pi}^n$ is similar. □

LEMMA 2.2. *As n goes to infinity, the probability of $\{\xi: (-n, -n/3)$ is ξ-vacant$\}$ converges to 0 under π^n and under $\tilde{\pi}^n$.*

PROOF. Notice that the result for π^n immediately implies the corresponding result for $\tilde{\pi}^n$. Now by Lemma 2.1,

$$
\pi^n((-n,-n/3) \text{ is vacant}) = \frac{\beta(2n) + \sum_{l=-n/3}^{n-n^\alpha-1}\beta(n+l)R_l^n}{C(n)},
$$

where

$$
R_l^n = \sum_{n-n^\alpha > r \ge l} A(r-l)\beta(n-r) \le \bar{\beta}(n^\alpha)
$$

and $C(n) \asymp \bar{\beta}(n^\alpha)^2$. So,

$$
\begin{aligned}
\pi^n((-n,-n/3) \text{ is vacant}\}) &\asymp \frac{\beta(2n)}{\bar{\beta}(n^\alpha)^2} + \frac{\bar{\beta}(n^\alpha)\sum_{l=-n/3}^{n-n^\alpha-1}\beta(n+l)}{\bar{\beta}(n^\alpha)^2} \\
&\le \frac{\beta(2n)}{\bar{\beta}(n^\alpha)^2} + \frac{n\beta(2n/3)}{\bar{\beta}(n^\alpha)},
\end{aligned}
$$

which converges to 0 by (2.4).

PROPOSITION 2.3. *Let f be any cylinder function. Then both $\langle\tilde{\pi}^n, f\rangle$ and $\langle\pi^n, f\rangle$ converge to $\langle\nu_\beta, f\rangle$ as $n\to\infty$.*

PROOF. We only write out the proof for $\langle \pi^n, f \rangle$. It is only necessary to prove this result for f increasing with $f(\mathbf{0}) = 0$. If $\nu_\beta^{-n,n}$ denotes the measure on $\{0,1\}^{(-n,n)}$ given by ν_β conditioned on $\xi(n) = \xi(-n) = 1$, then by attractiveness,

$$\langle \pi^n, f \rangle \le \langle \nu_\beta^{-n,n}, f \rangle \to \langle \nu_\beta, f \rangle.$$

So it is sufficient to show that

$$\liminf_{n\to\infty} \langle \pi^n, f \rangle \ge \langle \nu_\beta, f \rangle.$$

Let

$$l(\xi) = \inf\{i > -n : \eta(i) = 1\},$$
$$r(\xi) = \sup\{i < n : \eta(i) = 1\},$$

$$\begin{aligned}
\langle \pi^n, f \rangle &= \int \pi^n(d\xi) f(\xi) \\
&\ge \sum_{l < -n/2} \sum_{r > n/2} \int 1_{l(\xi)=l,\, r(\xi)=r} \pi^n(d\xi) \langle \nu_\beta^{l,r} f \rangle \\
&\ge \pi^n(\{\xi : l(\xi) < -n/2,\, r(\xi) > n/2\}) \langle \nu_\beta, f \rangle \\
&= (1 + o(1)) \langle \nu_\beta, f \rangle,
\end{aligned}$$

where the third step follows from attractiveness and the last step follows from Lemma 2.2. □

PROPOSITION 2.4. *Consider ν_β restricted to $(-n, n)$, which we now denote (abusing notation) also as ν_β. We can couple ν_β and π^n as the law of $(\xi, \eta) \in \{0,1\}^{(-n,n)} \times \{0,1\}^{(-n,n)}$ so that there is a set $B \subset \{0,1\}^{(-n,n)}$ with $\nu_\beta(B) \preccurlyeq n^{-1/2}$ such that $\xi \ge \eta$ when $\xi \notin B$.*

PROOF. Recall that $\mu^{-1} = \lim A(i)$. The chance under ν_β that the interval $[-n, -n + n^\alpha]$ is unoccupied is equal to

$$\begin{aligned}
\frac{1}{\mu} \sum_{x < -n} \bar\beta(-n + n^\alpha - x + 1) &\preccurlyeq \sum_{x=-\infty}^{-n-1} \sum_{i=-n+n^\alpha-x+1}^{\infty} \beta(i) \\
&= \sum_{i=n^\alpha+2}^{\infty} (i - (n^\alpha + 1)) \beta(i) \\
&\preccurlyeq \bar\beta(n^\alpha) n^\alpha \\
&\preccurlyeq n^{\alpha(-L+1+\delta)} n^\alpha \\
&\preccurlyeq n^{-1/2},
\end{aligned}$$

where the second to last step used (2.2) and (2.3) and the last step is valid using $\delta = L - 5$ and $\varepsilon < 1/4$, for instance. Let $B = \{\xi : l(\xi) > -n + n^\alpha$ or $r(\xi) < n - n^\alpha\}$. By the above calculation, $\nu_\beta(B) \preccurlyeq n^{-1/2}$. As before we can write ν_β conditioned on B^c as the convex sum $\sum_{l=-n}^{-n+n^\alpha} \sum_{r=n-n^\alpha}^{n} \nu_\beta 1_{l(\xi)=l,\, r(\xi)=r}$. By attractiveness, $\nu_\beta 1_{l(\xi)=l,\, r(\xi)=r}$ can be coupled stochastically above π^n and so the result follows. □

3. The spectral gap estimate. In this section, we establish the spectral gap estimates for the Markov chain $\tilde{X}^n$. Recall that $\tilde{X}^n$ is obtained by permanently fixing ones at $-n$ and n, permanently fixing zeros on the intervals $(-n, -n+n^\alpha)$ and $(n-n^\alpha, n)$ and letting the sites in $[-n+n^\alpha, n-n^\alpha]$ behave according to the nearest particle system, subject to the null configuration $\mathbf{0}$ never being hit. Further recall that

$$\tilde{\pi}^n(\eta) = c_n^{-1} \prod_{i=1}^{r} \beta(x_i - x_{i-1}),$$

where $-n = x_0 < x_1 < \cdots < x_{r-1} < x_r = n$ are the occupied sites of η and c_n is the appropriate normalizing constant which satisfies

$$c_n \asymp \bar{\beta}(n^\alpha)^2.$$

PROPOSITION 3.1. *Let* $\mathrm{Gap}(\tilde{X}^n)$ *be the spectral gap of the chain* $\tilde{X}^n$. *Then*

$$\mathrm{Gap}(\tilde{X}^n) \succcurlyeq n^{-2}.$$

As was mentioned in the introduction, the easier upper bound of $\mathrm{Gap}(\tilde{X}^n) \preccurlyeq n^{-2}$ when $L < \infty$ can be found in Sweet (1997). It uses the well-known fact [see Diaconis and Stroock (1991), for example] that

$$\mathrm{Gap}(\tilde{X}^n) = \inf \frac{\sum_{\eta, x}(f(\eta) - f(\eta^x))^2 \tilde{\pi}^n(\eta) c(\eta, x)}{2\,\mathrm{Var}(f)}, \tag{3.1}$$

where the infimum runs over nonconstant f. This paper will only be concerned with the more significant (and difficult) bound: $\mathrm{Gap}(\tilde{X}^n) \succcurlyeq n^{-2}$. Before discussing that proof, let us make two remarks. One is that the rate n^2 for the chain $\tilde{X}^n$ to reach stationary matches the intuition coming from the following result of Schinazi (1992): starting from renewal measure on the half line, the edge has Brownian fluctuations (and, in particular, the time for particles to reach sites beyond n is of order n^2). The second remark is that $\mathrm{Gap}(\tilde{X}^n)$ can be much smaller than n^{-2}, which is why we had to introduce a process which does not hit the null configuration. To see this, take $\beta(l) = \lambda / l^\rho$; apply (3.1) to X^n with $f = 1_{\{\mathbf{0}\}}$; then

$$\mathrm{Gap}(X^n) \preccurlyeq c(\{\mathbf{0}\}, \{\mathbf{0}\}^c)\pi^n(\{\mathbf{0}\}^c)^{-1} \asymp \bar{\beta}(n^\alpha),$$

where $c(\{\mathbf{0}\} \cdot \{\mathbf{0}\}^c)^{-1}$ is the rate of leaving the null configuration.

Returning to the method of proof for the lower bound, we use the abstract canonical paths technique for reversible chains of Jerrum and Sinclair (1990) as found in Diaconis and Stroock (1991). In considering critical NPS, it is natural to make use of this approach. For example, Griffeath and Liggett (1982) used the Dirichlet principle for reversible Markov chains and clever comparison arguments to show that a finite reversible NPS dies out almost surely iff $\sum \beta(n) \leq 1$. This approach has also been very successful in yielding results for the stochastic ising model [see, e.g., Schonmann (1994)] where dynamics are hard to describe but the invariant measures and invariant measures with boundary conditions are known.

The path argument technique has been around in some form for many years, but the paper of Jerrum and Sinclair (1990) has recently made its usefulness clear. The idea is to create a path from η to ξ for every ordered pair (η, ξ) of states in the state space. By a path from η to ξ of length m, we mean a sequence of states $\eta = \eta_0, \eta_1, \dots, \eta_m = \xi$ such that $q_{\eta(i-1)\eta(i)} > 0$ for all $i \le m$. We say that an edge (λ, λ^x) is in the path if $\lambda = \eta_{i-1}$ and $\lambda^x = \eta_i$ for some i. If paths which do not create many "bottlenecks" can be chosen, then the chain should be able to converge quickly. This is formalized into our setting by the following theorem, one of many similar results, where the aforementioned paths are randomized.

THEOREM 3.2 (Jerrum and Sinclair). *For each pair of configurations in the state space of $\tilde{X}^n$, choose a set of paths $\gamma_i(\eta, \xi)$ from η to ξ and assign probabilities $P_{(\eta,\xi)}(i)$. Then the spectral gap is bounded below by $(\max_{\lambda,x} \kappa^{(\lambda,x)})^{-1}$, where*

$$\kappa^{(\lambda,x)} = \sum_{(\eta,\xi,i)} |\gamma_i(\eta,\xi)| \frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)c(\lambda,x)} P_{(\eta,\xi)}(i),$$

with $|\gamma|$ denoting the length of the path and with the sum running over (η, ξ, i) such that $\eta \neq \xi$ and that (λ, λ^x) is in the path $\gamma_i(\eta, \xi)$.

So, provided that the paths are chosen to have lengths bounded uniformly by a constant multiple of n, the task here is to show

$$\sup_{(\lambda,x)} \sum_{(\eta,\xi)} \frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)c(\lambda,x)} P_{(\eta,\xi)}(\lambda,x) \preccurlyeq n, \tag{3.2}$$

where $P_{(\eta,\xi)}(\lambda, x)$ is the probability that (λ, x) is used in a randomly chosen path from η to ξ.

Before starting the proof by explicitly describing the paths to be used, let us take a moment to compare this situation to an easier one. In Sweet (1997), similar analysis is done to the Markov chain Z^n that just fixes particles on $-n$ and n and allows $(-n, n)$ to evolve. There, the paths from η to ξ are deterministic and easily described: move from left to right on the interval of sites, changing the value of a site if its values on η and ξ differ.

Such an approach cannot work on $\tilde{X}^n$. First, such a path may pass through the "forbidden" null configuration. Furthermore, the distance between the leftmost occupied site of ξ and that of η can be large for typical η and ξ. So it is likely that a "naive" path from η to ξ will pass through an edge (λ, λ^x) with a configuration that has a single isolated site far to the left of the other occupied sites. This would cause $\tilde{\pi}^n(\lambda)c(\lambda, x)$ to be small and the sum in (3.2) to be larger than order n for that edge.

In fact, any method which fixes the sites in some order will potentially have an edge that changes around a large gap (i.e., a large vacant interval between two occupied sites) next to the boundary. To prevent this, we temporarily "fill the gap" with occupied sites (that exist in neither η nor ξ)

until the gap is no longer a problem. It seems reasonable to expect that the best way to fill the gap is to place particles according to renewal measure (conditioned on particles at the endpoints).

PROOF OF PROPOSITION 3.1. We begin by describing our method of (randomly) selecting a path from η to ξ. There are three cases to consider. In each case, the path is formally described and an illustrated example of a sample path is given. (See Figures 1 and 2; the second case does not have an associated figure as that case is similar to the first case.)

CASE 1. Each occupied site of η lies to the left of the occupied sites of ξ. In this case, let

$$x = \sup\{u < n: \eta(u) = 1\}$$

and

$$y = \inf\{u > -n: \xi(u) = 1\}.$$

By assumption, $x < y$. Let the occupied sites of η in $(-n, n)$ be $x_1 < x_2 < \cdots < x_j = x$. Let the occupied sites of ξ in $(-n, n)$ be $y = y_1 < y_2 < \cdots < y_k$. Now, to choose a path, we first use renewal measure (with β) to pick a sequence of points

$$x = z_0 < z_1 < \cdots < z_{r-1} < z_r = y$$

with probability

$$A(y - x)^{-1} \prod_{i=1}^{r} \beta(z_i - z_{i-1}).$$

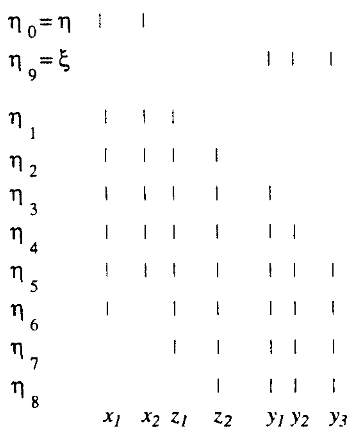

FIG. 1. *A realization of a path between two configurations of* $(-n, n)$; *the vertical lines indicate occupied sites. All the variables are as they appear in the description of Case* 1. *In particular, the sequence* z_i *is a realization of the sequence of renewal points between* $x = x_2$ *and* $y = y_1$, *conditioned on points at* x *and* y.

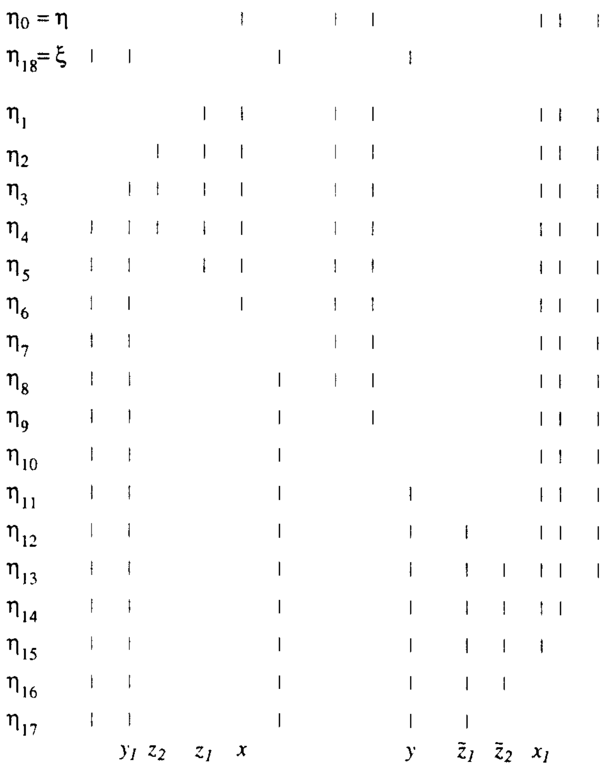

FIG. 2. *A realization of a path between two configurations on* $(-n, n)$; *the vertical lines indicate occupied sites. All the variables are as they appear in the description of Case* 3. *In particular, the sequence* z_i (*resp.* $\tilde{z}_i$) *is a realization of the sequence of renewal points between* y_1 *and* x (*resp.* y *and* x_1), *conditioned on points at* y_1 *and* x (*resp.* y *and* x_1). *In this realization, phase* (a) *ends with* η_6 *and phase* (b) *ends with* η_{11}.

Given this selection, the path $\eta = \eta_0, \eta_1, \dots, \eta_{2r+j+k-2} = \xi$ is as follows:

$$\eta_i = \begin{cases} \eta_{i-1} \cup \{z_i\}, & \text{if } 1 \le i \le r, \\ \eta_{i-1} \cup \{y_{i-r+1}\}, & \text{if } r+1 \le i \le r+k-1, \\ \eta_{i-1} \setminus \{x_{r+k+j-i}\}, & \text{if } r+k \le i \le r+k+j-1, \\ \eta_{i-1} \setminus \{z_{i+1-(r+k+j)}\}, & \text{if } r+k+j \le i \le r+k+j+r-2. \end{cases}$$

CASE 2. Each occupied site of η lies to the right of the occupied sites of ξ. Not surprisingly, the paths in this case are chosen in a manner similar to Case 1: A sequence of renewal points is added (from right to left) between the leftmost site of η and the rightmost site of ξ; then the sites of ξ are added (from right to left); then the sites of η are removed (from left to right); finally, the sequence of renewal points is removed (from right to left).

CASE 3. Some occupied sites of η lie to the left of some occupied sites of ξ *and* some occupied sites of η lie to the right of some occupied sites of ξ.

In this case, the construction of a path from η to ξ proceeds in three phases.

Phase a. This phase only takes place if there exists occupied sites of ξ that lie to the left of all occupied sites of η. In this phase, a sequence of renewal points is added (from right to left) to "move from leftmost site of η to those sites of ξ to its left." Then those sites of ξ are added (from right to left) and the sequence of renewal points is removed (from left to right). Specifically, let x be the leftmost occupied site of η and denote the occupied sites of ξ lying to the left of x by $y_s < \cdots < y_1 < x$. Randomly select a sequence of renewal points $y_1 = z_r < z_{r-1} < \cdots < z_1 < z_0 = x$ (as in Case 1). Then the configurations $\eta = \eta_0, \eta_1, \dots, \eta_{2r+s-2}$ that begin the random path from η to ξ are given by

$$\eta_i = \begin{cases} \eta_{i-1} \cup \{z_i\}, & \text{if } 1 \le i \le r, \\ \eta_{i-1} \cup \{y_{i-r+1}\}, & \text{if } r+1 \le i \le r+s-1, \\ \eta_{i-1} \setminus \{z_{2r+s-1-i}\}, & \text{if } r+s \le i \le 2r+s-2. \end{cases}$$

Phase b. In this phase, the sites from the leftmost site of η to the rightmost site of ξ are considered (from left to right) and are changed if necessary. Specifically, let x be the leftmost occupied site of η and let y be the rightmost occupied site of ξ. Notice that $x < y$ in this case. Let $x \le w_1 < \cdots < w_k \le y$ be all those sites $w \in [x, y]$ such that $\eta(w) \ne \xi(w)$. Let $\eta = \eta_0, \dots, \eta_K$ be the configurations in our random path already obtained from phase (a). Then the path continues with $\eta_{K+1}, \dots, \eta_{K+k}$ defined by $\eta_{K+i} = \eta_{K+i-1}^{w_i}$.

Phase c. Let $\eta = \eta_0, \dots, \eta_K$ be the configurations in our random path already obtained from phases (a) and (b). If $\eta_K = \xi$, then we are done and do not require this phase. If $\eta_K \ne \xi$, then there exist $y < x_1 < \cdots < x_k$ such that y is the rightmost occupied particle of ξ and $x_1 < \cdots < x_k$ are all those sites x such that $\eta_K(x) = 1$ and $\xi(x) = 0$; further, there are no sites where ξ is occupied and η_K is not. To continue our path, randomly select a sequence of renewal points (as before) $y = \tilde{z}_0 < \cdots < \tilde{z}_r = x_1$. Then our path ends with $\eta_{K+1}, \dots, \eta_{K+2r+k-2}$, where

$$\eta_{K+i} = \begin{cases} \eta_{K+i-1} \cup \{\tilde{z}_i\}, & \text{if } 1 \le i \le r-1, \\ \eta_{K+i-1} \setminus \{x_{r+k-i}\}, & \text{if } r \le i \le r+k-1, \\ \eta_{K+i-1} \setminus \{\tilde{z}_{2r+k-1-i}\}, & \text{if } r+k \le i \le 2r+k-2. \end{cases}$$

With the paths now specified, note that their lengths are indeed bounded uniformly by a constant multiple of n. So, as mentioned before, the task here is to show (3.2). In fact, the following reasoning reduces the supremum in (3.2) to only those (λ, x) with $\lambda(x) = 1$. For the moment, consider a second choice of paths formed from the above method with all notions of "left" and "right" (and, technically, of "$<$" and "$>$" as well) switched. The proof provided below for the given choice of paths [for $\lambda(x) = 1$ only] will naturally work for the second choice of paths. But the second choice of paths can also be described as stepping through the given choice of paths backwards (i.e., a given path from η to ξ corresponds to an alternative path from ξ to η). From this point of view, if a path from η to ξ uses the edge (λ, λ^x), then the alternative path from ξ to η uses (λ^x, λ). As reversibility tells us that $\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)\tilde{\pi}^n(\lambda)^{-1}c(\lambda, x)^{-1} = \tilde{\pi}^n(\xi)\tilde{\pi}^n(\eta)\tilde{\pi}^n(\lambda^x)^{-1}c(\lambda^x, x)^{-1}$, it is now clear that the supremum over $\lambda(x) = 1$ is the same as the supremum over $\lambda(x) = 0$.

We fix a λ and x with $\lambda(x) = 1$ and proceed to bound

$$\sum_{(\eta, \xi)} \frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)} P_{(\eta, \xi)}(\lambda, x), \tag{3.3}$$

noting that our bounds will not depend on the choice of λ. It is sufficient to separately consider the sum over (η, ξ) when using the edge (λ, λ^x) in Cases 1, 2 and 3 [and phases (a), (b), and (c)].

Case 1. If edge (λ, λ^x) is used in getting from η to ξ in Case 1, then there exists $y \in (x, n)$ such that

$$\xi(y) = 1 \quad \text{and} \quad \xi = 0 \text{ on } (-n, y)$$

and there exists $z \in (-n, y)$ such that

$$\eta(z) = 1 \quad \text{and} \quad \eta = 0 \text{ on } (z, n).$$

To analyze the quantity (3.3), fix $-n < z < y < n$, consider the sum over (η, ξ) corresponding to z and y as above, and sum over z and y.

We deal separately with $z < x$ and $x \geq z$. (See Figures 3 and 4.)

Subcase 1. If $z < x$, then denote the occupied sites of λ on $[x, y]$ by

$$x = x_0 < x_1 < \cdots < x_{r-1} < x_r = y$$

and denote the occupied sites of λ on $[y, n]$ by

$$y = y_0 < y_1 < \cdots < y_s = n.$$

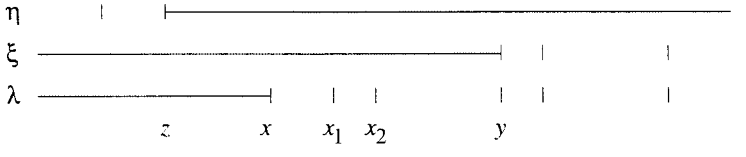

FIG. 3. *A pair (η, ξ) that uses the edge (λ, λ^x) in its corresponding path with probability $A(y - x)^{-1}A(x - z)\beta(x_1 - x)\beta(x_2 - x_1)\beta(y - x_2)$. The variables are as they appear in Subcase* 1 *of Case* 1. *The vertical lines indicate occupied sites, while the horizontal lines indicate a contribution to the right-hand side of* (3.4).

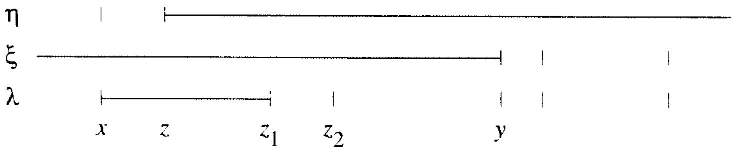

FIG. 4. *A pair* (η, ξ) *that uses the edge* (λ, λ^x) *in its corresponding path with probability* $A(y-z)^{-1}\beta(z_1 - z)\beta(z_2 - z_1)\beta(y - z_2)$. *The variables are as they appear in Subcase* 2 *of Case* 1. *The vertical lines indicate occupied sites, while the horizontal lines indicate a contribution to the right-hand side of* (3.5).

Notice that there is only one ξ in the sum and that its occupied sites are given by $\{y_i\}$. Then we have the following:

$$\tilde{\pi}^n(\xi) \asymp \bar{\beta}(n^\alpha)^{-2}\left(\beta(y+n)\prod_{i=1}^{s}\beta(y_i - y_{i-1})\right),$$

$$\begin{aligned}\tilde{\pi}^n(\eta(z) = 1, \eta = 0 \text{ on } (z, n))&\\ &\asymp \bar{\beta}(n^\alpha)^{-2}\left(\sum_{w=-n+n^\alpha}^{z}\beta(w+n)A(z-w)\beta(n-z)\right)\\ &\preccurlyeq \bar{\beta}(n^\alpha)^{-1}\beta(n-z),\end{aligned}$$

$$\tilde{\pi}^n(\lambda) \asymp \bar{\beta}(n^\alpha)^{-2}\left(\beta(x+n)\prod_{i=1}^{r}\beta(x_i - x_{i-1})\prod_{i=1}^{s}\beta(y_i - y_{i-1})\right)$$

and

$$\begin{aligned}P_{(\eta,\xi)}(\lambda, \lambda^x) &= A(y-z)^{-1}\left(A(x-z)\prod_{i=1}^{r}\beta(x_i - x_{i-1})\right)\\ &\asymp \prod_{i=1}^{r}\beta(x_i - x_{i-1}).\end{aligned}$$

So

$$\sum_{(\eta,\xi)}\frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)}P_{(\eta,\xi)}(\lambda, x) \preccurlyeq \frac{\beta(y+n)\beta(n-z)}{\beta(x+n)\bar{\beta}(n^\alpha)}, \tag{3.4}$$

where the sum runs over (η, ξ) associated with z and y. As $\beta(y+n) < \beta(x+n)$ and $\sum \beta(n-z) < \bar{\beta}(n^\alpha)$, we have our bound.

Subcase 2. If $x \le z$, then denote the occupied sites of λ on $[z, y]$ by

$$z = z_0 < z_1 < \cdots < z_{r-1} < z_r = y$$

and denote the occupied sites of λ on $[y, n]$ by

$$y = y_0 < y_1 < \cdots < y_s = n.$$

Notice that there is again only one ξ in the sum and that its occupied sites are given by $\{y_i\}$. Also notice that any η in the sum is not only fixed on $[z, n]$ as before, but is also fixed on $[x, z]$ since $\eta = \lambda$ on $[x, z]$. Denote the occupied sites of η on $[0, x]$ by

$$-n = x_0 < x_1 < \cdots < x_k = x.$$

Then we have the following:

$$\tilde{\pi}^n(\xi) \asymp \bar{\beta}(n^\alpha)^{-2}\left(\beta(y+n)\prod_{i=1}^{s}\beta(y_i - y_{i-1})\right),$$

$$\tilde{\pi}^n(\eta(z) = 1, \eta = 0 \text{ on } (z, n), \eta = \lambda \text{ on } [0, x])$$

$$\preccurlyeq \bar{\beta}(n^\alpha)^{-1}\prod_{i=1}^{k}\beta(x_i - x_{i-1})\beta(n - z),$$

$$\tilde{\pi}^n(\lambda) \asymp \bar{\beta}(n^\alpha)^{-2}\left(\prod_{i=1}^{k}\beta(x_i - x_{i-1})\beta(z_1 - x)\right.$$

$$\left.\times\prod_{i=2}^{r}\beta(z_i - z_{i-1})\prod_{i=1}^{s}\beta(y_i - y_{i-1})\right)$$

and

$$P_{(\eta,\xi)}(\lambda, \lambda^x) \asymp \prod_{i=1}^{r}\beta(z_i - z_{i-1}).$$

Case 2. The argument here is the same as in Case 1.

Case 3 (phase a). If edge (λ, λ^x) is used in getting from η to ξ in phase (a), then there exists $y > x$ such that

$$\eta(y) = 1, \qquad \eta = \lambda \text{ on } (y, n], \quad \text{and} \quad \eta = 0 \text{ on } (-n, y)$$

and

$$\xi = \lambda \text{ on } [-n, z] \quad \text{and} \quad \xi = 0 \text{ on } (z, y),$$

where $z = \sup\{i < x\colon \lambda(i) = 1\}$. See Figure 5. To analyze the quantity (3.3), fix $y \in (x, n)$, consider the sum over (η, ξ) corresponding to y as above, and sum over y. It suffices to show the following: the quantity (3.3), where the sum is over (η, ξ) corresponding to fixed y as above, is less than a constant.

Denote the occupied sites of λ on $[-n, z]$ by

$$-n = w_0 < w_1 < \cdots < w_s = z;$$

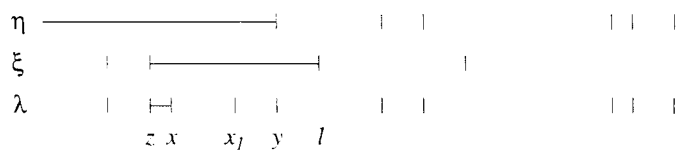

FIG. 5. *A pair* (η, ξ) *that uses the edge* (λ, λ^x) *in its corresponding path with probability* $A(y-z)^{-1}A(x-z)\beta(x_1 - x)\beta(y - x_1)$. *The variables are as they appear in phase* (a) *of Case* 3. *The variable l is the dummy variable for a sum in* (3.5). *The vertical lines indicate occupied sites, while the horizontal lines indicate a contribution to the right-hand side of* (3.5).

denote the occupied sites of λ on $[x, y]$ by

$$x = x_0 < x_1 < \cdots < x_r = y$$

and denote the occupied sites of λ on $[y, n]$ by

$$y = y_0 < y_1 < \cdots < y_k = n.$$

Then

$$\begin{aligned}
&\tilde{\pi}^n(\xi = \lambda \text{ on } [-n, z), \xi = 0 \text{ on } (z, y)) \\
&\qquad \preccurlyeq \bar{\beta}(n^\alpha)^{-1} \prod_{i=1}^{s} \beta(w_i - w_{i-1}) \sum_{l=y}^{\infty} \beta(l - z),
\end{aligned}$$

$$\tilde{\pi}^n(\eta) \asymp \bar{\beta}(n^\alpha)^{-2} \beta(y + n) \prod_{i=1}^{k} \beta(y_i - y_{i-1}),$$

$$\begin{aligned}
\tilde{\pi}^n(\lambda) \asymp \bar{\beta}(n^\alpha)^{-2} \Bigg(&\prod_{i=1}^{s} \beta(w_i - w_{i-1}) \beta(x - z) \\
&\times \prod_{i=2}^{r} \beta(x_i - x_{i-1}) \prod_{i=1}^{k} \beta(y_i - y_{i-1}) \Bigg)
\end{aligned}$$

and

$$P_{(\eta, \xi)}(\lambda, \lambda^x) \asymp \prod_{i=1}^{r} \beta(x_i - x_{i-1}).$$

So

$$\sum_{(\eta, \xi)} \frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)} P_{(\eta, \xi)}(\lambda, x) \preccurlyeq \frac{\beta(y + n)\sum_{l=y}^{\infty} \beta(l - z)}{\beta(x - z)\bar{\beta}(n^\alpha)}, \tag{3.5}$$

where the sum runs over (η, ξ) associated with y. As $\beta(y + n) < \beta(x - z)$ and $\sum_{l=y}^{\infty} \beta(l - z) < \bar{\beta}(n^\alpha)$, we have our constant bound.

Case 3 (phase b). If edge (λ, λ^x) is used in getting from η to ξ in phase (b), then

$$\begin{aligned}
\eta &= \lambda \quad \text{on } [x, n], \\
\xi &= \lambda \quad \text{on } [-n, x),
\end{aligned}$$

and ξ has at least one occupied site on (x, n). [Also notice that every possible path from η to ξ must use (λ, λ^x) in phase (b).] We seek to show that the quantity (3.3), where the sum is over (η, ξ) corresponding to λ and x as in this phase, is less than a constant multiple of n. This is the most standard phase and case and so the proof here follows a standard path argument technique. Since λ tells us what ξ looks like below x and what η looks like above x, we can encode (in an injective manner) the pair (η, ξ) into a single configuration ω, defined as follows:

$$\omega = \eta \text{ on } [-n, x) \quad \text{and} \quad \omega = \xi \text{ on } [x, n].$$

Then

$$\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi) = \frac{\beta(\hat{l}+r)\beta(l)}{\beta(l+r)\beta(\hat{l})}\tilde{\pi}^n(\lambda)\tilde{\pi}^n(\omega) \le \frac{\beta(l)}{\beta(l+r)}\tilde{\pi}^n(\lambda)\tilde{\pi}^n(\omega),$$

where

$$\hat{l} = x - \sup\{y < x: \xi(y) = 1\},$$
$$r_\omega = \inf\{y > x: \omega(y) = 1\} - x$$

and

$$l_\omega = x - \sup\{y < x: \omega(y) = 1\}.$$

So

$$\sum_{(\eta,\xi)} \frac{\tilde{\pi}^n(\eta)\tilde{\pi}^n(\xi)}{\tilde{\pi}^n(\lambda)} P_{(\eta,\xi)}(\lambda, x) \le \sum_\omega \tilde{\pi}^n(\omega)\frac{\beta(l_\omega)}{\beta(l_\omega + r_\omega)},$$

where the left-hand sum runs over (η, ξ) whose paths use (λ, λ^x) in phase (b) and where the right-hand sum runs over ω with at least one occupied site on (x, n). But

$$\begin{aligned}
\sum_\omega \tilde{\pi}^n(\omega)\frac{\beta(l)}{\beta(l+r)} &= \sum_{r=1}^{n-n^{\alpha-x}} \sum_{l=1}^{x+n-n^\alpha} \frac{\beta(l)}{\beta(l+r)}\tilde{\pi}^n(\lambda_\omega = l, r_\omega = r) \\
&\quad + \sum_{r=1}^{n-n^\alpha - x} \frac{\beta(x+n)}{\beta(x+n+r)}\tilde{\pi}^n(l_\omega = x+n, r_\omega = r) \\
&\preccurlyeq \sum_{r=1}^{n-n^\alpha - x} \sum_{l=1}^{x+n-n^\alpha} \frac{\beta(l)}{\beta(l+r)}\beta(l+r) \\
&\quad + \sum_{r=1}^{n-n^\alpha-x} \frac{\beta(x+n)}{\beta(x+n+r)}\frac{\beta(x+n+r)}{\bar{\beta}(n^\alpha)} \\
&\preccurlyeq \sum_{r=1}^{n-n^\alpha-x} 2 \\
&\preccurlyeq n.
\end{aligned}$$

Case 3 (phase c). The argument here is similar to that of phase (a). So (3.2) has been justified and Proposition 3.1 follows. □

This section concludes with a few corollaries. The first corollary is a direct application of typical spectral gap bounds on convergence rates. The other two corollaries adjust the first one to suit the needs of the application in Section 5; it is Corollary 3.5 that will be used in the proof of Theorem 1.1. In what follows, $\tilde{E}$ denotes expectation under the Markov chain $\tilde{X}$.

COROLLARY 3.3. *Let f be a cylinder function, that is, assume f depends on only finitely many sites. Then*

$$\int \tilde{\pi}^n(d\xi)\Big|\langle \tilde{\pi}^n, f\rangle - \tilde{E}^n_\xi f(\xi_{n^{2+\varepsilon}})\Big| \preccurlyeq \exp(-cn^\varepsilon)$$

where $c > 0$ is independent of n.

PROOF. We know that the spectral gap has the following property:

$$\int \tilde{\pi}^n(d\xi)\Big(\langle \tilde{\pi}^n f\rangle - \tilde{E}^n_\xi f(\xi_{n^{2+\varepsilon}})\Big)^2 \le \|f\|_2^2 \exp\Big(-\mathrm{Gap}(\tilde{X}^n)\, n^{2+\varepsilon}\Big).$$

By Proposition 3.1, we have

$$\int \tilde{\pi}^n(d\xi)\Big(\langle \tilde{\pi}^n f\rangle - \tilde{E}^n_\xi f(\xi_{n^{2+\varepsilon}})\Big)^2 \preccurlyeq \exp(-\bar{c}n^\varepsilon)$$

for some $\bar{c} > 0$ independent of n. The result now follows from Cauchy–Schwarz. □

Let $(\theta_t)_{t\in\mathbf{Z}}$ be the shift operators on $\{0,1\}^{\mathbf{Z}}$. For $f\colon \{0,1\}^{\mathbf{Z}} \to \mathbf{R}$, define $\theta_t f(\xi) = f(\theta_t \xi)$.

COROLLARY 3.4. *Let f be a cylinder function. Then for sufficiently large n, the chance under $\tilde{\pi}^n$ that there exists $|t| \le 2n/3$ such that*

$$\tilde{E}^n_\xi[\theta_t f(\xi_{n^{2+\varepsilon}})] \le \langle \tilde{\pi}^n, \theta_t f\rangle - \frac{1}{n}$$

is less than $K\exp(-cn^\varepsilon/2)$, where c is as in Corollary 3.3.

PROOF. The probability of the event in the corollary is bounded above by

$$\sum_{|t|\le 2n/3} \int \tilde{\pi}^n(d\xi) 1_{1/n \le \langle \tilde{\pi}^n, \theta_t f\rangle - \tilde{E}^n_\xi[\theta_t f(\xi_{n^{2+\varepsilon}})]}.$$

However,

$$\begin{aligned}
&\sum_{|t|\le 2n/3} \int \tilde{\pi}^n(d\xi) 1_{1/n \le \langle \tilde{\pi}^n, \theta_t f\rangle - \tilde{E}^n_\xi[\theta_t f(\xi_{n^{2+\varepsilon}})]} \\
&\quad \le n \sum_{|t|\le 2n/3} \int \tilde{\pi}^n(d\xi)\Big|\langle \tilde{\pi}^n, \theta_t f\rangle - \tilde{E}^n_\xi[\theta_t f(\xi_{n^{2+\varepsilon}})]\Big| \\
&\quad \preccurlyeq n \sum_{|t|\le 2n/3} \exp(-cn^\varepsilon) \\
&\quad \preccurlyeq \exp(-cn^\varepsilon/2).
\end{aligned}$$

□

The next corollary is the same as the previous one except that the probability is under ν_β instead of $\tilde{\pi}^n$. This creates a potential technical problem as $\tilde{E}_0^n(\cdot)$ has no meaning. However, the chance of this problem occurring is small and so we simply define $\tilde{E}_0^n(\cdot)$ arbitrarily, bound the event that it comes up, and avoid dealing with that event thereafter.

COROLLARY 3.5. *Let f be an increasing cylinder function. Let ξ on $(-n + n^\alpha, n - n^\alpha)$ be chosen according to ν_β. The probability that for some $|t| \le 2n/3$,*

$$\tilde{E}_\xi^n[\theta_t f(\xi_{n^{2+\varepsilon}})] \le \langle \tilde{\pi}^n, \theta_t f \rangle - 1/n$$

is less than K/n^ε (where K is independent of n).

PROOF. Let (ξ, η) be a coupling of ν_β (on $(-n, n)$) and π^n as in Proposition 2.4 and let B be the event with $\nu_\beta(B) \preccurlyeq n^{-1/2}$ such that $\xi \ge \eta$ for $\xi \notin B$. Let ζ be distributed according to $\tilde{\pi}^n$ and coupled with η so that $\eta = \zeta$ unless $\eta = \mathbf{0}$. So $\xi \ge \zeta$ outside of probability $P(B) + P(\eta = \mathbf{0})$. By attractiveness, the probability of the event in Corollary 3.5 is bounded by the probability of the event in Corollary 3.4 plus $P(B) + P(\eta = \mathbf{0})$. However, [by Lemma 2.1 and (2.4)] $P(B) + P(\eta = \mathbf{0}) \preccurlyeq n^{-1/2} + n^{-\varepsilon} \preccurlyeq n^{-\varepsilon}$. The result now follows using Corollary 3.4. □

4. Hitting the null configuration. As the section title suggests, the next proposition bounds the chance that $X_t^n = \mathbf{0}$ for $t < n^{2+\varepsilon}$. The reason for wanting to stay away from $\mathbf{0}$ is as mentioned before: it takes a long time to leave $\mathbf{0}$. But it is exactly that fact that we will use to prove that with huge probability we do not hit it. If we hit the null configuration in some interval, then a significant portion of that interval will be spent there; but the expected amount of time spent there is low. The proof below makes this precise.

PROPOSITION 4.1. *If $X_0^n =_D \pi^n$, the stationary distribution, then the probability that $X_t^n = \mathbf{0}$ for some $t \in [0, n^{2+\varepsilon}]$ is less than K/n^ε (where K is independent of n).*

PROOF. The mass assigned to $\mathbf{0}$ by the stationary distribution π^n is (by Lemma 2.1) less than $K\beta(2n)\overline{\beta}(n^\alpha)^{-2}$ (where K is independent of n). Consequently, the expected total amount of time spent by X^n at $\mathbf{0}$ during the time interval $[0, 2n^{2+\varepsilon}]$ is less than $Kn^{2+\varepsilon}\beta(2n)\overline{\beta}(n^\alpha)^{-2}$.

Let $\tau = \sum_{-n+n^\alpha+1}^{n-(n^\alpha+1)} \beta(l + n)\beta(n - l)\beta(2n)^{-1}$ be the total flip rate of X^n at the null configuration $\mathbf{0}$. Once X^n hits $\mathbf{0}$ it will remain there for an exponentially distributed amount of time with mean $1/\tau$. Consequently if X^n hits $\mathbf{0}$ in the time interval $[0, n^{2+\varepsilon}]$, then the conditioned expected amount of time spent at $\mathbf{0}$ in the time interval $[0, 2n^{2+\varepsilon}]$ is at least $C \min(n^{2+\varepsilon}, 1/\tau)$ (where C is independent of n). Therefore by Markov's inequality,

$$\begin{aligned} &P^{\pi^n}\big(X^n \text{ hits } \mathbf{0} \text{ in time interval } [0, n^{2+\varepsilon}]\big) \\ &\quad \preccurlyeq \frac{\beta(2n)}{\overline{\beta}(n^\alpha)^2} n^{2+\varepsilon}\left[\max\left(\frac{1}{n^{2+\varepsilon}}, \tau\right)\right] \\ &\quad \preccurlyeq \max\left(\frac{\beta(2n)}{\overline{\beta}(n^\alpha)^2}, \frac{n^{2+\varepsilon}}{\overline{\beta}(n^\alpha)^2} \sum_{l=-n+n^\alpha+1}^{n-n^\alpha-1} \beta(n+l)\beta(n-l)\right). \end{aligned}$$

We have already seen in (2.4) that

$$\frac{\beta(2n)}{\overline{\beta}(n^\alpha)^2} \preccurlyeq \frac{1}{n^\varepsilon}.$$

Meanwhile

$$\frac{n^{2+\varepsilon}}{\overline{\beta}(n^\alpha)^2} \sum_{l=-n+n^\alpha+1}^{n-n^\alpha-1} \beta(l+n)\beta(n-l) \preccurlyeq \frac{n^{2+\varepsilon}\beta(n)}{\overline{\beta}(n^\alpha)}.$$

If $L = \infty$, then by (2.4) and (2.3),

$$\frac{n^{2+\varepsilon}\beta(n)}{\overline{\beta}(n^\alpha)} \preccurlyeq n^2\overline{\beta}(n^\alpha) \preccurlyeq n^{-1000},$$

while if $L < \infty$, then by (2.1) and (2.2),

$$\frac{n^{2+\varepsilon}\beta(n)}{\overline{\beta}(n^\alpha)} \preccurlyeq n^{2+\varepsilon} n^{-(L-\delta)} n^{\alpha(L+\delta-1)} \preccurlyeq n^{-1/2},$$

using $\delta = (L-5)/3$ and $\varepsilon < 1/4$ for instance. □

COROLLARY 4.2. *If X_0^n is distributed as ν_β restricted to $(-n+n^\alpha, n-n^\alpha)$, then the probability that $X_t^n = \mathbf{0}$ for some t in $[0, n^{2+\varepsilon}]$ is less than K/n^ε.*

PROOF. This simply follows from Proposition 2.4 and the fact that $P_\xi(X_t^n$ hits $\mathbf{0}$ in $[0, n^{2+\varepsilon}))$ is a decreasing function of ξ. □

5. Proof of Theorem 1.1. This section contains the first of two applications of the work on $\tilde{X}^n$. Recall Theorem 1.1 appearing in the introduction (as well as the two subsequent paragraphs). The proof considers the event where the edge r_t has moved far to the right. Under this (increasing) event, various copies of X^n are started on various intervals of the integer line, relative to r_t. One copy—we don't know which one unless we condition on the location of r_t—will be centered near the origin. The starting configuration of this version of X^n will be very close to ν_β, since ν_β and the invariant measure for the process seen from the edge look arbitrarily alike far away from the edge [Schinazi (1992)]. So, Corollary 3.5 and Corollary 4.2 will imply the following: under "most" starting configurations, the expected behavior of this process at time $n^{2+\varepsilon}$ should be close to what we want, even when the starting configuration is conditioned on this copy of X^n being the relevant one.

The sketch given above ignores one key issue: Will the coupling hold long enough? We begin this section with a proposition that answers that question affirmatively. Proposition 5.1 considers the probability that there is a large gap of vacancies in the process ξ_s "close" to the rightmost particle $r_s = \sup\{x: \eta(x) = 1\}$. After the proof of the proposition (and a lemma it requires), we prepare for, and then provide, the proof of Theorem 1.1.

PROPOSITION 5.1. *Let γ, M be constants strictly greater than zero. Let $3/(2L) < \rho < 1/2$. The probability that in time interval $[0, t]$ that there exists a ξ_s-gap of size (at least) t^ρ intersecting the interval $[r_s - Mt^{1/2}, r_s - \gamma t^{1/2}]$ tends to zero as t tends to infinity.*

PROOF. *Case* 1. $L = \infty$. In this case we bound the probability that for some $s \in [0, t]$ there is a gap of size t^β in the interval $[r_s - Mt^{1/2}, r_s - \gamma t^{1/2}]$ by

$$\sum_{n \geq \rho \log_2(t)} P(A_{n,t}),$$

where

$$A_{n,t} = \Big\{\exists\ s \in [0, t],\ \xi_s \text{ has a gap with size in } [2^n, 2^{n+1}) \text{ intersecting } \big[r_s - Mt^{1/2}, r_s - \gamma t^{1/2}\big]\Big\}.$$

For fixed time s, the probability that a gap with size in $[2^n, 2^{n+1})$ intersects $[r_s - Mt^{1/2}, r_s - \gamma t^{1/2}]$ is less than the probability that such a gap intersects $[r_s - Mt^{1/2}, r_s]$. That probability is bounded by $Mt^{1/2} 2^n \beta(2^n)$ since renewal measure on the half line is invariant for the process seen from the edge [Schinazi (1992)]. Thus if

$$I^{n,t} = \int_0^{t+1} 1_{\text{there is a gap with size in } [2^n, 2^{n+1}) \text{ intersecting } [r_s - Mt^{1/2}, r_s]}\, ds,$$

then $E(I^{n,t}) \preccurlyeq t^{3/2} 2^n \beta(2^n)$.

Let τ be the infimum of times where a gap with size in $[2^n, 2^{n+1})$ intersects $[r_s - Mt^{1/2}, r_s]$. Then after τ such a gap will continue to exist at least until:

(a) there is a flip from 0 to 1 within the gap;
(b) there is a flip from 1 to 0 at boundary of gap;
(c) or r_s changes.

Now, the total flip rate for (a) is less than $2^{n+1}(\beta(1))^2/\beta(2)$, the total flip rate for (b) is 2 and the total flip rate for (c) is $1 + \sum \beta(m) = 2$. So the total flip rate is less than $C2^{n+1}$ for some C. We conclude $E[I^{n,t} | \tau \leq t] \succcurlyeq 2^{-(n+1)}$.

Thus by Markov's inequality and (2.3),

$$P(A_{n,t}) = P(\tau \le t) \preccurlyeq \frac{t^{3/2} 2^n \beta(2^n)}{2^{-(n+1)}} \preccurlyeq t^{3/2} 2^{-1000n}.$$

Case 2. $L < \infty$. We define

$$I_t = \int_0^{t+t^\rho} 1_{\xi_s \text{ has a gap of size } t^\rho/2 \text{ intersecting } [r_s - (M+\gamma)t^{1/2}, r_s]}\, ds.$$

Notice that, similarly to $I^{n,t}$ in the first case, $E(I^t) \preccurlyeq t^{3/2}\overline{\beta}(t^\rho)$. Let τ be the infimum of times where a gap of size t^ρ intersects $[r_s - Mt^{1/2}, r_s - \gamma t^{1/2}]$. Now on $\{\tau \le t\}$, ξ_τ has a gap of size t^ρ intersecting the interval $[r_\tau - Mt^{1/2}, r_\tau - \gamma t^{1/2}]$. For $s \ge \tau$, ξ_s will continue to have a gap of size $t^\rho/2$ intersecting $[r_s - (M+\gamma)t^{1/2}, r_s]$ at least until:

(a) sites in the gap become sufficiently occupied to reduce its size;
(b) or $|r_s - r_\tau| > \gamma t^{1/2}/4$

provided t is sufficiently large. We wish to analyze the chance that $\tau \le t$. We deal with possibility (b) first. Let B be the event $\{\sup\{|r_s - r_{s'}|: |s - s'| \le t^\rho, 0 \le s, s' \le t\} > \gamma t^{1/2}/4\}$. Then by the Brownian behavior of the right edge [Schinazi (1992)], $\lim_{t\to\infty} P(B) = 0$. Hence the expected time for possibility (b) to occur is greater than $t^\rho/2$. For possibility (a), we use the following lemma, which is proved below.

LEMMA 5.2. *Assume $L < \infty$. Let J_s be a stochastic process on intervals coupled with our nearest particle system in the following way: let $J_0 = (0, m)$ and if $\lim_{s \nearrow s_0} J_s = (u, v)$ and there is a birth in the particle system at $w \in (u, v)$, then J_s is equal to the larger of (w, v) and (u, w) (or a random choice if they are equal in size). Let σ be the expected time for the size of J to be smaller than $m/2$. Then*

$$E\sigma \succcurlyeq m.$$

Compare the event in (a) with σ in the lemma; clearly the expected time for possibility (a) to occur is at least $E\sigma$ where $m = t^\rho$. Now on $\{\tau \le t\}$, I^t is at least the time for (a) or (b) to occur. So

$$P(\tau \le t) \preccurlyeq \frac{E(I^t)}{t^\rho} \preccurlyeq t^{3/2} \frac{\overline{\beta}(t^\rho)}{t^\rho} \to 0$$

by (2.2) and since $3/(2L) < \rho$.

PROOF. Let J_s and σ be as in the statement of Lemma 5.2. Let $u_s \le v_s$ denote the endpoints of J_s. Then u_s jumps to $u_s + a$ at a rate $\beta(a)\beta(v_s - u_s - a)/\beta(v_s - u_s)$ for $a < (v_s - u_s)/2$ [and half that amount for $a = (v_s - u_s)/2$]. Let $K = \sup \beta(m)/\beta(2m)$. As $L < \infty$, $K < \infty$. Then

$$\frac{\beta(a)\beta(v_s - u_s - a)}{\beta(v_s - u_s)} \le K\beta(a)$$

for $a \le (v_s - u_s)/2$. That is, u_s is stochastically less than a process that increases at rate K by a random amount distributed according to $\beta(\cdot)$ (and similarly for $-v_s$). So the size of J_s is stochastically greater than a process that decreases at rate $2K$ by an amount given by $\beta(\cdot)$, from which the result follows. $\square$

In preparation for the proof of Theorem 1.1, fix $0 < \gamma < M < \infty$. Given t, choose n to be the largest integer such that $t^{1-a} > n^{2+\varepsilon}$ where $a = \varepsilon/4(\varepsilon + 2)$. The exact value of a is irrelevant; the point is that $n^{-(1+\varepsilon)}t^{1/2}$ converges to 0, while $n^{-(2+\varepsilon)}t$ converges to ∞. We introduce random intervals I_i, $i \in \mathbf{Z}$ (though we really only care about $i \ge 0$), defined by

$$I_i = \left[r_{t-n^{2+\varepsilon}} - (i+2)n, r_{t-n^{2+\varepsilon}} - in \right].$$

We are interested in the random interval that contains the origin "reasonably close" to its center.

Let $A(t, i, n) = \{i = \min\{j: |r_{t-n^{2+\varepsilon}} - (j+1)n| < 2n/3\}\}$. Fix $\delta > 0$ sufficiently small. Let

$$G = \left\{ i: in \in \left(\gamma t^{1/2}/2, 2Mt^{1/2} \right), P\left(A(t, i, n) \right) \ge \frac{\delta n}{t^{1/2}} \right\}.$$

From this definition, the following lemma is obvious.

LEMMA 5.3. *For sufficiently large t,*

$$P\left(r_{t-n^{2+\varepsilon}} \in \left(\gamma t^{1/2}, Mt^{1/2} \right), A(t, i, n) \text{ occurs for } i \notin G \right) \le (2M - \gamma/2)\delta.$$

This means we can neglect the event $\bigcup_{i \notin G} A(t, i, n)$ for $r_{t-n^{2+\varepsilon}} \in (\gamma t^{1/2}, Mt^{1/2})$. Its importance is that when conditioning on $A(t, i, n)$ for $i \in G$ the distribution of ξ_t around the origin cannot deviate too extremely from ν_β.

For $i \in G$ we define the Markov process $X_s^{i,n}$, $0 \le s \le n^{2+\varepsilon}$ on $\{0, 1\}^{I_i}$ as follows:

1. $X_0^{i,n}(r_{t-n^{2+\varepsilon}} - in) = X_0^{i,n}(r_{t-n^{2+\varepsilon}} - (i+2)n) = 1$;
2. $X_0^{i,n} \equiv 0$ within n^α of the endpoints of I_i;
3. on $(r_{t-n^{2+\varepsilon}} - (i+2)n + n^\alpha, r_{t-n^{2+\varepsilon}} - in - n^\alpha)$, $X_0^{i,n} \equiv \xi_{t-n^{2+\varepsilon}}$;
4. $X_s^{i,n}$ evolves on $[0, n^{2+\varepsilon}]$ according to the Harris construction of ξ_t over the time interval $[t - n^{2+\varepsilon}, t]$.

It follows that $X_s^{i,n} \le \xi_{t-n^{2+\varepsilon}+s}$ on I_i at least until there is a ξ gap of size n^α intersecting I_i. So $X_s^{i,n} \le \xi_{t-n^{2+\varepsilon}+s}$ on I_i at least until either:

1. $|r_{t-n^{2+\varepsilon}+v} - r_{t-n^{2+\varepsilon}}| \ge \gamma t^{1/2}/4$, or
2. $\xi_{t-n^{2+\varepsilon}+v}$ has a gap of size n^α intersecting

$$\left[r_{t-n^{2+\varepsilon}+v} - (2M + \gamma)t^{1/2}, r_{t-n^{2+\varepsilon}+v} - \gamma t^{1/2}/4 \right].$$

Just as we defined $\tilde{X}^n$ from X^n, we now define processes $\tilde{X}_s^{i,n}$ to be $X_s^{i,n}$ conditioned off $\mathbf{0}$. Of course, $\tilde{X}_0^{i,n}$ could itself be $\mathbf{0}$; that is, $\xi_{t-n^{2+\varepsilon}}$ could be null on $(r_t(i+2)n + n^\alpha, r_t - in - n^\alpha)$, which creates a potential problem similar to the one mentioned before Corollary 3.5. Again we arbitrarily define $\tilde{X}_s^{i,n}$ where $\tilde{X}_0^{i,n} = \mathbf{0}$ in the same way we did there. Notice that the distribution of $X_0^{i,n} = \tilde{X}_0^{i,n}$ is arbitrarily close to ν_β (despite the randomness of the interval on which it lives) and that the evolution of $X^{i,n}$ (resp. $\tilde{X}^{i,n}$) has the same behavior as X^n (resp. $\tilde{X}^n$).

PROOF OF THEOREM 1.1. It is sufficient to show the convergence with an increasing, cylinder function f satisfying $f(\mathbf{0}) = 0$. Using the Brownian behavior of the right edge, it is easy to see that

$$\begin{aligned}
\limsup_{t\to\infty} E[f(\xi_t)] &= \limsup_{t\to\infty} P\Big(r_{t-\sqrt{t}} < -k\sqrt{t}\Big) E\Big[f(\xi_t) | r_{t-\sqrt{t}} < -k\sqrt{t}\Big] \\
&\quad + P\Big(r_{t-\sqrt{t}} > -k\sqrt{t}\Big) E\Big[f(\xi_t) | r_{t-\sqrt{t}} > -k\sqrt{t}\Big] \\
&\le 0 + \tfrac{1}{2} \limsup_{t\to\infty} E\Big[f(\xi_t) | \xi_{t-\sqrt{t}}(x) = 1, \forall\, x\Big] \\
&= \tfrac{1}{2}\langle \nu_\beta, f\rangle.
\end{aligned}$$

So we seek to prove

$$\liminf_{t\to\infty} E[f(\xi_t)] \ge \tfrac{1}{2}\langle \nu_\beta, f\rangle.$$

By the above discussion

$$\begin{aligned}
Ef(\xi_t) &\ge \sum_{i\in G} E\Big[f(\xi_t) I_{A(t,i,n)}\Big] \\
&\ge \sum_{\substack{i\in G,\\ in\in(\gamma t^{1/2}, Mt^{1/2})}} E\Big[f\Big(X^{i,n}_{n^{2+\varepsilon}}\Big) | A(t,i,n)\Big] P(A(t,i,n)) \\
&\quad - \|f\|_\infty P\Bigg(\sup_{v\in[0,n^{2+\varepsilon}]} |r_{t-n^{2+\varepsilon}+v} - r_{t-n^{2+\varepsilon}}| \ge \gamma t^{1/2}/4\Bigg) \\
&\quad - \|f\|_\infty P\Big(\text{there exists a gap of size } n^\alpha \text{ intersecting} \\
&\qquad\qquad \big(r_s - (M+1)t^{1/2}, r_s - \gamma t^{1/2}/4\big) \text{ for some } s\in[0,t]\Big).
\end{aligned}$$

By Schinazi (1992), the second term converges to 0 as $t\to\infty$. For the third term, we will apply Proposition 5.1 to say that it converges to 0. The condition $3/(2L) < \rho$ in that proposition corresponds to the condition $\alpha/(2+\varepsilon) > 3/(2L)$ which is true if

$$\varepsilon < \frac{L-6}{2L+3}.$$

The condition $\varepsilon < (L(1)-6)/100 \wedge 1/4$ guarantees the above inequality and so Proposition 5.1 does apply.

It remains to examine $E[f(X^{i,n}_{n^{2+\varepsilon}}) | A(t,i,n)]$ for $i \in G$. By our definition of G, the conditioning event $A(t,i,n)$ has probability at least $\delta n/\sqrt{t}$. Thus by

Corollary 4.2, the probability that $X_s^{i,n}$ hits $\mathbf{0}$ in time $[0, n^{2+\varepsilon}]$ [conditioned on $A(t,i,n)$] is less than $Kn^{-\varepsilon}\sqrt{t}(\delta n)^{-1} = o(1)$ for large t. Therefore,

$$E\Big[f\Big(X_{n^{2+\varepsilon}}^{i,n}\Big)|A(t,i,n)\Big] - E\Big[f\Big(\tilde{X}_{n^{2+\varepsilon}}^{i,n}\Big)|A(t,i,n)\Big] = o(1) \tag{5.1}$$

as $t \to \infty$.

By Corollary 3.5, conditioning on $A(t,i,n)$,

$$\tilde{S}_{n^{2+\varepsilon}}^{i,n} f\Big(\tilde{X}_0^{i,n}\Big) \geq \langle \tilde{\pi}^n, f\rangle - 1/n,$$

outside of probability $Kn^{-\varepsilon}\sqrt{t}(\delta n)^{-1} = o(1)$ where $\tilde{S}^{i,n}$ is the semigroup for $\tilde{X}^{i,n}$ and $\tilde{\pi}^{i,n}$ is the stationary distribution for $\tilde{X}^{i,n}$. It follows from Proposition 2.3 that

$$E\Big[f\Big(\tilde{X}_{n^{2+\varepsilon}}^{i,n}\Big)|A(t,i,n)\Big] \geq \langle \nu_\beta, f\rangle(1+o(1))$$

and so by (5.1),

$$E\Big[f\Big(X_{n^{2+\varepsilon}}^{i,n}\Big)|A(t,i,n)\Big] \geq \langle \nu_\beta, f\rangle(1+o(1)).$$

Thus

$$\begin{aligned} Ef(\xi_t) &\geq \sum_{\substack{i\in G,\\ in\in(\gamma t^{1/2},\, M_t^{1/2})}} P(A(t,i,n))\langle \nu_\beta, f\rangle + o(1) \\ &\geq \Big(P\Big(r_{t-n^{2+\varepsilon}} \in \big(\gamma t^{1/2}, Mt^{1/2}\big)\Big) - (2M-\gamma/2)\delta\Big)\langle \nu_\beta, f\rangle + o(1), \end{aligned}$$

where the last step uses Lemma 5.3. This is enough by the Brownian behavior of the right edge and as δ, γ and M are arbitrary. □

6. Proof of Theorem 1.2. In this section we consider the NPS starting with initial configuration ξ_0 distributed according to ν_β. We wish to show Theorem 1.2: for almost all ξ_0 (with respect to ν_β) $S(t)\xi_0 \to_D \nu_\beta$. Since the space of continuous functions on $\{0,1\}^{\mathbf{Z}}$ is separable, it is sufficient to show that, for any fixed continuous function f, $S(t)f(\xi_0) \to_D \langle \nu_\beta, f\rangle$ for almost all ξ_0. Obviously it is sufficient to restrict attention to a fixed increasing cylinder function f. For such f, $S(t)f(\xi_0) \leq S(t)f(\mathbf{1})$ where $\mathbf{1}$ is the completely occupied configuration given by $\mathbf{1}(i) = 1$; therefore, as $S(t)f(\mathbf{1})$ converges to $\langle \nu_\beta, f\rangle$, we need only show that for almost all ξ_0,

$$\liminf_{t\to\infty} S(t)f(\xi_0) \geq \langle \nu_\beta, f\rangle. \tag{6.1}$$

In this section we consider $Y_t^n = X_t^{2^n}$ and $\tilde{Y}_t^n = \tilde{X}_t^{2^n}$. As before we have

$$\tilde{Y}_t^n(x) \leq \xi_t(x)$$

for $|x| \leq 2^n$ and for $t \leq \sigma_n \wedge \tau_n$, where

$$\tau_n = \inf\big\{t\colon \xi_t = 0 \text{ on either } (-2^n, -2^n + 2^{\alpha n}) \text{ or } (2^n - 2^{\alpha n}, 2^n)\big\}$$

and

$$\sigma_n = \inf\big\{t\colon Y_t^n = 0 \text{ on } (-2^n, 2^n)\big\}.$$

We now present two propositions. The first deals with the time τ_n, before which the coupling between Y^n and ξ will hold. The second deals with the time σ_n, before which Y^n and $\tilde{Y}^n$ are the same. They are analogous to earlier results and so one proof is merely sketched and the other is omitted.

PROPOSITION 6.1. *The random time* τ_n *satisfies*

$$P^{\nu_\beta}\big(\tau_n \le 2^{(2+\varepsilon)n}\big) \preccurlyeq 2^{-\varepsilon n}.$$

SKETCH OF PROOF OF PROPOSITION 6.1. First suppose $L < \infty$. (This proof follows the proof of Proposition 5.1 closely; the case $L = \infty$ follows it even more closely and so we do not address it here.) The probability under ν_β that either

$$J = \big(-2^n + (1/3)2^{\alpha n}, -2^n + (2/3)2^{\alpha n}\big)$$

or

$$J' = \big(2^n - (2/3)2^{\alpha n}, 2^n - (1/3)2^{\alpha n}\big)$$

is completely vacant is bounded by $k\sum_{x=-\infty}^{0}\sum_{i=2^{\alpha n}/3}^{\infty}\beta(i-x)$, which itself is bounded by $K2^{-(\alpha n)(L-2-\delta)}$ using (2.1) where $\delta > 0$ will be chosen later. Let I be the expected Lebesgue amount of time (under ν_β) in the time interval $[0, 2\cdot 2^{(2+\varepsilon)n}]$ that ξ_t spends with J or J' completely vacant. Then

$$E(I) \le 2K2^{(2+\varepsilon)n}2^{-(\alpha n)(L-2-\delta)}.$$

On the other hand, we can apply Lemma 5.2 to conclude

$$E\big[I|\tau_n \le 2^{(2+\varepsilon)n}\big] \succcurlyeq 2^{\alpha n}.$$

Hence

$$P^{\nu_\beta}\big(\tau_n \le 2^{(2+\varepsilon)n}\big) \preccurlyeq \frac{2^{(2+\varepsilon)n}2^{-(\alpha n)(L-2-\delta)}}{2^{\alpha n}}.$$

By choosing δ and ε appropriately [$\delta = (5-L)/2$ and $\varepsilon < (L-5)/4(2+L)$ will suffice] and recalling that $\alpha = 1/2 - \varepsilon$, we have

$$2 + \varepsilon - \alpha L + 1/2 + \delta/2 \le -\varepsilon,$$

which certainly implies

$$P^{\nu_\beta}\big(\tau_n \le 2^{(2+\varepsilon)n}\big) \preccurlyeq 2^{-\varepsilon n}. \qquad \square$$

The next proposition is similar to Proposition 4.1.

PROPOSITION 6.2. *The random time* σ_n *satisfies*

$$P^{\nu_\beta}\big(\sigma_n \le 2^{(2+\varepsilon)n}\big) \preccurlyeq 2^{-\varepsilon n}.$$

Just as the propositions in Sections 4 and 5 led to the proof of Theorem 1.1, the above propositions lead to the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Define $B_n \subset \{0,1\}^{\mathbf{Z}}$ by

$$B_n\big\{\xi\colon P^{\xi}\big(\sigma_n \wedge \tau_n \le 2^{(2+\varepsilon)n}\big) \ge 1/n\big\}.$$

By the preceding propositions, $\nu_\beta(B_n) \preccurlyeq n2^{-\varepsilon n}$. On the other hand, our coupling of ξ_t and $\tilde{Y}_t^n$ tells us that

$$S(t)f(\xi_0) \ge S_{\tilde{Y}^n}(t)f\big(\tilde{Y}_0^n\big) - 2\|f\|_\infty P^{\xi_0}\big(\tau_n \wedge \sigma_n \le 2^{(2+\varepsilon)n}\big),$$

where $S_{\tilde{Y}^n}$ denotes the semigroup for $\tilde{Y}^n$. So

$$\exists\ t,\ S(t)f(\xi_0) \le S_{\tilde{Y}^n}(t)f\big(\tilde{Y}_0^n\big) - 2\|f\|_\infty/n \Rightarrow \xi_0 \in B_n. \tag{6.2}$$

Now, define times t_i^n by

$$t_0^n = 2^{(2+\varepsilon/2)n}$$

and for $i = 1, 2, \dots, n(2^{(2+\varepsilon)n} - 2^{(2+\varepsilon/2)n})$,

$$t_i^n = t_{i-1}^n + 1/n.$$

Let $\pi_{\tilde{Y}^n} = \tilde{\pi}^{2^n}$ be the stationary distribution for $\tilde{Y}^n$. By Corollary 3.3 and since $\tilde{Y}_0^n(\xi_0)$ is stochastically larger under ν_β than under $\pi_{\tilde{Y}^n}$, we have that for each fixed t_i^n,

$$\nu_\beta\Big(S_{\tilde{Y}^n}(t_i^n)f\big(\tilde{Y}_0^n\big) \le \langle \pi_{\tilde{Y}^n}, f\rangle - 1/n\Big) \le kn\exp(-c2^{\varepsilon n}), \tag{6.3}$$

where k is independent of n and i.

Therefore, by (6.2) and (6.3) (and letting $a = 2\|f\|_\infty + 1$),

$$\begin{aligned}
&\nu_\beta\big(\exists\ t_i^n,\ S(t_i^n)f(\xi) \le \langle \pi_{\tilde{Y}^n}, f\rangle - a/n\big)\\
&\quad\preccurlyeq P(B_n) + \sum_{i=0}^{n(2^{(2+\varepsilon)n} - 2^{(2+\varepsilon/2)n})} n\exp(-c2^{\varepsilon n})\\
&\quad\preccurlyeq n2^{-\varepsilon n} + n^2 2^{(2+\varepsilon)n} e^{-c2^{\varepsilon n}}.
\end{aligned}$$

Now, choose $t \in [2^{(2+\varepsilon/2)n}, 2^{(2+\varepsilon)n}]$. Then for some i, $t \in [t_i^n, t^n + 1/n]$ and

$$\begin{aligned}
|S(t_i^n)f(\xi_0) - S(t)f(\xi_0)| &\le \left|\int_{t_i^n}^{t} S(s)\Omega f(\xi_0)\,ds\right|\\
&\preccurlyeq 1/n,
\end{aligned}$$

where Ω is the generator for $S(t)$; notice that Ωf is bounded as f depends on only finitely many sites. Therefore

$$S(t)f(\xi_0) \ge \langle \pi_{\tilde{Y}^n}, f\rangle - c/n \quad \text{for each } t \in \big[2^{(2+\varepsilon/2)n}, 2^{(2+\varepsilon)n}\big]$$

outside of ν_β probability $kn2^{-\varepsilon n} + kn^2 2^{(2+\varepsilon)n}\exp(-c2^{\varepsilon n})$ (where k is some constant independent of n and $c = 2\|f\|_\infty + 2$). Since $\lim_{n\to\infty}\langle \pi_{\tilde{Y}^n}, f\rangle = \langle \nu_\beta, f\rangle$, (6.1) now follows from Borel–Cantelli. □

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